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On the group-homological description of the second Johnson homomorphism

Yuji YOKOMIZO

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Abstract. The Johnson homomorphisms τ_k ($k \geq 1$) give abelian quotients of a series of certain subgroups of the mapping class group of a surface. Morita constructed the refinement $\widetilde{\tau}_k$ of τ_k in terms of group homology. In this paper, we describe $\widetilde{\tau}_2$ explicitly and show that the reduction of $\widetilde{\tau}_2$ to τ_2 does not lose any informations.

Key words: mapping class group; Johnson homomorphism; group homology.

1. Introduction

Let $\mathcal{M}_{g,1}$ be the mapping class group of a compact oriented surface $\Sigma_{g,1}$ of genus $g \geq 2$ with one boundary component. To investigate the structure of the Torelli group $\mathcal{I}_{g,1}$, which is the kernel of the classical representation

$$\mathcal{M}_{g,1} \longrightarrow \mathrm{Sp}(2g;\mathbb{Z}),$$

Johnson defined a surjective homomorphism

$$\tau_1 : \mathcal{I}_{g,1} \longrightarrow \Lambda^3 H_1(\Sigma_{g,1};\mathbb{Z})$$

in [2]. Moreover, he generalized it to a series of homomorphisms $\{\tau_k\}$ such that τ_{k+1} is defind on the kernel of τ_k and the target of τ_k is an abelian group denoted by $\mathcal{L}_{k+1} \otimes H$ for each k (see [3]).

As a clue to determine the image of τ_k, Morita constructed a refinement $\widetilde{\tau}_k$ of the Johnson homomorphism in terms of group homology. According to his work [6], the target of $\widetilde{\tau}_k$ is the third homology $H_3(N_k)$ of a nilpotent group N_k and there is an exact sequence

$$H_3(N_k) \longrightarrow \mathcal{L}_{k+1} \otimes H \longrightarrow \mathcal{L}_{k+2} \longrightarrow 0,$$

where the composition of $\widetilde{\tau}_k$ with the first map is equal to τ_k. This implies that $\mathrm{Im}\,\tau_k$ is included in the kernel of the projection $\mathcal{L}_{k+1} \otimes H \to \mathcal{L}_{k+2}$. It is a natural question to ask whether the reduction of $\widetilde{\tau}_k$ to τ_k lose any

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informations about the mapping class group or not. If $k = 1$, the answer is easily obtained from Johnson's fundamental work that $\operatorname{Im}\tau_1 = \Lambda^3 H = H_3(N_1)$. In this paper, describing $\widetilde{\tau}_2$ explicitly, we give an answer for $k = 2$. Actually, we see that the reduction $\operatorname{Im}\widetilde{\tau}_2 \to \operatorname{Im}\tau_2$ is an isomorphism.

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2. Definitions

First, we define the Johnson homomorphism introduced in [2, 3]. We write Γ_0 for the fundamental group $\pi_1(\Sigma_{g,1})$ of $\Sigma_{g,1}$ and H for the integral homology group $H_1(\Sigma_{g,1};\mathbb{Z})$. Let N_k be the kth nilpotent quotient Γ_0/Γ_k where Γ_k is inductively defined by $\Gamma_k = [\Gamma_{k-1}, \Gamma_0]$, and let $\mathcal{L}_k$ be the homogeneous part of degree k in the free Lie algebra $\mathcal{L}$ on H over $\mathbb{Z}$. Recall that the isomorphism $\Gamma_k/\Gamma_{k+1} \cong \mathcal{L}_{k+1}$ gives a central extension $0 \to \mathcal{L}_{k+1} \to N_{k+1} \to N_k \to 1$. If we write $\mathcal{M}(k)$ for the subgroup of $\mathcal{M}_{g,1}$ consisting of all the elements which act on N_k trivially, then we can define the kth Johnson homomorphism τ_k as follows. Take a lift $\eta \in N_{k+1}$ of $h \in H$. For each $\varphi \in \mathcal{M}(k)$, the correspondence $H \ni h \mapsto \varphi(\eta)\eta^{-1} \in \mathcal{L}_{k+1} \subset N_{k+1}$ gives a well-defined homomorphism, that is, an element of $\operatorname{Hom}(H, \mathcal{L}_{k+1})$, which is isomorphic to $\mathcal{L}_{k+1} \otimes H$ by the Poincaré duality. Then we obtain a map

$$\tau_k : \mathcal{M}(k) \longrightarrow \mathcal{L}_{k+1} \otimes H$$

and indeed this is a homomorphism and commutes with the action of $\mathcal{M}_{g,1}$.

