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Absence of resonances for semiclassical Schrödinger operators with Gevrey coefficients

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Abstract. We give lower bounds on resonance free domains for a Schrödinger operator $-h^2\Delta+V$ in the semi-classical limit $h\to 0$, near a non-trapping energy level E_0, when the potential V is dilation analytic at infinity, but only of Gevrey class in a compact set of $\mathbf{R}^n$.

Key words: microlocal spectral asymptotics, semi-classical analysis, h-pseudodifferential operators, Gevrey classes, resonances.

Introduction

In this paper, we give lower bounds on the width of resonance free domains near a non-trapping energy level, for semi-classical operators like

$$P=-h^2\Delta+V(x) \tag{0.1}$$

as V is long range, dilation analytic at infinity but may be only of Gevrey class on a compact set of $\mathbf{R}^n$.

As is well known, (see e.g. [Sj2]), many of the phenomena in semi-classical Quantum Mechanics have their counterpart in geometrical optics. Namely, if one considers the exterior Dirichlet problem for the Helmholtz equation

$$(\Delta+k^2)u=0 \quad \text{in} \quad \mathbf{R}^n\setminus\Omega\ (n \text{ odd}) \tag{0.2}$$

for a bounded domain Ω with smooth boundary, the resonances can be defined as the poles of the scattering matrix in the framework of the Lax-Phillips theory [LaPh], or as the set of $k\in\mathbf{C}$, $\operatorname{Im}k<0$, for which (0.2) has a non-trivial solution in some suitable Hilbert space. If the obstacle is non trapping and has a C^∞ boundary, it follows from the results of propagation of singularities of Melrose-Sjöstrand and Ivrïi (see [Hö Chap. 24]), that there are only finitely many resonances inside any logarithmic neighborhood of

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the real axis. For an analytic boundary, C. Bardos, G. Lebeau and J. Rauch [BaLeR] showed that there can be only finitely many resonances inside a parabolic neighborhood of the real axis of the form $\operatorname{Im} k \geq -C\langle \operatorname{Re} k\rangle^{1/3}$ (here we use the notation $\langle x\rangle = (1+x^2)^{1/2}$); this again follows from the results on propagation of G^3 singularities due to Lebeau [Le]. More recently, B. and R. Lascar [BRLas] showed in turn that if Ω has a $G^{s'}$ boundary, there are at most a finite number of resonances inside $\operatorname{Im} k \geq -C\langle \operatorname{Re} k\rangle^{1/s}$, for all $s > 2s'+1$.

Now, consider the Schrödinger operator (0.1). We say that $V \in C^\infty(\mathbf{R}^n)$ is dilation analytic outside a compact set $K \subset \mathbf{R}^n$ if V extends analytically in a domain:

$$\Gamma = \left\{x \in \mathbf{C}^n \mid |\operatorname{Im} x| \leq C\langle \operatorname{Re} x\rangle,\ \operatorname{Re} x \in \mathbf{R}^n \setminus K\right\} \tag{0.3}$$

where it satisfies:

$$\lim_{x\in\Gamma, |x|\to\infty} V(x) = 0 \tag{0.4}$$

Then $\inf \sigma_{\text{ess}}(P) = 0$, (actually P has only continuous spectrum above 0) and we may define the resonances of P near the energy level $E_0 > 0$ by the method of analytic distorsions (W. Hunziker [Hu], B. Simon [Si] where technics of exterior complex scaling were introduced). When $K = \emptyset$, this method reduces to the celebrated Aguilar-Balslev-Combes theory of analytic dilations (see [ReSi Chap. 13]). Indeed, let $M \subset \mathbf{C}^n$ be a real submanifold of dimension n, totally real (i.e. $\forall x \in M,\ T_xM \cap iT_xM = 0$) and $P = \sum_{|\alpha|\leq d} a_\alpha(z)(hD_z)^\alpha$ a differential operator with C^∞ coefficients in some suitable complex neighborhood $M^{\mathbf{C}}$ of M. (Here D_z denotes the holomorphic derivative with respect to coordinates in $M^{\mathbf{C}}$). Then we can define a differential operator $P_M : C^\infty(M) \to C^\infty(M)$ such that, if u is holomorphic, then $(Pu)_{|M} = P_M(u_{|M})$. Now assume P is dilation analytic outside a compact set K in the sense above. For $0 \leq \theta \leq \theta_0$, we let $M = M_\theta$ be parametrized by $f_\theta \in C^\infty(\mathbf{R}^n; \mathbf{C}^n)$ such that $f_\theta(x) = x$ for x in a neighborhood of K and $f_\theta(x) = e^{i\theta}x$ for large x. The corresponding family of operators $P_\theta = P_{M_\theta}$ on L^2 is known to be an analytic family of type (A); the essential spectrum of P_θ is now $e^{-2i\theta}\mathbf{R}^+$, and when $\theta > 0$, P_θ may also have discrete eigenvalues in the lower half plane near E_0, which are called (outgoing) resonances. The resonnant (or extended) states are the associated eigenfunctions. This presentation due to J. Sjöstrand and M. Zworski [SjZ] is adapted from [Hu] (see also [Co], or [HeSj] where a phase-space theory of resonances was introduced; various equivalences are proved in [HeMa]).

Let $E_0 > 0$ (i.e. an energy away from the threshold). Conditions (0.3) and (0.4) ensure that the underlying classical dynamical system is non-trapping near infinity for all energies $E \in I = [E_0 - \varepsilon_0, E_0 + \varepsilon_0]$ for $\varepsilon_0 > 0$ small enough. Actually, if $p(x,\xi) = \xi^2 + V(x)$ is the classical hamiltonian and $H_p = \frac{\partial p}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial}{\partial \xi}$ the hamiltonian vector field, then $\exp tH_p(x,\xi) \to \infty$, as t tends either to $+\infty$ or to $-\infty$ for all $(x,\xi) \in p^{-1}(I)$ if x is large enough. This is a consequence of the virial condition, which is in turn implied by (0.4) and Cauchy's inequalities for large x. Let

$$\Gamma_\pm(I) = \left\{(x,\xi) \in p^{-1}(I) \mid \exp tH_p(x,\xi) \not\to \infty \quad \text{as} \quad t \to \mp\infty\right\}$$

be the outgoing and incoming tails, and $K(I) = \Gamma_+(I) \cap \Gamma_-(I)$ be the set of trapped trajectories (see the Appendix of [GeSj]). When $K(I) = \emptyset$, we can construct a global escape function (in the terminology of [HeSj] and [MRS] for the exterior Dirichlet problem), i.e. construct $G \in C^\infty(\mathbf{R}^{2n})$ such that:

$$H_pG(x,\xi) \geq C_0 > 0 \;\forall (x,\xi) \in p^{-1}(I) \quad \text{for some} \quad C_0 > 0. \qquad (0.5)$$

When the potential is everywhere analytic and under some global virial conditions, which imply $K(I) = \emptyset$, it is known ([BrCoDu], [K], [Na1]) that there are no resonances for $h > 0$ small enough in a h-independent neighborhood of E_0 in $\operatorname{Im} z < 0$. This result was obtained already implicitely in [HeSj] for more general V and under the mere assumption $K(I) = \emptyset$.

A natural question arises when considering non analytic potentials; thus the distribution of poles for the S-matrix of short range Schrödinger (or more general) operators is discussed by W. Goodhue [Go] for a potential of Gevrey class. More recently S. Nakamura [Na2] proved the absence of resonances for some short range C^∞ potentials. Here we prove the following:

Theorem 0.1 *Let $P = -h^2\Delta + V(x)$, where $V \in G^s(\mathbf{R}^n)$ is dilation analytic outside a compact set $K \subset \mathbf{R}^n$ (in the sense above). Assume $E_0 > 0$ is a non-trapping energy, i.e. there is $\varepsilon_0 > 0$ such that $K(I) = \emptyset$ for $I = [E_0 - \varepsilon_0, E_0 + \varepsilon_0]$. Then there is $\delta > 0$ such that P has no resonances in $I - i[0, \delta h^{1-1/s}]$ for $h > 0$ sufficiently small.*

In [Ro] we have constructed a potential of class G^s such that there are actually resonances E near a non-trapping energy level $E_0 > 0$ with $\operatorname{Im} E \approx - \text{Const. } h^{1-1/s}$, for $h > 0$ small enough; so we can consider that our lower bound in Theorem 0.1 is optimal.

Let us say a few words about the proof. The main simple idea is to derive a priori estimates for an eigenfunction u associated to the resonance E, which in turn imply a lower bound on Im E. It is convenient to work in the Bargmann representation which describes u (or rather its image Tu by the Bargmann or FBI transform T), both in position and frequency variables. As a rule Tu is small away from the characteristics of P_θ. But where dilation is turned on (i.e. for large x), P_θ is elliptic; to give a global control on Tu, it suffices to gain some ellipticity in the "classically allowed phase-space region"; there, by an argument of non-characteristic, or lagrangian deformations in the directions of the hamiltonian vector field (for complex, small h-dependent times) and using almost analytic extensions of the potential, we show that condition (0.5) implies that Im P_θ becomes everywhere, but in some weak sense, non characteristic; this gives the control on Im E. Thus our proof heavily relies on the analytic machinery of [Sj] suitably adapted to the Gevrey classes. In particular, we deform a plurisubharmonic weight function. An alternative approach would consist in using the h-Pseudodifferential Weyl calculus of [Hö] in suitable $S(m, g)$ classes, but in the real domain (see [BRLas]). Note that the corresponding exterior Dirichlet problem treated in [BRLas] is considerably more difficult than ours, but our method is somewhat conceptually different. It may also help to reformulate in a simpler way the theory of [HeSj] when the potential decays at infinity as in (0.4).

Note also that the width $h^{1-1/s}$ is expected since this is precisely the order of magnitude of admissible deformations of weights in the FBI transform (and hence the size for the domain of ellipticity), for G^s functions: see e.g. [BLas], [Le], [Li], ... We include also an auxiliary result on almost analytic extensions of Gevrey functions which makes precise a well known theorem (but for which we did not find any references in the literature).

It should be worth looking at the C^∞ case, where we would expect the $h^{1-1/s}$ term in Theorem 0.1 to be replaced by $h \log 1/h$. It would be also natural to ask what happens if the potential is everywhere of class G^s; the resonances would then not be intrinsically defined, but maybe only with an exponentially small uncertainty (in some fractional power of h) if we assume some decrease of $\overline{\partial} V(x)$ near infinity. Se also [GeSig] for a time-dependent approach.

1. Review of h-pseudo differential calculus in Gevrey classes

We begin with some statements on Weyl calculus with small parameter $h > 0$, essentially due to B. Lascar [BLas].

Let V be a n-dimensional vector space over $\mathbf{R}$, V' its dual space, and $W = V \oplus V'$. The duality pairing is denoted simply by $(x, \xi) \in W \mapsto x.\xi$. Actually we shall use the results below in a different context, where W is replaced by a totally real submanifold Λ_Φ of $\mathbf{C}^n$. As the calculus is exactly the same, it is not suitable to carry it out in this setting, for it would uselessly complicate the notations. However, we shall show the link with the calculus as it will be used in this work in Section 2. The presentation below is adapted from Hörmander [Hö Chap. 18], and Ivrïi [Iv Chap. 1] in the C^∞ case.

a) Some classes of symbols

We recall the definition of G^s symbols, for $s > 1$. Such classes were introduced by L. Boutet de Monvel and P. Krée [BoK] for functions partially holomorphic with respect to x, then extended by R. Lascar [RLas], where a G^s regularity in the cotangent variable is permitted as well, for the purposes of microlocal analysis. For $m \in \mathbf{Z}$, we denote by $S(m, s)$ the set of $a \in C^\infty(W)$, such that $\exists A, C > 0$ with:

$$\sup_{(x,\xi)\in W} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi, h)| \leq C h^{-m} A^{|\alpha+\beta|} \alpha!^s \beta!^s \tag{1.1}$$

for all multi-indices $\alpha, \beta \in \mathbf{N}^n$. We call a a G^s symbol of order m, and emphasize that we do not require here any decrease in ξ. The variable in W will be denoted by $X = (x, \xi)$. As usual, if $\alpha = (\alpha_1, \ldots, \alpha_n)$, we note $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\alpha! = \alpha_1! \cdots \alpha_n!$. If $T = (T_x, T_\xi) \in (\mathbf{R}^+)^{2n}$, $\mathbf{R}^+ =]0, +\infty[$, we shall write $T \succ 0$, and $T^{(\alpha,\beta)} = T_{x_1}^{\alpha_1} \cdots T_{\xi_1}^{\beta_1} \cdots$. Let:

$$N_s(a, m, T)(X, h) = h^m \sum_{\alpha,\beta\in\mathbf{N}^n} |\partial_x^\alpha \partial_\xi^\beta a(X, h)| T^{(\alpha,\beta)} / \alpha!^s \beta!^s \tag{1.2}$$

and:

$$\overline{N}_s(a, m, T)(h) = \sup_{X\in W} N_s(a, m, T)(X, h)$$

which converges if $T \succ 0$ is small enough. (Note that we can replace the supremum norm by suitable Sobolev norms). Then $a \in S(m,s)$ iff $\overline{N}_s(a,m,T)(h) = \mathcal{O}(1)$, uniformly for $h>0$ small enough, for some $T \succ 0$.

Let $S(m,T,s)$ be the set of those $a \in S(m,s)$ such that $\overline{N}_s(a,m,T)$ is uniformly bounded for $h>0$ small enough. If $a \in S(m',T,s)$, $b \in S(m,T,s)$, then $ab \in S(m+m',T,s)$ and we have the easy multiplicative property:

$$\overline{N}_s(ab,m'+m,T)(h) \leq \overline{N}_s(a,m',T)(h)\overline{N}_s(b,m,T)(h)$$

In particular, $\Sigma(T,s) = \bigoplus_{m\in\mathbf{Z}} S(m,T,s)$ is a graded Banach algebra, and $\big(\Sigma(T,s)\big)_{T\succ 0}$, for the usual ordering on $(\mathbf{R}^+)^{2n}$ a decreasing family of Banach algebras, whose union is exactly $S(s) = \bigoplus_{m\in\mathbf{Z}} S(m,s)$. The topology of $S(m,s)$ is the inductive limit of the $S(m,T,s)$ topologies filtered by the norms $\overline{N}_s(\cdot,m,T)(h)$ as $T \succ 0$, $T \to 0$.

We next define the class of negligible symbols for our pseudo-differential calculus as:

$$\begin{aligned} S(-\infty,s) = \{a \in C^\infty(W) \mid \exists T \succ 0\ \forall m \in \mathbf{Z}:\ &\overline{N}_s(a,m,T)(h) \\ &\leq C\exp(-1/Ch^{1/s}) \text{ for some } C = C(m,T),\ 0<h<h_0\} \end{aligned}$$

Note that we can replace "for all m" by "there is some m" in the definition above.

