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Analytic foliations and center problem

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Abstract. We prove a real version of Lins Neto's synthesis Theorem. The technics used, allow us to give a foliation without Liouvillian first integral and which restricts to center on the fixed point set of many antiholomorphic involutions leaving $\mathcal{F}$ invariant.

Key words: holomorphic 1-forms - center - reduction of singularities - groups of diffeomorphisms - antiholomorphic involutions - Liouvillian first integral.

1. Introduction

Let $\mathcal{F}$ and σ be germs of holomorphic foliation and antiholomorphic involution at $0 \in \mathbf{C}^2$. It is well known that if $\sigma^*\mathcal{F} = \mathcal{F}$ then $\mathcal{F}$ restricts to a real foliation on the fixed point set of σ (Fix_σ). We say that $\mathcal{F}_{/\mathrm{Fix}_\sigma}$ is monodromic if to each germ of real analytic curve $\tau : \mathbf{R}_0^+ \to \mathrm{Fix}_{\sigma,\tau(0)}$ corresponds a Poincaré return map $\mathcal{P}$ (for t small enough the leaf of $\mathcal{F}$, which passes through $\tau(t)$ cuts again $\tau(\mathbf{R}_0^+)$ at $\mathcal{P}(\tau(t))$). When $\mathcal{P}$ is the germ of identity, we say that $\mathcal{F}_{/\mathrm{Fix}_\sigma}$ is a center. The simplest example of center is the one defined by the levels of the function $f(x,y) = x^2 + y^2$, or equivalently by the 1-form $\omega = x\,dx + y\,dy$. The complexification of $\mathcal{F}_\omega$, denoted $\mathcal{F}_\omega^{\mathbf{C}}$, is the germ of foliation at $0 \in \mathbf{C}^2$ defined by 1-form $\omega^{\mathbf{C}}$, whose restriction on $\mathbf{R}_0^2$ is ω. This example corresponds to the case where $\mathcal{F}_\omega^{\mathbf{C}}$ has two holomorphic invariant curves and has the following property (cf. 4.2):

1. for each antiholomorphic involution σ which does not fix any invariant curve of $\mathcal{F}_\omega^{\mathbf{C}}$ and such that $\sigma^*\mathcal{F}_\omega^{\mathbf{C}} = \mathcal{F}_\omega^{\mathbf{C}}$, $\mathcal{F}_{\omega\,/\mathrm{Fix}_\sigma}^{\mathbf{C}}$ is a center.

When $\mathcal{F}_\omega^{\mathbf{C}}$ has two invariant tangent curves (node), according to Brunella [Br], the assumption of center and some generic conditions on ω ensure that there exists an elementary mutiform first integral for ω [CM]. We are interested in centers whose complexification has four invariant curves. That is the simplest case after the one described above, since the complexification of a germ of real analytic foliation which is a center has an even number of

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invariant curves (the image of an invariant curve by the standard antiholomorphic involution ($\phi_0 : (x,y) \longmapsto (\bar{x},\bar{y})$) of $\mathbf{C}_0^2$ is an invariant curve and the presence of invariant curve fixed by ϕ_0 is an obstruction to the monodromicity). The foliation $\mathcal{F}_\omega^{\mathbf{C}}$, where $\omega = d(x^4+y^4)$, is fixed by ϕ_0 and $\sigma_0 : (x,y) \longmapsto (\bar{x}, \sqrt{-1}\bar{y})$ has a holomorphic first integral. More generally it is proved in [G] that each 1-form ω such that the reduction of singularities of $\mathcal{F}_\omega^{\mathbf{C}}$ is a blowing up, $\mathcal{F}_\omega^{\mathbf{C}}$ has four invariant curves and is fixed by the last two antiholomorphic involutions, has a holomorphic first integral. The involutions which leave the foliation fixed are the reason of the existence of multiform first integrals in this example. Thus we can hope that a foliation has an elementary or a Liouvillian [P] first integral as soon as it has "many sections" in which it restricts to centers. The following result proves, in general, we do not have such an answer:

Proposition 1.1 *There is a germ of foliation $\mathcal{F}$ at $0 \in \mathbf{C}^2$ which satisfies the following properties*:

1. *For any antiholomorphic involution σ which leaves $\mathcal{F}$ invariant and does not fix an invariant curve of $\mathcal{F}$, $\mathcal{F}_{/\,\mathrm{Fix}_\sigma}$ is a center.*
2. *There are two antiholomorphic involutions σ_1 and σ_2 which leave $\mathcal{F}$ invariant, which are not conjugate by a holomorphic diffeomorphism tangent to $\mathcal{F}$ and such that $\mathcal{F}_{/\,\mathrm{Fix}_{\sigma_k}}$, $k=1,2$, are centers.*
3. *$\mathcal{F}$ has not a Liouvillian first integral.*
4. *$\mathcal{F}$ has four invariant curves.*

Thus we have an example of foliation with a maximal number of "real sections", which are not conjugate by any diffeomorphism tangent to the foliation, while the foliation restricts to center and does not have a Liouvillian first integral. The exceptional divisor of the minimal reduction of singularities of a node center has at least two components and some corners which are linearizable and resonant. Thus the following question is natural: do there exist some multiform first integrals related to the types of singularities in the corners (linearizable and non-resonant) if $\mathcal{F}_\omega^{\mathbf{C}}$ has four invariant curves? The next result gives a negative answer to this question:

Proposition 1.2 *There is a real analytic center ω such that*:

1. *$\mathcal{F}_\omega^{\mathbf{C}}$ has four invariant curves,*
2. *$\mathcal{F}_\omega$ has no Liouvillian first integral,*
3. *the exceptional divisor of the minimal reduction of singularities of $\mathcal{F}_\omega^{\mathbf{C}}$*

has two components and its corner is linearizable and non resonant.

The proofs of these results consist of building some non dicritical holomorphic 1-forms which have some basic properties (cf. 2). For this we generalize a real version of Lins Neto's synthesis theorem, established by Berthier, Cerveau and Lins-Neto [BCL] in a special case. For the understanding of the assumptions of this, we will describe, in Section 2, the properties of blowing up of a foliation and antiholomorphic involution which leaves it invariant. Let $\pi : \tilde{\mathbf{C}}_0^2 \to \mathbf{C}_0^2$ be a finite sequence of blowings up, $\mathcal{S}$ a finite set of points consisting of the corners and those points of $\pi^{-1}(0)$, $C_1, \dots, C_n$ the irreductible components of $\pi^{-1}(0)$ and some group homomorphisms Hol_k from the Poincaré group $\pi_1(C_k \setminus \mathcal{S})$ to $\mathrm{Diff}(\mathbf{C}_0)$, where $\mathrm{Diff}(\mathbf{C}_0)$ is the group of holomorphic diffeomorphisms of $\mathbf{C}_0$. We assume that equivariance, compatibility and Chern conditions (Definition 2.5) hold. These conditions are natural properties of singularities and holonomy groups of the complexification of real analytic foliation. Then we have the following result which generalizes the special case of a blowing-up [BCL]:

Theorem 1.3 (Real synthesis theorem) *There is a germ of real analytic 1-form ω at 0 in $\mathbf{R}^2$ such that:*

1. *up to a homeomorphism π is the minimal reduction of singularities of $\mathcal{F}_\omega^{\mathbf{C}}$ and the set of singularities of $\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$, denoted $\mathrm{Sing}\,\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$, is equal to $\mathcal{S}$,*
2. *$\mathcal{F}_\omega^{\mathbf{C}}$ is not dicritical,*
3. *Hol_k is the holonomy representation of C_k.*

In Section 3, we prove the previous theorem which is the main result of this paper. Using this theorem, we will show Propositions 1.1 and 1.2 respectively in Sections 5 and 6.

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2. Properties of germs of real analytic foliations

Now we describe the blowing up of an antiholomorphic involution in order to understand the germs of foliations at 0 obtained by complexification of real analytic foliations. For this we fix the following notations: $i = \sqrt{-1}$ and $\phi_0 : (x, y) \longrightarrow (\bar{x}, \bar{y})$ for the standard antiholomorphic involution of $\mathbf{C}^2$.

