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Invariant currents and automorphic forms of an elementary Kleinian group

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Abstract. We compute the cohomology of currents invariant by some elementary Kleinian groups. The result is used to compute the cohomology of harmonic coclosed automorphic forms on the hyperbolic space of even dimension for such a group. This proves that for these groups the cohomology of these forms is isomorphic to the cohomology of the quotient space. This question is related to the Borel conjecture (*cf.* Corollaries 2.14 and 3.9).

Key words: current, distribution, Kleinian group, automorphic form.

Introduction

Let n be an integer $n \geqslant 2$ and let $\mathbb{H}^n$ be the hyperbolic space of dimension n. Its boundary $\partial\mathbb{H}^n$ can be identified to the unit sphere $\mathbb{S}^{n-1}$ of $\mathbb{R}^n$, or to $\mathbb{R}^{n-1} \cup \{\infty\}$. The group of orientation-preserving conformal transformations of $\mathbb{S}^{n-1}$ is denoted by $\mathrm{Conf}^+(\mathbb{S}^{n-1})$. It is isomorphic to the group $\mathrm{Iso}^+(\mathbb{H}^n)$ of positive isometries of $\mathbb{H}^n$. A discrete subgroup of $\mathrm{Conf}^+(\mathbb{S}^{n-1})$ is called a *Kleinian group*.

For a Kleinian group Γ, the cohomology of the quotient space $X := \Gamma\backslash\mathbb{H}^n$ is defined as the homology of the complex $\Omega^\bullet(X) := \Omega^\bullet_\Gamma(\mathbb{H}^n)$ of Γ-invariant differential forms on $\mathbb{H}^n$. This cohomology is isomorphic to the singular cohomology of X, and to its de Rham cohomology if X is a differentiable manifold.

A differential form $\alpha \in \Omega^p(X)$ is said to be *automorphic* if:

- α and $d\alpha$ have moderate growth:

$$\forall x_0 \in \mathbb{H}^n,\ \exists a, b > 0,\ \forall x \in \mathbb{H}^n,\ \max\{\|\alpha(x)\|, \|d\alpha(x)\|\} \leqslant a.e^{b.\delta(x,x_0)},$$

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where δ denotes the hyperbolic distance and $\|.\|$ the norm of p-linear forms on the tangent space $T_x\mathbb{H}^n$, and

- α vanishes under a non-trivial polynomial P with complex coefficients in the laplacian: $P(\Delta)(\alpha) = 0$.

The space of automorphic forms is a subcomplex, denoted by $A^\bullet(X)$, of $\Omega^\bullet(X)$. The *Borel conjecture* (*cf.* [B]) states that the cohomology of automorphic forms is isomorphic to the cohomology of X, and more precisely that the inclusion $A^\bullet(X) \hookrightarrow \Omega^\bullet(X)$ is a quasi-isomorphism.

Moreover, the space $\Omega^\bullet_{hc}(X)$ of harmonic coclosed differential forms and the space $A^\bullet_{hc}(X)$ of harmonic coclosed automorphic forms are also subcomplexes of $\Omega^\bullet(X)$.

We have the inclusions:

$$\begin{tikzcd}
A^\bullet_{hc}(X) \arrow[r, hook, "i_1"] \arrow[d, "i_2"'] & \Omega^\bullet_{hc}(X) \arrow[d] \\
A^\bullet(X) \arrow[r, hook, "i_3"] & \Omega^\bullet(X).
\end{tikzcd}$$

The following statement is a variant of the Borel conjecture, sometimes called *Borel-Harder conjecture.*

Conjecture 0.1 All the injective morphisms of complexes in the diagram above are quasi-isomorphisms, so the cohomology of each of these complexes is isomorphic to the cohomology of X.

The fact that the inclusion $\Omega^\bullet_{hc}(X) \hookrightarrow \Omega^\bullet(X)$ is a quasi-isomorphism is a consequence of the Hodge theorem for non compact manifolds established in [Ga2].

This conjecture is trivially true if Γ is a cocompact Kleinian group (*i.e.* X is compact), in which case every harmonic form is automorphic. Recently, J. Franke proved it when Γ is a lattice (*cf.* [Fra] and [Wal]). The goal of this work is to study the case of an elementary group of hyperbolic isometries, which is the simplest example where X has infinite volume.

As in [Ra] §5.5, we will call *elementary group* a Kleinian group having a finite orbit in the closure $\mathbb{H}^n \cup \partial\mathbb{H}^n$ of the hyperbolic space. The elementary groups are divided into three classes.

- Elementary groups of *elliptic* type are those for which there exists a finite orbit in $\mathbb{H}^n$. Equivalently, there exists a fixed point in $\mathbb{H}^n$ (*cf.* [Ra] Theorem 5.5.1). They are the finite subgroups of $\mathrm{Conf}^+(\mathbb{S}^{n-1})$.

- An elementary group is of *parabolic* type if the only finite orbit consists of one fixed point on $\partial\mathbb{H}^n$. Equivalently, it is conjugated to an infinite discrete subgroup of euclidian isometries of $\mathbb{R}^{n-1}$ (*cf.* [Ra] Theorem 5.5.5).
- Elementary groups of *hyperbolic* type are the elementary groups which are neither of elliptic nor parabolic type. The finite orbit is then made of two points of $\partial\mathbb{H}^n$ and such a group contains an infinite cyclic subgroup of finite index (*cf.* [Ra] Theorems 5.5.6 and 5.5.8).