Next, we summarize Morita's construction of the refinement

$$\widetilde{\tau}_k : \mathcal{M}(k) \longrightarrow H_3(N_k)$$

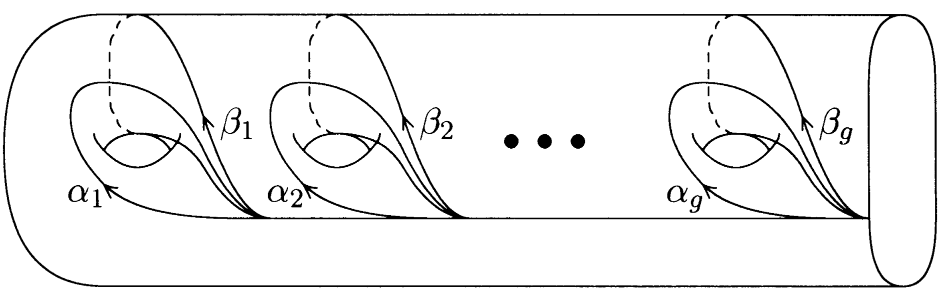

Fig. 1.

of the Johnson homomorphism, which is reduced to the original τ_k under a natural homomorphism $H_3(N_k) \to \mathcal{L}_{k+1} \otimes H$ obtained from a spectral sequence. See [6] for details. Take $2g$ elements $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ as in Figure 1, generating Γ_0 freely. We write a_i, b_i for the homology classes of α_i, β_i respectively. Let $\zeta \in \Gamma_0$ be the product $\prod_{i=1}^{g}[\alpha_i, \beta_i]$ of the commutators. Since the first homology $H_1(\Gamma_0)$ is naturally isomorphic to the abelianization of Γ_0, the 1-cycle $-(\zeta)$ is a 1-boundary and there exists a 2-chain σ_0 such that $\partial\sigma_0 = -(\zeta)$. In [6], Morita gave an explicit formula defining such a 2-chain as

$$\sigma_0 = \sum_{i=1}^{g}\Big\{(\alpha_i, \beta_i) - ([\alpha_i, \beta_i]\beta_i, \alpha_i) - ([\alpha_i, \beta_i], \beta_i)\Big\} + \sum_{i=1}^{g-1}\Big(\prod_{j=1}^{i}[\alpha_j, \beta_j], [\alpha_{i+1}, \beta_{i+1}]\Big).$$

For each $\varphi \in \mathcal{M}_{g,1}$, the difference $\sigma_0 - \varphi_*\sigma_0$ is a 2-cycle of Γ_0, because ζ represents the homotopy class of a simple closed curve parallel to the boundary of $\Sigma_{g,1}$. As is well known, the homology of the free group is always trivial except for degree 0 and 1 and hence there exists a 3-chain c_φ such that $\partial c_\varphi = \sigma_0 - \varphi_*\sigma_0$. We write $\bar{c}_\varphi$ for the image of c_φ in $C_3(N_k)$. If φ is an element of $\mathcal{M}(k)$, then $\bar{c}_\varphi$ is a 3-cycle on N_k. Now we obtain a well-defined homomorphism $\tilde{\tau}_k$ by the correspondence $\mathcal{M}(k) \ni \varphi \mapsto [\bar{c}_\varphi] \in H_3(N_k)$.

