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On strongly regular graphs with parameters $(k, 0, 2)$ and their antipodal double covers

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Abstract. Let Γ be a strongly regular graph with parameters $(k, \lambda, \mu) = (q^2 + 1, 0, 2)$ admitting $G(\cong PGL(2, q^2))$ as one point stabilizer for odd prime power q. We show that if G stabilizes a vertex x of Γ and acts on $\Gamma_2(x)$ transitively, then $q = 3$ holds and Γ is the Gewirtz graph. Moreover it is shown that an antipodal double cover whose diameter 4 of a strongly regular graph with parameters $(k, 0, 2)$ is reconstructed from a symmetric association scheme of class 6 with parameters $p^i_{j,k}$ $(0 \leq i, j, k \leq 6)$ in the Section 3.

Key words: antipodal cover of strongly regular graph, association scheme, finite transitive group.

1. Introduction

We are interested in the classification problems of distance regular graphs with $b_2 = 1$. Let Γ be a distance regular graph with $b_2 = 1$ and valency $k > 2$. If the diameter $d(\Gamma)$ of Γ is larger more than 4, then Γ is isomorphic to the dodecahedron ([3, p.182]). In [1], M. Araya, A. Hiraki and A. Jurišić showed that if $d(\Gamma) = 4$, then Γ is an antipodal double cover of a strongly regular graph with parameters $(k, \lambda, \mu) = (n^2 + 1, 0, 2)$ for an integer n not divisible by 4 and if $d(\Gamma) = 3$, then Γ is an antipodal cover of a complete graph. Obviously an antipodal cover of a complete graph is a distance regular graph with $b_2 = 1$ if it's diameter is 3.

The classification problems of antipodal covers of complete graphs are very difficult. Because the existence of an antipodal distance regular $(n-2)$-fold cover of the complete graph K_n claims the existence of a projective plane of order $(n-1)$ for an odd positive integer n, moreover an antipodal distance regular $(n-1)$-fold cover of K_n is equivalent to the existence of a Moore graph with the diameter 2 and the valency n. ([6], [7])

The strongly regular graphs with parameters $(k, \lambda, \mu) = (5, 0, 2)$ and $(10, 0, 2)$ are known, the former one has an antipodal double cover with $d = 4$, namely the Wells graph, the latter one (the Gewirtz graph) does not

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have an antipodal double cover with $d = 4$ ([3, p.372]). The existence or nonexistence of strongly regular graphs with $(n^2+1, 0, 2)$ for $n \geq 5$ are not known up to date. We have studied these graphs.

2. Strongly regular graphs with $(q^2+1, 0, 2)$ admitting $PGL(2, q^2)$ for $q = p^e$

In this section we prove the following theorem.

Theorem 2.1 *Let Γ be a strongly regular graph with parameters $(q^2+1, 0, 2)$ and G be a group isomorophic to $PGL(2, q^2)$ for an odd prime power $q = p^e$. Suppose that G acts on Γ as G stabilizes a vertex ∞ of Γ and G is transitive on $\Gamma_2(\infty)$. Then $q = 3$ and Γ is the Gewirtz graph.*

We denote the set of vertices of a graph Γ by $V(\Gamma)$, the set $\{y \in V(\Gamma) \mid d(x, y) = 1\}$ by $\Gamma(x)$ and the set $\{y \in V(\Gamma) \mid d(x, y) = i\}$ by $\Gamma_i(x)$ for $x \in V(\Gamma)$ and $i \geq 2$.

Lemma 2.1 *Let Γ be a strongly regular graph with parameters $(q^2+1, 0, 2)$ and ∞ be a vertex of Γ. Then the eigenvalues and their multiplicities of the induced subgraph $\Gamma_2(\infty)$ of Γ are the following.*

θ	q^2-1	-2	$q-1$	$-q-1$
$m(\theta)$	1	q^2	$\frac{(q^2+1)(q^2+q-2)}{4}$	$\frac{(q^2+1)(q^2-q-2)}{4}$

Proof. Let A and A_1 be adjacency matrices of Γ and $\Gamma_2(\infty)$ respectively. We note that the degrees of A and A_1 are $2 + q^2 + \frac{q^2(q^2+1)}{2}$ and $\frac{q^2(q^2+1)}{2}$ respectively. Then A is written as

$$A = \begin{pmatrix} A_1 & X & 0 \\ X^t & 0 & \mathbf{1} \\ 0^t & \mathbf{1}^t & 0 \end{pmatrix}$$

where X is a $\frac{q^2(q^2+1)}{2} \times (q^2+1)$ submatix indicating the adjacency relation between vertices of $\Gamma_2(\infty)$ and $\Gamma_1(\infty)$, and $\mathbf{1}$ is the $(q^2+1) \times 1$ all 1 matrix. Let I_n be the unit matrix of degree n and $J_{n,m}$ be the $n \times m$ all 1 matrix. Since $A^2 = (q^2+1)I_n + \lambda A + \mu(J_n - A - I_n)$ where $n = 2 + q^2 + \frac{q^2(q^2+1)}{2}$,

$\lambda = 0$ and $\mu = 2$, we have $A^2 = -2A + (q^2-1)I_n + 2J_n$. Therefore

$$A_1^2 + XX^t = -2A_1 + (q^2-1)I_m + 2J_m, \tag{2.1}$$

where $m = \frac{q^2(q^2+1)}{2}$. Moreover since $A_1X = 2J_{m,\ell} - 2X$ where $\ell = q^2+1$, we have

$$A_1XX^tA_1 = 4XX^t + 4(q^2-3)J_m. \tag{2.2}$$

Hence from (2.1) and (2.2),

$$\begin{aligned} &A_1^4 + 2A_1^3 - (q^2+3)A_1^2 - 8A_1 + 4(q^2-1)I_m \\ &\quad = 2(q^4 - 4q^2 + 3)J_m. \end{aligned} \tag{2.3}$$

We can calculate easily that $\{2, -2, q-1, -q-1\}$ are the roots of the equation $x^4 + 2x^3 - (q^2+3)x^2 - 8x + 4(q^2-1) = 0$. Therefore the eigenvalues of A_1 are these values and q^2-1 whose multiplicity 1.

Let the multiplicities of the eigenvalues 2, -2, $q-1$ and $-q-1$ be a, b, f and g respectively. Since the degree of A_1 is $\frac{q^2(q^2+1)}{2}$, $trace(A_1) = 0$, $trace(A_1^2) = \frac{q^2(q^2+1)(q^2-1)}{2}$, $trace(A_1^3) = 0$, we have $a = 0$, $b = q^2$, $f = \frac{(q^2+1)(q^2+q-2)}{4}$, $g = \frac{(q^2+1)(q^2-q-2)}{4}$ by solving the following linear equations.

$$\begin{cases} 1 + a + b + f + g = \dfrac{q^2(q^2+1)}{2} \\ q^2 - 1 + 2a - 2b + (q-1)f + (-q-1)g = 0 \\ (q^2-1)^2 + 4a + 4b + (q^2-2q+1)f + (q^2+2q+1)g = \dfrac{q^2(q^2+1)(q^2-1)}{2} \\ (q^2-1)^3 + 8a - 8b + (q^3-3q^2+3q-1)f + (-q^3-3q^2-3q-1)g = 0. \end{cases}$$

□

Let Γ be a strongly regular graph with parameters $(q^2+1, 0, 2)$ and G be a group isomorophic to $PGL(2, q^2)$ for an odd prime power $q = p^e$. Suppose that G acts on Γ as G stabilizes a vertex ∞ of Γ and transitively on $\Gamma_2(\infty)$. We put

$$z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \pmod{Z(GL(2,q^2))} \quad \text{and} \quad H = C_G(z).$$

Then H is a dihedral group of order $2(q^2-1)$ and $|G:H| = \frac{q^2(q^2+1)}{2}$. Since there is a unique conjugacy class of involutions in G and any two subgroups of index $\frac{q^2(q^2+1)}{2}$ are conjugate in G. Hence it holds that H is the stabilizer G_v of a vertex $v \in \Gamma_2(\infty)$. Throughout the section we fix this vertex v.