Neither of these three situations is part of the articles listed above. Remark that solvable subgroups of $\mathrm{Conf}^+(\mathbb{S}^{n-1})$ are elementary groups (*cf.* [Ra] Theorem 5.5.10).

By an averaging process, we will restrict to the case of infinite cyclic groups, generated by a loxodromy for the hyperbolic type or a translation for the parabolic type when $n = 2$. Note that one can easily prove Conjecture 0.1 by the same process for elementary groups of elliptic type.

The idea here is to send the objects acting on the boundary $\mathbb{S}^{n-1}$ inside $\mathbb{H}^n$. For that, we use the tool introduced by P.-Y. Gaillard in [Ga1]: the Poisson transformation, which is a generalization to differential forms of the usual Poisson transformation on functions. This idea was already expressed in [HB].

It was proved (*cf.* [Ga1] Theorems 1 and 2) that, if n is even, the Poisson transformation induces an isomorphism of the hyperforms complex (a p-hyperform on $\mathbb{S}^{n-1}$ is a continuous linear form on the space of $(n-1-p)$-analytic differential forms) on $\partial\mathbb{H}^n$ modified by the augmentation of distributions over the complex of harmonic coclosed forms on $\mathbb{H}^n$, and that it commutes to the action of $\mathrm{Conf}^+(\mathbb{S}^{n-1})$.

1. Invariant currents

Let M be an oriented C^∞ manifold of dimension n. We denote by $\mathcal{C}^p(M)$ the space of *currents of degree* p on M, defined as the topological dual of the space $\Omega_c^{n-p}(M)$ of compactly supported $(n-p)$-forms on M endowed with the Schwartz C^∞ topology. A p-form on M will be identified to the regular p-current it defines. By de Rham theorem, the cohomology of the complex of currents is isomorphic to the cohomology of M.

If a group Γ acts on M, we denote by $\mathcal{C}^p_\Gamma(M)$ the space of invariant p-currents on M. The invariant currents form a subcomplex whose cohomology is isomorphic to the cohomology of the quotient manifold $\Gamma\backslash M$ in the case the action of Γ is free and properly discontinuous.

Given a Kleinian group Γ, the *limit set of* Γ, denoted by Λ_Γ, is the trace on the boundary $\partial\mathbb{H}^n$ of the orbit closure of a point $z \in \mathbb{H}^n$:

$$\Lambda_\Gamma := \overline{\Gamma.z} \cap \partial\mathbb{H}^n.$$

This set does not depend on the choice of z (*cf.* [Ma]); its complementary D_Γ in $\partial\mathbb{H}^n$ is the *domain of discontinuity* of Γ. The group Γ acts properly discontinuously on D_Γ.

Let $z \in \mathbb{H}^n$ viewed as the unit ball of $\mathbb{R}^n$ and $s > 0$; we define the absolute Poincaré series of exponent s of Γ in z by

$$\Phi_s(z) := \sum_{\gamma\in\Gamma} \|\gamma'(z)\|^s,$$

where $\|.\|$ is the operator norm. If this series converges in z, it will converge uniformly on any compact set. The *critical exponent of* Γ is defined by

$$\delta(\Gamma) := \inf\{s > 0,\ \Phi_s(z) \text{ converges}\}.$$

Let $\mathcal{C}^p_\Gamma(\mathbb{S}^{n-1}, \Lambda_\Gamma)$ be the space of Γ-invariant currents on the sphere $\mathbb{S}^{n-1}$ with support in Λ_Γ, *i.e.* currents vanishing on every form whose support does not intersect Λ_Γ. For $p \in \{0, \dots, n-1\}$, the *localisation of p-currents into the domain of discontinuity of* Γ, denoted by $L^p : \mathcal{C}^p_\Gamma(\mathbb{S}^{n-1}) \longrightarrow \mathcal{C}^p_\Gamma(D_\Gamma)$ and also called *restriction of p-currents to* D_Γ, is defined by:

$$\forall T \in \mathcal{C}^p_\Gamma(\mathbb{S}^{n-1}),\quad \forall\omega \in \Omega^{n-1-p}(D_\Gamma),\quad \langle L^p(T), \omega\rangle = \langle T, \widetilde{\omega}\rangle,$$

where $\widetilde{\omega}$ is ω extended by 0 at the points of Λ_Γ.

The localization of currents is a morphism of complexes and its kernel is the space of currents with support in Λ_Γ. The following theorem is proved in [EMM].

Theorem 1.1 ([EMM] Theorem 2.2) *If $\Gamma\backslash D_\Gamma$ is compact, the localisation of currents of dimension greater than $\delta(\Gamma)$ is surjective; therefore, for $p < n - 1 - \delta(\Gamma)$, there is an exact sequence*:

$$0 \longrightarrow \mathcal{C}^p_\Gamma(\mathbb{S}^{n-1}, \Lambda_\Gamma) \xhookrightarrow{i^p} \mathcal{C}^p_\Gamma(\mathbb{S}^{n-1}) \xrightarrow{L^p} \mathcal{C}^p_\Gamma(D_\Gamma) \longrightarrow 0.$$

We will need a generalization of Theorem 3.1 in [EMM] about the localization of invariant distributions. We do not make any asumption on the number of connected components of $\Gamma\backslash D_\Gamma$.