We need also the existence of almost analytic extensions for G^s symbols. Indeed, if $a \in S(m,s)$, then Corollary a.4 (see Appendix) shows that for all $s' > s$, there exists an almost analytic extension $\widetilde{a}(x,\xi,h)$ of $a(x,\xi,h)$ to a neighborhood $\widetilde{W} = W + iB(0,c)$ of W in $W^{\mathbf{C}}$ (the complexification of W), such that $\widetilde{a} \in S(m,s')$ and

$$|\overline{\partial}_{(x,\xi)}\widetilde{a}(X,h)| \leq Ch^{-m}\exp(-|\operatorname{Im} X|^{-1/(s-1)}/C), \quad X = (x,\xi) \in \widetilde{W} \tag{1.3}$$

We also have to consider more general symbols. Following [BLas] we say that $a(x,\xi,h)$ is a G^s symbol of degree d and order m iff $a \in C^\infty(W)$, and $\exists A$, $C>0$ such that:

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi,h)| \leq Ch^{-m}A^{|\alpha+\beta|}\alpha!^s\beta!^s\langle\xi\rangle^{d-|\beta|} \tag{1.4}$$

(We could as well introduce more general spaces as $S(m,g)$ where m is a weight and $g(x,\xi)$ is a slowly varying metric, see [HöIII, Sect. 18] and [Iv]. Here $g(x,\xi) = dx^2 + \langle\xi\rangle^{-2}d\xi^2)$. We shall denote this class by $S((m,d),s)$, whose topology can be described exactly as the $S(m,s)$ topology. A careful inspection of the proof of Theorem a.3 and Corollary a.4 shows also that the almost analytic extension process carries over the $S((m,d),s)$ class. More precisely, if $a \in S((m,d),s)$, and $s' > s$, there exists $\widetilde{a} \in S((m,d),s')$ in a neighborhood $\widetilde{W} = W + iB(0,c)$ of W in $W^{\mathbf{C}}$ such that

$$|\overline{\partial}_{(x,\xi)}\widetilde{a}(X,h)| \leq Ch^{-m}\langle\xi\rangle^{d-1}\exp(-|\operatorname{Im} X|^{-1/(s-1)}/C), \qquad X = (x,\xi) \in \widetilde{W} \tag{1.5}$$

b) h-Pseudodifferential Weyl calculus

It is very convenient to work with Weyl quantization of G^s symbols. With any $a \in \mathcal{S}(W)$ (Schwartz space) we associate the operator

$$\begin{aligned} &a^w(x,hD,h)u(x,h) \\ &\quad = (2\pi h)^{-n}\iint e^{i(x-y)x/h}a((x+y)/2,\xi,h)u(y)dy\,d\xi, \quad u \in \mathcal{S}(V) \end{aligned} \tag{1.6}$$

(here dy is a Lebesgue measure in V and $d\xi$ is the dual one in V' such that Fourier inversion formula holds with the usual constant). If $a,b \in \mathcal{S}(W)$ then:

$$a^w(x,hD,h)b^w(x,hD,h) = c^w(x,hD,h) \tag{1.7}$$

where $c = a\sharp b$ is given by:

$$c(X,h) = (\pi h)^{-2n}\iint e^{-2i\sigma(Y,Z)/h}a(X+Y,h)b(X+Z,h)dYdZ. \tag{1.8}$$

Here, $X = (x,\xi)$, $Y = (y,\eta)$, $Z = (z,\zeta)$, σ is the canonical symplectic 2-form defined by $\sigma(Y,Z) = z\eta - y\zeta$. An arbitrary continuous linear map on $S(m',s) \times S(m,s)$ is not determined by its restriction to G^s functions in $\mathcal{S}(W) \times \mathcal{S}(W)$, since this is not a dense set; so following [Hö, Def. 18.4.9] we say that the bilinear continuous map $(a,b) \mapsto a\sharp b$ is weakly continuous iff its restriction to a bounded subset of $S(m',s) \times S(m,s)$ is continuous for the G^s topology. A weakly continuous map is then determined by its restriction to $\big(\mathcal{S}(W) \cap S(m',s)\big) \times \big(\mathcal{S}(W) \cap S(m,s)\big)$, since $a \in S(m,s)$

can be approximated by $\sum_{g\in\mathbf{Z}^{2n},|g|\leq N} X_g a$, $N\to\infty$ (here X_g are cutt-off functions; see Corollary a.4 below for notations). The main result on Weyl calculus of G^s symbols is the following:

Proposition 1.1 [BLas] *The composition formula* (1.8) *extends from* $\mathcal{S}(W)\times\mathcal{S}(W)\to\mathcal{S}(W)$ *to a bilinear map* $S(m',s)\times S(m,s)\to S(m+m',s)$, $(a,b)\to a\sharp b$. *More precisely, let* $\chi\in G_0^s(W^2)$ *be equal to* 1 *near* $(Y,Z)=0$,

$$\begin{aligned} &c_\chi(X,h)\\ &\quad=(\pi h)^{-2n}\iint e^{-2i\sigma(Y,Z)/h}a(X+Y,h)b(X+Z,h)\chi(Y,Z)dYdZ \end{aligned} \tag{1.9}$$

and $c_{\widehat{\chi}}(X,h)=c(X,h)-c_\chi(X,h)$. *Then the bilinear maps* $(a,b)\mapsto(a\sharp b)_\chi$ [*resp.* $(a,b)\mapsto(a\sharp b)_{\widehat{\chi}}$] *are weakly continuous from* $S(m',s)\times S(m,s)$ *to* $S(m+m'+2n,s)$ [*resp. to* $S(-\infty,s)$]. *Moreover, the map* $(a,b)\in S(m',s)\times S(m,s)$, *to the remainder term* $r_{a,b}$ *defined by*

$$\begin{aligned} r_{a,b}(X,h)&=c_\chi(X,h)\\ &\quad-\sum_{j<N}\frac{h^j}{j!}\big(i\sigma(D_x,D_\xi,D_y,D_\eta)/2\big)^j a(X,h)b(Y,h)_{|X=Y} \end{aligned}$$

is valued in $S(m'+m-N,s)$ *for all* $N\in\mathbf{N}$, (*but may not be weakly continuous*).

Remark Using stationary phase we can show that $(a,b)\in S(m',s)\times S(m,s)\to r_{a,b}\in S(m+m'-N,2s-1)$ is weakly continuous, that is (weak) continuity holds with a loss of $s-1$ Gevrey regularity. This fact will not be used in the sequel however.

We consider now a division problem. We say that $b\in S(0,s)$ is elliptic at infinity iff $|b(x,\xi,h)|\geq C>0$ for (x,ξ) outside a compact set $K_1\subset\subset W$. Let $c\in S(0,s)$ vanish in a neighborhood of K_1. We have:

Proposition 1.2 [BLas] *Let* b, c *be as above. Then there exist left and right parametrices* $a_L,a_R\in S(0,s)$ *and remainders* $r_L,r_R\in S(-\infty,s)$ *such that*:

$$\begin{aligned} a_L^w(x,hD,h)b^w(x,hD,h)&=c^w(x,hD,h)+r_L^w(x,hD,h)\\ b^w(x,hD,h)a_R^w(x,hD,h)&=c^w(x,hD,h)+r_R^w(x,hD,h) \end{aligned} \tag{1.10}$$

As an application of the notions and statements we have just reviewed we recall the:

Proof. We treat the case of the right parametrix a_R, the other one is similar. We may assume $b \in \mathcal{S}(W)$ as before. Let $\phi(x,\xi) \in G^s(W)$ to be equal to 0 on some neighborhood of K_1, to 1 on a neighborhood of $\operatorname{supp} c$. For fixed N, we construct by means of successive divisions and Proposition 1.1, a symbol $a' \in S(0,s)$ such that $b\natural a' = \phi - r$, with $r \in S(-N,s)$. By induction, define a sequence of functions r_j such that $r_0 = 1$, $r_j = (r\sharp r_{j-1})_\chi$, $j \geq 1$, where the cutoff function χ is chosen as in Proposition 1.1. Put $r' = 1 + r_1 + r_2 + \cdots$, and compute $(1-r)\sharp r'$. Using again Proposition 1.1 we see that the series defining r' is summable, and still with notations of Proposition 1.1: $(1-r)\sharp r' = 1 - \sum_{j\geq 0}(r\sharp r_j)_{\widehat{\chi}}$. We start with estimating r_j. For $T' \succ 0$ small enough, and $j \geq 1$ we have by Proposition 1.1:

$$\overline{N}_s(r_j,0,T')(h) \leq C_0 h^{-2n}\overline{N}_s(r,0,T')(h)\overline{N}_s(r_{j-1},0,T')(h)$$

where C_0 depends only on the diameter of $\operatorname{supp}\chi$. (The proof of Proposition 1.1 tells us that we can take the same T' on both sides of the inequality, although this is actually irrelevant). Since $\overline{N}_s(r,0,T')(h) \leq Ch^N$, we get for some new constant $C > 0$: $\overline{N}_s(r_j,0,T')(h) \leq C^{j+1}h^{(N-2n)j}$. On the other hand, from Proposition 1.1 and its proof, we can find $C_1 > 0$ depending on the diameter of $\operatorname{supp}\chi$ only, but not on r or r_j, and $0 \prec T \prec T'$ small enough, such that:

$$\overline{N}_s\big((r\sharp r_j)_{\widehat{\chi}},0,T\big)(h) \leq C_1\overline{N}_s(r,0,T')(h)\overline{N}_s(r_j,0,T')(h)e^{-1/C_1h^{1/s}}$$

From the above estimates there follows that for $N \geq 2n+1$ the series $r'' = \sum_{j\geq 1}(r\sharp r_j)_{\widehat{\chi}}$ converges absolutely in the $S(0,T,s)$ topology for $h > 0$ small enough, and:

$$\sum_{j\geq 1}\overline{N}_s\big((r\sharp r_j)_{\widehat{\chi}},0,T\big)(h) \leq C_2 e^{-1/C_1h^{1/s}}$$

So $r'' \in S(-\infty,s)$ and:

$$\begin{aligned} b\sharp(a'\sharp r'\sharp c) &= (b\sharp a')\sharp(r'\sharp c) = (\phi - r)\sharp(r'\sharp c) \\ &= \big((1-r)\sharp r'\big)\sharp c + (\phi-1)\sharp r'\sharp c = c - r''\sharp c + (\phi - 1)\sharp r'\sharp c \end{aligned}$$

Because $\operatorname{supp}(\phi-1) \cap \operatorname{supp} c = \emptyset$, we can choose $\operatorname{supp}\chi$ so small that $(\phi - 1)\sharp c \in S(-\infty,s)$, so the last 2 terms belong to $S(-\infty,s)$ and the right parametrix is given by $a_R = a'\sharp r'\sharp c$, and $r_R = -r''\sharp c + (\psi - 1)\sharp r'\sharp c$. □

There is a natural composition result as in Proposition 1.2 for G^s symbols of degree d, and order m, which tells essentially that the G^s symbols form a graded algebra for the composition law $\sharp$. Now let $d \geq 0$ and $b \in S((0,d),s)$ be a G^s symbol of degree d elliptic at infinity, that is, $|b(x,\xi,h)| \geq C\langle\xi\rangle^d$ for (x,ξ) outside a compact set $K_1 \subset\subset W$. Then for any $c \in S((0,0),s)$ vanishing in a neighborhood of K_1, there exist $a_L, a_R \in S((0,-d),s)$, and $r_L, r_R \in S(-\infty,s)$ such that

$$\begin{aligned} a_L^w(x,hD,h)b^w(x,hD,h) &= c^w(x,hD,h) + r_L^w(x,hD,h) \\ b^w(x,hD,h)a_R^w(x,hD,h) &= c^w(x,hD,h) + r_R^w(x,hD,h) \end{aligned} \tag{1.11}$$

Moreover, r_L and r_R are rapidly decreasing with respect to ξ (see [BLas]):

$$\begin{aligned} &\exists C > 0, \quad \forall(\alpha,\beta,N) \in \mathbf{N}^n \times \mathbf{N}^n \times \mathbf{N}, \\ &\exists C_{\alpha,\beta,N} > 0 : |\partial_x^\alpha \partial_\xi^\beta r(x,\xi,h)| \leq C_{\alpha,\beta,N}\langle\xi\rangle^{-N} \exp(-1/Ch^{1/s}) \end{aligned} \tag{1.12}$$

(these estimates can certainly be improved). We shall now study the relation between Weyl and standard pseudo differential calculus. Recall [Hö, Chap. 18.5] that if $a(x,\xi,h) \in \mathcal{S}(W)$ we can write $a(x,hD,h) = b^w(x,hD,h)$ where $b(x,\xi,h) \in \mathcal{S}(W)$ is given by:

$$\begin{aligned} b(X,h) = (\pi h)^{-n} \int a(X+Y,h) e^{2iy\eta/h} dY = e^{h<D_x,D_\xi>/2i} a(X,h), \\ X = (x,\xi), \quad Y = (y,\eta) \end{aligned} \tag{1.13}$$

In the same spirit as in Proposition 1.2 above, we have:

Proposition 1.3 *Formula* (1.13) *extends from* $\mathcal{S}(W)$ *to a weakly continuous linear map* $a \to b$ *from* $S(m,s)$ *to itself. More precisely, let* $c \in G_0^s(W)$ *be equal to* 1 *near* $Y = 0$,

$$b_\chi(X,h) = (\pi h)^{-n} \int a(X+Y,h) e^{2iy\eta/h} \chi(Y) dY$$

and $b_{\widehat{\chi}}(X,h) = b(X,h) - b_\chi(X,h)$. *Then the maps* $a \mapsto b_\chi$ [*resp.* $a \mapsto b_{\widehat{\chi}}$] *are weakly continuous from* $S(m,s)$ *to itself* [*resp from* $S(m,s)$ *to* $S(-\infty,s)$], *and the map from* $a \in S(m,s)$ *to the remainder term:*

$$b_\chi(X,h) - \sum_{j<N} \frac{(h/2)^j}{j!} \langle iD_x, D_\xi\rangle^j a(X,h)$$

is valued in $S(m-N,s)$ *for all* $N \in \mathbf{N}$.

Notice that we have a completely analogous result for G^s symbols of degree d as in (1.4). In particular if:

$$P(x,hD,h)=\sum_{|\alpha|\le d} a_\alpha(x,h)(hD_x)^\alpha, \quad a_\alpha(x,h)\in S(0,s) \tag{1.14}$$

is a differential operator of order d, the Weyl symbol of P is a G^s symbol of degree d. Let us close this section by recalling a wellknown L^2 estimate for h-Pseudo Differential operators, see [Hö], [Iv], … . We denote as usual by $W^{\sigma,2}(V)$ the Sobolev space with L^2-norm $\|u\|^2_{\sigma,2}=\int\langle\xi\rangle^{2\sigma}|\mathcal{F}_h u(\xi)|^2 d\xi$, $\sigma\in\mathbf{R}$, $\mathcal{F}_h u(\xi)=\int e^{-ix\xi/h}u(x)dx$. For $\sigma\in\mathbf{N}$, we have $\|u\|^2_{\sigma,2}=\sum_{|\alpha|\le\sigma}\|(hD)^\alpha u\|^2_{L^2}$.

Proposition 1.4 *If* $a\in S(0,s)$ *then* $a^w(x,hD,h)=\mathcal{O}(1): L^2(V)\to L^2(V)$, *and more generally* $a^w(x,hD,h)=\mathcal{O}(1): W^{\sigma,2}(V)\to W^{\sigma,2}(V)$. *For* $P(x,hD,h)$ *as in* (1.14) *we also have* $P(x,hD,h)=\mathcal{O}(1): W^{\sigma,2}(V)\to W^{\sigma-d,2}(V)$.