Let ω be a primitive element of the multiplicative group $GF(q^2)^*$. Put $D = \{\omega^i \mid 1 \leq i \leq \frac{q^2-1}{2}\}$. We may assume that $2^{-1} \in D$. We have $GF(q^2) = D \cup -D \cup \{0\}$. Set

$$I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{Z(GL(2,q^2))},$$

$$x_\alpha \equiv \begin{pmatrix} 1 & \alpha - 2^{-1} \\ 1 & \alpha + 2^{-1} \end{pmatrix} \pmod{Z(GL(2,q^2))},$$

where $\alpha \in D \cup \{0\}$. We can verify the following lemma.

Lemma 2.2 *The set $\{I\} \cup \{x_\alpha \mid \alpha \in D \cup \{0\}\}$ is a complete representative of double cosets $H\backslash G/H$. Moreover it follows that $Hx_\alpha H = Hx_{-\alpha}H$, $Hx_\alpha H = Hx_\alpha^{-1}H$ where $\alpha \in D \setminus \{2^{-1}\}$, and $|Hx_0H : H| = \frac{(q^2-1)}{2}$, $|Hx_{2^{-1}}H : H| = 2(q^2-1)$ and $|Hx_\alpha H : H| = (q^2-1)$ for any $\alpha \in D \setminus \{2^{-1}\}$.*

G acts naturally on $G/H = \{Hx \mid x \in G\}$. It is easily shown that $(G, V(\Gamma_2(\infty))) \cong (G, G/H)$ as the permutation groups.

An orbital graph Γ_α of the permutation group $(H, G/H)$ with respect to an orbit $Hx_\alpha H$ is defined as the following.

The set of vertices is G/H. A vertex Hx is adjacent to a vertex Hy if and only if $xy^{-1} \in Hx_\alpha H$. Now we have the following lemma.

Lemma 2.3 *The graph $\Gamma_2(\infty)$ is isomorphic to an orbital graph Γ_{α_0} for a suitable element $\alpha_0 \in D \setminus \{2^{-1}\}$.*

Proof. Take a vertex $w \in \Gamma_2(\infty)$ which is adjacent to v. There is an element $x \in G$ such that $w = v^x$ by our assumption. Pick up α_0 such that $x \in Hx_{\alpha_0}H$. Then a mapping f defined by $f(v^y) = Hy$ $(y \in G)$ gives an isomorphism from $\Gamma_2(\infty)$ onto Γ_{α_0}. □

Here we denote the adjacency matrix of the graph Γ_α by A_α. It is well known that the permutation character 1_H^G of $(G, G/H)$ has $\frac{q^2-5}{4}$ distinct irreducible characters of degree q^2+1, $\frac{q^2-1}{4}$ distinct irreducible characters of degree q^2-1, 2 distinct irreducible characters of degree q^2 and a trivial character as its irreducible constituent. G acts on $G/H \times G/H$ naturally. It is also known that $R_\Delta = \{(Hx, Hx) \mid x \in G\}$ and $R_\alpha = \{(Hx, Hy) \mid xy^{-1} \in$

$Hx_\alpha H\}$ $(\alpha \in D\cup\{0\})$ are orbits of the permutation group $(G, G/H\times G/H)$. Moreover $\mathcal{X}(G, G/H) = (G/H, \{R_\alpha \mid \alpha \in D\cup\{0, \Delta\}\})$ is a symmetric association scheme as 1_H^G is multiplicity free. Then A_α is the adjacency matrix of the association scheme $\mathcal{X}(G, G/H)$ corresponding to the relation R_α.

The eigenvalues and their multiplicities of A_α are found from the first eigenmatrix of $\mathcal{X}(G, G/H)$. To describe this matrix we define a certain partition of elements of $GF(q^2)$ and a number of sums of the quadratic character values of the multiplicative group $GF(q^2)^*$.

We set as the following.

$$\begin{aligned}
\Lambda &= \{\lambda \in GF(q^2) \mid \lambda \neq 0,\ \lambda \text{ is a nonsquare, } \lambda - 4 \text{ is a nonsquare}\}\\
\Theta &= \{\theta \in GF(q^2) \mid \theta \neq 0, 4,\ \theta \text{ is a square, } \theta - 4 \text{ is a square}\}\\
\Pi &= \{\pi \in GF(q^2) \mid \pi \neq 0, 4,\ \pi \text{ is a square, } \pi - 4 \text{ is a nonsquare}\}\\
\Xi &= \{\xi \in GF(q^2) \mid \xi \neq 0,\ \xi \text{ is a nonsquare, } \xi - 4 \text{ is a square}\}
\end{aligned}$$

Then we obtain $|\Theta| = \frac{q^2-5}{4}$, $|\Lambda| = |\Pi| = |\Xi| = \frac{q^2-1}{4}$ and $GF(q^2) = \{0, 4\} \cup \Theta \cup \Lambda \cup \Pi \cup \Xi$.

We set $\ell_1 = \frac{q^2-1}{4}$, $\ell_2 = \frac{q^2-5}{4}$, $\Lambda = \{\lambda_i \mid 1 \leq i \leq \ell_1\}$, $\Theta = \{\theta_i \mid 1 \leq i \leq \ell_2\}$, $\Pi = \{\pi_i \mid 1 \leq i \leq \ell_1\}$, and $\Xi = \{\xi_i \mid 1 \leq i \leq \ell_1\}$.

Let δ be a primitive (q^2+1)-th root of 1, ε be a primitive (q^2-1)-th root of 1 and χ be the character of order 2 of $GF(q^2)^*$ with $\chi(0) = 0$.

For $\alpha \in GF(q^2)$ and a positive integer m, we define $\mu_0(\alpha)$, $\mu_1(m, \alpha)$, $\mu_2(m, \alpha)$ as follows.

$$\begin{aligned}
\mu_0(\alpha) = \chi(2\alpha+2) &+ \chi(2\alpha-2)\\
&+ \sum_{i=1}^{\ell_1}\big(\chi(\lambda_i - 2\alpha - 2) + \chi(\lambda_i + 2\alpha - 2)\big)\\
&+ \sum_{i=1}^{\ell_2}\big(\chi(\theta_i - 2\alpha - 2) + \chi(\theta_i + 2\alpha - 2)\big)
\end{aligned}$$

$$\begin{aligned}
\mu_1(m, \alpha) &= 2 + \chi(2\alpha - 2) + \chi(2\alpha + 2)\\
&\quad + \frac{1}{2}\sum_{i=1}^{\ell_1}\big(2 - \chi(\lambda_i - 2\alpha - 2) - \chi(\lambda_i + 2\alpha - 2)\big)\big(\varepsilon^{(2i-1)m} + \varepsilon^{-(2i-1)m}\big)
\end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{\ell_2}\big(2+\chi(\theta_i-2\alpha-2)+\chi(\theta_i+2\alpha-2)\big)\big(\varepsilon^{(2i)m}+\varepsilon^{-(2i)m}\big)$$

$$\begin{aligned}\mu_2(m,\alpha) = {} & -\chi(2\alpha-2)-\chi(2\alpha+2) \\ & -\frac{1}{2}\sum_{i=1}^{\ell_1}\big(2-\chi(\xi_i-2\alpha-2)-\chi(\xi_i+2\alpha-2)\big)\big(\delta^{(2i-1)m}+\delta^{-(2i-1)m}\big) \\ & -\frac{1}{2}\sum_{i=1}^{\ell_1}\big(2+\chi(\pi_i-2\alpha-2)+\chi(\pi_i+2\alpha-2)\big)\big(\delta^{(2i)m}+\delta^{-(2i)m}\big)\end{aligned}$$

Now we have the following lemma.