Lemma 1.2 *Let Γ be a Kleinian group satisfying the following conditions: (1) $\delta(\Gamma) < 1$, (2) Γ acts freely on D_Γ and (3) $\Gamma\backslash D_\Gamma$ is the union of disjoint compact sets. Then the image $\operatorname{Im} L^{n-1}$ of the localization of Γ-invariant distributions contains the kernel of the linear map*

$$\begin{array}{rccc} \widetilde{\Theta}: & \mathcal{C}_\Gamma^{n-1}(D_\Gamma) & \longrightarrow & \mathbb{C}^C \\ & T & \longmapsto & (\langle T, f_c\rangle)_{c\in C} \end{array}$$

where C is the set of connected components of D_Γ and, for $c \in C$, f_c is a function of $\Omega^0(D_\Gamma)$ whose support is contained in c and satisfying the conditions of Lemma 1.4 in [EMM], namely:

- *For every compact $K \subset c$, $\{\gamma \in \Gamma,\ \operatorname{Supp}(f_c) \cap K \neq \varnothing\}$ is finite;*
- $\sum_{\gamma\in\Gamma} f_c \circ \gamma = \mathbf{1}_c$.

The proof is an immediate generalization of that of [EMM] Theorem 3.1.

2. The case of an elementary group of hyperbolic type

An elementary group of hyperbolic type contains an infinite cyclic subgroup of finite index. We shall therefore restrict to the case where Γ is generated by a single loxodromy, *i.e.* an isometry of $\mathbb{H}^n$ fixing exactly two points on $\partial\mathbb{H}^n$.

These two points form the limit set of Γ. Since the problem we examine is invariant by conjugation, we consider the particular case of a loxodromy $\gamma = \gamma_a \circ R$ where γ_a is the homothecy $z \longmapsto az$ with coefficient $a \in]0,1[$ and $R \in SO(n-1)$. (Observe that the subgroup of $\operatorname{Iso}^+(\mathbb{H}^n)$ generated by R need not be discrete.) When $n = 2$ we get $\gamma = \gamma_a$.

From now on let Γ be the subgroup of $\operatorname{Conf}^+(\mathbb{S}^{n-1})$ generated by γ; it is a Kleinian group. When identifying $\mathbb{S}^{n-1}$ with $\mathbb{R}^{n-1} \cup \{\infty\}$, its limit set is $\Lambda_\Gamma = \{0, \infty\}$: 0 is the attractor and ∞ is the repeller.

A fundamental domain for the action of Γ on $\mathbb{H}^n$ is the half-ring defined by $a \leqslant \|x\| \leqslant 1$. The quotient space $X = \Gamma\backslash\mathbb{H}^n$ has infinite volume and has the homotopy type of a circle.

Lemma 2.1 *The Γ-invariant p-currents on $\mathbb{S}^{n-1} = \partial\mathbb{H}^n$ with support in the limit set $\Lambda_\Gamma = \{0, \infty\}$ are:*

$$\mathcal{C}^p_\Gamma(\mathbb{S}^{n-1},\Lambda_\Gamma)=\begin{cases}0 & \textit{if } p\neq n-1\\ \mathbb{C}\,\delta_0\oplus\mathbb{C}\,\delta_\infty & \textit{if } p=n-1\end{cases}$$

where δ_0 and δ_∞ are the Dirac masses at points 0 *and* ∞.

Proof. Consider $\mathbb{R}^{n-1}$ as local map of the sphere $\mathbb{S}^{n-1}$ and let's determine the Γ-invariant currents with support in $\{0\}$.

Given a $(n-1-p)$-tangent vector ξ at point 0 to $\mathbb{R}^{n-1}$, the *Dirac p-current associated to* ξ, denoted by δ_ξ, is defined by the evaluation of $(n-1-p)$-forms on ξ. It is a p-current with support $\{0\}$.

A p-current $T\in\mathcal{C}^p(\mathbb{S}^{n-1})$ can locally be written as

$$T=\sum_{1\leqslant i_1<\dots<i_p\leqslant n-1}T_{i_1\cdots i_p}dx_{i_1}\wedge\dots\wedge dx_{i_p},$$

where the $T_{i_1\cdots i_p}$ are 0-currents (*cf.* [Rh] Chap.3 §8).

From the characterization of distributions with punctual support given in [Sch] p.100 Theorem XXXV, we can easily find out which p-currents have punctual support. They are the linear combinations of partial derivatives of Dirac p-currents associated to the $(n-1-p)$-vectors extracted from a fixed basis of $T_0\mathbb{R}^{n-1}$.