2.1. Blowing up of antiholomorphic involution

Let $\mathcal{F}$ be a foliation and ϕ be an antiholomorphic involution on a complex surface M, such that $\phi^*\mathcal{F} = \mathcal{F}$. Let m be a singular point and $\phi(m)$ its image by ϕ. We have the following lemma:

Lemma 2.1 *There are some local coordinates (x_m, y_m) and $(x_{\phi(m)}, y_{\phi(m)})$ at m and $\phi(m)$ respectively, such that:*

$$(x_{\phi(m)}, y_{\phi(m)}) \circ \phi \circ (x_m, y_m)^{-1} = \phi_0.$$

Proof. Let $(\mathbf{C}^2_m, \varphi)$ and $(\mathbf{C}^2_{\phi(m)}, \varphi')$ be two germs of local charts of M at m and $\phi(m)$ respectively. For a suitable choice of endomorphism (resp. endomorphisme with real coefficients) of $\mathbf{C}^2_0$, θ, (resp. R) $\lambda = \theta \circ (\mathrm{id}_{\mathbf{C}^2_0} + R \circ \phi_0 \circ (\varphi' \circ \phi \circ \varphi^{-1}))$ is a diffeomorphism which conjugates $\varphi' \circ \phi \circ \varphi^{-1}$ to ϕ_0. □

Let $\pi = \pi_{m,\phi(m)} : \tilde{M} = \tilde{M}_{m,\phi(m)} \to M$ be a morphism obtained by blowing up of M at m and $\phi(m)$ simultaneously. Using the previous lemma it is easy to prove:

Lemma 2.2 *There is only one antiholomorphic involution $\tilde{\phi}$ on $\tilde{M}$ such that $\phi \circ \pi = \pi \circ \tilde{\phi}$.*

Remark 2.3 The invariance of $\mathcal{F}$ under ϕ implies that the strict transform of $\mathcal{F}$ by π, $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_{m,\phi(m)}$, is invariant under $\tilde{\phi}$. In particular the singular points and the invariant curves are globally invariant. Furthermore, if Σ is a curve tranverse to $\tilde{\mathcal{F}}$ so is $\tilde{\phi}(\Sigma)$.

2.2. Properties of complexifications of real analytic foliations

Using the process above we obtain the following version of Seidenberg's reduction theorem of singularities [MM]:

Theorem 2.4 *Let $\mathcal{F}$ be a foliation on $\mathbf{C}^2_0$ invariant under ϕ_0. There is a holomorphic proper map $\pi : \tilde{\mathbf{C}}^2_0 \to \mathbf{C}^2_0$ and a foliation $\tilde{\mathcal{F}}^{\mathbf{C}}$ (strict transform of $\mathcal{F}^{\mathbf{C}}$ by π) such that:*

1. *π is given by a finite number of blowings-up,*
2. *$\pi^{-1}(0)$ is a divisor with normal crossings,*
3. *$\phi_0 \circ \pi = \pi \circ \tilde{\phi}_0$,*
4. *$\tilde{\mathcal{F}}^{\mathbf{C}}$ is reduced,*
5. *$\tilde{\phi}_0^* \tilde{\mathcal{F}}^{\mathbf{C}} = \tilde{\mathcal{F}}^{\mathbf{C}}$.*

2.2.1. Equivariance of the holonomy generators. Let C_k be a component of $\pi^{-1}(0)$ which is assumed to be non dicritical, $P \in C_k \setminus \operatorname{Sing} \tilde{\mathcal{F}}^{\mathbf{C}}$ and $\Sigma : \mathbf{C}_0 \to \tilde{\mathbf{C}}_0^2$ a germ of holomorphic transversal at P whose image will be also denoted Σ. The invariance of $\tilde{\mathcal{F}}^{\mathbf{C}}$ under $\tilde{\phi}_0$ implies that:

1. the map Σ' from $\mathbf{C}_0$ to $\tilde{\mathbf{C}}_0^2$ which associates $\tilde{\phi}_0 \circ \Sigma(\bar{y})$ to y is a holomorphic transversal of $\tilde{\mathcal{F}}^{\mathbf{C}}$ at $f_k(P)$,
2. for all closed path γ of $C_k \setminus \operatorname{Sing} \tilde{\mathcal{F}}^{\mathbf{C}}$, with base point P, the following diagram is commutative:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\tilde{\phi}_0} & \Sigma' \\ \Big\downarrow h_\gamma & & \Big\downarrow h_{f_k(\gamma)} \\ \Sigma & \xrightarrow{\tilde{\phi}_0} & \Sigma' \end{array}$$

where h_γ (resp. $h_{f_k(\gamma)}$) is the holonomy diffeomorphism associated to γ (resp. to $f_k(\gamma)$). If we identify Σ and Σ' to $\mathbf{C}_0$ the previous diagram implies the equivariance of the holonomy generators:

$$\bar{h}_\gamma(z) = h_{f_k(\gamma)}(\bar{z}).$$

2.2.2. Chern conditions. If $C_k \cap \operatorname{Sing} \tilde{\mathcal{F}}_\omega^{\mathbf{C}} = \{P_1, \ldots, P_r\}$, then we choose the closed path γ_j so that its index around P_l is δ_{jl}. We have the following property which will be called the Chern condition:

either 1. there is $0 \le j \le r$ such that h_{γ_j} is holomorphically linearizable,

or 2. for all $0 \le j \le r$ there is $\mu_j \le 0$ such that $\sum_{j=1}^r \mu_j = \operatorname{Chern}(C_k)$, where $(h_{\gamma_j})'(0) = \exp(2i\pi\mu_j)$ and $\operatorname{Chern}(C_k)$ is the Chern class [GH] of C_k.

Indeed according to Poincaré [ko] if condition 1 is not, then each $\exp(2i\pi\mu_j)$ has modulus 1, this means that μ_j is real. The Camacho-Sad index [CS] of $\tilde{\mathcal{F}}^{\mathbf{C}}$ at P_j relatively to C_k is written:

$$\mathcal{I}_{P_j}(\tilde{\mathcal{F}}^{\mathbf{C}}, C_k) = \mu_j + n_j,$$

where n_j is in $\mathbf{Z}$. Again, according to Poincaré [Po] if the germ of $\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ at P_j is not linearisable, then $\mu_j + n_j$ is negative. The Camacho-Sad's index theorem [CS] ensures:

$$\operatorname{Chern}(C_k) = \sum_{j=1}^r \mu_j + n_j.$$

Thus, up to to put $\mu_j + n_j$ in the place of μ_j, we have either condition 1 or 2.

Conversely, let $\pi : \tilde{\mathbf{C}}_0^2 \to \mathbf{C}_0^2$, be a morphism given by a finite number of blow up and let $C_1, \ldots, C_n$ be the irreductible components of $\pi^{-1}(0)$. Let $\mathcal{S}$ be a finite set, which consists of corners and those points of $\pi^{-1}(0)$ and for each k in $I = \{1, \ldots, n\}$, let $\mathrm{Hol}_k : \pi_1(C_k \setminus \mathcal{S}) \to \mathrm{Diff}(\mathbf{C}_0)$ be a group representation. The properties of foliations, obtained by complexifications of real analytic foliations, suggest us the following:

Definition 2.5

1. We say that the equivariance condition holds if there are an involution a on I and an antiholomorphic diffeomorphism $f_k : C_k \to C_{a(k)}$ such that:
   (a) $f_k(C_k) \cap \mathcal{S} = C_{a(k)} \cap \mathcal{S}$,
   (b) if $C_k = C_{a(k)}$, then $a(k) = k$,
   (c) $C_k \cap C_j = \emptyset$ if and only if $C_{a(k)} \cap C_{a(j)} = \emptyset$,
   (d) $\forall \gamma \in \pi_1(C_k \setminus \mathcal{S})$, $\mathrm{Hol}_k(\gamma)(z) = \overline{\mathrm{Hol}}_{a(k)}(f_k(\gamma))(\bar{z})$ (equivariance of the holonomy generators).
2. We say that the Chern conditions hold when: If $C_k \cap \mathcal{S} = \{P_1, \ldots, P_r\}$ and γ_j^k is a closed path in $C_k \setminus \mathcal{S}$ whose index around P_l is δ_{jl}, then we have:
   (a) either there is $0 \leq j \leq r$ such that $\mathrm{Hol}_k(\gamma_j^k)$ is holomorphically linearizable,
   (b) or for all $0 \leq j \leq r$ there is $\mu_j \leq 0$ such that $\sum_{j=1}^r \mu_j = \mathrm{Chern}(C_k)$, where $(\mathrm{Hol}_k(\gamma_j^k))'(0) = \exp(2i\pi\mu_j)$.
3. We say that the compatibility conditions, in the corners, hold when: If $C_k \cap C_j = \{P\}$, there is a holomorphic reduced 1-form

$$\omega = \lambda_1 x(1 + a(x,y))dy + \lambda_2 y(1 + b(x,y))dx$$

   so that the holonomy generator of the invariant curve $x = 0$ (resp. $y = 0$) is conjugate to $\mathrm{Hol}_k(\gamma_k)$ (resp. $\mathrm{Hol}_j(\gamma_j)$), where γ_k (resp. γ_j) is a closed path in $C_k \setminus \mathcal{S}$ (resp. $C_j \setminus \mathcal{S}$) whose index around P is one.