To describe the reduction of the refinement $\tilde{\tau}_k$ to the original τ_k, we consider the Hochschild-Serre spectral sequence $\{E^r_{p,q}\}$ for the homology of the central extension $0 \to \mathcal{L}_{k+1} \to N_{k+1} \to N_k \to 1$. More explicitly, $\{E^r_{p,q}\}$ is the one associated to the increasing filtration C_* defined by $C^j_n =$ (a submodule of $C_n(N_{k+1})$ generated by n-chains $(\eta_1, \ldots, \eta_n)$ where at least $n - j$ of the elements η_i belong to $\mathcal{L}_{k+1} \subset N_{k+1}$). If we define

$$C^r_{p,q} = \{c \in C^p_{p+q} \mid \partial c \in C^{p-r}_{p+q-1}\},$$

then

$$E^r_{p,q} = C^r_{p,q}/(C^{r-1}_{p-1,q+1} + \partial C^{r-1}_{p+r-1,q-r+2})$$

and the differential $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ is induced by the boundary operator ∂ of the chain complex on N_{k+1}. Furthermore, we have

$$E^2_{p,q} = H_p(N_k) \otimes H_q(\mathcal{L}_{k+1}),$$

$$\bigoplus_{p+q=n} E^\infty_{p,q} = H_n(N_{k+1}).$$

Now we consider the differential $d^2 : E^2_{3,0} = H_3(N_k) \to E^2_{1,1} = \mathcal{L}_{k+1} \otimes H$. According to [6], the composition $d^2 \circ \widetilde{\tau}_k$ coincides with τ_k.

3. An answer to the question

Johnson proved in [4] that $\mathcal{M}(2) = \mathcal{K}_{g,1}$, where $\mathcal{K}_{g,1}$ denotes the subgroup of $\mathcal{M}_{g,1}$ which is generated by all the Dehn twists along separating simple closed curves. Now we consider the second Johnson homomorphism

$$\tau_2 : \mathcal{K}_{g,1} \longrightarrow \mathcal{L}_3 \otimes H$$

and its refinement

$$\widetilde{\tau}_2 : \mathcal{K}_{g,1} \longrightarrow H_3(N_2).$$

We naturally identify $\mathcal{L}_2$ with $\Lambda^2 H$ by the correspondence $[a, b] \mapsto a \wedge b$ and $\mathcal{L}_3$ with $\Lambda^2 H \otimes H/\Lambda^3 H$ by the surjective homomorphism $\Lambda^2 H \otimes H \to \mathcal{L}_3$ given by $(a \wedge b) \otimes c \mapsto [[a, b], c]$ with the kernel $\Lambda^3 H$. Let T be the symmetric power $S^2\Lambda^2 H$ included in $\Lambda^2 H \otimes \Lambda^2 H \subset \Lambda^2 H \otimes H^2$ and let $\overline{T}$ be its image under the projection $\Lambda^2 H \otimes H^2 \to \Lambda^2 H \otimes H^2/\Lambda^3 H \otimes H = \mathcal{L}_3 \otimes H$. In [5], Morita proved that $\operatorname{Im} \tau_2$ is a submodule of $\overline{T}$ of index a power of 2. On the other hand, the target of $\widetilde{\tau}_2$ is

$$H_3(N_2) = \bigoplus_{p+q=3} E^\infty_{p,q},$$

where $E^\infty_{p,q}$ is the E^∞-term of the Hochschild-Serre spectral sequence $\{E^r_{p,q}\}$ for the homology of the central extension $0 \to \mathcal{L}_2 \to N_2 \to H \to 1$.

Lemma 1 *$E^\infty_{2,1}$ is isomorphic to $\overline{T}$.*

Proof. Since the differential $d^2 : E^2_{2,1} = \Lambda^2 H \otimes \Lambda^2 H \to E^2_{0,2} = \Lambda^2\Lambda^2 H$ is the natural surjection with the kernel $S^2\Lambda^2 H$, it suffices to show that

$$\operatorname{Im}(d^2 : E^2_{4,0} \to E^2_{2,1}) = S^2\Lambda^2 H \cap \Lambda^3 H \otimes H$$

as a submodule of $S^2\Lambda^2 H \subset \Lambda^2 H \otimes H^2$.