We now determine which of these currents are γ-invariant. Let $\xi\in\Lambda^{n-1-p}(T_0\mathbb{R}^{n-1})$ be a $(n-1-p)$-vector and $s=(s_1,\dots,s_{n-1})\in\mathbb{N}^{n-1}$ a multi-index, denote by $|s|=s_1+\dots+s_{n-1}$ its length and let $T=D^s\delta_\xi$ where D^s is the differential operator $\frac{\partial^{|s|}}{\partial x_1^{s_1}\cdots\partial x_{n-1}^{s_{n-1}}}$. We have $\gamma^*T=R^*\circ\gamma_a^*(T)$ and, for $\omega\in\Omega^{n-1-p}(\mathbb{R}^{n-1})$:

$$\langle\gamma_a^*(D^s\delta_\xi),\omega\rangle=\langle D^s\delta_\xi,\gamma_a^*\omega\rangle=(-1)^{|s|}\langle\delta_\xi,D^s(\gamma_a^*\omega)\rangle$$

Writing locally $D^s(\gamma_a^*\omega)$ we can directly verify $\gamma_a^*T=a^{n-1-p+|s|}T$. So for every form $\omega\in\Omega^{n-1-p+|s|}(\mathbb{R}^{n-1})$, we have

$$\langle\gamma^*T,\omega\rangle=a^{n-1-p+|s|}\langle T,R^*\omega\rangle.$$

Choosing a R-invariant form ω such that $\langle T,\omega\rangle\neq0$, we get $a^{n-1-p+|s|}=1$, so $p=n-1$ and $s=0$.

Consider $\mathbb{R}^{n-1}$ as a local map of the manifold $\mathbb{R}^{n-1}\cup\{\infty\}\simeq\mathbb{S}^{n-1}$ close to ∞ via the inversion map $x\mapsto\frac{x}{\|x\|^2}$, we get the same computation with a^{-1} instead of a, which leads to the same result. $\square$

It will be necessary later to work in spherical coordinates. Let's write

$\mathbb{R}^{n-1} \setminus \{0\} \simeq \mathbb{R}_+^* \times \mathbb{S}^{n-2}$ and denote by π_1 and π_2 the projections over the first and second factors of this decomposition.

Definition 2.2 The principal value of Cauchy current is the 1-current on $\mathbb{S}^{n-1}$ defined by

$$\forall \varphi \in \Omega^{n-2}(\mathbb{S}^{n-1}), \quad \langle Vp, \varphi \rangle = \begin{cases} \lim\limits_{\varepsilon \to 0} \displaystyle\int_{[-\frac{1}{\varepsilon}, -\varepsilon] \cup [\varepsilon, \frac{1}{\varepsilon}]} \frac{\varphi(x)}{x} \, dx & \text{if } n = 2 \\ \displaystyle\int_{\mathbb{S}^{n-1}} \pi_1^* \left(\frac{dr}{r} \right) \wedge \varphi & \text{if } n \geqslant 3. \end{cases}$$

Lemma 2.3 *The* 1*-current* Vp *is well-defined and is* Γ*-invariant.*

Proof. First suppose that $n = 2$. Writing a Taylor expansion near 0 and an asymptotic expansion near ∞ of a function φ, we can easily prove that the limit in the definition of $\langle Vp, \varphi \rangle$ exists. The continuity of Vp does not cause problems either. The invariance of Vp by γ is a consequence of that of the 1-form $\frac{dx}{x}$ and of the symmetry of the domain of integration.

Suppose now that $n \geqslant 3$. We can write, in $\mathbb{R}^{n-1}$ seen as local map of $\mathbb{S}^{n-1} = \mathbb{R}^{n-1} \cup \{\infty\}$ near 0:

$$\varphi = \sum_{i=1}^{n-1} \varphi_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n-1}$$

where the φ_i are functions. Since

$$\pi_1^* \left(\frac{dr}{r} \right) = \frac{\displaystyle\sum_{i=1}^{n-1} x_i dx_i}{\displaystyle\sum_{i=1}^{n-1} x_i^2}$$

it remains to prove that the function

$$\psi = \frac{\displaystyle\sum_{i=1}^{n-1} (-1)^{i-1} x_i \varphi_i}{\displaystyle\sum_{i=1}^{n-1} x_i^2}.$$

is integrable near 0. We use the spherical coordinates, which can be written as:

$$\begin{cases} x_1 &= r\cos\theta_1\cos\theta_2\cdots\cos\theta_{n-3}\cos\theta_{n-2} \\ x_2 &= r\cos\theta_1\cos\theta_2\cdots\cos\theta_{n-3}\sin\theta_{n-2} \\ \vdots & \qquad\qquad\qquad \vdots \\ x_{n-2} &= r\cos\theta_1\sin\theta_2 \\ x_{n-1} &= r\sin\theta_1 \end{cases}$$

and whose jacobian is $r^{n-2}\cos^{n-3}\theta_1\cos^{n-4}\theta_2\cdots\cos\theta_{n-3}$. We get the following function (taking $\theta_{n-1}=\frac{\pi}{2}$):

$$r^{n-3}\sum_{i=1}^{n-1}(-1)^{i-1}\cos\theta_1\cdots\cos\theta_{n-i-1}\sin\theta_{n-i}\cos^{n-3}\theta_1\cos^{n-4}\theta_2 \\ \cdots\cos\theta_{n-3}\varphi_i(x_1,\dots,x_{n-1}).$$

Since $n\geqslant 3$, this function is integrable near 0.