Remark 2.6 1. Let π be the reduction of singularities of the complexification of a real foliation and $C_1, \ldots, C_n$ the irreductibles components of $\pi^{-1}(0)$. If we define $a(k)$ by $\tilde{\phi}_0(C_k) = C_{a(k)}$ and f_k as the restriction of $\tilde{\phi}_0$ to C_k we trivially have the properties (a), (b) and (c) of the Definition 2.5.

2. The compatibility conditions are trivially satisfied by the holonomy

generators associated to the components of the exceptional divisor of the reduction of singularities of a non dicritical foliation.

3. Proof of theorem 1.3

For each k, $1 \leq k \leq n$, the synthesis theorem [L] asserts that there are a complex surface M_k, a disk bundle $F_k : M_k \to C_k$, a foliation $\mathcal{F}_k$ and a point m_k in M_k such that:

1. there is a diffeomorphism φ_k from the zero section, S_k, of F_k to C_k,
2. the set of singularities of $\mathcal{F}_k$ is $\varphi_k^{-1}(C_k \cap \mathcal{S})$,
3. $\mathcal{F}_k$ has not a node singularity
4. $S_k \setminus \operatorname{Sing} \mathcal{F}_k$ is a leaf of $\mathcal{F}_k$,
5. F_k is transverse to all the leaves of $\mathcal{F}_k$ except the special fibers which pass through $\operatorname{Sing} \mathcal{F}_k$,
6. m_k is in $S_k \setminus \operatorname{Sing} \mathcal{F}_k$ and $f_k(\varphi_k(m_k)) = \varphi_{a(k)}(m_k)$,
7. the Chern class of S_k is equal to the one of C_k,
8. the holonomy representation of S_k is:

$$\begin{array}{rcl}\operatorname{Hol}(\mathcal{F}_k, S_k) : \pi_1(S_k \setminus \operatorname{Sing} \mathcal{F}_k, m_k) & \longrightarrow & \operatorname{Diff}(\mathbf{C}_0) \\ \gamma & \longmapsto & \operatorname{Hol}_k([\varphi_k(\gamma)]) = h_\gamma .\end{array}$$

Remark 3.1 The equivariance of the holonomy generators implies that there is an antiholomorphic diffeomorphism ϕ_k from $F_k^{-1}(\varphi_k(m_k))$ to $F_{a(k)}^{-1}(\varphi_k(m_{a(k)}))$ with $\phi_{a(k)} = \phi_k^{-1}$ inherited from the standard antiholomorphic involution of $\mathbf{C}$ such that:

1. $\phi_k(m_k) = m_{a(k)}$,
2. $\phi_k \circ h_\gamma \circ \phi_{a(k)} = h_{f_k(\gamma)} \quad \forall \gamma \in \pi_1(S_k \setminus \operatorname{Sing} \mathcal{F}_k)$.

The following results must be understood up the choice of a suitable neighbourhood of S_k in M_k. We have the fondamental:

Lemma 3.2 *For each k there is a antiholomorphic diffeomorphism, $\sigma_k : M_k \to M_{a(k)}$, which carries the leaves of $\mathcal{F}_k$ (resp. $\mathcal{F}_{a(k)}$) to the one of $\mathcal{F}_{a(k)}$ (resp. $\mathcal{F}_k$). Furthermore $\sigma_k^{-1} = \sigma_{a(k)}$.*

Proof. We are going to extend the germ of diffeomorphism ϕ_k given in Remark 3.1. We use essentially the ideas of [BCL]. Let m be a point of $S_k \setminus \operatorname{Sing} \mathcal{F}_k$ and let γ be a path in $S_k \setminus \operatorname{Sing} \mathcal{F}_k$ connecting m to m_k.

For z in $F_k^{-1}(\varphi_k(m))$ close enough to m, the lifting $\tilde{\gamma}$ of γ, in the leaf of $\mathcal{F}_k$ passing through z, following the fiber bundle F_k ends at the point $h_\gamma(z)$ of $F_k^{-1}(\varphi_k(m_k))$. We define $\sigma_\gamma(z)$ as the end of the path obtained by lifting $f_k(\gamma^{-1})$ from $\phi_k(h_\gamma(z))$ in the leaves of $\mathcal{F}_{a(k)}$ following $F_{a(k)}$. It does not depend on the choice of the path. Indeed let γ' be another path in $S_k \setminus \operatorname{Sing}\mathcal{F}_k$. The compound $\gamma_1 = \gamma'^{-1} \cdot \gamma$ is an element of $\pi_1(S_k \setminus \operatorname{Sing}\mathcal{F}_k, m)$ which corresponds to a diffeomorphism h_{γ_1}. The image $\sigma_{\gamma'}(z)$ of z is obtained by lifting $f_k \circ (\gamma'^{-1})$ from $\phi_k(h_{\gamma'}(z))$. From the equivariance of the holonomy generators, we get that:

$$\begin{aligned}\phi_k \circ h_{\gamma'} &= \phi_k \circ h_{\gamma'\gamma^{-1}\gamma} = \phi_k \circ h_{\gamma_1\gamma^{-1}} = \phi_k \circ h_{\gamma_1} \circ h_{\gamma^{-1}} \\ &= \phi_k \circ h_{\gamma_1} \circ \phi_k^{-1}(\phi_k \circ h_\gamma) \\ &= h_{f_i \circ \phi_i(\gamma_1)} \circ \phi_i^{-1}(\phi_k \circ h_\gamma) \\ &= \phi_k \circ h_\gamma.\end{aligned}$$

Thus $\sigma_{\gamma'}(z)$ is also obtained by lifting $f_k \circ \phi(\gamma'^{-1})$ from $\phi_k \circ h_\gamma(z)$. As $\phi_k(\gamma_1) = \phi_k(\gamma_1^{-1}) \cdot \phi_k(\gamma^{-1})$, we have $\sigma_{\gamma'}(z) = \sigma_\gamma(z)$. Using this process we find an antiholomorphic diffeomorphism σ_k from $M'_k = M_k \setminus \{\phi_k^{-1}(m) \,/\, m \in \operatorname{Sing} \cap C_k\}$ to $M'_{a(k)} = M_{a(k)} \setminus \{\phi_{a(k)}^{-1}(m) \,/\, m \in \operatorname{Sing} \cap C_{a(k)}\}$, such that $\sigma_k^* \mathcal{F}_{a(k)} = \mathcal{F}_k$. Now we have to extend σ_k to the invariant curves. Let m be a singular point of $\mathcal{F}_k$ and $m' = \varphi_{a(k)}^{-1} \circ f_k \circ \varphi_k(m)$. We are going to define an antiholomorphic involution in order to use a result of [BCL]. Let $(V_m, (u,v))$ and $(V_{m'}, (u',v'))$ be two local trivializations of F_k and $F_{a(k)}$ respectively at the points m and m' such that:

1. $m = (0,0)$ (resp. $m' = (0,0)$) is in (u,v) (resp. (u',v'))
2. $v = 0$ (resp. $v' = 0$) is a local equation of S_k (resp. $S_{a(k)}$).