Now we compute $d^2 : E^2_{4,0} = \Lambda^4 H \to E^2_{2,1} = \Lambda^2 H \otimes \Lambda^2 H$, which is given by

$$\begin{aligned}\Lambda^4 H &\cong H_4(H) \cong \{c \in C_4(N_2) \mid \partial c \in C_3^2\}/\sim \\ &\xrightarrow{\partial} \{c \in C_3^2 \mid \partial c \in C_2(\mathcal{L}_2)\}/\sim \cong H_1(\mathcal{L}_2) \otimes H_2(H) \cong \Lambda^2 H \otimes \Lambda^2 H.\end{aligned}$$

For each element $h_1 \wedge h_2 \wedge h_3 \wedge h_4 \in \Lambda^4 H$, we put

$$c = \sum_\sigma \operatorname{sgn}\sigma(\eta_{\sigma(1)}, \eta_{\sigma(2)}, \eta_{\sigma(3)}, \eta_{\sigma(4)}) \in C_4(N_2),$$

where $\eta_i \in N_2$ is a lift of $h_i \in H$. Although its boundary

$$\begin{aligned}\partial c = \sum_{\sigma:\,\text{even}} \Big\{&-(\eta_{\sigma(1)}\eta_{\sigma(2)}, \eta_{\sigma(3)}, \eta_{\sigma(4)}) + (\eta_{\sigma(2)}\eta_{\sigma(1)}, \eta_{\sigma(3)}, \eta_{\sigma(4)}) \\ &+ (\eta_{\sigma(1)}, \eta_{\sigma(2)}\eta_{\sigma(3)}, \eta_{\sigma(4)}) - (\eta_{\sigma(1)}, \eta_{\sigma(3)}\eta_{\sigma(2)}, \eta_{\sigma(4)}) \\ &- (\eta_{\sigma(1)}, \eta_{\sigma(2)}, \eta_{\sigma(3)}\eta_{\sigma(4)}) + (\eta_{\sigma(1)}, \eta_{\sigma(2)}, \eta_{\sigma(4)}\eta_{\sigma(3)})\Big\}\end{aligned}$$

does not belong to C_3^2, we can modify this chain as

$$\begin{aligned}c' = c + \sum_{\sigma:\,\text{even}} \Big\{&-([\eta_{\sigma(1)}, \eta_{\sigma(2)}], \eta_{\sigma(2)}\eta_{\sigma(1)}, \eta_{\sigma(3)}, \eta_{\sigma(4)}) \\ &+ ([\eta_{\sigma(2)}, \eta_{\sigma(3)}], \eta_{\sigma(1)}, \eta_{\sigma(3)}\eta_{\sigma(2)}, \eta_{\sigma(4)}) \\ &- (\eta_{\sigma(1)}, [\eta_{\sigma(2)}, \eta_{\sigma(3)}], \eta_{\sigma(3)}\eta_{\sigma(2)}, \eta_{\sigma(4)}) \\ &+ (\eta_{\sigma(1)}, \eta_{\sigma(2)}, \eta_{\sigma(3)}\eta_{\sigma(4)}, [\eta_{\sigma(4)}, \eta_{\sigma(3)}])\Big\}\end{aligned}$$

so that

$$\begin{aligned}\partial c' = \sum_{\sigma:\,\text{even}} \Big\{&-([\eta_{\sigma(1)}, \eta_{\sigma(2)}], \eta_{\sigma(2)}\eta_{\sigma(1)}\eta_{\sigma(3)}, \eta_{\sigma(4)}) \\ &+ ([\eta_{\sigma(1)}, \eta_{\sigma(2)}], \eta_{\sigma(2)}\eta_{\sigma(1)}, \eta_{\sigma(3)}\eta_{\sigma(4)}) \\ &- ([\eta_{\sigma(1)}, \eta_{\sigma(2)}], \eta_{\sigma(2)}\eta_{\sigma(1)}, \eta_{\sigma(3)}) \\ &+ ([\eta_{\sigma(2)}, \eta_{\sigma(3)}], \eta_{\sigma(1)}\eta_{\sigma(3)}\eta_{\sigma(2)}, \eta_{\sigma(4)}) \\ &- ([\eta_{\sigma(2)}, \eta_{\sigma(3)}], \eta_{\sigma(1)}, \eta_{\sigma(3)}\eta_{\sigma(2)}\eta_{\sigma(4)}) \\ &+ ([\eta_{\sigma(2)}, \eta_{\sigma(3)}], \eta_{\sigma(1)}, \eta_{\sigma(3)}\eta_{\sigma(2)}) \\ &- ([\eta_{\sigma(2)}, \eta_{\sigma(3)}], \eta_{\sigma(3)}\eta_{\sigma(2)}, \eta_{\sigma(4)}) \\ &+ (\eta_{\sigma(1)}, [\eta_{\sigma(2)}, \eta_{\sigma(3)}], \eta_{\sigma(3)}\eta_{\sigma(2)}\eta_{\sigma(4)}) \\ &- (\eta_{\sigma(1)}, [\eta_{\sigma(2)}, \eta_{\sigma(3)}], \eta_{\sigma(3)}\eta_{\sigma(2)})\end{aligned}$$