Changing the 1-form $\pi_1^*(\frac{dr}{r})$ by its conjugate by the inversion $x\mapsto\frac{x}{\|x\|^2}$, the integrability near infinity is true as well. Once again the continuity and the γ_a-invariance of Vp can easily be checked. Finally, Vp is R-invariant because R preserves the euclidian norm: $\pi_1\circ R=\pi_1$. □

Proposition 2.4 *The image* $\operatorname{Im}L^{n-1}$ *of the localization of* Γ*-invariant distributions on* $\mathbb{S}^{n-1}$ *is the kernel of the linear form*

$$\widehat{\theta}=\begin{cases}\widetilde{\Theta} & \text{if } n\geqslant 3\\ \widehat{\theta}_+-\widehat{\theta}_- & \text{if } n=2\end{cases}$$

where $\widehat{\theta}_+$ *and* $\widehat{\theta}_-$ *denote the two components of* $\widetilde{\Theta}$ *when* $n=2$ *(cf. Lemma 1.2), associated respectively to the two connected components* $\mathbb{R}_+^*$ *and* $\mathbb{R}_-^*$ *of* $D_\Gamma=\mathbb{R}^*$.

Proof. The case $n\geqslant 3$ is proved in [EMM] Theorem 3.2; we shall adapt the proof here to the case $n=2$. From Lemma 1.2 we have

$$\ker\widetilde{\Theta}\subset\operatorname{Im}L^1.$$

We also have $\ker\widetilde{\Theta}\subset\ker\widehat{\theta}$ and, more precisely

$$\ker\widehat{\theta}=\ker\widetilde{\Theta}\oplus\mathbb{C}\frac{dx}{x}.$$

Indeed, $\frac{dx}{x}$ is an invariant distribution on $\mathbb{R}^*$, $\ker\widetilde{\Theta}$ has codimension 1 in

$\ker \widehat{\theta}$, and

$$\widehat{\theta}_+\left(\frac{dx}{x}\right) = \int_{\mathbb{R}^*} \frac{f_+(x)}{x}\,dx > 0$$

but, choosing $f_-(x) = f_+(-x)$ (which satisfies the conditions of Lemma 1.4 in [EMM]):

$$\widehat{\theta}\left(\frac{dx}{x}\right) = \int_{\mathbb{R}^*} \frac{f_+(x)}{x}\,dx - \int_{\mathbb{R}^*} \frac{f_-(x)}{x}\,dx = 0.$$

Since we have $\frac{dx}{x} = L^1(Vp)$ where Vp is the element of $\mathcal{C}^1_\Gamma(\mathbb{S}^1)$ defined above, we conclude that $\mathbb{C}\frac{dx}{x} \subset \operatorname{Im} L^1$ and so

$$\ker \widehat{\theta} \subset \operatorname{Im} L^1.$$

Now we prove the inclusion $\operatorname{Im} L^1 \subset \ker \widehat{\theta}$ by constructing a generator of a supplementary of $\ker \widehat{\theta}$ in $\mathcal{C}^1_\Gamma(D_\Gamma)$ which does not belong to $\operatorname{Im} L^1$. Consider therefore the distribution

$$T = \sum_{m\in\mathbb{Z}} \delta_{a^m}.$$

Since $\widehat{\theta}(T) = 1$, T generates such a supplementary. One can then take without modification the arguments of the proof of Theorem 3.2 in [EMM], implying that T cannot be extended into an invariant distribution at 0 and ∞. This ends the proof of Proposition 2.4. □

Theorem 2.5 *The cohomology of Γ-invariant currents on $\partial\mathbb{H}^n$ is:*

$$H^p(\mathcal{C}^\bullet_\Gamma(\mathbb{S}^{n-1})) = \begin{cases} \mathbb{C}\,[\mathbf{1}] & \text{if } p=0 \\ \mathbb{C}\,[Vp] & \text{if } p=1 \text{ and } n\geqslant 3 \\ \mathbb{C}\,[\delta_0+\delta_\infty] & \text{if } p=n-1 \\ 0 & \text{else.} \end{cases} \qquad (n\geqslant 3)$$

$$H^p(\mathcal{C}^\bullet_\Gamma(\mathbb{S}^{1})) = \begin{cases} \mathbb{C}\,[\mathbf{1}] & \text{if } p=0 \\ \mathbb{C}\,[Vp]\oplus\mathbb{C}\,[\delta_0+\delta_\infty] & \text{if } p=1 \\ 0 & \text{else.} \end{cases}$$

Proof. The case $n = 2$ is special since the discontinuity domain is not connected. First suppose that $n = 2$. We compute the critical exponent of Γ using the complex structure of $\mathbb{H}^2$.

We need the expression of the loxodromy γ_a in the unit disc $\mathbb{D}^2$ of $\mathbb{C}$. Let's denote by $\sigma : z \longmapsto \frac{z-i}{z+i}$ the biholomorphism taking the upper half plane $\mathbb{H}^2$ to $\mathbb{D}^2$. Let $m \in \mathbb{Z}$; we have

$$\varphi_a^m = \sigma \circ \gamma_a^m \circ \sigma^{-1} : w \longmapsto \frac{(a^m - 1)w + a^m + 1}{(a^m + 1)w + a^m - 1}.$$

Choose $w = 0$ to test the convergence of the absolute Poincaré series, by computing the derivative of φ_a^m as a holomorphic function on $\mathbb{D}^2$:

$$(\varphi_a^m)'(0) = \frac{-4a^m}{(a^m - 1)^2}.$$

For $s > 0$, we have

$$\|(\varphi_a^m)'(0)\|^s \underset{m \to \pm\infty}{\sim} 4^s a^{|m|s},$$

which implies that $\Phi_s(0)$ converges for each $s > 0$, so that $\delta(\Gamma) = 0$. Thus, by Theorem 1.1, the localization of Γ-invariant 0-currents is surjective. Proposition 2.4 says that the image of the localization of distributions is the kernel of the linear form $\widehat{\theta}$.

Consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{C}_\Gamma^0(\mathbb{S}^1, \Lambda_\Gamma) & \xhookrightarrow{i^0} & \mathcal{C}_\Gamma^0(\mathbb{S}^1) & \xrightarrow{L^0} & \mathcal{C}_\Gamma^0(D_\Gamma) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & & & \\
0 & \longrightarrow & \mathcal{C}_\Gamma^1(\mathbb{S}^1, \Lambda_\Gamma) & \xhookrightarrow{i^1} & \mathcal{C}_\Gamma^1(\mathbb{S}^1) & \xrightarrow{L^1} & \ker \widehat{\theta} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

and set:

$$\mathcal{C}^p := \mathcal{C}_\Gamma^p(\mathbb{S}^1, \Lambda_\Gamma) \qquad \mathcal{S}^p := \mathcal{C}_\Gamma^p(\mathbb{S}^1) \qquad \left\{ \begin{array}{l} \mathcal{Q}^0 := \mathcal{C}_\Gamma^0(D_\Gamma) \\ \mathcal{Q}^1 := \ker \widehat{\theta} \end{array} \right.$$

Endowed with the differential d, $\mathcal{C}^\bullet$ et $\mathcal{S}^\bullet$ are differential complexes. But we still have to prove that $d(\mathcal{Q}^0) \subset \mathcal{Q}^1$ to add the missing vertical arrow in the above diagram.

The sum $\sum_{\gamma\in\Gamma} f_+ \circ \gamma = \mathbf{1}_{\mathbb{R}_+^*}$ is locally finite, we can then write $\sum_{\gamma\in\Gamma} d(f_+ \circ \gamma) = \sum_{\gamma\in\Gamma} \gamma^*(df_+) = 0$. Given $T \in Q^0$, we get $\sum_{\gamma\in\Gamma} \langle T, \gamma^*(df_+)\rangle = 0$ and, since T is Γ-invariant, $\sum_{\gamma\in\Gamma}\langle T, df_+\rangle = 0$. This proves that $\langle T, df_+\rangle = 0$, *i.e.* $\widehat{\theta}_+(dT) = \langle dT, f_+\rangle = 0$.

The same holds for f_- and $\widehat{\theta}_-$, which shows that $dT \in \ker\widetilde{\Theta} \subset Q^1$. Finally, we get the desired commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0 & \xhookrightarrow{i^0} & S^0 & \xrightarrow{L^0} & Q^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^1 & \xhookrightarrow{i^1} & S^1 & \xrightarrow{L^1} & Q^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the rows are exact sequences and the columns are complexes. This gives rise to an exact sequence in cohomology:

$$0 \longrightarrow H^0(C^\bullet) \xrightarrow{i^0_*} H^0(S^\bullet) \xrightarrow{L^0_*} H^0(Q^\bullet) \xrightarrow{c^0} H^1(C^\bullet) \xrightarrow{i^1_*} H^1(S^\bullet)$$
$$\xrightarrow{L^1_*} H^1(Q^\bullet) \longrightarrow 0.$$

From Lemma 2.1, we have $C^0 = 0$ and $C^1 = \mathbb{C}\,\delta_0 \oplus \mathbb{C}\,\delta_\infty$, implying that $H^0(C^\bullet) = 0$ and $H^1(C^\bullet) = \mathbb{C}\,[\delta_0] \oplus \mathbb{C}\,[\delta_\infty]$.

Since $D_\Gamma \longrightarrow \Gamma\backslash D_\Gamma$ is a regular covering, $\mathcal{C}^0_\Gamma(D_\Gamma)$ and $\mathcal{C}^1_\Gamma(D_\Gamma)$ are respectively isomorphic to $\mathcal{C}^0(\Gamma\backslash D_\Gamma)$ and $\mathcal{C}^1(\Gamma\backslash D_\Gamma)$. By de Rham theorem, the cohomology of $\mathcal{C}^\bullet(\Gamma\backslash D_\Gamma)$ is isomorphic to that of $\Omega^\bullet(\Gamma\backslash D_\Gamma)$. Here, $\Gamma\backslash D_\Gamma$ is made of two circles, its cohomology is isomorphic to $\mathbb{C}^2$ in degrees 0 and 1.

From this it follows that $H^0(Q^\bullet) = \mathbb{C}\,[\mathbf{1}_{\mathbb{R}_+^*}] \oplus \mathbb{C}\,[\mathbf{1}_{\mathbb{R}_-^*}]$. The generators of $H^1(\mathcal{C}^\bullet_\Gamma(D_\Gamma))$ are the regular 1-currents $\frac{dx}{x}$ on each connected component of $\mathbb{R}^*$. We have $\frac{dx}{x} = L^1(Vp) \in Q^1$, but $\frac{dx}{x}\mathbf{1}_{\mathbb{R}_+^*} \notin \ker\widehat{\theta} = \operatorname{Im} L^1$, so $H^1(Q^\bullet) = \mathbb{C}\left[\frac{dx}{x}\right]$.