Let fix a point P in $S_k \cap V_m$ close enough to m but distinct from it. It is easy to see that one can choose (u,v) and (u',v') so that $u' \circ f_k \circ u^{-1}$ and $v' \circ \sigma_{k/F_k^{-1}(\varphi_k(P))} \circ v^{-1}$ are equal to the standard antiholomorphic involution of $\mathbf{C}$. Up to this choice we define ϕ as follows: $(u',v') \circ \phi \circ (u,v)^{-1} = (\bar{u}, \bar{v})$. The holonomy generators equivariance and Proposition 2.1 of [MR] imply that $\phi^* \mathcal{F}_{a(k)/V_m}$ and $\mathcal{F}_{k/V_m}$ are holomorphically conjugate by a diffeomorphism which leaves a fiber bundle invariant [Me]. Thus, up to a chart change, we assume that $\phi^* \mathcal{F}_{a(k)/V_m} = \mathcal{F}_{k/V_m}$. It is enough to show that $\psi = \sigma_{a(k)} \circ \phi$ is bounded to obtain the extension of σ_k. Note that ψ is holomorphic on $V_m \setminus F_k^{-1}(\varphi_k(P))$ and conjugates $\mathcal{F}_{k/V_m}$ to itself. The germ of $\mathcal{F}_{k/V_m}$ at m

is defined by a reduced 1-form ω_m which is not a node. In order to extend ψ in a neighbourhood of m we proceed as in [BCL]. $\square$

Lemma 3.3 *There are a complex surface M, obtained by gluing together the M_k, a foliation $\tilde{\Im}$ on M, such that for any k in $\{1,\dots,n\}\tilde{\Im}_{/M_k} = \Im_k$, and an antiholomorphic involution $\tilde{\sigma}$ on M which leaves $\tilde{\Im}$ invariant.*

Proof. Let $\{m\}$ be the intersection of two components C_k and C_j of $\pi^{-1}(0)$ and put $\{m'\} = C_{a(k)} \cap C_{a(j)}$. We deal with the case m is different from m'. Let (V_k, ρ_k) be a local chart of M_k at $\varphi_k^{-1}(m)$ and whose values are in a polydisk Δ_k of $\mathbf{C}^2$. Put $V_{a(k)} = \sigma_k(V_k)$ and $\rho_{a(k)} = \phi_0 \circ \rho_k \circ \sigma_{a(k)}$. By construction $(V_{a(k)}, \rho_{a(k)})$ is a chart of $M_{a(k)}$ which contains $\varphi_{a(k)}^{-1}(m')$, with values in $\phi_0(\Delta_k)$, and such that $\rho_{a(k)} \circ \sigma_k \circ \rho_k^{-1} = \phi_{0/\Delta_k}$. In the same way we can choose a local chart (V_j, ρ_j) (resp. $(V_{a(j)},\ \rho_{a(j)})$) at $\varphi_j^{-1}(m)$ (resp. $\varphi_{a(j)}^{-1}(m')$ with values in a poly disk Δ_j (resp. $\Delta_{a(j)} = \phi_0(\Delta_j)$) such that $\rho_{a(j)} \circ \sigma_j \circ \rho_j^{-1}$ is equal to ϕ_{0/Δ_j}. Let θ be the diffeomorphism from $\Delta_k \cup \Delta_{a(k)}$ to $\Delta_j \cup \Delta_{a(j)}$ defined by:

1. $\rho_j \circ \theta \circ \rho_k^{-1}(u_k, v_k) = (u_k, v_k)$ and $\rho_{a(j)} \circ \theta \circ \rho_{a(k)}^{-1}\ (u_{a(k)}, v_{a(k)}) = (u_{a(k)}, v_{a(k)})$.

The compatibility conditions (Definition 2.5) in the corners imply that $\mathcal{G}_j = (\rho_j^{-1})^*\mathcal{F}_j$ and $\mathcal{G}_k = (\rho_k^{-1} \circ \theta^{-1})^*\mathcal{F}_k$ (resp. $\mathcal{G}_{a(j)} = (\rho_{a(j)}^{-1})^*\mathcal{F}_{a(j)}$ and $\mathcal{G}_{a(k)} = (\rho_{a(k)}^{-1} \circ \theta^{-1})^*\mathcal{F}_{a(k)}$) are holomorphically conjugate. As ϕ_0 carries the leaves of $\mathcal{G}_k$ (resp. $\mathcal{G}_j$) to those of $\mathcal{G}_{a(k)}$ (resp. $\mathcal{G}_{a(j)}$), there is a holomorphic diffeomorphism $\psi : \Delta_j \cup \Delta_{a(j)} \to \Delta_j \cup \Delta_{a(j)}$ which commutes with ϕ_0 and so that:

$$\psi^*\mathcal{G}_k = \mathcal{G}_j \quad \text{and} \quad \psi^*\mathcal{G}_{a(k)} = \mathcal{G}_{a(j)}.$$

Identifying each element of V_k (resp. V_j) with its image by $\rho_j^{-1} \circ \psi^{-1} \circ \theta \circ \rho_k^{-1}$ (resp. $\rho_{a(j)}^{-1} \circ \psi^{-1} \circ \theta \circ \rho_{a(k)}^{-1}$) we glue simultaneously:

1. M_k and M_j (resp. $M_{a(k)}$ and $M_{a(j)}$),
2. $\mathcal{F}_k$ and $\mathcal{F}_j$ (resp. $\mathcal{F}_{a(k)}$ and $\mathcal{F}_{a(j)}$),
3. σ_k and σ_j (resp. $\sigma_{a(k)}$ and $\sigma_{a(j)}$).

Assume now that $\{m\}$ is equal to $\{m'\}$. This means that $a(k) = k$, $a(j) = j$ and $\varphi_k^{-1}(m)$ (resp. $\varphi_j^{-1}(m)$) is a fixed point of σ_k (resp. σ_j). From Lemma 2.1 we choose a local chart (V_k, ρ_k) (resp. (V_j, ρ_j)) of M_k (resp. M_j) at

$\varphi_k^{-1}(m)$ (resp. $\varphi_j^{-1}(m)$) with values in a polydisk Δ of $\mathbf{C}^2$ such that:

$$\phi_0(\Delta) = \Delta \quad \text{and} \quad \rho_l \circ \sigma_l \circ \rho_l^{-1} = \phi_{0/\Delta} \quad \text{for} \quad l = k, j.$$

As before the compatibility conditions in the corners imply that $\mathcal{G}_k = (\rho_k^{-1})^* \mathcal{F}_k$ and $\mathcal{G}_j = (\rho_j^{-1})^* \mathcal{F}_j$ are holomorphically conjugate. As ϕ_0 leaves $\mathcal{G}_k$ and $\mathcal{G}_j$ fixed there is a complexification ψ of a real analytic diffeomorphism, from Δ to Δ, such that $\psi^* \mathcal{G}_k$ is equal to $\psi^* \mathcal{G}_j$. Identifying each element of V_k with its image by $\rho_j^{-1} \circ \psi^{-1} \circ \rho_k^{-1}$ we glue M_k and M_j (resp. $\mathcal{F}_k$ and $\mathcal{F}_j$, σ_k and σ_j). Gluing as above at each corner, we have done the proof of Lemma 3.3. □

As the self intersections of S_k are the same as the ones of the C_k (components of $\pi^{-1}(0)$), we have a morphism $\bar{\pi} : M \to V$ composed by a finite number of blowings down, where V is a neighbourhood of 0 in $\mathbf{C}^2$. By Hartog's theorem one can extend σ, defined by $\sigma \circ \bar{\pi} = \bar{\pi} \circ \tilde{\sigma}$, at 0 in $\mathbf{C}^2$. From Lemme 2.1 σ is holomorphically conjugate to ϕ_0. Thus, up a conjugation, the foliation $\mathcal{F}$, whose strict transform is $\tilde{\mathcal{F}}$, is invariant under ϕ_0 and is therefore a real analytic foliation.