$$
\begin{aligned}
&+ \left(\eta_{\sigma(2)}, \eta_{\sigma(3)}\eta_{\sigma(4)}, [\eta_{\sigma(4)}, \eta_{\sigma(3)}]\right)\\
&- \left(\eta_{\sigma(1)}\eta_{\sigma(2)}, \eta_{\sigma(3)}\eta_{\sigma(4)}, [\eta_{\sigma(4)}, \eta_{\sigma(3)}]\right)\\
&+ \left(\eta_{\sigma(1)}, \eta_{\sigma(2)}\eta_{\sigma(3)}\eta_{\sigma(4)}, [\eta_{\sigma(4)}, \eta_{\sigma(3)}]\right)\Big\} \in C_3^2
\end{aligned}
$$

and $c = c'$ as a cycle on H. Since the isomorphism $\{c \in C_3^2 \mid \partial c \in C_2(\mathcal{L}_2)\}/\sim \,\cong \Lambda^2 H \otimes \Lambda^2 H \subset \Lambda^2 H \otimes H^2$ is given by the correspondence

$$
C_3^2 \ni (\alpha, \beta, \gamma) \mapsto \begin{cases} \alpha \otimes [\beta] \otimes [\gamma] & (\text{if } \alpha \in \mathcal{L}_2) \\ 0 & (\text{otherwise}), \end{cases}
$$

the image of $\partial c'$ in $\Lambda^2 H \otimes \Lambda^2 H$ is

$$
\begin{aligned}
&- \sum_{\sigma:\,\text{even}} h_{\sigma(1)} \wedge h_{\sigma(2)} \otimes h_{\sigma(3)} \otimes h_{\sigma(4)}\\
&\qquad = - \sum_{\substack{\sigma:\,\text{even}\\ \sigma(1)<\sigma(2)}} h_{\sigma(1)} \wedge h_{\sigma(2)} \otimes h_{\sigma(3)} \wedge h_{\sigma(4)}
\end{aligned}
$$

and therefore $\operatorname{Im}(d^2 : E_{4,0}^2 \to E_{2,1}^2)$ is generated by elements of this form, which also generate $S^2\Lambda^2 H \cap \Lambda^3 H \otimes H$. This completes the proof. □

Remark The fact that the differential $d^2 : E_{2,2}^2 \to E_{0,3}^2$ is the natural surjection implies that $E_{0,3}^\infty = 0$. We can see also that $E_{3,0}^\infty = 0$ as follows. Consider the first Johnson homomorphism

$$
\tau_1 : \mathcal{I}_{g,1} \longrightarrow \Lambda^2 H \otimes H
$$

and its refinement

$$
\widetilde{\tau}_1 : \mathcal{I}_{g,1} \longrightarrow \Lambda^3 H.
$$

Since $\operatorname{Im} \tau_1 = \Lambda^3 H$ (see [2]), the image of the differential $d^2 : E_{3,0}^2 = \Lambda^3 H \to E_{1,1}^2 = \Lambda^2 H \otimes H$, which satisfies $d^2 \circ \widetilde{\tau}_1 = \tau_1$, is $\Lambda^3 H$. So this differential is injective and hence $E_{3,0}^\infty = 0$. Thus we can write

$$
H_3(N_2) = E_{2,1}^\infty \oplus E_{1,2}^\infty.
$$

Here the latter term $E_{1,2}^\infty$ is not trivial. Actually, we can estimate the rank of $E_{1,2}^\infty$ as

$$
\operatorname{rank} E_{1,2}^\infty \geq \operatorname{rank} E_{1,2}^2 - \operatorname{rank} E_{3,1}^2 - \operatorname{rank} E_{4,0}^2
$$

$$\begin{aligned}&= \operatorname{rank} \Lambda^2\Lambda^2 H \otimes H - \operatorname{rank} \Lambda^2 H \otimes \Lambda^3 H - \operatorname{rank} \Lambda^4 H \\ &= \frac{1}{6} g(2g-1)(4g^3+4g^2-5g-3) \\ &\geq 35\end{aligned}$$

for all $g \geq 2$.