Finally, the exact sequence becomes:

$$0 \longrightarrow H^0(S^\bullet) \xrightarrow{L^0_*} \mathbb{C}\,[\mathbf{1}_{\mathbb{R}_+^*}] \oplus \mathbb{C}\,[\mathbf{1}_{\mathbb{R}_-^*}] \xrightarrow{c^0} \mathbb{C}\,[\delta_0] \oplus \mathbb{C}\,[\delta_\infty]$$

$$\xrightarrow{i^1_*} H^1(S^\bullet) \xrightarrow{L^1_*} \mathbb{C}\left[\frac{dx}{x}\right] \longrightarrow 0.$$

The only closed 0-currents on a connected manifold being constants (which are Γ-invariant), we get $H^0(S^\bullet) = \mathbb{C}\,[\mathbf{1}]$.

The distribution $\delta_0 + \delta_\infty$ is not exact (it does not vanish on constants), so we have $0 \neq \mathbb{C}\,[\delta_0 + \delta_\infty] \subset H^1(S^\bullet)$. Moreover, $L^1(Vp) = \frac{dx}{x}$. Since $\frac{dx}{x}$ is not exact in $Q^\bullet$ and $L^\bullet$ is a morphism of complexes, the distribution Vp is not exact in the complex $S^\bullet$. So we get $0 \neq \mathbb{C}\,[Vp] \subset H^1(S^\bullet)$.

We have $\mathbb{C}\,[\delta_0 + \delta_\infty] \oplus \mathbb{C}\,[Vp] \subset H^1(S^\bullet)$, and since the alternated sum of dimensions in an exact sequence is 0, we have the inclusion $H^1(S^\bullet) \subset \mathbb{C}\,[\delta_0 + \delta_\infty] \oplus \mathbb{C}\,[Vp]$. Without using the characterization of closed 0-currents, we could also conclude by describing the connecting homomorphism c^0. This argument will be detailed in the following proof of the theorem for $n \geqslant 3$. This ends the proof of the theorem when $n = 2$.

Now suppose $n \geqslant 3$. As in the case $n = 2$, the critical exponent of Γ is 0. We get the following diagram, in which the rows are exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0 & \xhookrightarrow{i^0} & S^0 & \xrightarrow{L^0} & Q^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^1 & \xhookrightarrow{i^1} & S^1 & \xrightarrow{L^1} & Q^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{n-2} & \xhookrightarrow{i^{n-2}} & S^{n-2} & \xrightarrow{L^{n-2}} & Q^{n-2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{n-1} & \xhookrightarrow{i^{n-1}} & S^{n-1} & \xrightarrow{L^{n-1}} & Q^{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

the complexes $C^\bullet$, $S^\bullet$ et $Q^\bullet$ being defined by

$$C^p = \mathcal{C}^p_\Gamma(\mathbb{S}^{n-1}, \Lambda_\Gamma) \quad S^p = \mathcal{C}^p_\Gamma(\mathbb{S}^{n-1}) \quad Q^p = \begin{cases} \mathcal{C}^p_\Gamma(D_\Gamma) & \text{if } p \in \{0, \dots, n-2\} \\ \ker \widehat{\theta} & \text{if } p = n-1. \end{cases}$$

We then know by Proposition 2.4 that the map L^{n-1} is surjective and we prove as above that $d(Q^{n-2}) \subset Q^{n-1}$. We get an exact sequence in cohomology:

$$\begin{aligned}
0 \longrightarrow H^0(C^\bullet) &\xrightarrow{i_*^0} H^0(S^\bullet) \xrightarrow{L_*^0} H^0(Q^\bullet) \xrightarrow{c^0} \\
H^1(C^\bullet) &\xrightarrow{i_*^1} H^1(S^\bullet) \xrightarrow{L_*^1} H^1(Q^\bullet) \xrightarrow{c^1} \cdots \\
\xrightarrow{c^{p-1}} H^p(C^\bullet) &\xrightarrow{i_*^p} H^p(S^\bullet) \xrightarrow{L_*^p} H^p(Q^\bullet) \xrightarrow{c^p} \cdots \quad (p \in \{2, \dots, n-3\}) \\
\cdots \xrightarrow{c^{n-3}} H^{n-2}(C^\bullet) &\xrightarrow{i_*^{n-2}} H^{n-2}(S^\bullet) \xrightarrow{L_*^{n-2}} H^{n-2}(Q^\bullet) \\
\xrightarrow{c^{n-2}} H^{n-1}(C^\bullet) &\xrightarrow{i_*^{n-1}} H^{n-1}(S^\bullet) \xrightarrow{L_*^{n-1}} H^{n-1}(Q^\bullet) \longrightarrow 0
\end{aligned}$$

From Lemma 2.1, $C^p = 0$ for $p \in \{0, \dots, n-2\}$ so $H^p(C^\bullet) = C^p \cap \ker d$ for each p, which implies that $H^p(C^\bullet) = 0$ for $p \leqslant n-2$ and $H^{n-1}(C^\bullet) = \mathbb{C}[\delta_0] \oplus \mathbb{C}[\delta_\infty]$.