4. Examples

4.1. Example

We give here some informations on the set, $\text{Ahi}(\mathcal{F}_\omega^{\mathbf{C}})$, of antiholomorphic involutions which leave $\mathcal{F}_\omega^{\mathbf{C}}$ invariant, where $\omega = d(H)$, with $H = x^4 + y^4$. Remark that $\text{Ahi}(\mathcal{F}_\omega^{\mathbf{C}})$ is not a group (the compound of two antiholomorphic involution is holomorphic). Let ϕ_0 still be the standard antiholomorphic involution of $\mathbf{C}^2$ and let us put:

$$\text{Iso}(\mathcal{F}_\omega^{\mathbf{C}}) = \{\psi \in \text{Diff}(\mathbf{C}_0^2) \,/\, \exists \varphi \in \text{Diff}(\mathbf{C}_0) \;\text{ so that }\; H \circ \psi = \varphi \circ H\}$$

and

$$\text{Fix}(\mathcal{F}_\omega^{\mathbf{C}}) = \{\psi \in \text{Diff}(\mathbf{C}_0^2) \,/\, H \circ \psi = H\},$$

where $\text{Diff}(\mathbf{C}_0^2)$ is the group of holomorphic diffeomorphisms of $\mathbf{C}_0^2$. We assert that:

1. each element of $\text{Ahi}(\mathcal{F}_\omega^{\mathbf{C}})$ tangent to the identity is of the form $\phi_0 \circ \exp[\tau(x,y)]\mathcal{X}$, where τ is the complexification of an analityc function of $\mathbf{R}_0^2$,

2. there is a bijection between the set of elements of $\mathrm{Ahi}(\mathcal{F}_\omega^{\mathbf{C}})$ which are not tangent to the identity and $\{\varphi \in \mathrm{Diff}(\mathbf{C}_0) \,/\, \varphi^{-1}(z) = \bar{\varphi}(\bar{z})\}$.

The morphism L_H from $\frac{\mathrm{Iso}(\mathcal{F}_\omega^{\mathbf{C}})}{\mathrm{Fix}(\mathcal{F}_\omega^{\mathbf{C}})}$ to $\mathrm{Diff}(\mathbf{C}_0)$, which associates to ψ the element φ of $\mathrm{Diff}(\mathbf{C}_0)$ such that $H \circ \psi = \varphi \circ h$ [BCM] is well defined and injective. According to [BCM] an element ψ of $\mathrm{Fix}(\mathcal{F}_\omega^{\mathbf{C}})$ is tangent to the identity (that is $\psi(x,y) = (x + h.o.t,\ y + h.o.t))$ if and only $\psi(x,y) = \exp[\tau(x,y)]\mathcal{X}$, where τ is a holomorphic function and $\mathcal{X}(x,y) = y^3\frac{\partial}{\partial x} - x^3\frac{\partial}{\partial y}$. Now let σ be an element of $\mathrm{Ahi}(\mathcal{F}_\omega^{\mathbf{C}})$. We have two cases:

1. $\phi_0 \circ \sigma$ is an element of $\mathrm{Fix}(\mathcal{F}_\omega^{\mathbf{C}})$. There is a holomorphic function τ such that $\sigma = \phi_0 \circ \exp[\tau(x,y)]\mathcal{X}$. As σ is an involution, an easy computation shows that τ must be the complexification of a real analytic function. Conversely for each complexification of real analytic function τ, $\sigma = \phi_0 \circ \exp[\tau(x,y)]\mathcal{X}$ is in $\mathrm{Ahi}(\mathcal{F}_\omega^{\mathbf{C}})$.

2. $\phi_0 \circ \sigma$ is not an element of $\mathrm{Fix}(\mathcal{F}_\omega^{\mathbf{C}})$. Let us put $\varphi = L_H(\phi_0 \circ \sigma)$. From the following equations:

$$\begin{aligned} H &= H \circ (\phi_0 \circ \sigma) \circ (\sigma \circ \phi_0) = \varphi \circ H \circ (\sigma \circ \phi_0) \\ &= \varphi \circ \bar{H} \circ (\phi_0 \circ \sigma) \circ \phi_0 = \varphi \circ \bar{\varphi} \circ H \circ \phi_0 = \varphi \circ \bar{\varphi} \circ \bar{H} \end{aligned}$$

we get that $\varphi^{-1}(x) = \bar{\varphi}(\bar{x})$. Conversely let φ be a diffeomorphism of $\mathbf{C}_0$ such that $\varphi^{-1}(x) = \bar{\varphi}(\bar{x})$. We are going to show that there exists an element σ of $\mathrm{Ahi}(\mathcal{F}_\omega^{\mathbf{C}})$ which verifies $H \circ \phi_0 \circ \sigma = \varphi \circ H$. Put $\tilde{\varphi}(x) = (\varphi(x^4))^{\frac{1}{4}}$ and $\phi(x) = \tilde{\varphi}(\bar{x})$. Since $\varphi^{-1}(x) = \bar{\varphi}(\bar{x})$ an easy computation shows that ϕ is an involution. Let $\pi : \tilde{\mathbf{C}}_0^2 \to \mathbf{C}_0^2$ be a blowing up, (x,t) and (s,y) the charts of $\tilde{\mathbf{C}}_0^2$ glued together by $y = tx$ and $st = 1$. Let us remark that the Hopf's fiber bundle (that is the fiber bundle $\tilde{\mathbf{C}}_0^2 \to \pi^{-1}(0)$ given by $t =$ constante) is transverse to each leaf of $\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ except the special ones which pass through the singularities and that the holonomy group computed on the transversal $t = 0$ is generated by $h(x) = ix$. Moreover for each γ, in $\pi_1(\pi^{-1}(0) \setminus \mathrm{Sing}(\tilde{\mathcal{F}}_\omega^{\mathbf{C}}))$, we have:

$$(*) \qquad h_\gamma(z) = \phi \circ h_{\tilde{\phi}_0(\gamma)} \circ \phi(z)$$

We build an antiholomorphic involution $\tilde{\sigma}$ on $\tilde{\mathbf{C}}_0^2$ which leaves $\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ invariant as follows: let (x,t) be a point of $\tilde{\mathbf{C}}_0^2$ such that x is close enough to 0 and $(0,t)$ is not a singularity, and γ is a path in $\pi^{-1}(0) \setminus \mathrm{Sing}\,\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ connecting $(0,t)$ to $(0,0)$. The lifting of γ in the leaf of $\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ passing through (x,t)

following Hopf's fiber bundle ends at $(x_\gamma, 0)$. We define $\tilde{\sigma}_\gamma(x,t)$ to be the end of the path obtained by lifting $f(\gamma^{-1})$ from $\phi(x_\gamma, 0)$. As in the proof of Theorem 1.3 we find, using equality $(*)$, that $\tilde{\sigma}_\gamma$ does not depend on the choice of the path, extends to invariant curves and induces an element σ of $\mathrm{Ahi}(\mathcal{F}_\omega^{\mathbf{C}})$. By construction, in the local coordinates $(z = x(1+t^4)^{\frac{1}{4}}, t)$, $\tilde{\sigma} \circ \tilde{\phi}_0(z,t)$ is equal to $(\tilde{\varphi}(z), t)$ and $\tilde{H}(z,t)$ is equal to z^4. We have:

$$\tilde{H} \circ (\tilde{\phi}_0 \circ \tilde{\sigma})(z,t) = (\tilde{\varphi}(z))^4 = \varphi(z^4) = \tilde{H} \circ \tilde{\phi}_0 \circ \tilde{\sigma}(z,t)$$

which implies that $H \circ (\phi_O \circ \sigma) = \varphi \circ H$. Thus the assertion.