2. Bargmann transform and Weyl calculus in the complex domain

a) Bargmann transform

It will be convenient to replace L^2 functions by holomorphic ones; this can be achieved by performing a Bargmann transform (or FBI transform, in the terminology of [Sj] which we follow here, see also [BLasSj]). Let

$$Tu(x,h)=C_0h^{-3n/4}\int e^{-(x-y)^2/2h}u(y)dy, \quad u\in C_0^\infty(\mathbf{R}^n) \tag{2.1}$$

For a suitable choice of C_0, T extends to a unitary operator from $L^2(\mathbf{R}^n)$ to H_Φ, the space of entire functions in $\mathbf{C}^n$ which are L^2 with respect to $e^{-2\Phi(x)/h}L(dx)$ where $L(dx)=(2i)^{-n}dx\wedge d\overline{x}$ is the standard Lebesgue measure on $\mathbf{C}^n\approx\mathbf{R}^{2n}$, and $\Phi(x)=(\operatorname{Im}x)^2/2$ the weight function associated to T. Then the adjoint of T is:

$$T^*v(y,h)=\overline{C_0}h^{-3n/4}\int_{\mathbf{C}^n} e^{-(\overline{z}-y)^2/2h}v(z)e^{-2\Phi(z)/h}L(dz)$$

The canonical relation associated to $\varphi(x,y)=i(x-y)^2/2$ is given by κ_T :

$(y,\eta) \mapsto (x,\xi)$, with $\eta = -\frac{\partial \varphi}{\partial y}$, $\xi = \frac{\partial \varphi}{\partial x}$. In other words $(x,\xi) = \kappa_T(y,\eta)$ if and only if $\xi = \eta$ and $x = y - i\eta$. κ_T maps isomorphically $T^*\mathbf{R}^n$ endowed with its symplectic structure given by the 2-form $dy \wedge d\eta$, onto the I-lagrangian linear manifold $\Lambda_\Phi \subset T^*\mathbf{C}^n$ (i.e. $\operatorname{Im} dx \wedge d\xi_{|\Lambda_\Phi=0}$) given by $\Lambda_\Phi = \{\xi = \xi(x) = \frac{2}{i}\frac{\partial \Phi}{\partial x}(x) = -\operatorname{Im} x\}$. The space Λ_Φ is a totally real manifold of $T^*\mathbf{C}^n$, i.e. $\Lambda_\Phi \cap i\Lambda_\Phi = 0$, which allows to define its complexification $\Lambda_\Phi^{\mathbf{C}}$ in a canonical way. Thus, T can be viewed as a Wave-Packet Transform (in the sense of A. Cordoba and Ch. Feffermann [CorFe]), which localizes on Λ_Φ. We shall identify the symbols on Λ_Φ with symbols on $\mathbf{R}^{2n}$ via the canonical isomorphism κ_T. Let

$$\begin{aligned}\mathcal{S}_\Phi &= \mathcal{S}(\Lambda_\Phi) = T(\mathcal{S}(\mathbf{R}^n)) \\ &= \{u \in H_\Phi : |u(x)| \leq C_N(1+|x|)^{-N} e^{\Phi(x)/h},\ N = 1,2,\dots\}\end{aligned}$$

be the Schwartz space on Λ_Φ.

b) Pseudo-differential calculus

If $a \in \mathcal{S}(\Lambda_\Phi)$, we put:

$$\begin{aligned}&a^w(x,hD,h)u(x,h) \\ &\quad = (2\pi h)^{-n} \int_{\Gamma(x)} e^{i(x-y)\theta/h} a((x+y)/2,\theta,h) u(y,h) dy \wedge d\theta \qquad (2.2)\end{aligned}$$

for $u \in H_\Phi$. Here $\Gamma(x)$ is the contour:

$$\Gamma(x) = \left\{\theta = \xi((x+y)/2) = \frac{2}{i}\frac{\partial \Phi}{\partial x}((x+y)/2) : y \in \mathbf{C}^n\right\} \approx \Lambda_\Phi. \qquad (2.3)$$

We can check that there is, for instance, a one to one correspondance between the h-quantization of symbols in $\mathcal{S}(\mathbf{R}^{2n})$ and those in $\mathcal{S}(\Lambda_\Phi)$, given by: $Q = TPT^{-1}$ where $P = p^w(x,hD,h)$, $p \in \mathcal{S}(\mathbf{R}^{2n})$ and $Q = q^w(x,hD,h)$, $q \in \mathcal{S}(\Lambda_\Phi)$ with the quantization rule (2.2). The symbols are related by $p = q \circ \kappa_T$. The remarkable fact (wellknown when T is any metaplectic operator, see [Hö Theorem 18-5-9]) is that this relation is exact, i.e. no remainder term occurs in the formula $Q = TPT^{-1}$. The h-quantization rule (2.2) extends to symbols in the class $S(m,s)$ defined on Λ_Φ. We now consider the composition of 2 operators of the form (2.2); because of the result in $\mathbf{R}^{2n}$ and the symplectic invariance, we get:

Proposition 2.1 *Let $a \in S(m', s)$, $b \in S(m, s)$ be Gevrey s symbols of order m', m respectively, defined on Λ_Φ and $a^w(x, hD, h)$, $b^w(x, hD, h)$ their corresponding Weyl h-quantization given by (2.2). Then $a^w(x, hD, h)b^w(x, hD, h) = c^w(x, hD, h)$ with:*

$$c(x, \xi, h)_{|\Lambda_\Phi} = (\pi h)^{-2n} \iint e^{-2i\sigma(Y,Z)/h} a(X+Y, h) b(X+Z, h) L(dY) L(dZ) \quad (2.4)$$

Here we put: $X = (x, \xi) \in \Lambda_\Phi$, $Y = (y, \eta) \in \Lambda_\Phi$, $Z = (z, \zeta) \in \Lambda_\Phi$. Again, $L(dY) = (2i)^{-n} dy \wedge d\overline{y}$ is the Lebesgue measure on Λ_Φ and $\sigma(Y, Z) = z\eta - y\zeta$ the symplectic 2-form on $\mathbf{C}^{2n}$ (because Λ_Φ is I-lagrangian, $\sigma|_{\Lambda_\Phi}$ is real and non degenerate).

c) Method of stationary phase

For real analytic phase functions and G^s symbols (in the real domain), the method of stationary phase is discussed by Gramchev [Gr]. A more general situation is considered in [BLas]. It is known in particular that stationary phase method gives a loss of Gevrey smoothness equal to $s - 1$. For the case at hand, we content ourselves to give a few terms in the expansion, and mimick the proof of [Sj2 Lemma 1.1], but it should not be too hard to give the full expansion along the same lines using the results in Appendix. Let $\psi \in G_0^s(\mathbf{C}^n)$ be real valued, μ a small parameter we will eventually set to $h^{1-1/s}$ and:

$$\Phi_\mu(x) = \Phi(x) + \mu\psi(x) \quad (2.5)$$

be the weight function. We denote by $L^2_{\Phi_\mu}$ the space of functions on $\mathbf{C}^n$ which are L^2 with respect to $e^{-2\Phi_\mu(x)/h} L(dx)$ and by $H_{\Phi_\mu} \subset L^2_{\Phi_\mu}$ the (closed) subspace of entire functions. As Hilbert spaces we have, for instance: $H_{\Phi_\mu} = H_\Phi$ but the corresponding norms depend on h. We also need:

$$\mathcal{S}_{\Phi_\mu} = \{u \in H_{\Phi_\mu} : |u(x)| \leq C_N (1 + |x|)^{-N} e^{\Phi_\mu(x)/h},\ N = 1, 2, \ldots\}. \quad (2.6)$$

As Fréchet spaces we have $\mathcal{S}_{\Phi_\mu} = \mathcal{S}_\Phi$ but the best constants C_N in the respective definitions depend on h. Let $b \in S(m, s)$ be defined on Λ_Φ; by the same letter we denote an almost analytic extension to $T^*\mathbf{C}^n$ as in

Corollary a.4. Note that inequality (1.3) becomes:

$$|\overline{\partial}_{(x,\theta)}b(x,\theta,h)| \le Ch^{-m}\exp(-|\theta-\xi(x)|^{-1/(s-1)}/C),$$
$$(x,\theta)\in T^*\mathbf{C}^n,\quad |\theta-\xi(x)|\le c \tag{2.7}$$

Let $\chi\in C_0^\infty\mathbf{C}^n$ be equal to one in a neighborhood of 0. We quantize b in the following way:

$$b_{\chi,0}u(x,h)$$
$$=(2\pi h)^{-n}\int_{\Gamma_0(x)} e^{i(x-y)\theta/h}b((x+y)/2,\theta,h)\chi(x-y)u(y,h)dy\wedge d\theta \tag{2.8}$$

where $\Gamma_0(x)$ is the integration contour parametrized by $y\in\mathbf{C}^n$: $\theta=\xi_\mu(x)+ic\,\overline{x-y}$, $(c>0)$, and $u\in H_{\Phi_\mu}$. We have set $\xi_\mu(x)=\frac{2}{i}\frac{\partial\Phi_\mu}{\partial x}(x)$. Since $u\in H_{\Phi_\mu}$ is microlocalized near $(x,\xi_\mu(x))$, the action of a h-Pseudo differential operator on u is approximated to every order in $h^{1/2}$ by expanding its symbol in powers of $hD_x-\xi_\mu(x)$. The following is a straightforward generalization of Proposition 4.4 of [GeSj] and Lemma 1.1 of [Sj2].

Proposition 2.2 *For $u\in H_{\Phi_\mu}$, $b\in S(m,s)$, $\mu=h^{1-1/s}$ and $x\in\mathbf{C}^n$, we have*:

$$b_{\chi,0}u(x,h)=b(x,\xi_\mu(x),h)u(x,h)$$
$$+d_\xi b(x,\xi_\mu(x),h)(hD_x-\xi_\mu(x))u(x,h)+Ru(x,h) \tag{2.9}$$

where R is of norm $\mathcal{O}(h^{1-m})$ from H_{Φ_μ} to $L^2_{\Phi_\mu}$.

Proof. Although the only thing really new compared to [Sj2] (beside the fact we use Weyl quantization), is the control on the anti-holomorphic terms, we recall the argument. We make the Taylor expansion:

$$b((x+y)/2,\theta,h)$$
$$=b(x,\xi_\mu(x),h)+d_\xi b(x,\xi_\mu(x),h)(\theta-\xi_\mu(x))$$
$$+d_{\overline{\xi}}b(x,\xi_\mu(x),h)\overline{(\theta-\xi_\mu(x))}+d_xb(x,\xi_\mu(x),h)(y-x)/2$$
$$+d_{\overline{x}}b(x,\xi_\mu(x),h)\overline{(y-x)}/2+R_0(x,y,\theta,h) \tag{2.10}$$

where $R_0(x,y,\theta,h)\in S(m,s)$ is defined for $x,y\in\mathbf{C}^n$, and verifies $R_0(x,y,\theta,h)=\mathcal{O}(|x-y|^2+|\theta-\xi_\mu(x)|^2)=\mathcal{O}(|x-y|^2)$ for $\theta\in\Gamma_0(x)$.

The corresponding operator, that we also denote by R_0, is of the form:

$$R_0u(x,h)$$
$$= h^{-n-m}\int \mathcal{O}(1)e^{-c|x-y|^2/h}e^{i(x-y)\xi_\mu(x)/h}|x-y|^2\chi(x-y)u(y)L(dy)$$

The phase function in the reduced kernel $e^{-\Phi_\mu(x)/h}R_0(x,y)e^{\Phi_\mu(y)/h}$ is given by:

$$\varphi(x,y,\mu) = -c|x-y|^2 + i(x-y)\xi_\mu(x) + \Phi_\mu(y) - \Phi_\mu(x)$$

Hence:

$$\operatorname{Re}\varphi(x,y,\mu) \le -c|x-y|^2 + (\operatorname{Im}(x-y))^2/2 + C_1\mu|x-y|^2, \quad C_1 > 0$$

(here C_1 is chosen so that $|\psi''(x)| \le C_1$). So when $c > 1/2$ and $\mu > 0$ is small enough, we have:

$$\operatorname{Re}\varphi(x,y,\mu) \le -\delta|x-y|^2, \quad \delta > 0$$

As in [Sj2], using the very first terms in the expansion of $R_0u(x,h)$ by stationary phase (in the C^∞ sense), we see that $R_0 = \mathcal{O}(h^{1-m}) : H_{\Phi_\mu} \to L^2_{\Phi_\mu}$. Now we examine the contribution to $b_{\chi,0}$ of the various terms in (2.10). First we know from [Sj1] that:

$$(2\pi h)^{-n}\int_{\Gamma_0(x)} e^{i(x-y)\theta/h}u(y,h)\chi(x-y)dy\wedge d\theta = u(x,h) + R_1u(x,h) \tag{2.11}$$

where $R_1 = \mathcal{O}(e^{-1/Ch}) : H_{\Phi_\mu} \to L^2_{\Phi_\mu}$ (this is Fourier's inversion formula in the H_{Φ_μ} spaces), so the first term in (2.10) gives $b(x,\xi_\mu(x),h)u(x,h)$ with an error that enters into R. Next we have

$$d_\xi b(x,\xi_\mu(x),h)(2\pi h)^{-n}\int_{\Gamma_0(x)} e^{i(x-y)\theta/h}(\theta - \xi_\mu(x))u(y,h)\chi(x-y)dy\wedge d\theta$$
$$= d_\xi b(x,\xi_\mu(x),h)(2\pi h)^{-n}$$
$$\times \int_{\Gamma_0(x)} [(-hD_y - \xi_\mu(x))e^{i(x-y)\theta/h}]u(y,h)\chi(x-y)dy\wedge d\theta$$

so by an argument similar to this leading to (2.11) the second term in (2.10) gives: $d_\xi b(x,\xi_\mu(x),h)$ $(hD_x - \xi_\mu(x))u(x,h)$ up to an error that enters also into R. Similarly one has

$$d_x b(x,\xi_\mu(x),h)(2\pi h)^{-n}\int_{\Gamma_0(x)} e^{i(x-y)\theta/h}(y-x)u(y,h)\chi(x-y)dy\wedge d\theta$$
$$= R_2u(x,h) + ih\sum_{j=1}^{n}\partial_{x_j}b(x,\xi_\mu(x),h)(2\pi h)^{-n}$$
$$\times\int_{\Gamma_0(x)} d\big(e^{i(x-y)\theta/h}u(y,h)\chi(x-y)dy\wedge d\theta_1\cdots\wedge\widehat{d\theta_j}\cdots\wedge d\theta_n\big)$$

where $R_2 = \mathcal{O}(e^{-1/Ch}) : H_{\Phi_\mu} \to L^2_{\Phi_\mu}$ (here d denotes the exterior derivative, and $\widehat{d\theta_j}$ means as usual that this factor has been omitted). So we are left with the antiholomorphic terms. For instance we have, for $\mu = h^{1-1/s}$:

$$d_{\bar\xi}b(x,\xi_\mu(x),h) = h^{-m}\mathcal{O}(\exp(-|\xi_\mu(x)-\xi(x)|^{-1/(s-1)}/C)$$
$$= \mathcal{O}(\exp(-1/Ch^{1/s})),\quad C>0$$

and we see easily that the corresponding operator enters also into R, which brings the proof to an end. □

Actually, we shall use the following quantization of b:

$$b_{\chi,1}u(x,h)$$
$$= (2\pi h)^{-n}\int_{\Gamma_1(x)} e^{i(x-y)\theta/h}b((x+y)/2,\theta,h)\chi(x-y)u(y,h)dy\wedge d\theta \tag{2.12}$$

where $\Gamma_1(x)$ is given by $\theta = \xi_\mu((x+y)/2) + ic\,\overline{x-y}$.

Proposition 2.3 *Let $b \in S(m,s)$ as before. For $u \in H_{\Phi_\mu}$, $\mu = h^{1-1/s}$ and $x \in \mathbf{C}^n$ we have:*

$$b_{\chi,1}u(x,h) = b(x,\xi_\mu(x),h)u(x,h)$$
$$+ d_\xi b(x,\xi_\mu(x),h)(hD_x - \xi_\mu(x))u(x,h) + Ru(x,h) \tag{2.13}$$

where $R = \mathcal{O}(h^{1-m}) : H_{\Phi_\mu} \to L^2_{\Phi_\mu}$.