Lemma 2.4 *The first eigenmatrix of the association scheme $\mathcal{X}(G,G/H)$ is the following.* (Here $\alpha \in D\setminus\{2^{-1}\}$)

	R_Δ	$R_{2^{-1}}$	R_0	R_α
ρ_1	1	$2(q^2-1)$	$\frac{q^2-1}{2}$	q^2-1
$\rho_{q^2}^{(1)}$	1	q^2-3	-1	-2
$\rho_{q^2}^{(2)}$	1	-2	$\frac{\mu_0(0)}{2}$	$\mu_0(\alpha)$
$\rho_{q^2+1}^{(m)}$ $1\le m\le \frac{q^2-5}{2}$ $(m:\text{even})$	1	-2	$\frac{\mu_1(m,0)}{2}$	$\mu_1(m,\alpha)$
$\rho_{q^2-1}^{(m)}$ $1\le m\le \frac{q^2-3}{2}$ $(m:\text{odd})$	1	-2	$\frac{\mu_2(m,0)}{2}$	$\mu_2(m,\alpha)$

Proof. W.M. Kwok gave the first eigenmatrix of the association scheme corresponding to the permutation group $(O(3,q), O(3,q)/O^+(2,q))$ in [5]. It follows that $O(3,q)\cong\{\pm1\}\times SO(3,q)$ and $SO(3,q)\cong PGL(2,q)$, $(G,G/H)$ is isomorphic to $(O(3,q)/\{\pm1\},\ O(3,q)/(\{\pm1\}\times O^+(2,q)))$ as permutation groups. Therefore we can compute the first eigenmatrix of the association scheme $\mathcal{X}(G,G/H)$ as the table of the lemma by using the table in [5].

(cf. [5], p.48) $\square$

From Lemma 2.3, $\Gamma_2(\infty) \cong \Gamma_{\alpha_0}$. Concerning the element α_0, we have the following lemma.

Lemma 2.5 *It follows that $\mu_0(\alpha_0) = q-1$ and $\mu_2(m, \alpha_0) = q-1$ for any odd number m such that $1 \leq m \leq \frac{q^2-3}{2}$.*

Proof. We compare eigenvalues and their multiplicities in the table in Lemma 2.1 with those in the table of Lemma 2.4 for $\alpha = \alpha_0$. They coincide as a whole. Hence we have

$$\frac{(q^2+1)(q^2+q-2)}{4} = s(q^2+1) + t(q^2-1)$$

where s is the number of m such that $\mu_1(m, \alpha_0) = q-1$ and t is the number of m such that $\mu_2(m, \alpha_0) = q-1$ in the case $\mu_0(\alpha_0) = -q-1$ or

$$\frac{(q^2+1)(q^2-q-2)}{4} = s(q^2+1) + t(q^2-1)$$

where s is the number of m such that $\mu_1(m, \alpha_0) = -q-1$ and t is the number of m such that $\mu_2(m, \alpha_0) = -q-1$ in the case $\mu_0(\alpha_0) = q-1$.

Suppose that $\frac{(q^2+1)(q^2+q-2)}{4} = s(q^2+1) + t(q^2-1)$. Then for each odd prime divisor r of q^2+1, r divides t because that the greatest common divisor of r and q^2-1 is 1. Therefore $\frac{q^2+1}{2}$ divides t. However it is impossible if $t \neq 0$ as $t \leq \frac{q^2-1}{4}$. Hence $t = 0$. Then $s = \frac{q^2+q-2}{4}$. It contradicts to the fact that $s \leq \frac{q^2-5}{4}$.

Suppose that $\frac{(q^2+1)(q^2-q-2)}{4} = s(q^2+1) + t(q^2-1)$. Then similarly we have $t = 0$. Then $\mu_0(\alpha_0) = q-1$ and $\mu_2(m, \alpha_0) = q-1$ for any odd number m. The lemma is proved. $\square$

Lemma 2.6 *Let ℓ and k be positive integers. Then the following equations hold.*

$$\sum_{1 \leq m \leq \frac{q^2-1}{2},\, m:\text{odd}} (\delta^{(2\ell-1)m} + \delta^{-(2\ell-1)m}) = 1 \tag{2.4}$$

$$\sum_{1 \leq m \leq \frac{q^2-1}{2},\, m:\text{odd}} (\delta^{(2k)m} + \delta^{-(2k)m}) = -1 \tag{2.5}$$

Proof. Concerning the first equation, let d be the greatest common divisor of $2\ell-1$ and q^2+1. Put $s=\frac{q^2+1}{d}$ and $\eta=\delta^{(2\ell-1)}$. We note that d is odd and s is even. Obviously η is a primitive s-th root of 1. Hence $\eta^{\frac{s}{2}}=-1$. Then we have

$$\begin{aligned}\sum_{1\leq m\leq\frac{q^2-1}{2},\, m:\mathrm{odd}} &(\delta^{(2\ell-1)m}+\delta^{-(2\ell-1)m})\\ &=\left(\sum_{1\leq i\leq sd-1,\, i:\mathrm{odd}}\eta^i\right)-\eta^{\frac{sd}{2}}\\ &=(\eta+\eta^{s+1}+\cdots+\eta^{(d-1)s+1})(1+\eta^2+\eta^4+\cdots+\eta^{s-2})+1=1\end{aligned}$$

as $1+\eta^2+\eta^4+\cdots+\eta^{s-2}=0$.

Concerning the second equation, let d be the greatest common divisor of $2k$ and q^2+1. Put $s=\frac{q^2+1}{d}$ and $\eta=\delta^{2k}$. Then d is even and s is odd and η is a primitive s-th root of 1. We have

$$\begin{aligned}\sum_{1\leq m\leq\frac{q^2-1}{2},\, m:\mathrm{odd}} &(\delta^{(2k)m}+\delta^{-(2k)m})\\ &=\left(\frac{d}{2}-1\right)(1+\eta+\eta^2+\cdots+\eta^{s-1})+(\eta+\eta^2+\cdots+\eta^{s-1})=-1\end{aligned}$$

as $1+\eta^1+\eta^2+\cdots+\eta^{s-1}=0$. □

The following lemma can be easily verified from Lemma 2.6.

Lemma 2.7 *It follows that*

$$\begin{aligned}2&\left(\sum_{1\leq m\leq\frac{q^2-1}{2},\, m:\mathrm{odd}}\mu_2(m,\alpha_0)\right)\\ &=-\frac{(q^2-1)}{2}(\chi(2\alpha_0+2)+\chi(2\alpha_0-2))\\ &\quad+\sum_{i=1}^{\ell_1}(\chi(\xi_i-2\alpha_0-2)+\chi(\xi_i+2\alpha_0-2))\\ &\quad+\sum_{i=1}^{\ell_1}(\chi(\pi_i-2\alpha_0-2)+\chi(\pi_i+2\alpha_0-2)).\end{aligned}$$

Proof of Theorem 2.1. From the definition of $\mu_0(\alpha_0)$ and Lemma 2.7 we have

$$\begin{aligned}
\mu_0(\alpha_0) + 2 &\left(\sum_{1 \le m \le \frac{q^2-1}{2},\, m:\text{odd}} \mu_2(m,\alpha_0) \right) \\
&= -\frac{(q^2-3)}{2}(\chi(2\alpha_0+2)+\chi(2\alpha_0-2)) \\
&\quad + \sum_{i=1}^{\ell_1}(\chi(\lambda_i-2\alpha_0-2)+\chi(\lambda_i+2\alpha_0-2)) \\
&\quad + \sum_{i=1}^{\ell_2}(\chi(\theta_i-2\alpha_0-2)+\chi(\theta_i+2\alpha_0-2)) \\
&\quad + \sum_{i=1}^{\ell_1}(\chi(\xi_i-2\alpha_0-2)+\chi(\xi_i+2\alpha_0-2)) \\
&\quad + \sum_{i=1}^{\ell_1}(\chi(\pi_i-2\alpha_0-2)+\chi(\pi_i+2\alpha_0-2)).
\end{aligned}$$

However since $\sum_{y\in GF(q^2)}\chi(y)=0$ and $GF(q^2)=\{0,4\}\cup\Theta\cup\Lambda\cup\Pi\cup\Xi$, it follows that

$$\mu_0(\alpha_0) + 2 \left(\sum_{1 \le m \le \frac{q^2-1}{2},\, m:\text{odd}} \mu_2(m,\alpha_0) \right) = -\frac{(q^2+1)}{2}(\chi(2\alpha_0+2)+\chi(2\alpha_0-2)).$$

On the other hand from Lemma 2.5,

$$\mu_0(\alpha_0) + 2 \left(\sum_{1 \le m \le \frac{q^2-1}{2},\, m:\text{odd}} \mu_2(m,\alpha_0) \right) = (q-1)+2\left(\frac{q^2-1}{4}\right)(q-1) = \frac{q^2+1}{2}(q-1).$$

Hence we obtain that $|\chi(2\alpha_0+2)+\chi(2\alpha_0-2)| = q-1$. Therefore $q \le 3$.