Lemma 2 $\operatorname{Im} \widetilde{\tau}_2$ *is included in* $E^{\infty}_{2,1}$.

Proof. $E^{\infty}_{2,1}$ is generated by homology classes of 3-cycles $\sum(\alpha_i, \beta_i, \gamma_i)$ on N_2 such that exactly one of the elements α_i, β_i and γ_i belongs to $\mathcal{L}_2$ for each i. We compute $\overline{c}_{\varphi}$ explicitly and show that it is homologous to a cycle of above form for each $\varphi \in \mathcal{K}_{g,1}$. Johnson proved in [1] that $\mathcal{K}_{g,1}$ is generated by all the Dehn twists along separating simple closed curves of genus 1 and 2. Hence we have only to prove it for these twists. Moreover, since we can replace α_i, β_i appearing in the definition of σ_0 with $f_*\alpha_i$, $f_*\beta_i$ for each φ which is a twist along a separating simple closed curve γ of genus k ($k = 1, 2$) where f is a diffeomorphism on $\Sigma_{g,1}$ such that $f_*\gamma_k = \gamma$ if γ_1 and γ_2 are defined as in Figure 2, it suffices to check for only two elements φ_1, φ_2 which are twists along γ_1, γ_2 respectively. Indeed, we can easily see that

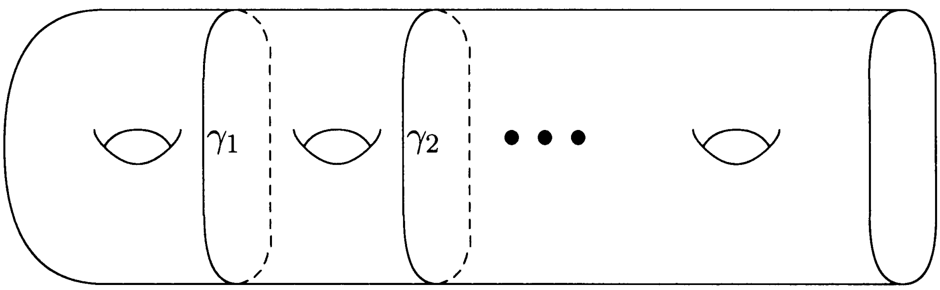

Fig. 2.

$$\begin{aligned}\overline{c}_{\varphi_1} &= -(\zeta_1, \zeta_1^{-1}\alpha_1, \beta_1) + (\zeta_1, \beta_1, \alpha_1) \\ &\quad -(\beta_1, \zeta_1, \zeta_1^{-1}\alpha_1) + (\zeta_1^{-1}\alpha_1, \zeta_1, \beta_1) \\ &\quad -(\zeta_1^{-1}\alpha_1, \beta_1, \zeta_1) + (\beta_1\zeta_1, \zeta_1^{-1}\alpha_1, \zeta_1), \\ \overline{c}_{\varphi_2} &= -(\zeta_1\zeta_2, \alpha_1, \beta_1) + (\zeta_1\zeta_2, \beta_1, \alpha_1) - (\zeta_1\zeta_2, \alpha_2, \beta_2) + (\zeta_1\zeta_2, \beta_2, \alpha_2) \\ &\quad -(\beta_1, \zeta_1\zeta_2, \alpha_1) + (\alpha_1, \zeta_1\zeta_2, \beta_1) - (\beta_2, \zeta_1\zeta_2, \alpha_2) + (\alpha_2, \zeta_1\zeta_2, \beta_2) \\ &\quad -(\alpha_1, \beta_1, \zeta_1\zeta_2) + (\beta_1, \alpha_1, \zeta_1\zeta_2) - (\alpha_2, \beta_2, \zeta_1\zeta_2) + (\beta_2, \alpha_2, \zeta_1\zeta_2)\end{aligned}$$