Proof. We shall reduce $b_{\chi,1}u(x,h)$ to $b_{\chi,0}u(x,h)$ by a deformation of contours. For $t \in [0,1]$, consider: $\Gamma_t(x) = \{\theta = (1-t)\xi_\mu(x) + t\xi_\mu(\widehat{x}) + ic\,\overline{x-y} =_{\text{def}} \gamma_x(t,y),\ y \in \mathbf{C}^n\}$ where for short we have put $\widehat{x} = (x+$

$y)/2$. By Stokes' formula,

$$b_{\chi,1}u(x,h) = b_{\chi,0}u(x,h) + (2\pi h)^{-n} \int_{[0,1]\times\mathbf{C}^n} e^{i(x-y)\theta/h} u(y,h)\gamma_x^*(d\omega)$$

where $\omega = b(\widehat{x},\theta,h)\chi(x-y)dy \wedge d\theta$. (Here we have used that u is holomorphic). As above, we estimate the reduced kernel $e^{-\Phi_\mu(x)/h}\gamma_x^*(d\omega)e^{\Phi_\mu(y)/h}$. We have:

$$\begin{aligned}\gamma_x^*(d\omega) = \sum_{j=1}^n \big(i\overline{\partial}_{x_j} b(\widehat{x},\theta,h)J_j(x,y,t) \\ + \overline{\partial}_{\xi_j} b(\widehat{x},\theta,h)K_j(x,y,t)\big)\chi(x-y)dy\wedge d\overline{y}\wedge dt \\ + \omega_1(y,\overline{y},t)\end{aligned}$$

where $\omega_1(y,\overline{y},t)$ contains the derivatives of χ, and:

$$J_j(x,y,t) = -\det\left[\frac{\partial(\theta_1,\ldots,\theta_j,\ldots,\theta_n)}{\partial(\overline{y}_1,\ldots,t,\ldots,\overline{y}_n)}\right] = \mathcal{O}(1) \tag{2.14}$$

$$K_j(x,y,t) = -\det\left[\frac{\partial(\theta_1,\ldots,\theta_j,\overline{\theta}_j,\theta_{j+1},\cdots\theta_n)}{\partial(\overline{y}_1,\ldots,\overline{y}_j,t,\overline{y}_{j+1},\ldots,\overline{y}_n)}\right] = \mathcal{O}(1) \tag{2.15}$$

uniformly wich respect to all variables. We have: $\omega_1(y,\overline{y},t) = \mathcal{O}(e^{-1/Ch})L(dy)dt$. On the other hand (1.3) gives:

$$\overline{\partial}_{(x,\theta)} b(\widehat{x},\theta,h) \le Ch^{-m}\exp\big(-|\operatorname{Im}\widehat{x}+\operatorname{Re}\theta,\operatorname{Im}\theta|^{-1/(s-1)}/C\big)$$

uniformly for $x\in\mathbf{C}^n$, that is, for $\mu>0$ small enough

$$\overline{\partial}_{(x,\theta)} b(\widehat{x},\theta,h) \le Ch^{-m}\exp\Big(-\Big(\mu\Big|2\frac{\partial\psi}{\partial x}(x)\Big| + c'|x-y|\Big)^{-1/(s-1)}/C\Big)$$

with a new constant c'. Let

$$\varphi_\mu(x,y,\theta) = i(x-y)\theta + \Phi_\mu(y) - \Phi_\mu(x) \tag{2.16}$$

and

$$\begin{aligned}F_t(x,y) = \operatorname{Re}\Big(\psi(y)-\psi(x) + 2\frac{\partial\psi}{\partial x}(\widehat{x})(x-y) \\ + 2(1-t)\Big(\frac{\partial\psi}{\partial x}(\widehat{x}) - \frac{\partial\psi}{\partial x}(x)\Big)(y-x)\Big)\end{aligned}$$

so that:

$$\operatorname{Re}\varphi_\mu(x,y,\theta) = \mu F_t(x,y) - c|x-y|^2 - (1-t)(\operatorname{Im}(x-y))^2/2$$

We have $|F_t(x,y)| \leq C\|\psi\|_{2,\infty}|x-y|^2$ (here $\|\psi\|_{2,\infty}$ denotes the norm of ψ in the C^2-topology), so when $c > 1/2$ and $\mu > 0$ is small enough $\mathrm{Re}\,\varphi_\mu(x,y,\theta) \leq -\delta|x-y|^2$, $\delta > 0$. So we need a (positive) lower bound for:

$$\phi(x,y,\mu) = \Big(\mu\Big|2\frac{\partial\psi}{\partial x}(x)\Big| + c'|x-y|\Big)^{-1/(s-1)} + \delta|x-y|^2/h$$

Let $\alpha = |x-y|$. We have:

$$\phi(x,y,\mu) \geq \text{Const.}\left(\frac{\alpha^2}{h} + (\alpha + C\mu)^{-1/(s-1)}\right)$$

for some $C > 0$. Let $f(\alpha) = \frac{\alpha^2}{h} + (\alpha + C\mu)^{-1/(s-1)}$. It is easy to see that $f''(\alpha) > 0$ on $[0,1]$. Rescaling α by $\alpha = \beta h^{(s-1)/(2s-1)}$ we find that $f'(\alpha)$ vanishes for some $\beta = \beta_0$ verifying $1/C_1 \leq \beta_0 \leq C_1$, where $C_1 > 0$ is independent of h. The corresponding value for $f(\alpha)$ is $\sim$ Const. $h^{-1/(2s-1)}$. We have proved that:

$$\begin{aligned}&|e^{i(x-y)\theta/h}e^{-\Phi_\mu(x)/h}(\gamma_x^*(d\omega) - \omega_1(y,\overline{y},t))e^{\Phi_\mu(y)/h}|\\ &\quad = \mathcal{O}(\exp(-1/Ch^{1/(2s-1)}))\end{aligned}$$

uniformly for $x \in \mathbf{C}^n$, $t \in]0,1]$, and $x - y \in \operatorname{supp}\chi$. As we clearly have:

$$|e^{i(x-y)\theta/h}e^{-\Phi_\mu(x)/h}\omega_1(y,\overline{y},t)e^{\Phi_\mu(y)/h}| = \mathcal{O}(\exp(-1/Ch))$$

we can conclude by Proposition 2.2. □

Remark 2.4 Propositions 2.2 and 2.3 also hold, *mutatis mutandis*, when b is a G^s symbol of degree d and order m as in (1.4). The remainder satisfies then: $R = \mathcal{O}(h^{1-m}) : W^{d,2}_{\Phi_\mu} \to L^2_{\Phi_\mu}$, where the $W^{d,2}_{\Phi_\mu}$-norm of a function $u \in H_{\Phi_\mu}$ is given by:

$$\|u\|^{d,2}_\mu = \sum_{|\alpha|\leq d} \|(hD)^\alpha u\|_\mu.$$

(see the proof of Proposition 3.1 below). The Schwartz space $\mathcal{S}_{\Phi_\mu}$ introduced above is of course dense in every $W^{d,2}_{\Phi_\mu}$.

The link between quantizations (2.2), (2.8) and (2.12) will be explained in the next Section.

3. Phase space distorsions or Lagrangian deformations

Here we use the method of non-characteristic deformations in a variant due to Sjöstrand [Sj1, Sect. 10], where plurisubharmonic weight functions are distorted rather than hypersurfaces or domains as in the classical techniques of Holmgren. More explicitly, the method consists in deforming the integration contour in (2.2) in order to gain ellipticity when operating on certain weighted spaces. At the same time we are in a situation where the method of stationary phase applies as in Sect. 2.

Let $b(x,\theta,h) \in S((0,d),s)$ be a G^s symbol of degree $d \geq 0$ and order 0 (for simplicity) as in (1.4) defined for $(x,\theta) \in \Lambda_\Phi$, and $B = b^w(x,hD,h)$ as in (2.2). As before, the same letter denotes an almost analytic extension in the sense of Corollary a.4. The main result of this section is the following:

Proposition 3.1 *With the notations above, if $\|\psi\|_{2,\infty}$ is small enough, B extends as a continuous operator $W^{d,2}_{\Phi_\mu} \to L^2_{\Phi_\mu}$ and $B = B_1 + R$, with:*

$$B_1 u(x,h) = (2\pi h)^{-n} \iint_{\widetilde{\Gamma}_1(x)} e^{i(x-y)\theta/h} b((x+y)/2,\theta,h) u(y,h) dy \wedge d\theta \quad (3.1)$$

Here $\widetilde{\Gamma}_1(x)$ is the contour given by:

$$\widetilde{\Gamma}_1(x) = \Big\{\theta = \frac{2}{i}\frac{\partial \Phi_\mu}{\partial x}((x+y)/2) + ic(|x-y|)\,\overline{x-y},\ y \in \mathbf{C}^n\Big\}$$

and $c : \mathbf{R}^+ \to \mathbf{R}^+$ is a smooth, positive decreasing function such that $c(\rho) = c_0 > 0$ for $\rho \leq \lambda$, $(\lambda > 0)$ and $c(\rho) = c_0\lambda'/(2\rho)$ for $\rho \geq \lambda' > \lambda$. Moreover:

$$R = \mathcal{O}\big(\exp -\delta/h^{1/(2s-1)}\big) : W^{d,2}_{\Phi_\mu} \to L^2_{\Phi_\mu},\quad h \to 0, \quad (3.2)$$

for some $\delta > 0$ and:

$$B_1 = \mathcal{O}(1) : W^{d,2}_{\Phi_\mu} \to L^2_{\Phi_\mu},\quad h \to 0. \quad (3.3)$$

Proof. As before we set $\widehat{x} = (x+y)/2$. For $t \in [0,1]$ we consider the contours:

$$\widetilde{\Gamma}_t(x) = \Big\{\theta = (1-t)\frac{2}{i}\frac{\partial \Phi}{\partial x}(\widehat{x}) + t\Big(\frac{2}{i}\frac{\partial \Phi_\mu}{\partial x}(\widehat{x}) + ic(|x-y|)\,\overline{x-y}\Big) =_{\mathrm{def}} \gamma_x(t,y),\ y \in \mathbf{C}^n\Big\} \quad (3.4)$$

By Stokes' formula, for $u \in \mathcal{S}_\Phi$ we have:

$$Bu(x,h) = B_1 u(x,h) - (2\pi h)^{-n} \int_{[0,1]\times\mathbf{C}^n} e^{i(x-y)\theta/h} u\gamma_x^*(d\omega) \quad (3.5)$$

where $\omega = b((x+y)/2, \theta, h) dy \wedge d\theta$. (Here we have used that u is holomorphic). We will estimate the reduced kernels. Let $\varphi_\mu(x,y,\theta)$ be as in (2.16). As in the proof of Proposition 2.3, we see that along $\widetilde{\Gamma}_t(x)$:

$$\begin{aligned}\operatorname{Re}\varphi_\mu(x,y,\theta) = \mu t F(x,y) + \mu(1-t)(\psi(y) - \psi(x)) \\ - tc(|x-y|)|x-y|^2\end{aligned} \quad (3.6)$$

where

$$F(x,y) = \operatorname{Re}\Big(\psi(y) - \psi(x) + 2\frac{\partial\psi}{\partial x}(\widehat{x})(x-y)\Big) \quad (3.7)$$

verifies

$$|F(x,y)| \leq \text{Const. } \|\psi\|_{2,\infty}|x-y|^2 \quad (3.8)$$

We first analyse the remainder term in the right hand side of (3.5). We have:

$$\begin{aligned}&\int_{[0,1]\times\mathbf{C}^n} e^{i(x-y)\theta/h} u\gamma_x^*(d\omega) = \int_0^1 dt \\ &\times \Big[\sum_{j=1}^n \int_{\mathbf{C}^n} e^{i(x-y)\theta/h} \overline{\partial}_{y_j} b((x+y)/2, \theta, h)_{|\theta=\gamma_x(t,y)} J_j(x,y,t) u(y,h) dy \wedge d\overline{y} \\ &+ \sum_{j=1}^n \int_{\mathbf{C}^n} e^{i(x-y)\theta/h} \overline{\partial}_{\theta_j} b((x+y)/2, \theta, h)_{|\theta=\gamma_x(t,y)} K_j(x,y,t) u(y,h) dy \wedge d\overline{y}\Big]\end{aligned} \quad (3.9)$$

where $J_j(x,y,t)$ and $K_j(x,y,t)$ defined in (2.14) and (2.15), satisfy $J_j(x,y,t) = \mathcal{O}(1)$ and $K_j(x,y,t) = \mathcal{O}(1)$ uniformly in all variables. For $\theta = \gamma_x(t,y)$

$$|\theta - \xi(\widehat{x})| = t\left|2\mu\frac{\partial\psi}{\partial x}(\widehat{x}) - c(|x-y|)\,\overline{x-y}\right| \quad (3.10)$$

so that by (1.4):

$$|\partial_{\overline{y}_j} b(\widehat{x},\theta,h)| + |\partial_{\overline{\theta}_j} b(\widehat{x},\theta,h)| \leq C_1 \langle\theta\rangle^d$$
$$\times \exp\Big(-t^{-1/(s-1)}\Big|2\mu\frac{\partial\psi}{\partial x}(\widehat{x}) - c(x-y)\overline{x-y}\Big|^{-1/(s-1)}/C_1\Big) \quad (3.11)$$

uniformly for $x,y\in\mathbf{C}^n$, $\theta\in\widetilde{\Gamma}_t(x)$. The worst singularity of the reduced kernel occurs in the neighborhood of the diagonal $|x-y|<\lambda$ of $\mathbf{C}^{2n}$.

Lemma 3.2 *If* $\varepsilon = \|\psi\|_{2,\infty}$ *is small enough* (*depending on* c_0 *and* λ) *then there exists* $C_2>0$ *such that for all* $t\in]0,1]$:

$$e^{\mathrm{Re}\,\varphi_\mu(x,y,\theta)/h}\big(|\partial_{\overline{y}_j} b(\widehat{x},\theta,h)| + |\partial_{\overline{\theta}_j} b(\widehat{x},\theta,h)|+\big)$$
$$\leq C_2\langle\theta\rangle^d \exp(-1/C_2 h^{1/(2s-1)}), \quad h\to 0 \quad (3.12)$$

uniformly for $\theta=\gamma_x(t,y)$, $|x-y|<\lambda$.

Proof. Using (3.6), (3.8) and (3.11) we are led to find a (positive) lower bound for:

$$\phi(x,y,t,\mu,h)$$
$$= -\frac{\mu}{h}(1-t)(\psi(y)-\psi(x)) + \frac{t}{h}c(x-y)|x-y|^2 - \frac{\mu t}{h}F(x,y)$$
$$+ t^{-1/(s-1)}\Big|2\mu\frac{\partial\psi}{\partial x}(\widehat{x}) - c(|x-y|)\overline{x-y}\Big|^{-1/(s-1)}/C_1$$

For $|x-y|\leq\lambda$, we have $c(|x-y|)=c_0$. Let $\alpha=|x-y|$. In (3.8) we can assume, without loss of generality, that Const. $=1$, so for $\mu=h^{1-1/s}$:

$$h^{1/s}\phi(x,y,t,\mu,h) \geq -\varepsilon(1-t)\alpha + tc_0\Big(1-\frac{\mu\varepsilon}{c_0}\Big)\frac{\alpha^2}{\mu}$$
$$+ t^{-1/(s-1)}\Big(1+\frac{\alpha c_0}{2\mu\varepsilon}\Big)^{-1/(s-1)}/\big(\varepsilon^{1/(s-1)}C_1\big)$$

For $\frac{\mu\varepsilon}{c_0}\leq\frac{1}{2}$, we get for a new $C_1>0$, changing 2ε to ε:

$$2h^{1/s}\phi(x,y,t,\mu,h)$$
$$\geq -\varepsilon(1-t)\alpha + tc\frac{\alpha^2}{\mu} + t^{-1/(s-1)}\Big(1+\frac{\alpha c_0}{\mu\varepsilon}\Big)^{-1/(s-1)}/\big(\varepsilon^{1/(s-1)}C_1\big)$$

Then we set $c_0/\varepsilon = c'$, factor out ε, and put $\varepsilon' = C_1\varepsilon^{s/(s-1)}$. We get:

$$2\varepsilon^{-1}h^{1/s}\phi(x,y,t,\mu,h) \\ \geq -(1-t)\alpha + tc'\frac{\alpha^2}{\mu} + t^{-1/(s-1)}\Big(1+\frac{\alpha c'}{\mu}\Big)^{-1/(s-1)}/\varepsilon'$$

We shall remember that we can take ε' as small as we want, and drop the primes to simplify the notations. Then we study the comparison function:

$$g(t,\alpha,\mu) = -(1-t)\alpha + tc\frac{\alpha^2}{\mu} + t^{-1/(s-1)}\Big(1+\frac{\alpha c}{\mu}\Big)^{-1/(s-1)}/\varepsilon$$

for $t,\ \alpha,\ \mu \in]0,1] \times [0,\lambda] \times]0,1]$. Actually, μ will appear as a parameter. We first show that g has only one critical point in the open set: $(t,\alpha) \in]0,1[\times]0,\lambda[$. Indeed, $\frac{\partial g}{\partial t}(t,\alpha,\mu) = 0$ iff $t = t(\alpha) = (\varepsilon(s-1)\alpha)^{(1-s)/s}(1+\frac{c\alpha}{\mu})^{-1}$. But:

$$\frac{\partial g}{\partial \alpha}(t,\alpha,\mu) = \Big(1+\frac{2c\alpha}{\mu}\Big)t - \frac{c}{\mu\varepsilon(s-1)}\Big(1+\frac{\alpha c}{\mu}\Big)^{-s/(s-1)}t^{-1/(s-1)} - 1$$