Then $q = 3$ and Γ is the Gewirtz graph. ([3, p.372]). Thus Theorem 2.1 is proved. $\square$

3. Reconstruction of the antipodal double cover Γ^* of a strongly regular graph with $\lambda = 0$ and $\mu = 2$

We give the definition of association schemes.

Let Y be a finite set. An symmetric association scheme with d class is a pair $(Y, \mathcal{R})$ such that

(i) $\mathcal{R} = \{R_0, R_1, \dots, R_d\}$ is a partition of $Y \times Y$;

(ii) $R_0 = \{(x,x) \mid x \in Y\}$;

(iii) If $(x,y) \in R_i$, then $(y,x) \in R_i$ for all $i \in \{0, 1, \dots, d\}$;

(iv) There are numbers $p^j_{h,i}$ such that for any pair $(x,y) \in R_j$ the number of $z \in Y$ with $(x,z) \in R_h$ and $(z,y) \in R_i$ equal $p^j_{h,i}$.

The number $n_j = p^0_{j,j}$ of $z \in Y$ with $(x,z) \in R_j$ (which is independent on $x \in Y$) is called the valency of R_j, moreover for any fixed j $(1 \le j \le d-1)$ intersection numbers c_j, a_j and b_j is defined as $c_j = p^j_{j-1,1}$, $a_j = p^j_{j,1}$ and $b_j = p^j_{j+1,1}$.

Now let Γ be a strongly regular graph with parameters $(k, 0, 2)$. In this section we study about the structure of the second neighbourhood of Γ and antipodal double covers of them with $d = 4$. E.R. van Dam and A. Munemasa proved the following theorem independently. ([4, pp.13–14], [8])

Theorem 3.1 *Let Γ be a strongly regular graph with $\lambda = 0$, $\mu = 2$ of valency k with $k > 5$. Then the second neighbourhood of Γ with respect to any vertex generates a 3-class association scheme. Furthermore any scheme with the same parameters can be constructed in this way from a strongly regular graph with the same parameters as Γ.*

Now we consider the antipodal double cover Γ^* with $d(\Gamma^*) = 4$ of Γ. From now on we assume that $k > 7$ through this section. The intersection array of Γ^* is the following.

$$\iota(\Gamma^*) = \begin{pmatrix} 0 & 1 & 1 & k-1 & k \\ 0 & 0 & k-2 & 0 & 0 \\ k & k-1 & 1 & 1 & 0 \end{pmatrix}$$

Put $\Omega = \{1, 2, \dots, k\}$. Let ∞^+ be a vertex of Γ^* and ∞^- be the unique vertex in Γ^* such that $d(\infty^+, \infty^-) = 4$. We set $\Gamma^*(\infty^+) = \{1^+, 2^+, \dots, k^+\}$

and $\Gamma^*(\infty^-) = \{1^-, 2^-, \ldots, k^-\}$. Then we may assume that $d(i^+, i^-) = 4$ for any $i \in \Omega$ without loss of generallity. We denote the set of vertices of the subgraph $\Gamma_2^*(\infty^+)$ by X. For each $x \in X$, $|\Gamma^*(\infty^+) \cap \Gamma^*(x)| = 1$ and $|\Gamma^*(\infty^-) \cap \Gamma^*(x)| = 1$ as $c_2 = b_3 = 1$. Set $\Gamma^*(\infty^+) \cap \Gamma^*(x) = \{i^+\}$ and $\Gamma^*(\infty^-) \cap \Gamma^*(x) = \{j^-\}$. There exists a bijection φ from X onto $(\Omega \times \Omega) \setminus \{(i,i) \mid i \in \Omega\}$ defined by $\varphi(x) = (i,j)$. Then we put $i = \varphi(x)_1$ and $j = \varphi(x)_2$. We denote a unique element of $\Gamma_4^*(x)$ by x', then $\varphi(x)_1 = \varphi(x')_2$ and $\varphi(x)_2 = \varphi(x')_1$. Moreover we set as follows.

$$
\begin{aligned}
A(x) &= \{y \in X \mid d(x,y) = 1\},\\
B(x) &= \{y \in X \mid \varphi(y)_1 = \varphi(x)_2 \text{ or } \varphi(y)_2 = \varphi(x)_1,\ y \neq x'\}\\
A'(x) &= \{y \in X \mid d(x',y) = 1\},\\
B'(x) &= \{y \in X \mid \varphi(y)_1 = \varphi(x)_1 \text{ or } \varphi(y)_2 = \varphi(x)_2,\ x \neq y\}\\
C(x) &= X \setminus \big(A(x) \cup B(x) \cup A'(x) \cup B'(x) \cup \{x, x'\}\big)
\end{aligned}
$$

We have the following theorem.

Theorem 3.2 *We define relations on X as follows.*

$$
\begin{aligned}
R_0 &= \{(x,x) \mid x \in X\}, & R_1 &= \{(x,y) \mid y \in A(x)\},\\
R_2 &= \{(x,y) \mid y \in B(x)\}, & R_3 &= \{(x,y) \mid y \in C(x)\},\\
R_4 &= \{(x,y) \mid y \in B'(x)\}, & R_5 &= \{(x,y) \mid y \in A'(x)\},\\
R_6 &= \{(x,x') \mid x \in X\}
\end{aligned}
$$

Then $\mathcal{X} = (X, R_i (0 \le i \le 6))$ is a symmetric 6-association scheme whose parameters are $p^j_{h,i} (0 \le h,j,i \le 6)$ where $(B_h)_{i,j} = p^j_{h,i}$ in the following matrices $B_h (h = 0, 1, \ldots, 6)$.

$$
B_0 = I,\ B_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0\\
k-2 & 0 & 1 & 1 & 0 & 0 & 0\\
0 & 2 & 1 & 2 & 1 & 0 & 0\\
0 & k-5 & k-5 & k-8 & k-5 & k-5 & 0\\
0 & 0 & 1 & 2 & 1 & 2 & 0\\
0 & 0 & 0 & 1 & 1 & 0 & k-2\\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix},
$$

$$B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 & 0 \\ 2k-4 & 2 & 1 & 2 & k-3 & 2 & 0 \\ 0 & 2k-10 & k-5 & 2k-12 & k-5 & 2k-10 & 0 \\ 0 & 2 & k-3 & 2 & 1 & 2 & 2k-4 \\ 0 & 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_3 =$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & k-5 & k-5 & k-8 & k-5 & k-5 & 0 \\ 0 & 2k-10 & k-5 & 2k-12 & k-5 & 2k-10 & 0 \\ (k-2)(k-5) & (k-5)(k-8) & (k-5)(k-6) & k^2-13k+48 & (k-5)(k-6) & (k-5)(k-8) & (k-2)(k-5) \\ 0 & 2k-10 & k-5 & 2k-12 & k-5 & 2k-10 & 0 \\ 0 & k-5 & k-5 & k-8 & k-5 & k-5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$(B_4)_{i,j} = (B_2)_{i,(6-j)}, \quad (B_5)_{i,j} = (B_1)_{i,(6-j)},$$
$$(B_6)_{i,j} = (B_0)_{i,(6-j)} \quad \textit{for } 0 \le i \le 6, \ 0 \le j \le 6.$$

The following theorem asserts that the inverse of the statement in Theorem 3.2 is also true.