4.2. Relation between Poincaré return map and holonomy

Let $\mathcal{F}$ be a germ of holomorphic foliation on $\mathbf{C}_0^2$, σ an antiholomorphic involution which leaves it invariant and π the minimal reduction of singularities of $\mathcal{F}$. We assume that $\mathcal{F}_{/\,\mathrm{Fix}_\sigma}$ is monodromic and for simplicity that σ is the standard antiholomorphic involution of $\mathbf{C}^2$. Let β be the "interval" of leaf which begins at $(x, 0)$ and ends at $(\mathcal{P}(x), 0)$ where $0 \leq x$ is close enough to 0 and $\mathcal{P}$ is the Poincaré return map corresponding to $\mathcal{F}_{/\,\mathrm{Fix}_\sigma}$. Let us denote $\beta^+ = \beta \cap \{0 < y\}$, $\beta^- = \beta \cap \{y < 0\}$ and $\tilde{\beta}$ (resp. $\tilde{\beta}^+$, $\tilde{\beta}^-$) be the strict transform of β (resp. β^+, β^-) by π. When π is a blowing up, as $\mathcal{F}_{/\,\mathrm{Fix}_\sigma}$ is monodromic, the Hopf's fiber bundle is transverse to $\mathcal{F}$ in a neighbourhood of $\tilde{\mathrm{Fix}}_\sigma \cap \pi^{-1}(0)$. The Hopf's fiber bundle gives a homeomorphism between $\tilde{\beta}^+$ (resp. $\tilde{\beta}^-$) and $\gamma(]0,1[)$, where γ is a path $\gamma : [0,1]$ such that $\gamma([0,1]) = \tilde{\mathrm{Fix}}_\sigma \cap \pi^{-1}(0)$. Thus in this case $\mathcal{P}$ is the holomomy diffeomorphism corresponding to $\gamma \cdot \gamma$. When π is not a blowing up $\pi^{-1}(0) \cap \tilde{\mathbf{R}}_0^2$ is not smooth in general and we have not a fiber bundle transverse to $\mathcal{F}$. But as $\mathcal{F}_{/\,\mathrm{Fix}_\sigma}$ is monodromic $\pi^{-1}(0) \cap \tilde{\mathbf{R}}_0^2$ has only saddle points and we can use as Brunella [Br] the Dulac real maps in the corners or those technics like Berthier and Moussu [BM]. In the particular case of $\mathcal{F}_\omega^{\mathbf{C}}$, where $\omega = x\,dx + y\,dy$ for for each antiholomorphic involution σ which does not fix any invariant curve of $\mathcal{F}_\omega^{\mathbf{C}}$ and such that $\sigma^*\mathcal{F}_\omega^{\mathbf{C}} = \mathcal{F}_\omega^{\mathbf{C}}$, $\mathcal{F}_{\omega\,/\,\mathrm{Fix}_\sigma}^{\mathbf{C}}$ is a center since $\pi^{-1}(0) \cap \tilde{\mathrm{Fix}}_\sigma$ is homotopic to $\pi^{-1}(0) \cap \tilde{\mathbf{R}}_0^2$.

5. Proof of proposition 1.1

We build here a germ of foliation satisfying the properties 1, 2 and 3 of Proposition 1.1 and whose minimal reduction of singularities is a blowing up. Furthermore the holonomy group of its exceptional divisor is not resoluble; thus according to [BCL], [P], it does not have a Liouvillian first integral.

For this we introduce:

1. $h_1(x) = iK^{-1}(\frac{x}{1-\varepsilon x})$, where $\varepsilon = \frac{1-i}{2}$ et $K(x) = x + x^5$,
2. $h_2(x) = -h_1^{-1}(x) = \frac{iK(x)}{1-\varepsilon iK(x)}$,
3. $h_3(x) = \bar{h}_2^{-1}(\bar{x}) = iK^{-1}(\frac{x}{1-\bar{\varepsilon}x})$,
4. $h_4(x) = \bar{h}_1^{-1}(\bar{x}) = \frac{iK(x)}{1+\bar{\varepsilon}iK(x)}$.

Obviously $h_4 \circ h_3 \circ h_2 \circ h_1 = \mathrm{id}_{\mathbf{C}}$. Let $\pi : \tilde{\mathbf{C}}_0^2 \to \mathbf{C}_0^2$ be a blowing up at 0 in $\mathbf{C}_0^2$, (x,t) and (s,y) the charts of $\tilde{\mathbf{C}}_0^2$ glued together by $x = sy$ and $st = 1$. Denote by m_0 (resp. m_1, m_2, m_3, m_4) the point of coordinates $(0,0)$ (resp. $(0, 1+i)$, $(0, -1+i)$, $(0, -1-i)$, $(0, 1-i)$) in the chart (x,t) and γ_1 (resp. γ_2, γ_3, γ_4) a closed path in $\pi_1(\pi^{-1}(0) \setminus \{m_1, m_2, m_3, m_4\}, m_0)$ whose index around m_1 (resp. m_2, m_3, m_4) is one. According to Theorem 1.3, there is a real analytic 1-form ω such that:

1. π is a minimal reduction of singularities of $\mathcal{F}_\omega^{\mathbf{C}}$,
2. $\mathcal{F}_\omega^{\mathbf{C}}$ is not dicritical,
3. $\mathrm{Sing}\, \tilde{\mathcal{F}}_\omega^{\mathbf{C}} = \{m_1, m_2, m_3, m_4\}$,
4. the holonomy diffeomorphism h_{γ_k} of γ_k is h_k.

Moreover, according to synthesis theorem [L], we can assume that the strict transform of $\mathcal{F}_\omega^{\mathbf{C}}$ by π is transverse to Hopf's fiber bundle and all the invariant curves, except $\pi^{-1}(0) \setminus \mathrm{Sing}\, \tilde{\mathcal{F}}_\omega^{\mathbf{C}}$, are fibers of Hopf's fiber bundle. As $h_2 \circ h_1(x) = -x$, we see that ω is a real center since its Poincaré return map is the square of $h_2 \circ h_1$. Let now show that there is an antiholomorphic involution, σ, with $\tilde{\mathrm{Fix}}_\sigma \cap \pi^{-1}(0) = \{(0, it)/t \in \mathbf{R} \cup \{\infty\}\}$ and $\mathcal{F}_\omega^{\mathbf{C}}{}_{/\mathrm{Fix}_\sigma}$ is a center. Remark that $\mathrm{Sing}\, \tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ is invariant under the map f on $\pi^{-1}(0)$, defined by $t \longmapsto -\bar{t}$ $s \longmapsto -\bar{s}$, and that:

$$h_{\gamma_k} = \phi \circ h_{f(\gamma_k)} \circ \phi, \quad \forall\, k = 1,2,3,4 \quad \text{and} \quad \phi(x) = i\bar{x}.$$

We deduce from the above equation that for any γ in $\pi_1(\pi^{-1}(0) \setminus \mathrm{Sing}\, \tilde{\mathcal{F}}_\omega^{\mathbf{C}}, m_0)$ we have:

$$(**) \qquad h_\gamma = \phi \circ h_{f(\gamma)} \circ \phi.$$

We build an antiholomorphic involution $\tilde{\sigma}$ on $\tilde{\mathbf{C}}_0^2$ which leaves $\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ invariant as follows: let (x,t) be a point of $\tilde{\mathbf{C}}_0^2$ such that x is close enough to 0 and $(0,t)$ is not a singularity, and such that γ is a path in $\pi^{-1}(0) \setminus \mathrm{Sing}\, \tilde{\mathcal{F}}_\omega^{\mathbf{C}}$

connecting $(0,t)$ to $(0,0)$. The lifting of γ in the leaf of $\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ passing through (x,t) following Hopf's fiber bundle ends at $(x_\gamma, 0)$. We define $\tilde{\sigma}_\gamma(x,t)$ to be the end of the path obtained by lifting $f(\gamma^{-1})$ from $\phi(x_\gamma, 0)$. As in the proof of Theorem 1.3 we find, using the equality (**), that $\tilde{\sigma}_\gamma$ does not depend on the choice of the path, extends to invariant curves and induces an antiholomorphic involution of $\mathbf{C}_0^2$ so that:

$$\sigma^* \mathcal{F}_\omega^{\mathbf{C}} = \mathcal{F}_\omega^{\mathbf{C}} \text{ and } \mathrm{Fix}_\sigma \cap \pi^{-1}(0) = \gamma([0,1]) = \{(0,it)/t \in \mathbf{R} \cup \{\infty\}\}$$

where the homology class of γ is $\gamma_2 \cdot \gamma_1$. Furthermore $\mathcal{F}_\omega^{\mathbf{C}}$ restricts to a center on Fix_σ since $h_\gamma^2 = \mathrm{id}_{\mathbf{C}}$. Indeed we have:

$$\begin{aligned} h_\gamma^2(x) &= (h_1 \circ h_2)^2(x) = \left[iK^{-1}\Big(\frac{K(ix)}{1+iK(ix)}\Big)\right]^2 \\ &= -K^{-1}\left[\Big(\frac{K}{1-K}\Big)\right]^2(x) = -K^{-1}\left[\frac{K\big(-K^{-1}\circ\frac{K}{1-K}\big)}{1-K\big(-K^{-1}\circ\frac{K}{1-K}\big)}\right](x) \\ &= -K^{-1}\left[\frac{\big(\frac{-K}{1-K}\big)}{\big(1+\frac{K}{1-K}\big)}\right](x) = -K^{-1}(-K)(x) = x. \end{aligned}$$

Let us show that there is no diffeomorphism, ψ, tangent to $\mathcal{F}_\omega^{\mathbf{C}}$ which conjugates ϕ_0 and σ. Assume for instance the converse. On one hand $\tilde{\psi}_{/\pi^{-1}(0)}$ is different from the identity since $\tilde{\sigma}_{/\pi^{-1}(0)}$ and $\tilde{\phi}_{0/\pi^{-1}(0)}$ are distinct. On the other hand we have:

a) either $\tilde{\psi}(m_1) = m_2$, $\tilde{\psi}(m_2) = m_3$, $\tilde{\psi}(m_3) = m_4$ and $\tilde{\psi}(m_4) = m_1$
b) or $\tilde{\psi}(m_1) = m_4$, $\tilde{\psi}(m_2) = m_1$, $\tilde{\psi}(m_3) = m_2$ and $\tilde{\psi}(m_4) = m_3$.