After some straightforword calculation, we find that $\frac{\partial g}{\partial \alpha}(t_0,\alpha;\mu) = 0$ iff $\alpha = \alpha_0 = (\varepsilon(s-1))^{-1}$. Then $t_0 = t(\alpha_0) = (1+\frac{c}{\mu\varepsilon(s-1)})^{-1}$ and $g(t_0,\alpha_0,\mu) = \varepsilon^{-1}$. If this values is a minimum of g for $(t,\alpha) \in]0,1] \times [0,\lambda]$, then we have $\phi(x,y,t,\mu,h) \geq \delta h^{-1/s}$ (actually we can check that this never holds when $s \geq 2$). Otherwise we have to look for a minimum of g on $\partial(]0,1] \times [0,\lambda])$. We have:

$$\begin{aligned} g(t,0;\mu) &= t^{-1/(s-1)}/\varepsilon \geq \varepsilon^{-1} \\ g(1,\alpha;\mu) &= \frac{c\alpha^2}{\mu} + \Big(1+\frac{\alpha c}{\mu}\Big)^{-1/(s-1)}/\varepsilon \\ g(0,\alpha;\mu) &= +\infty \\ g(t,\lambda;\mu) &= -(1-t)\lambda + \frac{ct\lambda^2}{\mu} + t^{-1/(s-1)}\Big(1+\frac{c\lambda}{\mu}\Big)^{-1/(s-1)}/\varepsilon \end{aligned}$$

Let $f(\alpha) = \frac{c\alpha^2}{\mu} + (1+\frac{\alpha c}{\mu})^{-1/(s-1)}/\varepsilon$. Then $f'(\alpha) = \frac{2c\alpha}{\mu} - \frac{c}{\mu(s-1)}(1+\frac{\alpha c}{\mu})^{-s/(s-1)}/\varepsilon$, and $f'(\alpha) = 0$ iff $(1+\frac{\alpha c}{\mu})^{-s/(s-1)} = 2\varepsilon\alpha(s-1)$. We notice that $f'' > 0$ so f' is strictly increasing. Let $\alpha = \beta\mu^{s/(2s-1)}$; we define:

$$\begin{aligned} f_1(\beta) &= \varepsilon(s-1)\mu c^{-1}f'(\alpha) \\ &= \mu^{s/(2s-1)}\big[2\beta\alpha_0^{-1} - \big(c\beta + \mu^{(s-1)/(2s-1)}\big)^{-s/(s-1)}\big] \end{aligned}$$

Then it is easy to see that $f_1(\beta)$ changes its sign for $\beta = \beta_0$ in the interval

$[C_0^{-1}, C_0]$ if C_0 is large enough, but independent of $\mu \in]0,1]$. Therefore, $f(\alpha)$ reaches its minimum value for $\alpha = \alpha_1 = \beta_0 \mu^{s/(2s-1)}$ and:

$$f(\alpha_1) = \mu^{1/(2s-1)} \big(c\beta_0^2 + (\mu^{(s-1)/(2s-1)} + c\beta_0)^{-1/(s-1)}/\varepsilon \big)$$

i.e. $f(\alpha_1) \geq \delta \mu^{1/(2s-1)}$, with $\delta > 0$. At last, we examine $h(t) = g(t, \lambda, \mu)$. We find: $h'(t) = 0$ iff $t = t_1 = \big(\varepsilon\lambda(s-1)\big)^{-(s-1)/s}\big(1 + \frac{c\lambda}{\mu}\big)^{-1}$ and

$$h(t_1) = \lambda^{1/s}\big(\varepsilon(s-1)\big)^{(s-1)/s} + \lambda^{1/s}(s-1)^{1/s}\varepsilon^{-(s-1)/s} - \lambda$$

For given $\lambda > 0$, we have $h(t_1) > \delta > 0$ for ε small enough. We eventually proved that there is $\delta_0 > 0$ such that:

$$\phi(x, y, t, \mu, h) \geq \delta_0 h^{-1/s} \mu^{1/(2s-1)} = \delta_0 h^{-1/(2s-1)}$$

uniformly for $(t, \alpha) \in]0,1] \times [0, \lambda]$, which proves the Lemma. □

Next we examine the reduced kernel outside the diagonal, i.e. for $|x - y| > \lambda$, in the following easy:

Lemma 3.3 *If $\varepsilon = \|\psi\|_{2,\infty}$ is small enough (depending on c_0 and λ, then there exists $C_3 > 0$ such that, uniformly for $t \in]0,1]$ and $\theta \in \Gamma_t(x)$:*

$$\begin{aligned} &e^{\mathrm{Re}\, \varphi_\mu(x,y,\theta)/h} \big(|\partial_{\overline{y}_j} b(\widehat{x}, \theta, h)| + |\partial_{\overline{\theta}_j} b(\widehat{x}, \theta, h)| \big) \\ &\qquad \leq C_3 \langle\theta\rangle^d \exp(-|x-y|^{1/s}/C_3 h^{1/s}), \quad h \to 0 \end{aligned} \tag{3.13}$$

uniformly for $\theta = \gamma_x(t, y)$, $|x - y| > \lambda$

Proof. It suffices to find a positive lower bound for:

$$\begin{aligned} \phi(x, y, t, h) = &- \mathrm{Re}\, \varphi_\mu(x, y, \theta)/h \\ &+ t^{-1/(s-1)} \Big| 2\mu \frac{\partial\psi}{\partial x}(\widehat{x}) - c(x-y)\overline{x-y} \Big|^{-1/(s-1)} / C_1 \end{aligned}$$

when $\theta = \gamma_x(t, y)$, $|x - y| > \lambda$. We have $|2\mu \frac{\partial\psi}{\partial x}(\widehat{x}) - c(x-y)\overline{x-y}| = \mathcal{O}(1)$ when $|x - y| > \lambda$, so by (3.6) and (3.8):

$$\begin{aligned} \phi(x, y, t) \geq \Big[&-\mu(\psi(y) - \psi(x)) - 2\mu t\, \mathrm{Re}\, \frac{\partial\psi}{\partial x}(\widehat{x})(x - y) \\ &+ tc(x-y)|x-y|^2 \Big] / h + C t^{-1/(s-1)} \end{aligned}$$

for some $C > 0$. If we denote by $f(t)$ the right hand side of the inequality above, we find that $f(t)$ reaches its minimum for $t = t_0 \sim$ Const. $\big(\frac{h}{|x-y|}\big)^{1-1/s}$, and this minimum verifies $f(t_0) \sim$ Const. $\left(\frac{|x-y|}{h}\right)^{1/s}$ when $\mu > 0$ and $\|\psi\|_{2,\infty}$ are small enough. $\square$

We use integration by parts and a simple variant of Schur's lemma to deduce (3.2) from Lemmas 3.2 and 3.3, getting rid of the factor $\langle\theta\rangle^d$. Let $\chi \in G_0^s(\mathbf{C}^n)$ be equal to 1 in a neighborhood of 0, $\widehat{\chi} = 1 - \chi$ and:

$$^tL = \chi(\theta) - \frac{\widehat{\chi}(\theta)}{|\theta|^2} \sum_{j=1}^{n} \overline{\theta_j} h D_{y_j}$$

We have $^tLe^{i(x-y)\theta/h} = e^{i(x-y)\theta/h}$, so $(^tL)^d e^{i(x-y)\theta/h} = e^{i(x-y)\theta/h}$, and integrating by parts, we can rewrite formula (3.5) in the form:

$$\begin{aligned} Bu(x,h) &= B_1 u(x,h) \\ &\quad - (2\pi h)^{-n} \sum_{|\alpha|\leq d} \int_{[0,1]\times\mathbf{C}^n} e^{i(x-y)\theta/h}((hD_y)^\alpha u)\gamma_x^*(d\omega_\alpha) \end{aligned} \tag{3.14}$$

where $\omega_\alpha = b_\alpha(\widehat{x},\theta,h)dy \wedge d\theta$. Here $b_\alpha(x,\theta,h)$ is a G^s symbol of degree 0, of the form:

$$b_\alpha(x,\theta,h) = \sum_{\beta\leq\alpha} a_{\alpha,\beta}(\theta)\partial_x^\beta b(x,\theta,h) \tag{3.15}$$

where $a_{\alpha,\beta}(\theta)$ is a G^s symbol of degree $-d$. Each integral in the sum can be treated with the same arguments as above, and inequalities similar to (3.12) and (3.13) but without the factor $\langle\theta\rangle^d$. So we proved that R can be written in the form:

$$Ru(x,h) = \sum_{|\alpha|\leq d} \int_{\mathbf{C}^n} K_\alpha(x,y,h)(hD_y)^\alpha u(y,h)L(dy)$$

where each of the kernels $K_\alpha(x,y,h)$ satisfies:

$$e^{(\Phi_\mu(y)-\Phi_\mu(x))/h}|K_\alpha(x,y,h)| \leq C_2 \exp(-1/C_2 h^{1/(2s-1)}), \quad h \to 0 \tag{3.16}$$

when $|x-y| < \lambda$, and:

$$e^{(\Phi_\mu(y)-\Phi_\mu(x))/h}|K_\alpha(x,y,h)| \leq C_3 \exp(-|x-y|^{1/s}/C_3 h^{1/s}), \quad h \to 0 \tag{3.17}$$

when $|x-y|>\lambda$. It follows easily that R is continuous as an operator $W^{d,2}_{\Phi_\mu}\to L^2_{\Phi_\mu}$ and that (3.2) holds (because also of the density of $\mathcal{S}_{\Phi_\mu}$ in $W^{d,2}_{\Phi_\mu}$).

We eventually estimate B_1. For $t=1$, (3.6) gives $\operatorname{Re}\varphi_\mu(x,y,\theta)=\mu F(x,y)-c(|x-y|)|x-y|^2$ for $\theta\in\Gamma_t(x)$. For $|x-y|<\lambda$ and $h>0$ small enough we get $\operatorname{Re}\varphi_\mu(x,y,\theta)\sim-\text{Const.}\ |x-y|^2$ while for $|x-y|>\lambda$, $\operatorname{Re}\varphi_\mu(x,y,\theta)\sim-\text{Const.}\ |x-y|$. We can conclude as above that $B_1=\mathcal{O}(1):W^{d,2}_{\Phi_\mu}\to L^2_{\Phi_\mu}$ and the Proposition is proved. □

Now we give a "local" version of Proposition 3.1 that will be directly used in the sequel. Let $b(x,\theta,h)\in S((0,d),s)$, and $\chi_1\in C_0^\infty(\mathbf{C}^n)$; with the notations of Proposition 3.1, we have:

Proposition 3.4 *Let $B=b^w(x,hD,h)$ as above. Then $\chi_1 B$ extends to a continuous operator $H_{\Phi_\mu}\to L^2_{\Phi_\mu}$ and $\chi_1 B=\chi_1 B_1+\chi_1 R$, where*

$$\chi_1 B_1=\mathcal{O}(1):H_{\Phi_\mu}\to L^2_{\Phi_\mu},\quad h\to 0 \tag{3.18}$$

and

$$\chi_1 R=\mathcal{O}(\exp-\delta/h^{1/(2s-1)}):H_{\Phi_\mu}\to L^2_{\Phi_\mu},\quad h\to 0,\quad (\delta>0) \tag{3.19}$$

Proof. We can conclude directly from the proof of Proposition 3.1 and usual Schur's lemma that (3.18) and (3.19) hold, since x varies in a compact set and thus the factor $\langle\theta\rangle^d$ can be ignored in Lemmas 3.3 and 3.4. □

4. Reduction of waves packets and weighted energy estimates

The method was initiated by A. Cordoba and Ch. Fefferman [CorFe], (see also [Sj2,3]) and adapted to the present context by C. Gérard and J. Sjöstrand [GeSj]. It is called the method of reduction of wave packets, since it reduces a pseudo-differential operator to a "multiplier".

Let $b(x,\theta,h)\in S((0,d),s)$ be elliptic at infinity on Λ_Φ, uniformly for $h>0$ small enough, i.e. $|b(x,\theta,h)|\geq C\langle\theta\rangle^d$ for x outside a compact set $K_1\subset\subset\mathbf{C}^n$ (we identify Λ_Φ with its projection $\pi_x(\Lambda_\Phi)$ on $\mathbf{C}^n$, and a function χ on Λ_Φ with $\chi\circ\pi_x^{-1}$).

Let $\psi\in G_0^s(\mathbf{C}^n)$ be real valued, supported in a neighborhood of K_1 and Φ_μ as in (2.5). Let $\chi_1\in C_0^\infty(\mathbf{C}^n)$ be equal to 1 in a neighborhood of K_1 and such that $\operatorname{supp}\chi_1\subset\subset\operatorname{supp}\psi$. The function χ_1 will be fixed later. We denote also by $b(x,\theta,h)$ an almost analytic extension of $b(x,\theta,h)$ as in

(1.5). First we compute $\chi_1 Bu(x,h),\ u \in H_{\Phi_\mu}$ using Proposition 3.4. Let B_1 be associated with B as in Proposition 3.1 or 3.4. In the integral defining $B_1 u(x,h)$, we insert a cut-off function $\chi \in C_0^\infty(\mathbf{C}^n)$. With the notations of Sect. 2 we have: $\chi_1 B_1 = \chi_1(B_1)_\chi + \chi_1(B_1)_{\widehat{\chi}}$. From now on, we work modulo error terms $\mathcal{O}(h)$. Propositions 3.4 and 2.3 give:

$$\begin{aligned}\chi_1(B_1)_\chi u(x,h) &= \chi_1(x) b(x,\xi_\mu(x),h) u(x,h) \\ &\quad + \chi_1(x) d_\xi b(x,\xi_\mu(x),h)(hD_x - \xi_\mu(x)) u(x,h) \\ &\quad + Ru(x,h), \quad u \in H_{\Phi_\mu} \end{aligned} \tag{4.1}$$

where $R = \mathcal{O}(h) : H_{\Phi_\mu} \to L^2_{\Phi_\mu}$. We observe next that the contribution of $\chi_1(B_1)_{\widehat{\chi}}$ to $\chi_1 B_1$ gives an error term that also enters into R.

Proposition 4.1 *With the hypotheses above, we have for* $(u,v) \in H_{\Phi_\mu} \times H_{\Phi_\mu}$:

$$\begin{aligned}(\chi_1 Bu|v)_\mu &= \int b(x,\xi_\mu(x),h) u(x,h) \overline{v(x,h)} \chi_1(x) e^{-2\Phi_\mu(x)/h} L(dx) \\ &\quad + \mathcal{O}(h) \|u\|_\mu \|v\|_\mu, \quad h \to 0 \end{aligned} \tag{4.2}$$

(Here $(\cdot\,|\,\cdot)_\mu$ *denotes the scalar product in* H_{Φ_μ}*).*

Proof. From (4.1) and the following discussion, we have:

$$\begin{aligned}(\chi_1 Bu|v)_\mu &= \int b(x,\xi_\mu(x),h)) u(x,h) \overline{v(x,h)} \chi_1(x) e^{-2\Phi_\mu(x)/h} L(dx) \\ &\quad + \sum_{j=1}^n \int \partial_{\xi_j} b(x,\xi_\mu(x),h)) \left[(hD_{x_j} - \xi_{\mu,j}(x)) u(x,h)\right] \\ &\quad \overline{v(x,h)} \chi_1(x) e^{-2\Phi_\mu(x)/h} L(dx) \\ &\quad + \mathcal{O}(h) \|u\|_\mu \|v\|_\mu \end{aligned}$$

In the last sum, we integrate by parts; we have $(-hD_{x_j} - \xi_{\mu,j}(x)) e^{-2\Phi_\mu(x)/h} = 0$, while $hD_{x_j} \overline{v(x,h)} = 0$ (since v is holomorphic) and $hD_{x_j}(\chi_1(x) \partial_{\xi_j} b(x,\xi_\mu(x),h)) = \mathcal{O}(h)$. So this sum is again $\mathcal{O}(h)\ \|u\|_\mu \|v\|_\mu$. □

Thus it follows that for any $u \in H_{\Phi_\mu}$:

$$\operatorname{Im}(\chi_1 Bu|u)_\mu = \int \operatorname{Im}(b(x,\xi_\mu(x),h)\chi_1(x)|u(x,h)|^2 e^{-2\Phi_\mu(x)/h}L(dx) + \mathcal{O}(h)\|u\|_\mu^2 \tag{4.3}$$