Theorem 3.3 *Let $\mathcal{X} = (X, R_i(0 \le i \le 6))$ be a symmetric 6-association scheme with the same parameters as $p^j_{h,i}$ in Theorem 3.2. Then the antipodal double cover Γ^* with $d(\Gamma^*) = 4$ of a strongly regular graph with parameters $(k, 0, 2)$ can be constructed from $\mathcal{X}$. Moreover the graph (X, R_1) is isomorphic to the second neighbourhood of Γ^* with respect to any vertex.*

We now start with a short sketch of the proof. First, we consider the graph $\widetilde{\Gamma} = (X, R_4)$. It is shown that the parameters of this graph are those of the graph deleting the diagonal vertices of the $k \times k$-grid. We reconstruct the graph $\widehat{\Gamma}$ isomorphic to the $k \times k$-grid from $\widetilde{\Gamma}$ by adding a set of pairs of maximal cliques as new vertices to the vertices of $\widetilde{\Gamma}$. Lastly using the graph $\widehat{\Gamma}$, an extended graph Γ^* of the graph (X, R_1) is constructed.

We use the following notation here. Let $\Gamma' = (V(\Gamma'), E(\Gamma'))$ be a finite connected graph and d' be the metric of Γ'. For two vertices x, y of Γ' such that $d'(x, y) = i$, we denote the cardinalities of the sets $\{z \in V(\Gamma') \mid$

$d'(x,z) = i-1,\ d'(z,y) = 1\}$, $\{z \in V(\Gamma') \mid d'(x,z) = i+1,\ d'(z,y) = 1\}$ and $\{z \in V(\Gamma') \mid d'(x,z) = i,\ d'(z,y) = 1\}$ by $c_i(x,y)$, $b_i(x,y)$ and $a_i(x,y)$ respectively. Moreover we denote the valency of a vertex x by $k(x)$, and if Γ' is regular we denote the valency of Γ' by $k(\Gamma')$.

We state four lemmas to prove the theorem.

We note that $k_0 = k_6 = 1$, $k_1 = k_5 = k-2$, $k_2 = k_4 = 2k-4$ and $k_3 = (k-2)(k-5)$. Therefore we have $|X| = k(k-1)$. We note $p^j_{h,i} = (B_h)_{i,j} = (B_{6-h})_{i,6-j} = p^{6-j}_{6-h,i}$ for $\forall j$, h, i. For any element $x \in X$ there exists a unique element $x' \in X$ such that $(x,x') \in R_6$ as $p^6_{0,6} = 1$. We consider a bijection ψ on X defined by $\psi(x) = x'$ for any $x \in X$. It is clear that $\psi^2 = id_X$. We denote the metric of $\widetilde{\Gamma}$ by $\tilde{\rho}$.

Lemma 3.1 *The graph $\widetilde{\Gamma}$ is a regular graph with the valency $2k-4$, $d(\widetilde{\Gamma}) = 3$, $a_1(\widetilde{\Gamma}) = k-3$, $b_1(\widetilde{\Gamma}) = k-2$ and $a_2(\widetilde{\Gamma}) = 2k-6$. Suppose that $\tilde{\rho}(x,y) = 2$. If $y \notin \widetilde{\Gamma}(\psi(x))$, then $c_2(x,y) = 2$ and if $y \in \widetilde{\Gamma}(\psi(x))$, then $c_2(x,y) = 1$. We have also $\widetilde{\Gamma}_3(x) = \{\psi(x)\}$ for any $x \in X$.*

Proof. It is easily verified that $\widetilde{\Gamma}$ is a regular graph of the valency $2k-4$ as $p^0_{4,4} = 2k-4$. Now $p^i_{4,4} \neq 0$ for $i \in \{1,2,3,4,5\}$. Therefore there is an element $z \in X$ such that $\tilde{\rho}(x,z) = 1$ and $\tilde{\rho}(z,y) = 1$ for elements x, y such that $(x,y) \in R_i$ $(i = 1,2,3,4$ or $5)$. Moreover $\tilde{\rho}(x,\psi(x)) = 3$ holds. Therefore we have $d(\widetilde{\Gamma}) = 3$ and $\tilde{\rho}(x,y) = 3$ holds if and only if $y = \psi(x)$.

Here we note that

$$(x,y) \in R_4 \quad \text{if and only if} \quad (\psi(x),y) \in R_2 \tag{3.1}$$

as $p^6_{4,i} = 0$ for $i \neq 2$ and $p^6_{2,i} = 0$ for $i \neq 4$. Therefore we have

$$\tilde{\rho}(x,y) = 1 \quad \text{if and only if} \quad \tilde{\rho}(\psi(x),\psi(y)) = 1. \tag{3.2}$$

We have $a_1(\widetilde{\Gamma}) = k-3$ and $b_1(\widetilde{\Gamma}) = k-2$ as $p^4_{4,4} = k-3$ and $\sum_{1\le i\le 5(i\neq 4)} p^4_{i,4} = k-2$. Let x, y be elements in X such that $\tilde{\rho}(x,y) = 2$ and $y \notin \widetilde{\Gamma}(\psi(x))$. Then $c_2(x,y) = 2$, $a_2(x,y) = 2k-6$ and $b_2(x,y) = 0$ as $p^i_{4,4} = 2$ and $\sum_{1\le h\le 5(h\neq 4)} p^i_{h,4} = 2k-6$ for $i = 1,3,5$ and from (3.1). Let x, y be elements in X such that $\tilde{\rho}(x,y) = 2$ and $y \in \widetilde{\Gamma}(\psi(x))$. Then $(x,y) \in R_2$ from (3.1). Therefore we have $c_2(x,y) = 1$, $b_2(x,y) = 1$ and $a_2(x,y) = 2k-6$ as $p^2_{4,4} = p^2_{6,4} = 1$ and $\sum_{1\le h\le 5(h\neq 4)} p^2_{h,4} = 2k-6$. We also have $c_3(x,\psi(x)) = 2k-4$ for any $x \in X$. This completes the proof of the lemma. □

Lemma 3.2 *Let x be an element of X. Then $\widetilde{\Gamma}(x)$ is a disjoint union of two cliques of the same cardinality $k-2$.*

Proof. Let $x \in X$ and $y \in \widetilde{\Gamma}(x)$. Since $a_1(x,y) = k-3$ and $k(\widetilde{\Gamma}) = 2k-4$, we may assume

$$\widetilde{\Gamma}(x) = \{y, y_1, y_2, \ldots, y_{k-3}, z_1, z_2, \ldots, z_{k-2}\},\ \{y_1, y_2, \ldots, y_{k-3}\} \subset \widetilde{\Gamma}(y)$$

and $\{z_1, z_2, \ldots, z_{k-2}\} \subset \widetilde{\Gamma}_2(y)$. Set $S = \{y, y_1, y_2, \ldots, y_{k-3}\}$ and $T = \{z_1, z_2, \ldots, z_{k-2}\}$. Let z be any element of T. Since $c_2(y,z) \le 2$, $\tilde{\rho}(x,y) = \tilde{\rho}(x,z) = 1$ and $S \cap \widetilde{\Gamma}(z) \subset \widetilde{\Gamma}(y) \cap \widetilde{\Gamma}(z)$, it follows that $|S \cap \widetilde{\Gamma}(z)| \le 1$. Then we have $|T \cap \widetilde{\Gamma}(z)| \ge k-4$ since $a_1(x,z) = k-3$. However it follows that $k-3$ elements excluding z are contained in T. Therefore $|T \cap \widetilde{\Gamma}_2(z)| \le 1$. Suppose that $T \cap \widetilde{\Gamma}_2(z) \ne \emptyset$. Then there exists an element $u \in T$ where $\tilde{\rho}(z,u) = 2$. Therefore $T \setminus \{z,u\} \subset \widetilde{\Gamma}(z)$. Moreover $|T \cap \widetilde{\Gamma}_2(u)| \le 1$ as same as $|T \cap \widetilde{\Gamma}_2(z)| \le 1$. Hence we obtain $T \cap \widetilde{\Gamma}_2(u) = \{z\}$. Therefore it follows that x and every elements of T except z and u are contained in $\widetilde{\Gamma}(z) \cap \widetilde{\Gamma}(u)$. It implies $k-3 \le 2$ since $c_2(z,u) \le 2$. Thus $k \le 5$, which contradicts our assumption. Hence we have $T \cap \widetilde{\Gamma}_2(z) = \emptyset$ and any element of T except z is adjacent to z. However since z is any element of T, T is a clique. Similarly S is a clique. Thus the lemma is proved. □