$$\begin{aligned}
&+(\alpha_2\beta_2,\zeta_1,\beta_1\alpha_1)-(\beta_2\alpha_2,\zeta_1\zeta_2,\beta_1\alpha_1)+(\beta_2\alpha_2,\zeta_2,\alpha_1\beta_1)\\
&+(\alpha_1\beta_1,\zeta_2,\beta_2\alpha_2)-(\beta_1\alpha_1,\zeta_1\zeta_2,\beta_2\alpha_2)+(\beta_1\alpha_1,\zeta_1,\alpha_2\beta_2)\\
&-(\zeta_1\alpha_1\beta_1,\zeta_2,\alpha_1\beta_1)+(\alpha_1\beta_1,\zeta_1\zeta_2,\alpha_1\beta_1)-(\alpha_1\beta_1,\zeta_1,\zeta_2\alpha_1\beta_1)\\
&-(\zeta_2\alpha_2\beta_2,\zeta_1,\alpha_2\beta_2)+(\alpha_2\beta_2,\zeta_1\zeta_2,\alpha_2\beta_2)-(\alpha_2\beta_2,\zeta_2,\zeta_1\alpha_2\beta_2)\\
&-(\alpha_1,\beta_1,\zeta_2)+(\zeta_1\alpha_1,\beta_1,\zeta_2)+(\zeta_1,\alpha_1,\zeta_2\beta_1)-(\zeta_1,\alpha_1,\beta_1)\\
&-(\beta_1,\alpha_1,\zeta_1)+(\zeta_1\beta_1,\alpha_1,\zeta_1)+(\zeta_1,\beta_1,\zeta_1\alpha_1)-(\zeta_1,\beta_1,\alpha_1)\\
&-(\alpha_2,\beta_2,\zeta_1)+(\zeta_2\alpha_2,\beta_2,\zeta_1)+(\zeta_2,\alpha_2,\zeta_1\beta_2)-(\zeta_2,\alpha_2,\beta_2)\\
&-(\beta_2,\alpha_2,\zeta_2)+(\zeta_2\beta_2,\alpha_2,\zeta_2)+(\zeta_2,\beta_2,\zeta_2\alpha_2)-(\zeta_2,\beta_2,\alpha_2)\\
&\hspace{22em}\bmod \partial C_4(N_2)
\end{aligned}$$

where $\zeta_i = [\alpha_i, \beta_i]$ and this shows that $\widetilde{\tau}_2(\varphi_1), \widetilde{\tau}_2(\varphi_2) \in E^{\infty}_{2,1}$. This completes the proof. □

Theorem *The restriction of* $d^2 : H_3(N_2) \to \mathcal{L}_3 \otimes H$ *to* $\mathrm{Im}\,\widetilde{\tau}_2$ *is an isomorphism onto* $\mathrm{Im}\,\tau_2$.

Proof. According to the previous lemmas, we can regard the values $\tau_2(\varphi)$ and $\widetilde{\tau}_2(\varphi)$ as elements of the quotient module of $S^2\Lambda^2 H$. Using the cycles $\overline{c}_{\varphi_1}$ and $\overline{c}_{\varphi_2}$ computed in the proof of Lemma 2, we have

$$\begin{aligned}
\widetilde{\tau}_2(\varphi_1) &= -(a_1 \wedge b_1)^{\otimes 2},\\
\widetilde{\tau}_2(\varphi_2) &= -(a_1 \wedge b_1 + a_2 \wedge b_2)^{\otimes 2},
\end{aligned}$$

which coincide with the values $\tau_2(\varphi_1)$, $\tau_2(\varphi_2)$ in $S^2\Lambda^2 H/\sim$ computed in [5]. It follows that the homomorphisms τ_2 and $\widetilde{\tau}_2$ have the same image in $S^2\Lambda^2 H/\sim$. This completes the proof. □

Remark It is an open problem to determine the abelianization of $\mathcal{K}_{g,1}$. It was expected that the refinement $\widetilde{\tau}_2$ would give a new abelian quotient of $\mathcal{K}_{g,1}$, but the above theorem shows that $\widetilde{\tau}_2$ has no informations about $\mathcal{K}_{g,1}$ which τ_2 loses.

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Department of Mathematical Sciences
University of Tokyo
Komaba, Meguro, Tokyo 153-8914
Japan
E-mail: yokomizo@ms.u-tokyo.ac.jp