Now we estimate u in the elliptic zone, i.e. outside K_1. Let $\chi_0 \in S(0,s)$ be equal to 1 near K_1 and $\operatorname{supp}\chi_0 \subset\subset \operatorname{supp}\psi$. We also denote by χ_0 its almost analytic extension in the sense of Theorem a.3. If $\widehat{\chi}_0 = 1 - \chi_0$, this relation makes sense also after almost analytic extension. Let $\widehat{\chi}_0^w$ be the corresponding h-quantized Weyl operator. Since $b(x,\theta,h)$ is elliptic on $\operatorname{supp}\widehat{\chi}_0$, Proposition 1.2 shows there exists $a \in S(0,s)$, such that $A = a^w(x,hD,h)$ satisfies: $AB = \widehat{\chi}_0^w + r^w$, where $r \in S(-\infty,s)$. In particular, $r^w = \mathcal{O}(\exp -1/Ch^{1/s}) : H_\Phi \to L^2_\Phi$ for some $C > 0$ and if the perturbation ψ is chosen small enough in the C^1-topology, we also have $r^w = \mathcal{O}(\exp -1/C'h^{1/s}) : H_{\Phi_\mu} \to L^2_{\Phi_\mu}$, with some $C' > 0$. So:

$$(ABu|v)_\mu = (\widehat{\chi}_0^w u|v)_\mu + \mathcal{O}(\exp -1/C'h^{1/s})\ \|u\|_\mu\|v\|_\mu, \quad u,v \in H_{\Phi_\mu} \tag{4.4}$$

Then, we estimate $(\widehat{\chi}_0^w u|v)_\mu$, with Propositions 3.1 and 4.1. We get:

$$(ABu|v)_\mu = \int \widehat{\chi}_0(x,\xi_\mu(x))u\bar{v}e^{-2\Phi_\mu(x)/h}L(dx) + \mathcal{O}(h)\ \|u\|_\mu\|v\|_\mu \tag{4.5}$$

We would like to set "$\chi_1(x) = \chi_0(x,\xi_\mu(x))$" but this cannot be done since we shall require $\chi_1(x) \geq 0$. So we need another estimate. Let $\chi_4 \in C_0^\infty(\mathbf{C}^n;\mathbf{R}^+)$ be supported in $\operatorname{supp}\psi$. As in [Sj2, Th. 1.3] and Proposition 4.1 we have:

$$(\chi_4 Bu|Bv)_\mu = \int |b(x,\xi_\mu(x),h)|^2 u(x,h)\overline{v(x,h)}\chi_4(x)e^{-2\Phi_\mu(x)/h}L(dx) + \mathcal{O}(h)\|u\|_\mu\|v\|_\mu, \quad h \to 0 \tag{4.6}$$

So let $\chi_1(x) = \chi_0(x,\xi(x))$ and $\chi_5(x) = \chi_0(x,\xi_\mu(x)) - \chi_0(x,\xi(x))$. As follows from Theorem a.3, $\chi_5(x)$ is supported in a small neighborhood of $\operatorname{supp}\nabla\chi_0$ modulo a function which is $\mathcal{O}(\exp -1/Ch^{1/s})$ uniformly in $\mathbf{C}^n$. So if χ_4 is conveniently chosen with support in a small neighborhood of $\operatorname{supp}\nabla\chi_0$, we have:

$$(\chi_5 u|v)_\mu = (\chi_5\chi_4 u|v)_\mu + \mathcal{O}(\exp -1/C'h^{1/s})\ \|u\|_\mu\|v\|_\mu, \quad u,v \in H_{\Phi_\mu} \tag{4.7}$$

Using that $|b(x,\xi_\mu(x),h)|^2 \geq c > 0$ on $\operatorname{supp}\chi_4$ (where $c > 0$ can be chosen independent of h), we deduce from (4.6) and (4.7):

$$(\chi_5 u|u)_\mu \leq \frac{1}{c}\|\chi_5\|_\infty(\chi_4 Bu|Bu)_\mu + \mathcal{O}(h)\|u\|_\mu^2, \quad u \in H_{\Phi_\mu} \tag{4.8}$$

Formulae (4.3), (4.5) and (4.8) are basic estimates for Theorem 0.1.

5. Proof of Theorem 0.1

We shall be working from now on with the Schrödinger operator $P_0 = -h^2\Delta + V$ although our arguments could apply to a more general situation.

a) Distorsion and global escape function

We return for a while to the real phase-space variables $(y,\eta) \in \mathbf{R}^{2n}$. The symbol of P_0 (for the standard or Weyl quantization) is $p_0(y,\eta) = \eta^2 + V(y)$. First we recall from [GeMa] how to construct a global escape function $G(y,\eta)$, under the non-trapping hypothesis $K(I) = \emptyset$, $I = [E_0 - \varepsilon_0, E_0 - \varepsilon_0]$ (see (0.5)).

Recall from (0.3) that V is dilation analytic outside a compact set $K_0 \subset \mathbf{R}^n$. Then by Cauchy's inequalities, there exists $R > 0$ such that $H_{p_0}(y\eta) = 2\eta^2 - y\nabla V(y) \geq E_0$ on $\Sigma_I = \{(y,\eta) \mid p_0(y,\eta) \in I\}$ for $|y| \geq R$. Thus, $y\eta$ is an escape function for $|y| \geq R$ and we may assume $K_0 \subset B(0,R)$. Now let $g(y) \in G_0^s(\mathbf{R}^n)$ be supported in a neighborhood of $B(0,R)$, equal to 1 on $B(0,R)$ and satisfying $0 \leq g(y) \leq 1$. We set

$$f(y,\eta) = -\int_0^\infty g\big(\pi_y(\exp tH_{p_0}(y,\eta)\big)dt$$

where $\pi_y : T^*\mathbf{R}^n \to \mathbf{R}^n$ denotes the natural projection. Because V is non trapping near energy E_0, the integrand is compactly supported for $(y,\eta) \in \Sigma_I$, $f \in G^s(\Sigma_I)$ is bounded, and moreover $H_{p_0} f = g$. Let now $\chi_2 \in G_0^s(\mathbf{R}^n)$ be equal to 1 on $\operatorname{supp} g$ and $C_2 > 0$ a constant to be chosen. We set $G(y,\eta) = C_2\chi_2(y)f(y,\eta) + y\eta$, and compute $H_{p_0}G = C_2 g + C_2 f H_{p_0}\chi_2 + H_{p_0}(y\eta)$. Choosing $C_2 > 0$ large enough, and then taking $\operatorname{supp}\chi_2$ sufficiently large, we have $H_{p_0}G(y,\eta) \geq C_0$ on Σ_I for some $C_0 > 0$. Moreover, $G(y,\eta) - y\eta \in S(0,s)$.

Now, for $|y| \geq R$, we can construct an analytic distorsion for P_0 in configuration variables as in [SjZ] as follows.

Take polar coordinates $y = r\omega$, with $(r,\omega) \in \mathbf{R}^+ \times S^{n-1}$ and for $0 \leq \theta \leq \theta_0$, define a family of G^s functions $f_\theta :]0, +\infty] \to \mathbf{C}$ such that $f_\theta(r) = r$ for $r < R < R'$ and $f_\theta(r) = e^{i\theta}r$ for $r > R'$. The distorsions are defined by: $U_\theta u(r\omega) = (f'_\theta(r))^{n/2}u(f_\theta(r)\omega)$ so that the distorted hamiltonian takes the form $P_\theta = U_\theta P_0 U_\theta^{-1}$. This is a differential operator of the form (1.17) for $d = 2$, and we can check

$$\begin{aligned} &\exists \widetilde{a}_\alpha(x) \in G^s, \quad |\alpha| \leq 2, \quad \text{such that} \quad a_\alpha(y,h) - \widetilde{a}_\alpha(y) = \mathcal{O}(h) \\ &\qquad \text{uniformly for} \quad y \in \mathbf{R}^n, \quad 0 < h < h_0. \end{aligned} \tag{5.1}$$

We call $\widetilde{p}_\theta(y,\eta) = \sum_{|\alpha|\leq 2} \widetilde{a}_\alpha(y)\eta^\alpha$ the "principal symbol" of $P_\theta(y, hD_y, h)$.

The resonances of P_0 are actually the discrete eigenvalues of P_θ when $\theta > 0$ (it is wellknown that they are independent of the distorsion, see e.g. [HeMa]).

Everything will essentially depend on the principal symbol of P_θ only. If (r^*, ω^*) stand for the dual variables of (r,ω) in polar coordinates, this principal symbol is given by:

$$p_\theta(r,\omega,r^*,\omega^*) = (f'_\theta(r))^{-2}r^{*2} + (f_\theta(r))^{-2}\omega^{*2} + V((f_\theta(r)\omega)$$

Then it can be easily shown that we can choose f_θ such that $\operatorname{Im} p_\theta(r,\omega,r^*,\omega^*) \leq 0$ on a neighborhood of $\operatorname{Re} p_\theta(r,\omega,r^*,\omega^*) = E_0$, the inequality being strict for r sufficiently large, e.g. $r \geq R$.

Operator P_θ is elliptic outside a compact neighborhood K_1 of $\Sigma_I \cap \{r \leq R\}$, i.e. its principal symbol verifies $|p_\theta(y,\eta)| \geq C\langle\eta\rangle^2$ for (y,η) outside K_1. Because of (5.1), this holds also for the full symbol, either for the classical, or the Weyl quantization.

Let $\chi_0 \in G_0^s(\mathbf{R}^{2n})$ be equal to 1 on a small neighborhood of K_1, and $\chi_6 \in G_0^s(\mathbf{R}^{2n})$ be equal to 1 in a neighborhood of $\operatorname{supp}\chi_0$. We let $\delta > 0$ be very small but independent of h. If $\widetilde{G} = \delta\chi_6 G$, then

$$H_{p_\theta}\widetilde{G}(y,\eta) = H_{p_0}\widetilde{G}(y,\eta) \geq \delta' > 0 \quad \text{on} \quad \operatorname{supp}\chi_0. \tag{5.2}$$

We take an almost analytic extension of P_θ as in Corollary a.4 and make the FBI transformation T. We denote by $B_\theta(x, hD_x, h)$ the Weyl operator corresponding to $P_\theta(y, hD_y, h)$. All what we have said extends trivially to the complex phase space, after applying κ_T, so we use generally the same notations for objects belonging either to $T^*\mathbf{R}^n$ or Λ_Φ.

Put $\psi = \widetilde{G} \circ \kappa_T^{-1}$. As usual, we identify Λ_Φ with $\mathbf{C}^n$, so that $\psi(x, \xi(x)) = \delta(\chi_6 G) \circ \kappa_T^{-1}(x, \xi(x))$ can be viewed as a function of x alone. Let $\widetilde{b}_\theta(x, \xi(x))$ be the principal symbol of $b_\theta(x, \xi(x), h)$ as in (5.1). By a Taylor expansion, we have:

$$\operatorname{Im} \widetilde{b}_\theta(x, \xi_\mu(x)) = \operatorname{Im} \widetilde{b}_\theta(x, \xi(x)) - 2\mu \operatorname{Re}\Big[d_\xi \widetilde{b}_\theta(x, \xi(x)) \frac{\partial \psi}{\partial x}(x)\Big] + \mathcal{O}(\mu^2) \tag{5.3}$$

uniformly for $x \in \mathbf{C}^n$ (so the 2 first terms on the RHS of (5.3) are evaluated on the "real" domain), and with $\chi_1(x) = \chi_0(x, \xi(x))$:

$$2\chi_1(x) \operatorname{Re}\Big[d_\xi \widetilde{b}_\theta(x, \xi(x)) \frac{\partial \psi}{\partial x}(x)\Big] = \chi_1(x)(H_{p_0} \widetilde{G}) \circ \kappa_T^{-1}(x, \xi(x)) \tag{5.4}$$

By the discussion above, we have: $\chi_1(x) \operatorname{Im} \widetilde{b}_\theta(x, \xi(x)) \leq 0$ when χ_1 is supported by some sufficiently narrow neighborhood of $p_0(y, \eta) = E_0$ that we call again K_1. Then (5.3) gives:

$$\chi_1(x)(\operatorname{Im} \widetilde{b}_\theta(x, \xi_\mu(x)) - \operatorname{Im} E) \leq \chi_1(x)(-\mu\delta' - \operatorname{Im} E + \mathcal{O}(\mu^2)) \tag{5.5}$$

for E in a small complex neighborhood of E_0. In (5.5) of course, $\chi_1(x) \operatorname{Im} \widetilde{b}_\theta(x, \xi_\mu(x))$ does not depend on θ since distorsion is turned on for $|x| \geq R$ and $\operatorname{supp} \chi_1 \subset \{|x| \leq R\}$. Now we are ready to conclude with the energy estimates of Sect. 4.

b) End of the proof

We prove an a priori estimate for an eigenfunction associated to E. Let $u_\theta \in L^2(\mathbf{R}^n)$ be such that $(P_\theta - E)u_\theta = 0$, and $u = Tu_\theta \in H_{\Phi_\mu}$ be the corresponding eigenfunction for B_θ, that is $(B_\theta - E)u = 0$. We normalize u in such a way that $\|u\|_\mu = 1$. Let A be a parametrix of $B_\theta - E$ outside K_1 as above. Because of (5.1), we can replace in (4.3) the full symbol $b(x, \xi_\mu(x), h)$ of B by its principal symbol $\widetilde{b}(x, \xi_\mu(x))$. By (4.5):

$$(\widehat{\chi}_0(x, \xi_\mu(x)) u | u)_\mu = \mathcal{O}(h) \ \|u\|_\mu^2$$

On the other hand (4.8) and (5.5) give respectively:

$$(\chi_5(x) u | u)_\mu = \mathcal{O}(h) \ \|u\|_\mu^2$$

and:

$$\big(\mu\delta' + \operatorname{Im} E + \mathcal{O}(\mu^2)\big)(\chi_1(x) u | u)_\mu = \mathcal{O}(h) \ \|u\|_\mu^2$$

These estimates, together with the identity

$$(\chi_1(x)u|u)_\mu + (\chi_5(x)u|u)_\mu + (\widehat{\chi}_0(x,\xi_\mu(x))u|u)_\mu = \|u\|_\mu^2 = 1$$

clearly show that $\operatorname{Im} E \leq -\delta_1\mu$, as h is sufficiently small, for some $\delta_1 > 0$, which brings the proof to an end. □

Appendix

We prove existence of almost analytic extensions for G^s functions. Although such results are of course wellknown among specialists, it might be useful to give self-contained proofs.

a) The 1-d case

The most complete result holds in one variable and will not apply to our problem, we just give it for the sake of completeness. We follow the Borel argument of L. Carleson.

Theorem a.1 [Ca] *Given $A > 0$, let $G^s_A(\mathbf{R})$ be the set of $u \in C^\infty(\mathbf{R})$ such that $\exists C > 0$ with:*

$$\sup_{x\in\mathbf{R}} |\partial_x^\alpha u(x)| \leq CA^\alpha \alpha!^s, \quad \alpha \in \mathbf{N}$$

endowed with its natural topology. Then for any $A > 0$, there exists $B > 0$ and a continuous almost analytic extension operator $G^s_A(\mathbf{R}) \to G^s_B(\mathbf{C})$, $u \mapsto \widetilde{u}$ such that:

$$|\overline{\partial}_z \widetilde{u}(z)| \leq C \exp(-|\operatorname{Im} z|^{-1/(s-1)}/C), \quad C > 0.$$

Proof. We make the general argument of [Ca] more explicit in this case. Introduce the weight function

$$W(\tau) = \sup_{\nu\in\mathbf{N}} \frac{|\tau|^\nu}{(\nu!)^s}, \quad \tau \in \mathbf{R}\setminus 0, \quad W(0) = 1$$

We have $W(\tau) \geq 1$, and $|\tau|^n = o(W(\tau))$, $|\tau| \to \infty$, any n; we may choose instead $W(\tau) = \exp(a|\tau|^{1/s})$, $a > 0$, or $W(\tau) = \sum_{\nu\in\mathbf{N}} \frac{|\tau|^\nu}{(\nu!)^s}$. Since G^s is non-quasianalytic, Denjoy-Carleman theorem [Ma] tells us that $\int_{\mathbf{R}} \frac{\log W(\tau)}{1+\tau^2} d\tau < +\infty$. We set $w(\tau) = W(\tau)^{-1}$. Let $Q(\tau)$ be a polynomial such that $\int_{\mathbf{R}} |Q(\tau)|^2 w(\tau) d\tau \leq 1$. Then, by a simple argument from harmonic analysis using the maximum principle [Ca], there is a numerical constant K_0, independant of Q, such that:

$$|Q^{(\nu)}(0)| \leq K_0 \nu! M_\nu^{-1} \tag{a.1}$$

where:

$$M_\nu = \sup_{r>0} \exp\Big(\Big(\nu + \frac{1}{2}\Big) \log r - \mu(r)\Big)$$

$$\mu(r) = \frac{1}{2\pi} \int_{\mathbf{R}} \frac{r}{r^2+\tau^2} \log W(\tau) d\tau$$

We can easily show:

$$K_1 = \liminf_{\nu\to+\infty} \Big(\frac{M_\nu}{(\nu!)^s}\Big)^{1/\nu} > 0 \tag{a.2}$$

Now we look for an almost analytic extension operator between some weighted L^2-spaces. For $j \in \mathbf{N}$, consider the sets:

$$E_j = \Big\{ f : \mathbf{R}\times\mathbf{R} \to \mathbf{C} \text{ measurable} : \quad \|f\|_j = \sup_{x\in\mathbf{R}} \Big(\int_{\mathbf{R}} |\partial_x^j f(\tau,x)|^2 w(\tau) d\tau\Big)^{1/2} < \infty \Big\}$$

For a given sequence $\lambda = (\lambda_\nu)_{\nu\in\mathbf{N}}$ of positive numbers, we consider also the sets of sequences of functions:

$$S_j(\lambda) = \Big\{ s = (s_\nu)_{\nu\in\mathbf{N}}, s_\nu : \mathbf{R} \to \mathbf{C}, \quad |s|_j = \sup_{x\in\mathbf{R}} \Big(\sum_{\nu\geq 0} |\partial_x^j s_\nu(x)|^2 \lambda_\nu^{-2}\Big)^{1/2} < \infty \Big\}$$

For any N, the sets $E^N = \bigcap_{j=0}^N E_j$ and $S^N(\lambda) = \bigcap_{j=0}^N S_j(\lambda)$ are Banach spaces, and we can form the inductive limit as $N \to \infty$.