We denote the set $S \cup \{x\}$ and $T \cup \{x\}$ by $C_1(x)$ and $C_2(x)$. We note that $|C_1(x)| = |C_2(x)| = k-1$. Obviously $C_i(x)$ is a maximal clique of $\widetilde{\Gamma}$ for $i = 1, 2$. Any maximal clique of $\widetilde{\Gamma}$ is equal to $C_i(x)$ for an element $x \in X$ and $i \in \{1,2\}$. We denote the set of maximal cliques of $\widetilde{\Gamma}$ by $MC(\widetilde{\Gamma})$. Put $\mathcal{D} = \{C \cup \psi(C) \mid C \in MC(\widetilde{\Gamma})\}$.

We note that $C \cap \psi(C) = \emptyset$ for any $C \in MC(\widetilde{\Gamma})$. For $i \in \{1,2\}$ we have $y \in C_i(x)$ if and only if $C_i(x) = C_j(y)$ for $j = 1$ or 2, as we see in the proof of Lemma 3.2. Hence we have $|MC(\widetilde{\Gamma})| = \frac{2|X|}{k-1} = 2k$ and $|\mathcal{D}| = k$. For $i \in \{1,2\}$ we have $\psi(C_i(x)) = C_j(\psi(x))$ for some j from (3.2). Hence we may assume $\psi(C_i(x)) = C_i(\psi(x))$ without loss of generality. We have the following lemma about $\mathcal{D}$.

Lemma 3.3 (1) *Let x be any element of X. Then there exist exactly two elements of $\mathcal{D}$ containing x.*

(2) *Let x, y be any elements of X such that $\tilde{\rho}(x,y) = 1$. Then there exists exactly one element of $\mathcal{D}$ containing x and y.*

(3) *Let x, y be any elements of X such that $\tilde{\rho}(x,y) = 2$ and $y \in$*

$\widetilde{\Gamma}(\psi(x))$. *Then there exists exactly one element of* $\mathcal{D}$ *containing* x *and* y.

(4) *Let* D_1 *and* D_2 *be distinct elements of* $\mathcal{D}$. *Then* $|D_1 \cap D_2| = 2$.

(5) *Let* D *be an element of* $\mathcal{D}$ *and* x *be an element of* X *such that* $x \notin D$. *Then* $|\widetilde{\Gamma}(x) \cap D| = 2$.

Proof. (1), (2) and (3) are trivial from Lemma 3.2.

(4): Let D_1 and D_2 be distinct elements of $\mathcal{D}$. Then there are elements a and b of X such that $D_1 = C_1(a) \cup \psi(C_1(a))$ and $D_2 = C_1(b) \cup \psi(C_1(b))$. Now we will prove that $D_1 \cap D_2 \neq \emptyset$.

Firstly suppose that $\tilde{\rho}(a, b) = 1$. If $a \in C_1(b)$, then $a \in C_1(a) \cap C_1(b)$. Therefore $D_1 \cap D_2 \neq \emptyset$. Similarly if $b \in C_1(a)$, then also $D_1 \cap D_2 \neq \emptyset$. Hence we may assume $a \in C_2(b)$ and $b \in C_2(a)$. Then $\tilde{\rho}(a, \psi(b)) = 2$ and $\tilde{\rho}(\psi(a), \psi(b)) = 1$. Therefore there is a unique element $u \in X$ which is adjacent to a and $\psi(b)$ from Lemma 3.1. If $u \in C_1(a) \cap C_1(\psi(b))$, then $D_1 \cap D_2 \neq \emptyset$. Hence we may assume $u \in C_2(a)$ or $u \in C_2(\psi(b))$. If $u \in C_2(a)$, then u is adjacent to b. Therefore $\tilde{\rho}(b, \psi(b)) = 2$, a contradiction. Similarly if $u \in C_2(\psi(b))$, then $\tilde{\rho}(a, \psi(a)) = 2$, also a contradiction.

Secondly suppose that $\tilde{\rho}(a, b) = 2$ and $\tilde{\rho}(a, \psi(b)) = 2$. Then there are exactly two elements $u, v \in X$ which are adjacent to both a and b and there are exactly two elements $u', v' \in X$ which are adjacent to both a and $\psi(b)$ from Lemma 3.1. Then u is not adjacent to v and u' is not adjacent to v'. Therefore we may assume (i): $u \in C_1(a)$, $v \in C_2(a)$, $u \in C_1(b)$ and $v \in C_2(b)$ or (ii): $u \in C_1(a)$, $v \in C_2(a)$, $u \in C_2(b)$ and $v \in C_1(b)$. If the case (i) occurs, then we have $D_1 \cap D_2 \neq \emptyset$. Thus we may assume the case (ii). Similarly we may assume $u' \in C_1(a)$, $v' \in C_2(a)$, $u' \in C_2(\psi(b))$ and $v' \in C_1(\psi(b))$. Then u is adjacent to u' and u is adjacent to $\psi(u')$. Therefore $\tilde{\rho}(u', \psi(u')) = 2$. This is a contradiction. Thus it is proved that $D_1 \cap D_2 \neq \emptyset$.

Since $C_1(a) \neq C_1(b)$, $|C_1(a) \cap C_1(b)| \leq 1$ from (2). However $C_1(a) \cap C_1(b) \neq \emptyset$ and $C_1(a) \cap C_1(\psi(b)) \neq \emptyset$ are not compatible. Because if compatible, there is an element $u \in C_1(a) \cap C_1(b)$ and an element $v \in C_1(a) \cap C_1(\psi(b))$. Then $\tilde{\rho}(u, \psi(u)) = 2$, a contradiction. Moreover $\psi(D_i) = D_i$ for $i = 1, 2$, and $D_1 \cap D_2 = (C_1(a) \cap C_1(b)) \cup (C_1(a) \cap C_1(\psi(b))) \cup (C_1(\psi(a)) \cap C_1(b)) \cup (C_1(\psi(a)) \cap C_1(\psi(b)))$. Hence $|D_1 \cap D_2| = 2$. Thus (4) is proved.

(5): Let $D \in \mathcal{D}$ and $x \in X$ such that $x \notin D$. For any $j \in \{1, 2\}$, $D \neq C_j(x) \cup \psi(C_j(x))$ as $x \notin D$. Therefore $|D \cap (C_j(x) \cup \psi(C_j(x)))| = 2$ from (4). Moreover $\psi(D) = D$. Hence $|D \cap C_j(x)| = 1$, which means that $|D \cap \widetilde{\Gamma}(x)| = 2$. Thus (5) is proved. $\square$

We now define a graph $\widehat{\Gamma}$ on the set $X \cup \mathcal{D}$ as follows:

two elements of X are adjacent in $\widehat{\Gamma}$ if and only if they are adjacent in $\widetilde{\Gamma}$, $x \in X$ is adjacent to $D \in \mathcal{D}$ in $\widetilde{\Gamma}$ whenever $x \in D$, and no distinct two elements of $\mathcal{D}$ are adjacent in $\widetilde{\Gamma}$.