Let $\Sigma : (t=0)$, let $\Sigma' = \tilde{\Psi}(\Sigma)$, let α be a path in $\pi^{-1}(0) \backslash \mathrm{Sing}\, \tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ connecting the intersection of Σ and $\pi^{-1}(0)$ to the one of Σ' and $\pi^{-1}(0)$, and let $h_\alpha : \Sigma \to \Sigma'$ be the holonomy diffeomorphism associated to it. Up to assume a), the invariance of $\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ under $\tilde{\psi}$, $\tilde{\phi}_0$ and $\tilde{\sigma}$ implies that $\varphi(x) = h_\alpha^{-1} \circ \tilde{\psi}(x)$ satisfies:

1. $\varphi^{-1} \circ \bar{\varphi}(x) = i\bar{x}$,
2. $h_1 = \varphi^{-1} \circ h_2 \circ \varphi(x)$.

We have $h_2(x) = ix - \varepsilon x^2 + h.o.t$ and let $\varphi(x) = a_1 x + a_2 x^2 + h.o.t$, with $a_1 \neq 0$. From 1. we get:

$$\frac{\bar{a}_1}{a_1}\bar{x} + \left[\frac{\bar{a}_2}{a_1} - a_2\frac{\bar{a}_1^2}{a_1^3}\right]\bar{x}^2 = i\bar{x}.$$

This is equivalent to say that a_1 (resp. $a_2 + \bar{a}_2$) is equal to $\alpha \exp(\frac{2i\pi}{8})$ (resp. 0), where α is in $\mathbf{R}$. Thus a_2 is in $i\mathbf{R}$. From 2. we find:

$$ix + \frac{1}{a_1}[ia_2 - \varepsilon a_1^2 + a_2]x^2 = ix + \varepsilon\, ix^2.$$

Thus a_2 is equal to $(\frac{i\alpha^2}{2} + \frac{\sqrt{2}\alpha}{2})$ which is not in $i\mathbf{R}$, contradiction. If σ' is another antiholomorphic involution which does not fix any invariant curve and leaves $\mathcal{F}_\omega^{\mathbf{C}}$ invariant then either $\tilde{\sigma}'_{/\pi^{-1}(0)}$ is equal to $\tilde{\phi}_{0/\pi^{-1}(0)}$ or $\tilde{\sigma}_{0/\pi^{-1}(0)}$ because $\operatorname{Sing}\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ is invariant under $\tilde{\sigma}$ and $\tilde{\phi}_0$. Thus $\gamma' = \tilde{\operatorname{Fix}}_{\sigma'} \cap \pi^{-1}(0)$ is equal to $\tilde{\mathbf{R}}_0^2 \cap \pi^{-1}(0)$ or $\tilde{\operatorname{Fix}}_\sigma \cap \pi^{-1}(0)$, and therefore $\mathcal{F}_{\omega\,/\,\operatorname{Fix}_{\sigma'}}^{\mathbf{C}}$ is a center since its Poincaré return map is given by:

$$\begin{array}{rcl}\mathcal{P} : \tilde{\operatorname{Fix}}_{\sigma'} \cap \{t=0\} & \to & \tilde{\operatorname{Fix}}_{\sigma'} \cap \{t=0\} \\ m & \longmapsto & h_{\gamma'}^2(m).\end{array}$$

As the holonomy group of $\pi^{-1}(0)$ contains the following two elements which are not tangent to the identity at the same order [CM] it is not resoluble:

1. $-h_1 \circ h_4(x) = x + \sqrt{2}x^2 + h.o.t,$
2. $h_1^4(x) = x - 4\sqrt{2}x^5 + h.o.t.$

□

Remark 5.1 Let $g_1(x) = ix + x^7$, $g_2(x) = -g_1^{-1}(x)$, $g_3(x) = ix - x^7$, and $g_4(x) = -g_3^{-1}(x)$. If we substitute g_1 (resp. g_2, g_3, g_4) to h_1 (resp. h_2, h_3, h_4) in the previous proof we obtain a real analytic center, ω', and an antiholomorphic involution, σ, at 0 in $\mathbf{C}^2$ such that:

1. $\sigma^*\mathcal{F}_{\omega'}^{\mathbf{C}} = \mathcal{F}_{\omega'}^{\mathbf{C}}$
2. $\tilde{\operatorname{Fix}}_\sigma \cap \pi^{-1}(0) = \{(0, it)/t \in \mathbf{R} \cup \{\infty\}\}.$

As $\operatorname{Sing}\tilde{\mathcal{F}}_{\omega'}^{\mathbf{C}} \cap \tilde{\operatorname{Fix}}_\sigma$ is empty, the flow box theorem implies that $\mathcal{F}_{\omega'\,/\,\operatorname{Fix}_\sigma}^{\mathbf{C}}$ is monodromic. $\tilde{\mathcal{F}}_{\omega'}^{\mathbf{C}} \cap \tilde{\operatorname{Fix}}_\sigma$ is not a center since its Poincaré return map is given by:

$$\begin{array}{rcl}\mathcal{P} : \exp^{\frac{2i\pi}{8}}\mathbf{R} & \longrightarrow & \exp^{\frac{2i\pi}{8}}\mathbf{R} \\ x & \longmapsto & h_\gamma^2(x\exp^{\frac{2i\pi}{8}}) = \exp^{\frac{2i\pi}{8}}(x - 4x^7) + h.o.t.\end{array}$$

6. Proof of proposition 1.2

We are going to build, using 1.3, a real analytic 1-form whose complexification has the properties prescribed. For this we express the following result [BCL]: there are an irrational λ and a germ of real analytic diffeomorphism of $\mathbf{R}$ at 0 such that the group $\langle \exp^{2i\pi\lambda} x, \varphi(x)\rangle$ is free. Moreover, up a ramification by $x \longmapsto x^2$ and to change λ by $\frac{\lambda}{2}$, we assume that $\varphi(-x) = -\varphi(x)$ for x small enough. Let π be a morphism given by a finite sequence of blowings-up and such that:

1. $\pi^{-1}(0)$ has two irreductibles components C_1 and C_2 whose Chern numbers are respectively -2 and -1,
2. the complex structure of $\tilde{\mathbf{C}}_0^2$ is given by the charts (x_2, y), (x_1, t_2) and (x, t_1) glued together by $x_2t_2 = 1$, $x_1t_1 = 1$, $x_1 = yx_2$ and $x = yx_1$.

Let m_0(resp. m_1, $\tilde{m}_1$, m_2, $\tilde{m}_2$) be the point whose coordinates are $(0,0)$ (resp. $(1,0)$, $(-i,0)$, $(0,i)$, $(0,-i)$) in the chart (x_1,t_1). Let us choose a chart (u,v) of $\tilde{\mathbf{C}}_0^2$ at m_0 which does not contain neither m_k, nor $\tilde{m}_k$, $k = 1,2$, and such that $u = 0$ (resp. $v = 0$) is a local equation of C_1 (resp. of C_2). Choose a positive real ε enough close to 0 and let introduce the following paths, which does not contain m_0:

1. γ_1 (resp. γ_2) is the real path which begins at the point whose coordinates are $(0,\varepsilon)$ (resp. $(\varepsilon,0)$) and ends to the one whose coordinates are $(0,-\varepsilon)$ (resp. $(-\varepsilon,0)$) in (u,v),
2. γ_1' (resp. γ_2') is the image of the map $t \in [0,1] \longmapsto (0, -\varepsilon \exp i\pi t)$ (resp. $t \in [0,1] \longmapsto (-\varepsilon \exp i\pi t, 0)$).