Let $u \in G_A^s(\mathbf{R})$. Since the extension $\widetilde{u} \in G^s(\mathbf{C})$ should verify $\overline{\partial}_z \widetilde{u}(z) = \mathcal{O}(|\operatorname{Im} z|^\infty)$, we obviously have, in the sense of formal power series:

$$\widetilde{u}(z) = \sum_{n\geq 0} \frac{u^{(n)}(x)}{n!} (iy)^n, \quad z = x + iy. \tag{a.3}$$

We then look for $\widetilde{u}(z)$ in the form:

$$\widetilde{u}(z) = \int_{\mathbf{R}} e^{iby\tau} f(\tau, x) w(\tau) d\tau \tag{a.4}$$

where $b > 0$ and $f \in \bigcap_N E^N$, are the unknowns. Identifying the derivatives at $y = 0$, we find $u^{(n)}(x) = b^n \int_{\mathbf{R}} \tau^n f(\tau,x) w(\tau) d\tau$. Let $s_n(x) = u^{(n)}(x) b^{-n}$. Since $u \in G^s_A(\mathbf{R})$, we have $|s_n(x)| \leq C\big(\frac{A}{b}\big)^n (n!)^s$ for some $C > 0$; we choose

$$\lambda_n = \left(\frac{2^{s+1/2}A}{b}\right)^n (n!)^s. \tag{a.5}$$

We consider the following moment problem: find an operator $L : s \mapsto f$ which is continuous $S^N(\lambda) \to E^N$, any N, (as well for the inductive limit topology), and such that: $s_n(x) = \int_{\mathbf{R}} \tau^n f(\tau,x) w(\tau) d\tau$. Following [Ca], we let $P_n(\tau)$ be a complete orthonormal family of polynomials with respect to the weight $w(\tau)$, $P_n(\tau) = \sum_{\nu=0}^n \alpha_{\nu,n} \tau^\nu$, and decompose $f(\tau,x)$ on this basis as $f(\tau,x) = \sum_{n\geq 0} b_n(x) P_n(\tau)$. We find $b_n(x) = \sum_{\nu=0}^n \alpha_{\nu,n} s_\nu(x)$, and by Parseval identity, $\|f\|_0^2 = \sup_{x\in\mathbf{R}} \sum_{n\geq 0} |b_n(x)|^2$. Applying (a.1) to the sequence $Q_{N,\nu}(\tau) = \sum_{n=0}^N a_{n,\nu} P_n(\tau)$, $a_{n,\nu} = \big(\sum_{k=0}^N |P_k^{(\nu)}(0)|^2\big)^{-1/2} P_n^{(\nu)}(0)$, we let $N \to \infty$ and find: $\sum_{n\geq 0} |\alpha_{\nu,n}|^2 \leq K_0^2 M_\nu^{-2}$, and so $\|f\|_0 \leq K_0 |s|_0 \big(\sum_{\nu\geq 0} \big(\frac{\lambda_\nu}{M_\nu}\big)^2\big)^{1/2}$, where K_0 as in (a.1) and λ_ν as in (a.5). Estimate (a.2) then shows that $K_2 = \big(\sum_{\nu\geq 0}\big(\frac{\lambda_\nu}{M_\nu}\big)^2\big)^{1/2} < \infty$ if $b > \frac{2^{s+1/2}A}{K_1}$, so we have determined a continuous operator $L : S_0(\lambda) \to E_0, s \mapsto f$. Reasoning similarly for the derivatives, we find that $L : S^N(\lambda) \to E^N$, any N, with

$$\|f\|_j \leq K_0 K_2 |s|_j \tag{a.6}$$

We then observe that

$$|s|_j \leq C A^j 2^{sj+1/2} (j!)^s. \tag{a.7}$$

By Cauchy-Schwarz inequality,

$$|\partial_x^j \partial_y^k \widetilde{u}(x+iy)| \leq b^k \Big(\int_{\mathbf{R}} |\partial_x^j f(\tau,x)|^2 w(\tau) d\tau\Big)^{1/2} \Big(\int_{\mathbf{R}} \tau^{2k} w(\tau) d\tau\Big)^{1/2}$$

The last integral is estimated by $K_3^{k+1}(k!)^s$, where $K_3 > 0$ is a numerical constant, while (a.6) and (a.7) show that the first integral can be estimated by $CK_0K_2A^j2^{sj+1/2}(j!)^s$. So we get $\widetilde{u}(z) \in G^s_B(\mathbf{C})$, with $B = \sup(bK_3, 2^sA)$. There remains to show the estimate on $\overline{\partial}_z \widetilde{u}(z)$. Since by construction $\overline{\partial}_z \widetilde{u}(z)$ vanishes of infinite order on $\mathbf{R}$, Taylor formula shows:

$$\overline{\partial}_z \widetilde{u}(z) = \frac{1}{(k-1)!} \int_0^1 \langle D^k \overline{\partial}_z \widetilde{u}(x+ity), (iy)^k \rangle (1-t)^{k-1} dt \tag{a.8}$$

Using $\overline{\partial}_z \widetilde{u} \in G^s_B(\mathbf{C})$, we choose k of the same order of magnitude as $|y|^{\frac{-1}{s-1}}$, and we get the Theorem with a constant $C > 0$ depending on u. □

b) The general case

The very elegant proof of [Ca] relies on properties of harmonic functions in the upper-half plane, which we were not able to extend to several variables. Maybe one could try to use Poisson integrals and maximum principles in tubes ([HöI, Sect. 9], [Ko],...), or Dynkin's method ([ChaCho],...). Borrowing an idea presumably due to J. Mather (as was reported to us by J. Sjöstrand), we prove instead a weaker result, since we loose (arbitrarily small) Gevrey smoothness, and the extension is not given by a continuous linear operator. But all what we really need is the right decay on $\overline{\partial}\widetilde{u}$ near the real axis. Our first result applies to compactly supported functions $G^s_0(\mathbf{R}^n)$.

Proposition a.2 *Let $u \in G^s_0(\mathbf{R}^n)$. Then for all $s' > s$, there exists an extension $\widetilde{u} \in G^{s'}(\mathbf{C}^n)$, such that $\widetilde{u}(z) = u(z)$ for real z and for some $C >$ 0 :*

$$N_{s'}\big(\overline{\partial}_z \widetilde{u}, T\big)(z) \leq C \exp(-|\operatorname{Im} z|^{-1/(s-1)}/C), \quad z \in \mathbf{C}^n$$

when $T \succ 0$ is small enough. (Notations here are those of Sect. 1, where we have omitted the variable h).

Proof. Let $\delta = s' - s$, $r > 1$, and $\psi \in G^{1+\delta}_0(\mathbf{R}^n)$ be equal to 1 in $B(0,1)$ and vanishing outside $B(0,r)$. For instance, we can take $\psi(y) = \widetilde{\psi}(y_1) \otimes \cdots \otimes \widetilde{\psi}(y_n)$, $y = (y_1, \ldots, y_n) \in \mathbf{R}^n$. For $\xi \in \mathbf{R}^n$, $z \in \mathbf{C}^n$ and $C' > 0$ to be chosen later, depending on u, we set $\chi(z,\xi) = \psi(C'\langle\xi\rangle^{(s-1)/s} \operatorname{Im} z)$. Define:

$$\widetilde{u}(z) = (2\pi)^{-n} \int e^{iz\xi} \widehat{u}(\xi) \chi(z,\xi) d\xi \tag{a.9}$$

where $\widehat{\cdot}$ denotes Fourier transform. This defines an extension $\widetilde{u}(z) \in C^\infty(\mathbf{C}^n)$ of u, because χ is compactly supported in ξ when $\operatorname{Im} z \neq 0$. There remains to show that $\widetilde{u} \in G^{s'}(\mathbf{C}^n)$ and so we compute:

$$. \quad N_{s'}(\widetilde{u}, T)(z) = \sum_{\alpha,\beta \in \mathbf{N}^n} |D^\alpha \overline{\partial}^\beta \widetilde{u}(z)| T^{(\alpha,\beta)} / (\alpha!)^{s'} (\beta!)^{s'} \tag{a.10}$$

We have:

$$D^\alpha \overline{\partial}^\beta \widetilde{u}(z) = (2\pi)^{-n} \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \int e^{iz\xi} \xi^{\alpha-\alpha'} \widehat{u}(\xi) D^{\alpha'} \overline{\partial}^\beta \chi(z,\xi) d\xi \tag{a.11}$$

where $D = -i\partial_z$ and $\overline{\partial} = \overline{\partial}_z$ denote respectively the holomorphic and anti-holomorphic derivatives. As $\psi \in G_0^{1+\delta}(\mathbf{R}^n)$ we get the uniform estimates:

$$|D^{\alpha'} \overline{\partial}^\beta \chi(z,\xi)| \leq C_1^{|\alpha'+\beta|+1} (\alpha' + \beta)!^{(1+\delta)} \langle \xi \rangle^{(s-1)|\alpha'+\beta|/s}$$

for some $C_1 > 0$. On the other hand, we know that $|\widehat{u}(\xi)| \leq C_0 \exp -\langle \xi \rangle^{1/s}/C_0$ for some $C_0 > 0$ depending on u. If we write:

$$\begin{aligned} |\xi^{\alpha-\alpha'} \widehat{u}(\xi)| \leq C_0 \langle \xi \rangle^{(s-1)|\alpha-\alpha'|/s} \exp(-\langle \xi \rangle^{1/s}/2C_0) \\ \times \langle \xi \rangle^{|\alpha-\alpha'|/s} \exp(-\langle \xi \rangle^{1/s}/2C_0) \end{aligned}$$

use the inequalities

$$\langle \xi \rangle^{|\alpha-\alpha'|/s} \exp(-\langle \xi \rangle^{1/s}/2C_0) \leq |\alpha - \alpha'|^{|\alpha-\alpha'|} (2C_0/e)^{|\alpha-\alpha'|} \tag{a.12}$$

and $(|\alpha - \alpha'|)^{|\alpha-\alpha'|} \leq C^{|\alpha-\alpha'|+1} |\alpha - \alpha'|!$ we see that the right hand side of (a.11) can be estimated by

$$\begin{aligned} \int \exp(-\xi \operatorname{Im} z - \langle \xi \rangle^{1/s}/2C_0) \langle \xi \rangle^{(s-1)|\alpha+\beta|/s} \\ \times C_2^{|\alpha+\beta|+1} \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} |\alpha - \alpha'|!(\alpha' + \beta)!^{(1+\delta)} \chi_{\alpha',\beta,z}(\xi) d\xi \end{aligned} \tag{a.13}$$

where $\chi_{\alpha',\beta,z}$ is the characteristic function of $\operatorname{supp} D^{\alpha'} \overline{\partial}^\beta \chi(z,\cdot)$ and $C_2 > 0$ a constant depending on u. Now we use again (a.12) which gives:

$$\exp(-\langle \xi \rangle^{1/s}/4C_0) \langle \xi \rangle^{(s-1)|\alpha+\beta|/s} \leq C_3^{|\alpha+\beta|+1} |\alpha + \beta|^{(s-1)|\alpha+\beta|}$$

On the other hand, one can check that:

$$\begin{aligned} (C_2C_3)^{|\alpha+\beta|+1} |\alpha + \beta|^{(s-1)|\alpha+\beta|} \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} |\alpha - \alpha'|!(\alpha' + \beta)!^{(1+\delta)} \\ \leq C_4^{|\alpha+\beta|+1} |\alpha + \beta|^{(s-1)|\alpha+\beta|} (\alpha!)^{1+\delta} (\beta!)^{1+\delta} \\ \leq C_5^{|\alpha+\beta|+1} (\alpha!)^{s+\delta} (\beta!)^{s+\delta} = C_5^{|\alpha+\beta|+1} (\alpha!)^{s'} (\beta!)^{s'} \end{aligned} \tag{a.14}$$

where $C_5 > 0$ depends on u, ψ, s and the dimension n only. We distinguish between the cases $\beta \neq 0$ and $\beta = 0$. For $\beta \neq 0$, we need to estimate the integral:

$$\int \exp(-\xi \operatorname{Im} z - \langle \xi \rangle^{1/s}/4C_0)d\xi$$

over a shell:

$$(C'|\operatorname{Im} z_j|)^{-s/(s-1)} \leq \langle \xi \rangle \leq (C'|\operatorname{Im} z_j|/r)^{-s/(s-1)} \tag{a.15}$$

for some $j \in \{1, \ldots, n\}$. Choosing $C' > rC_0$, we see easily that this integral is bounded by $C_6 \exp(-|\operatorname{Im} z|^{-1/(s-1)}/C_6)$. So (a.11) to (a.14) give for $\beta \neq 0$:

$$|D^\alpha \overline{\partial}^\beta \widetilde{u}(z)| \leq C_7 C_5^{|\alpha+\beta|+1} (\alpha!)^{s'} (\beta!)^{s'} \exp(-|\operatorname{Im} z|^{-1/(s-1)}/C_6) \tag{a.16}$$

For $\beta = 0$, $\chi_{\alpha',0,z}$ is supported in a shell (a.15) when $\alpha' \neq 0$, or in the whole ball $\langle \xi \rangle \leq \sup_j (C'|\operatorname{Im} z_j|/r)^{-s/(s-1)}$ when $\alpha' = 0$. In the first case we get the same estimate as (a.16) with $\beta = 0$, $\alpha = \alpha'$. For $\alpha' = 0$, choosing $C' > 4C_0$ we get

$$\left| \int e^{iz\xi} \widehat{u}(\xi) \xi^\alpha \chi(z, \xi) d\xi \right| \leq C_8^{|\alpha|+1} |\alpha|^{s|\alpha|}$$

and the Proposition easily follows. □

Next we show that $\widetilde{u}$ can be chosen to be compactly supported if this is the case of u.

Theorem a.3 *Let $u \in G_0^s(\mathbf{R}^n)$. Then for all $s' > s$, there exists an extension $\widetilde{u} \in G_0^{s'}(\mathbf{C}^n \cap \{|\operatorname{Im} z| < c\})$, $c > 0$ small enough, such that $\widetilde{u}(z) = u(z)$ for real z and for some $C > 0$:*

$$N_{s'}(\overline{\partial}\widetilde{u}, T)(z) \leq C \exp(-|\operatorname{Im} z|^{-1/(s-1)}/C), \quad z \in \mathbf{C}^n, \quad |\operatorname{Im} z| < c$$

when $T \succ 0$ is small enough.