The metric of the graph $\widehat{\Gamma}$ is denoted by $\hat{\rho}$.

Lemma 3.4 *The graph $\widehat{\Gamma}$ is isomorphic to the Hamming graph $H(2,k)$*

Proof. Let x be any element of X, then there exist exactly two elements of $\mathcal{D}$ containing x and $\psi(x)$. Therefore $\hat{\rho}(x, \psi(x)) = 2$ by the definition above. Hence we have $d(\widehat{\Gamma}) = 2$.

For any $x \in X$, there exist exactly two elements of $\mathcal{D}$ containing x from (1) of Lemma 3.3. Moreover, since $k(\widetilde{\Gamma}) = 2k - 4$, the valency of x in the graph $\widehat{\Gamma}$ is $2k-2$. Moreover for any $D \in \mathcal{D}$, since D contains exactly $2(k-1)$ elements of X, the valency of D in $\widehat{\Gamma}$ is $2k - 2$. Thus $k(\widehat{\Gamma}) = 2k - 2$.

Let x, y be elements of X such that $\hat{\rho}(x,y) = 1$. Then there exists exactly one element of $\mathcal{D}$ containing x and y from (2) of Lemma 3.3. On the other hand exactly $k - 3$ elements of X are adjacent to x and y as $a_1(\widetilde{\Gamma}) = k-3$. Hence it follows $a_1(x,y) = k-2$ in $\widehat{\Gamma}$. Let $x \in X$ and $D \in \mathcal{D}$ be adjacent in $\widehat{\Gamma}$. Then $x \in D$. We have $|D \cap \widetilde{\Gamma}(x)| = k-2$. Hence it follows $a_1(x, D) = k - 2$ in $\widehat{\Gamma}$. Thus $a_1(\widehat{\Gamma}) = k - 2$.

Let x, y be elements of X such that $\hat{\rho}(x,y) = 2$. If $y = \psi(x)$, then obviously $c_2(x,y) = 2$ in $\widehat{\Gamma}$ because there are exactly two elements of $\mathcal{D}$ containing x and $\psi(x)$. If $y \in \widetilde{\Gamma}(\psi(x))$, then there exists exactly one element of $\mathcal{D}$ containing x and y from (3) of Lemma 3.3.

Moreover there exists exactly one element of X which is adjacent to x and y as $c_2(x,y) = 1$ in $\widetilde{\Gamma}$ from Lemma 3.1. Therefore $c_2(x,y) = 2$ in $\widehat{\Gamma}$. If $y \notin \widetilde{\Gamma}(\psi(x))$, then there is no element of $\mathcal{D}$ containing x and y. However there exist exactly two elements of X which are adjacent to x and y as $c_2(x,y) = 2$ in $\widetilde{\Gamma}$. Therefore $c_2(x,y) = 2$ in $\widehat{\Gamma}$. Let D_1, D_2 be distinct elements of $\mathcal{D}$. Then $|D_1 \cap D_2| = 2$ from (4) of Lemma 3.3. Therefore $c_2(D_1, D_2) = 2$ in $\widehat{\Gamma}$. Let D be an element of $\mathcal{D}$ and x be an element of X such that $x \notin D$. Then $|\widetilde{\Gamma}(x) \cap D| = 2$ from (5) of Lemma 3.3. Therefore $c_2(D, x) = 2$ in $\widehat{\Gamma}$. Hence $c_2(\widehat{\Gamma}) = 2$.

Thus the graph $\widehat{\Gamma}$ has the same parameters as those of the Hamming graph $H(2,k)$. (cf. [9]). This completes the proof of the lemma. □

Proof of Theorem 3.3. From Lemma 3.4 there exists a bijection φ: $X \cup$

$\mathcal{D} \longmapsto \Omega \times \Omega$ such that $\varphi(\mathcal{D}) = \{(i,i) \mid i \in \Omega\}$ and $(x,y) \in R_4$ if and only if $\varphi(x)_1 = \varphi(y)_1$ or $\varphi(x)_2 = \varphi(y)_2$ for any $x, y \in X$ $(x \neq y)$.

We can now construct the antipodal double cover Γ^* of a strongly regular graph with parameters $(k, 0, 2)$. Let Ω^+ be the set $\{1^+, 2^+, \ldots, k^+\}$ and Ω^- be the set $\{1^-, 2^-, \ldots, k^-\}$. The set of vertices of Γ^* is $V(\Gamma^*) = X \cup \Omega^+ \cup \Omega^- \cup \{\infty^\pm\}$.

The adjacency of Γ^* is defined as the follows:

∞^+ adjacent to i^+ and ∞^- adjacent to i^- for any $i \in \Omega$,
for $x, y \in X$, x and y are adjacent iff $(x,y) \in R_1$,
$x \in X$ and $i^+ \in \Omega^+$ are adjacent iff $\varphi(x)_1 = i$,
$x \in X$ and $j^- \in \Omega^-$ are adjacent iff $\varphi(x)_2 = j$.

The metric of the graph Γ^* is denoted by ρ. Then we have the following statement.

$$\rho(x,y) = 2 \quad \text{if} \quad (x,y) \in R_4 \tag{3.3}$$

We can verify that Γ^* is a distance regular graph whose intersection array is $(k, k-1, 1, 1; 1, 1, k-1, k)$ in the sequel. For any $x \in \{\infty^\pm\} \cup \Omega^+ \cup \Omega^-$, it is clear that $k(x) = k$ from the definitions. For any $x \in X$, there are exactly $k-2$ elements of X which are adjacent to x as $p^0_{1,1} = k-2$. Moreover x is adjacent to only one element $\varphi(x)^+_1$ in Ω^+ and $\varphi(x)^-_2$ in Ω^- respectively. Therefore $k(x) = k$. Thus $k(\Gamma^*) = k$.

We note that the bijection φ is a graph isomorphism from $\widehat{\Gamma}$ onto the Hamming graph $H(2,k)$ on $\Omega \times \Omega$ such that $\varphi(\mathcal{D}) = \{(i,i) \mid i \in \Omega\}$. Moreover in the subgrph of $H(2,k)$ which is deleted the vertices $\{(i,i) \mid i \in \Omega\}$, there exists exactly one vertex at distance 3 from a vertex (i,j), namely (j,i). This implies the following statement.

$$\varphi(x) = (i,j) \quad \text{if and only if} \quad \varphi(\psi(x)) = (j,i) \tag{3.4}$$

We have the following lemma.

Lemma 3.5 *Let x, y be elements of X such that $\varphi(x) = (i,j)$ and $\varphi(y) = (\ell,h)$. Then the following* (1) *and* (2) *hold.*

(1) *If $\rho(x,y) = 1$, then $\{i,j\} \cap \{\ell,h\} = \emptyset$.*

(2) *If $t \notin \{i,j\}$, then there exists exactly one element $u \in X$ such that $\rho(x,u) = 1$ and $\varphi(u)_1 = t$ and exactly one element $v \in X$ such that $\rho(x,v) = 1$ and $\varphi(v)_2 = t$.*

Proof. (1): Suppose that $\varphi(x) = (i, j)$, $\varphi(y) = (\ell, h)$ and $\rho(x, y) = 1$. Then $(x, y) \in R_1$. If $i = \ell$ or $j = h$, then $(x, y) \in R_4$. This is a contradiction. If $i = h$ or $j = \ell$, then $(x, \psi(y)) \in R_4$ from (3.4). Therefore $(x, y) \in R_2$ from (3.1), also a contradiction. Therefore $\{i, j\} \cap \{\ell, h\} = \emptyset$.