The Theorem 1.3 allows us to build a real analytic 1-form, ω, such that:

1. π is a minimal reduction of singularities of $\mathcal{F}_\omega^{\mathbf{C}}$,
2. the singularities of $\mathcal{F}_\omega^{\mathbf{C}}$ are m_0, m_1, $\tilde{m}_1$, m_2 and $\tilde{m}_2$,
3. C_1 and C_2 are not dicritical,
4. the holonomy diffeomorphism associated to $\gamma_1.\gamma_1'$ (resp. $\gamma_2 \cdot \gamma_2'$) is $h_{\gamma_1\cdot\gamma_1'}(u) = \exp^{\frac{-i\pi}{\lambda_1}} u$ (resp. $h_{\gamma_2\cdot\gamma_2'}(v) = \exp^{2i\pi\lambda}\varphi(v)$), where $\lambda_1 > 0$ and $\exp^{\frac{-i\pi}{\lambda_1}} \times \exp 2i\pi\lambda = 1$,
5. the germ of $\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ at m_0 is $\tilde{\omega}_m = u\,dv + \lambda_1 v\,du$.

By construction the holonomy group of C_2 is $G = \langle \exp^{\frac{-i\pi}{\lambda_1}} v, \exp^{2i\pi\lambda}\varphi(v)\rangle = \langle \exp^{4i\pi\lambda} v, -\exp^{2i\pi\lambda}\varphi(v)\rangle$ which is free because it is isomorphic to the free group $\langle \exp^{2i\pi\lambda} v,\ \varphi(v)\rangle$. According to [BCL], [P] ω has no liouvillian first integral. Let show that ω is a center. Since the only real singularity of $\tilde{\mathcal{F}}_\omega^{\mathbf{C}}$ is the corner m_0 which is a saddle and since C_1 and C_2 are non dicritical, the flow box theorem assures that $\mathcal{F}_\omega$ is monodromic. In order to prove that it is a center, let us introduce the Dulac real maps in the corner:

$$\begin{array}{rcl}
D_1 : \{\varepsilon\} \times \mathbf{R}^+ \setminus (u=\varepsilon) \cap \tilde{\mathbf{R}}_0^2 & \longrightarrow & \mathbf{R}^+ \times \{\varepsilon\} \setminus (v=\varepsilon) \cap \tilde{\mathbf{R}}_0^2 \\
(\varepsilon, v) & \longmapsto & (\varepsilon^{1-\frac{1}{\lambda_1}} v^{\frac{1}{\lambda_1}}, \varepsilon), \\
D_2 : \mathbf{R}^+ \times \{-\varepsilon\} \setminus (v=-\varepsilon) \cap \tilde{\mathbf{R}}_0^2 & \longrightarrow & \{\varepsilon\} \times \mathbf{R}^- \setminus (v=\varepsilon) \cap \tilde{\mathbf{R}}_0^2 \\
(u, \varepsilon) & \longmapsto & (\varepsilon, \varepsilon^{1-\lambda_1} u^{\lambda_1}), \\
D_3 : \{-\varepsilon\} \times \mathbf{R}^+ \setminus (u=-\varepsilon) \cap \tilde{\mathbf{R}}_0^2 & \longrightarrow & \mathbf{R}^- \times \{\varepsilon\} \setminus (v=\varepsilon) \cap \tilde{\mathbf{R}}_0^2 \\
(-\varepsilon, v) & \longmapsto & (-\varepsilon^{1-\frac{1}{\lambda_1}} v^{\frac{1}{\lambda_1}}, \varepsilon), \\
D_4 : \mathbf{R}^- \times \{-\varepsilon\} \setminus (v=-\varepsilon) \cap \tilde{\mathbf{R}}_0^2 & \longrightarrow & \{-\varepsilon\} \times \mathbf{R}^- \setminus (v=-\varepsilon) \cap \tilde{\mathbf{R}}_0^2 \\
(u, -\varepsilon) & \longmapsto & (-\varepsilon, -\varepsilon^{1-\lambda_1}(-u)^{\lambda_1}).
\end{array}$$

The Poincaré return map is given by:

$$v \longmapsto p_2 \circ h_{\gamma_2}^{-1} \circ D_4 \circ h_{\gamma_1} \circ D_3 \circ h_{\gamma_2} \circ D_2 \circ h_{\gamma_1} \circ D_1(\varepsilon, v),$$
$$p_2(u,v) = v.$$

We easily find that $h_{\gamma_1'}(u) = e^{-i\pi/\lambda_1} u$ and $h_{\gamma_2'}(v) = e^{2i\pi/\lambda} v$ so:

$$\begin{array}{rcl}
h_{\gamma_1} : u \times \{\varepsilon\} \subset (u,v) & \longrightarrow & u \times \{\varepsilon\} \subset (u,v) \\
(u, \varepsilon) & \longmapsto & (u, \varepsilon)
\end{array}$$

and

$$\begin{array}{rcl}
h_{\gamma_2} : \{\varepsilon\} \times v \subset (u,v) & \longrightarrow & \{-\varepsilon\} \times v \subset (u,v) \\
(\varepsilon, v) & \longmapsto & (-\varepsilon, \varphi(v)).
\end{array}$$

Thus $\mathcal{P}$ is:

$$\mathcal{P}(v) = \varphi^{-1} \circ (-\varphi(-v)).$$

As $\varphi(x) = -\varphi(-x)$ $\mathcal{P}$ is equal to the identity and ω is a center. □

References

[BCL] Berthier M., Cerveau D. and Lins Neto A., *Sur les feuilletages analytiques réels et le problème du centre.* Journal of differential equations, **131**, n°2, November 1, (1996), 244–266.

[BCM] Berthier M., Cerveau D. and Meziani R., *Transformations isotropes des germes de feuilletages holomorphes.* J. Math. Pures Appl. **78** (1999), 701–722.

[BM] Berthier M. and Moussu R., *Reversibilité et classification des centres nilpotents.* Ann. de l'Institut Fourrier **44**, 2, (1994), 465–494.

[Br] Brunella M., *Vanishing holonomy and monodromy of certain centers and foci.* Astérique **222** (1994), 37–48.

[CS] Camacho C. and Sad P., *Invariant varieties through singularities of holomorphic vector fields.* Annals of Maths. **115** (1982), 579–595.

[CM] Cerveau D. and Mattei J.-F., *Formes intégrables singulières.* Astérisque **97** (1982), 1–193.

[G] Garba Belko D., *Trois études sur les feuilletages holomorphes: cohomologie relative, rigidité en famille et problème du centre.* Thèse de l'université de Rennes1, 1998.

[GH] Griffiths P. and Harris J., *Principles of algebraic geometry.* John Wiley and Sons. New York, 1978.

[Ko] Koenigs G., *Recherches sur les intégrales de certaines équations fonctionnelles.* Annales scientifiques de l'école normale supérieur, 3^{ieme} série, **1** supplément, 1994, 1–14.

[L] Lins Neto A., *Construction of singular holomorphic vector fields and foliations in dimension two.* J. Differential Geometry **26** (1987), 1–31.

[MR] Martinet J. and Ramis J.-P., *Classification analytique des équations différentielles non linéaires résonnantes du premier ordre.* Annales de l'institut Fourier **44**, 2, (1994), 571–621.

[MM] Mattei J.F. and Moussu R., *Holonomie et intégrales premières.* Annales scientifiques de l'école normale supérieur, 4^{ieme} série, t. 13, 1980, 469–523.

[Me] Meziani R., *Classification analytique des équations différentielles $y\,dx + \cdots =$ et espaces de modules.* Bol. Soc. da. Bras. Math. **27**, 1, (1996), 23–53.

[P] Paul E., *Feuilletages holomorphes singulier à holonomie résoluble.* J. reine angrew. Math. **514** (1999), 9–70.

[Po] Poincaré H., *Mémoire sur les courbes définies par une équation différentielle.* J. Math. Pures Appl. **3**, 7, 1881 et **4**, 1, 1885, 167–244.

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