Proof. We implement the previous construction with a Paley-Wiener type argument. So let again $\delta = s' - s$, and $\psi \in G_0^{1+\delta}(\mathbf{C}^n)$ be equal to 1 in the polydisc $\widetilde{B}(0,1) = \{z = (z_1, \ldots, z_n) \in \mathbf{C}^n : |z_1| < 1, \ldots, |z_n| < 1\}$ and vanishing outside $\widetilde{B}(0,2)$. For instance, we can take $\psi(y) = \widetilde{\psi}(y_1) \otimes \cdots \otimes \widetilde{\psi}(y_n)$, $y = (y_1, \ldots, y_n) \in \mathbf{C}^n$, and $\widetilde{\psi} \in G_0^{1+\delta}(\mathbf{C})$ rotation invariant. For $\zeta \in \mathbf{C}^n$, $z \in \mathbf{C}^n$ and $C' > 0$ to be chosen later, depending on u, we set $\chi(z, \zeta) = \psi(C' \langle \zeta \rangle^{(s-1)/s} \operatorname{Im} z)$, where $\langle \zeta \rangle^{(s-1)/s}$ denotes an analytic branch of $\exp\left(\frac{s-1}{2s}\log(1+\zeta_1^2+\cdots+\zeta_n^2)\right)$ near a point $\zeta \in \mathbf{C}^n$ such that $1+\zeta_1^2+\cdots+\zeta_n^2 \neq 0$. Assume that $\operatorname{supp} u \subset B(x_0, r)$, and let $Y \in G_0^s(\mathbf{C}^n)$ be equal to 1 near $B(x_0, r)$. By translation, we can assume $x_0 = 0$. We define as above:

$$\widetilde{u}(z) = (2\pi)^{-n} Y(z) \int e^{iz\xi}\widehat{u}(\xi)\chi(z,\xi)d\xi \tag{a.17}$$

By Proposition a.2, this defines an extension $\widetilde{u}(z) \in G^{s'}(\mathbf{C}^n)$, i.e. $\widetilde{u}(z) = u(z)$, $z \in \mathbf{R}^n$, with compact support. There remains to show that $\overline{\partial}\widetilde{u}$ has the right decrease near the real domain, and so we need estimate $\int e^{iz\xi}\widehat{u}(\xi)\chi(z,\xi)d\xi$ over $\operatorname{supp}\overline{\partial}Y$. Set $z = x+iy$, and $\rho = |y|^{\frac{1}{s-1}}$. If $|x| > r$ we shift the integral into the complex domain. For $t \in [0,1]$, consider the contours

$$\Gamma_t(z) = \left\{\zeta_t = \xi + itC_0\rho^{-1}\frac{x}{|x|^2} =_{\text{def}} \gamma_z(t,\xi),\ \xi \in \mathbf{R}^n\right\}$$

where $C_0 > 0$ will be chosen suitably. By Stokes' formula, using that $\widehat{u}$ is holomorphic,

$$\int e^{iz\xi}\widehat{u}(\xi)\chi(z,\xi)d\xi$$
$$= \int_{\Gamma_1(z)} e^{iz\zeta}\widehat{u}(\zeta)\chi(z,\zeta)d\zeta + \int_{[0,1]\times\mathbf{R}^n} e^{iz\zeta_t}\widehat{u}(\zeta_t)\gamma_z^*(d\omega) \tag{a.18}$$

where $\omega = \chi(z,\zeta)d\zeta$. First we estimate $u'(z) = \int_{\Gamma_1(z)} e^{iz\zeta}\widehat{u}(\zeta)\chi(z,\zeta)d\zeta$, and notice that if $u \in G_0^s(\mathbf{R}^n)$ is supported in $B(0,r)$, then $\widehat{u}$ an entire function in $\mathbf{C}^n$ and:

$$|\widehat{u}(\zeta)| \le C\exp\left(-(1+|\zeta|)^{1/s}/C\right)\exp(r|\eta|), \quad \zeta = \xi + i\eta \tag{a.19}$$

Using this estimate, we find, possibly changing C:

$$|e^{iz\zeta}\widehat{u}(\zeta)| \le C\exp\left(-C_0\rho^{-1}\left(1-\frac{r}{|x|}\right)\right)$$
$$\exp\left(-(1+|\xi|+C_0\rho^{-1}|x|^{-1})^{1/s}/C\right)\exp(-y\xi) \tag{a.20}$$

If $|\xi| \le \rho^{-1}$, and $r < |x| < 2r$, then we may replace $\exp\left(-(1+|\xi|+C_0\rho^{-1}|x|^{-1})^{1/s}/C\right)$ by $\exp\left(-\rho^{-1/s}/C\right)$ (with another constant C) on the right hand side of (a.20) and if $|y| > 0$ is small enough, then

$$\begin{aligned}&\exp\bigl(-(1+|\xi|+C_0\rho^{-1}|x|^{-1})^{1/s}/C\bigr)\exp(-y\xi)\\&\quad=\mathcal{O}(\exp\bigl(-\rho^{-1/s}/C\bigr)\le 1,\end{aligned}$$

and

$$|e^{iz\zeta}\widehat{u}(\zeta)|\le C\exp-\Bigl(C_0\rho^{-1}\Bigl(1-\frac{r}{|x|}\Bigr)\Bigr)$$

This quantity is bounded uniformly by

$$\exp\bigl(-|\operatorname{Im} z|^{-1/(s-1)}/C_1\bigr),\quad z\in\operatorname{supp}\overline{\partial}_zY,\quad 2r\ge|x|\ge r'>r. \tag{a.21}$$

If $|\xi|\ge\rho^{-1}$, and $r<|x|<2r$, but $|\xi|\le\rho^{-s}/C_2$, $C_2>0$ large enough, then we may replace $\exp\bigl(-(1+|\xi|+C_0\rho^{-1}|x|^{-1})^{1/s}/C\bigr)$ by $\exp\bigl(-|\xi|^{1/s}/C\bigr)$ (with a new constant C) on the right hand side of (a.20), and we get the same conclusion as in (a.21). Now we can choose $C'>0$ large enough (see the proof of Proposition a.2), so that $(z,\zeta)\notin\operatorname{supp}\chi$ for $|\xi|\ge\rho^{-s}/C_2$. From these estimates, and integrating over ξ, it follows that:

$$\begin{aligned}&|u'(z)|\le C_1\exp\bigl(-|\operatorname{Im} z|^{-1/(s-1)}/C_1\bigr),\\&z\in\operatorname{supp}\overline{\partial}_zY,\quad 2r\ge|x|\ge r'>r.\end{aligned} \tag{a.22}$$

Now we consider the remainder term in (a.18), and compute

$$u''(z)=\int_{[0,1]\times\mathbf{R}^n}e^{iz\zeta_t}\widehat{u}(\zeta_t)\gamma_z^*(d\omega)$$

We have:

$$\gamma_z^*(d\omega)=C'\sum_{j,k=1}^n\overline{\partial}_{z_k}\psi\bigl(C'\langle\zeta_t\rangle^{\frac{s-1}{s}}\operatorname{Im} z\bigr)\operatorname{Im} z_k\bigl(\overline{\partial}_{\zeta_j}\langle\overline{\zeta}_t\rangle^{\frac{s-1}{s}}\bigr)J_j(z,\xi,t)d\xi\wedge dt \tag{a.23}$$

where $J_j(z,\xi,t)=\mathcal{O}(1)$ is a jacobian. Replacing ζ by ζ_t, we get as in (a.20)

$$\begin{aligned}|e^{iz\zeta_t}\widehat{u}(\zeta_t)|\le C\exp\Bigl(-tC_0\rho^{-1}\Bigl(1-\frac{r}{|x|}\Bigr)\Bigr)&\\\exp\bigl(-(1+|\xi|+tC_0\rho^{-1}|x|^{-1})^{1/s}/C\bigr)\exp(-y\xi)&\end{aligned} \tag{a.24}$$

and we need to check that, on $\operatorname{supp}\gamma_z^*(d\omega)$, this can be estimated by

$\exp\big(-|\operatorname{Im} z|^{-1/(s-1)}/C_1\big)$ for some $C_1 > 0$. Using the rotation invariance of $\widetilde{\psi}$, we can bound $\overline{\partial}_{z_k}\psi\big(C'\langle\zeta_t\rangle^{\frac{s-1}{s}}\operatorname{Im} z\big)$ in (a.23), by the absolute value of:

$$\begin{aligned}
&\overline{\partial}_{|z_k|}\widetilde{\psi}\big(C'\langle\zeta_t\rangle^{\frac{s-1}{s}}\operatorname{Im} z_k\big)\\
&\quad= \overline{\partial}_{|z_k|}\widetilde{\psi}\Big(C'\exp\Big(\frac{s-1}{4s}\log\Big(\Big(1+|\xi|^2-(t\rho^{-1})^2\frac{C_0^2}{|x|^2}\Big)^2\\
&\qquad\qquad + 4(t\rho^{-1})^2\Big(\frac{C_0 x\xi}{|x|^2}\Big)^2\Big)\Big)\operatorname{Im} z_k\Big) \qquad \text{(a.25)}
\end{aligned}$$

The intuition is supported by the following fact: when $t = 0$, (a.24) and (a.25) show that we can choose $C' > 0$ so large that, for some $C_1 > 0$

$$\begin{aligned}
&|e^{iz\zeta_t}\widehat{u}(\zeta_t)\overline{\partial}_{|z_k|}\widetilde{\psi}\big(C'\langle\zeta_t\rangle^{\frac{s-1}{s}}\operatorname{Im} z_k\big)|\\
&\quad\le C_1\exp\big(-|\operatorname{Im} z|^{-1/(s-1)}/C_1\big), \quad t=0,
\end{aligned}$$

and, by what we just said, this holds again for $t = 1$. So as above, we discuss according to the range of values of ξ. When $|\xi| \le C_3 t\rho^{-1}$, where $C_3 > 0$ to be fixed, we have

$$\begin{aligned}
&\exp\big(-(1+|\xi|+tC_0\rho^{-1}|x|^{-1})^{1/s}/C\big)\exp(-y\xi)\\
&\quad= \mathcal{O}\big(\exp\big(-(t\rho^{-1})^{1/s}/C\big)\big) \le 1.
\end{aligned}$$

When $t\rho^{-1} \le 1$, the exponential in (a.25) is bounded, so $\overline{\partial}_{z_k}\psi\big(C'\langle\zeta_t\rangle^{(s-1)/s}\operatorname{Im} z\big) = 0$ for all k when $\operatorname{Im} z$ is small enough, so we may assume $t\rho^{-1} \ge 1$ and $\langle\xi\rangle \le C_3 t\rho^{-1}$. If we choose $C_0 > 0$ large enough (depending on C_3), then $\big(1+|\xi|^2-(t\rho^{-1})^2\frac{C_0^2}{|x|^2}\big)^2 + 4(t\rho^{-1})^2\big(\frac{C_0 x\xi}{|x|^2}\big)^2$ is comparable to $\big(t\rho^{-1}\frac{C_0}{|x|}\big)^4$, and we may replace the RHS of (a.25) by

$$\overline{\partial}_{|z_k|}\widetilde{\psi}\Big(C'\Big(t\rho^{-1}\frac{C_0}{|x|}\Big)^{(s-1)/s}\operatorname{Im} z_k\Big)$$

Assume this is non zero for some k. Then, using $\operatorname{supp}\widetilde{\psi} \subset \{1 \le |y| \le 2\}$, we get

$$c(x) \le (t\rho^{-1})^{(s-1)/s}|\operatorname{Im} z_k| \le 2c(x)$$

where for $r < |x| < 2r$, $c(x) = \big(\frac{|x|}{C_0}\big)^{(s-1)/s}/C'$ is comparable to 1. It is easy to see that this does not hold for y small enough. So the contribution to $u''(z)$ of $|\xi| \le C_3 t\rho^{-1}$ is zero.

Let now $\langle\xi\rangle \geq C_3 t\rho^{-1}$. If we choose $C_3 > 0$ large enough, then $\exp\big(-(1+|\xi|+tC_0\rho^{-1}|x|^{-1})^{1/s}/C\big)$ is comparable to $\exp(-\langle\xi\rangle^{1/s}/C)$ (possibly changing C a little). On the other hand, the argument of the log in (a.25) is also comparable to $\langle\xi\rangle^4$, and we may replace the RHS of (a.25) by

$$\overline{\partial}_{|z_k|}\widetilde{\psi}\big(C'\langle\xi\rangle^{(s-1)/s}\operatorname{Im} z_k\big)$$

Assume this is non zero for some k. Then, it is easy to see that choosing C' large enough, we have, for some $C > 0$

$$|e^{iz\zeta_t}\widehat{u}(\zeta_t)| \leq C\exp(-|y|^{-1/(s-1)}/C).$$

So far we proved the right estimate on (a.25) everywhere. It is easy to see that we still have a sufficient decrease in $\langle\xi\rangle$ at infinity to ensure convergence of the integral defining $u''(z)$, and

$$\begin{aligned}&|u''(z)| \leq C_1\exp\big(-|\operatorname{Im} z|^{-1/(s-1)}/C_1\big),\\ &z \in \operatorname{supp}\overline{\partial}_z Y,\quad 2r \geq |x| \geq r' > r.\end{aligned} \tag{a.26}$$

Now, differentiating (a.17), estimates (a.22) and (a.26) give

$$|\overline{\partial}\widetilde{u}(z)| \leq C_1\exp\big(-|\operatorname{Im} z|^{-1/(s-1)}/C_1\big),\quad z\in\mathbf{C}^n,\quad |\operatorname{Im} z| < c$$

To get the estimate on $N_{s'}(\overline{\partial}\widetilde{u},T)(z)$ it suffices to apply (a.11) where we have replaced $\chi(z,\xi)$ by $Y(z)\chi(z,\xi)$, and then by Stokes' formula split each integral

$$v_{\alpha,\alpha',\beta}(z) = \int e^{iz\xi}\xi^{\alpha-\alpha'}\widehat{u}(\xi)D^{\alpha'}\overline{\partial}^{\beta}Y(z)\chi(z,\xi)d\xi$$

into $v'_{\alpha,\alpha',\beta}(z)$ and $v''_{\alpha,\alpha',\beta}(z)$ as in (a.18). We estimate these terms as above, which amounts to add the factor $\xi^{\alpha-\alpha'}$ in the integrals, and then mimic the combinatorics of Proposition a.2. So the theorem is proved. □

The extension of Theorem a.3 to general G^s functions then goes as follows. Let $A > 0$ and $u \in G^s_A(\mathbf{R}^n)$ (where we have extended trivially to the multidimensional case the definition given in Theorem a.1). Consider a partition of unity $\sum_{g\in\mathbf{Z}^n} X_g = 1$ on $\mathbf{R}^n$, where $X_g(x) = X(x-g)$ and $0 \leq X \in G^s_0(\mathbf{R}^n)$ is supported in a small neighborhood of the cube $K = \{x \in \mathbf{R}^n : |x_j| \leq 1/2,\ j = 1\cdots n\}$ (see [HöI, Thm 1.4.6]). For each g, let also $Y_g(z) = Y(z-g)$, where $Y \in G^s_0(\mathbf{C}^n)$ is chosen as in the proof of

Theorem a.3, equal to 1 near K. We set:

$$\widetilde{u}(z) = (2\pi)^{-n} \sum_{g \in \mathbf{Z}^n} Y_g(z) \int e^{iz\xi} \widehat{X_g u}(\xi) \chi(z,\xi) d\xi \tag{a.27}$$

Because of the cut-off Y_g, the sum is locally finite. Because $u \in G^s_A(\mathbf{R}^n)$, the estimate (a.19) with $X_g u$ instead of u is uniform with respect to $g \in \mathbf{Z}^n$. So Theorem a.3 gives the

Corollary a.4 *Let $u \in G^s_A(\mathbf{R}^n)$. Then for all $s' > s$, there exists $B > 0$, an extension $\widetilde{u} \in G^{s'}_B(\mathbf{C}^n \cap \{|\operatorname{Im} z| < c\})$, $c > 0$ small enough, such that $\widetilde{u}(z) = u(z)$ for real z and for some $C > 0$:*

$$N_{s'}\big(\overline{\partial}\widetilde{u}, T\big)(z) \leq C \exp(-|\operatorname{Im} z|^{-1/(s-1)}/C), \quad z \in \mathbf{C}^n, \quad |\operatorname{Im} z| < c$$

when $T \succ 0$ is small enough.

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