(2): Suppose that $\varphi(x) = (i, j)$ and $t \notin \{i, j\}$. If $\rho(x, a) = 1$ and $\rho(x, b) = 1$, then $\varphi(a)_1 \neq \varphi(b)_1$ and $\varphi(a)_2 \neq \varphi(b)_2$ as $p^4_{1,1} = 0$. On the other hand $|\{z \in X \mid \rho(x, z) = 1\}| = k - 2$ as $p^0_{1,1} = k - 2$. Hence from (1), $\Omega \setminus \{i, j\} = \{\varphi(z)_1 \mid z \in X,\ \rho(x, z) = 1\}$. It means that there exists exactly one $u \in X$ such that $\varphi(u)_1 = t$. Similarly there exists exactly one $v \in X$ such that $\varphi(v)_2 = t$. The lemma is proved. $\square$

Lemma 3.6 *It follows that* $c_1(\Gamma^*) = b_3(\Gamma^*)$, $c_2(\Gamma^*) = b_2(\Gamma^*)$, $c_3(\Gamma^*) = b_1(\Gamma^*)$ *and* $c_4(\Gamma^*) = b_0(\Gamma^*)$. *Moreover the diameter of* Γ^* *is* 4.

Proof. Since $p^2_{1,1} \neq 0$, $p^3_{1,1} \neq 0$ and $p^5_{1,1} = 0$ we have

$$\rho(x, y) = 2 \quad \text{if} \quad (x, y) \in R_2 \cup R_3 \tag{3.5}$$

$$\rho(x, y) > 2 \quad \text{if} \quad (x, y) \in R_5 \tag{3.6}$$

Fix any $x \in X$, we set as the following.

$$\begin{aligned}
&A(x) = \{y \in X \mid (x, y) \in R_1\}, \\
&B(x) = \{y \in X \mid y \neq \psi(x),\ \varphi(y)_1 = \varphi(x)_2 \text{ or } \varphi(y)_2 = \varphi(x)_1\}, \\
&B'(x) = \{y \in X \mid y \neq x,\ \varphi(y)_1 = \varphi(x)_1 \text{ or } \varphi(y)_2 = \varphi(x)_2\}, \\
&A'(x) = \{y \in X \mid (x, y) \in R_5\} \text{ and} \\
&C(x) = X \setminus (A(x) \cup B(x) \cup B'(x) \cup A'(x) \cup \{\psi(x)\}).
\end{aligned}$$

We note that $y \in B'(x)$ if and only if $(x, y) \in R_4$ and $y \in B(x)$ if and only if $(x, y) \in R_2$ from (3.1) and (3.4). Hence it follows that $y \in C(x)$ if and only if $(x, y) \in R_3$.

Now since $p^i_{1,6} = p^{6-i}_{5,6} = 0$ for $i \in \{0, 1, 2, 3, 4, 6\}$, we obtain the following statement.

$$(x, y) \in R_1 \quad \text{if and only if} \quad (x, \psi(y)) \in R_5 \tag{3.7}$$

Suppose that $\varphi(x) = (i, j)$. Then we have $\Gamma^*(x) = A(x) \cup \{i^+, j^-\}$ and $\Gamma^*_2(x) = B(x) \cup C(x) \cup B'(x) \cup (\Omega^+ \setminus \{i^+, j^+\}) \cup (\Omega^- \setminus \{i^-, j^-\}) \cup \{\infty^\pm\}$ from (3.3), (3.5) and (2) of Lemma 3.5. Moreover $\Gamma^*_2(x) \cap \Gamma^*(y) \neq \emptyset$ for any $y \in A'(x)$. Hence $\Gamma^*_3(x) = A'(x) \cup \{i^-, j^+\}$ from (3.6) and $p^6_{1,2} = p^6_{1,3} =$

$p^6_{1,4} = 0$. We also have $\Gamma^*_4(x) = \{\psi(x)\}$.

For any $i \in \Omega$, $\Gamma^*(i^+) = \{x \in X \mid \varphi(x)_1 = i\} \cup \{\infty^+\}$, $\Gamma^*_2(i^+) = \{x \in X \mid \varphi(x)_1 \neq i$ and $\varphi(x)_2 \neq i\} \cup (\Omega^+ \setminus \{i^+\}) \cup (\Omega^- \setminus \{i^-\})$ as $p^3_{4,1} \neq 0$ and $p^5_{4,1} \neq 0$. Moreover $\Gamma^*_3(i^+) = \{x \in X \mid \varphi(x)_2 = i\} \cup \{\infty^-\}$ as $p^2_{1,1} \neq 0$ and $\Gamma^*_4(i^+) = \{i^-\}$. We obtain the similar results concerning $\Gamma^*_t(i^-)$ for $t = 1,2,3,4$. Therefore $d(\Gamma^*) = 4$.

From the facts above and (3.1), (3.4) and (3.7), $\Gamma^*(x) = \Gamma^*_3(\psi(x))$, $\Gamma^*_2(x) = \Gamma^*_2(\psi(x))$ and $\Gamma^*_3(x) = \Gamma^*(\psi(x))$ for any $x \in X$. Therefore we have $c_1(\Gamma^*) = b_3(\Gamma^*)$, $c_2(\Gamma^*) = b_2(\Gamma^*)$, $c_3(\Gamma^*) = b_1(\Gamma^*)$ and $c_4(\Gamma^*) = b_0(\Gamma^*)$. The lemma is proved. □

Theorem 3.3 follows from Lemma 3.6 if we prove that $b_1(\Gamma^*) = k - 1$ and $b_2(\Gamma^*) = 1$. There are no triangle whose vertices are all in X as $p^1_{1,1} = 0$. Suppose that $\rho(x,y) = 1$ for $x, y \in X$. Then $\psi(x)_1 \neq \psi(y)_1$ and $\psi(x)_2 \neq \psi(y)_2$ from (1) of Lemma 3.5. Thus there are no triangle containing x, y as the vertices. Hence $a_1(\Gamma^*) = 0$, which implies $b_1(\Gamma^*) = k - 1$.

Suppose that $\rho(x,y) = 2$ for $x, y \in X$. Then $y \in B(x) \cup C(x) \cup B'(x)$. If $y \in B(x) \cup C(x)$, then $c_2(x,y) = 1$ as $p^2_{1,1} = p^3_{1,1} = 1$. If $y \in B'(x)$, then also $c_2(x,y) = 1$ though $p^4_{1,1} = 0$ since either $\varphi(x)_1 = \varphi(y)_1$ or $\varphi(x)_2 = \varphi(y)_2$. Suppose that $\rho(x, i^+) = 2$ for $x \in X$. Then there is a unique element $u \in X$ such that $\rho(x,u) = 1$ and $\rho(u, i^+) = 1$ from (2) of Lemma 3.5. Therefore $c_2(x, i^+) = 1$. Similarly $c_2(x, j^-) = 1$ for x, j such that $\rho(x, j^-) = 2$. Obviously $c_2(\infty^+, x) = 1$ and $c_2(\infty^-, x) = 1$ for any $x \in X$, $c_2(i^+, j^+) = 1$, $c_2(i^-, j^-) = 1$ and $c_2(i^+, j^-) = 1$ for any $i \neq j$. Hence $c_2(\Gamma^*) = 1$, which implies $b_2(\Gamma^*) = 1$ from Lemma 3.6. This completes the proof of Theorem 3.3. □

Remark Let x be any vertex of Γ^*. Eigenvalues of the subgraph $\Gamma^*_2(x)$ are $k-2$, $\sqrt{k}$, $-1+\sqrt{k-1}$, 0, -2, $-\sqrt{k}$ and $-1-\sqrt{k-1}$. Moreover their multiplicities are 1, $\frac{(k-1)(k-2)}{4}$, $\frac{k(\sqrt{k-1}+2)(\sqrt{k-1}-1)}{4}$, $k-1$, $k-1$, $\frac{(k-1)(k-2)}{4}$ and $\frac{k(\sqrt{k-1}-2)(\sqrt{k-1}+1)}{4}$ respectively.

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