Since $D_\Gamma \longrightarrow \Gamma\backslash D_\Gamma$ is a regular covering the cohomology of $\mathcal{C}_\Gamma^\bullet(D_\Gamma)$ is isomorphic to that of $\mathcal{C}^\bullet(\Gamma\backslash D_\Gamma)$ and then to the de Rham cohomology of $\Gamma\backslash D_\Gamma$. A fundamental domain for the action of Γ on $D_\Gamma = \mathbb{R}^{n-1} - \{0\}$ is:

$$R_D = \{x \in \mathbb{R}^{n-1},\ a \leqslant \|x\| \leqslant 1\} \simeq \mathbb{S}^{n-2} \times [a, 1]$$

so $\Gamma\backslash D_\Gamma \simeq \mathbb{S}^{n-2} \times \mathbb{S}^1$ whose cohomology is isomorphic to $\mathbb{C}$ in degrees 0, 1, $n-2$ and $n-1$ if $n \geqslant 4$, $\mathbb{C}$ in degrees 0 and 2 and $\mathbb{C}^2$ in degree 1 if $n = 3$. However, notice that a generator of $H^{n-1}(\mathcal{C}_\Gamma^\bullet(D_\Gamma))$ is the $(n-1)$-differential form

$$\pi_1^*\left(\frac{dr}{r}\right) \wedge \pi_2^*\omega$$

where ω is the volume form on $\mathbb{S}^{n-2}$. But this form does not belong to $\ker \widehat{\theta}$. This implies that $H^{n-1}(Q^\bullet) = 0$ and the exact sequence becomes, for $n \geqslant 4$:

$$\begin{aligned}
0 \longrightarrow H^0(S^\bullet) \xrightarrow{L_*^0} \mathbb{C}[\mathbf{1}] \longrightarrow 0 \longrightarrow H^1(S^\bullet) \xrightarrow{L_*^1} \mathbb{C}\left[\pi_1^*\left(\frac{dr}{r}\right)\right] \longrightarrow 0 \\
\longrightarrow \cdots \longrightarrow 0 \longrightarrow H^p(S^\bullet) \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow H^{n-2}(S^\bullet) \\
\xrightarrow{L_*^{n-2}} \mathbb{C}[\pi_2^*(\omega)] \xrightarrow{c^{n-2}} \mathbb{C}[\delta_0] \oplus \mathbb{C}[\delta_\infty] \xrightarrow{i_*^{n-1}} H^{n-1}(S^\bullet) \longrightarrow 0.
\end{aligned}$$

The first line implies that L_*^0 et L_*^1 are isomorphisms. Since $\pi_1^*(\frac{dr}{r}) = L^1(Vp)$, we have $H^1(S^\bullet) = \mathbb{C}[Vp]$. The second line gives $H^p(S^\bullet) = 0$ for each $p \in \{2, \dots, n-3\}$, and the third one implies that

$$\dim H^{n-2}(S^\bullet) + 1 = \dim H^{n-1}(S^\bullet) \in \{1, 2\}.$$

Now let's describe the connecting homomorphism $c^{n-2} : H^{n-2}(Q^\bullet) \longrightarrow H^{n-1}(C^\bullet)$. Let $T \in Q^{n-2}$ such that $H^{n-2}(Q^\bullet) = \mathbb{C}\,[T]$, for example $T = \pi_2^*(\omega)$. Since the localization L^{n-2} is surjective, there exists $S \in S^{n-2}$ such that $T = L^{n-2}(S)$. By commutativity of the diagram, we have

$$L^{n-1}(dS) = dT = 0$$

so that $dS \in \ker L^{n-1} = \operatorname{Im} i^{n-1}$, that is $dS \in C^{n-1} = \mathbb{C}\,\delta_0 \oplus \mathbb{C}\,\delta_\infty$. Then there exist constants $\alpha, \beta \in \mathbb{C}$ such that

$$dS = \alpha\,\delta_0 + \beta\,\delta_\infty.$$

Moreover, we have $\langle dS, \mathbf{1}\rangle = 0 = \alpha + \beta$. Suppose $\alpha = 0$. Then S is closed and, since $H^{n-2}\big(C^\bullet(\mathbb{S}^{n-1})\big) \simeq H^{n-2}(\mathbb{S}^{n-1}) = 0$, there exists $U \in C^{n-3}(\mathbb{S}^{n-1})$ such that $S = dU$. So we have:

$$\pi_2^*(\widetilde{\omega}) = T = L^{n-2}(S) = L^{n-2}(dU) = d\big(L^{n-3}(U)\big)$$

which is absurd. We can therefore conclude that $c\big(H^{n-2}(S^\bullet)\big) = \mathbb{C}\,[\delta_0 - \delta_\infty]$. Finally, we have:

$$\dim H^{n-1}(S^\bullet) = rg(i_*^{n-1}) = 2 - \dim\ker(i_*^{n-1}) = 2 - rg(c^{n-2}) = 1$$

which ends the proof in the case $n \geqslant 4$.

Now suppose $n = 3$. We have $H^0(Q^\bullet) = \mathbb{C}\,[\mathbf{1}]$, $H^1(Q^\bullet) = \mathbb{C}\left[\pi_1^*\left(\frac{dr}{r}\right)\right] \oplus \mathbb{C}\,[\pi_2^*(\omega)]$ and $H^2(Q^\bullet) = 0$, the exact sequence gives

$$\begin{aligned}
&0 \longrightarrow H^0(S^\bullet) \xrightarrow{L_*^0} \mathbb{C}\,[\mathbf{1}] \longrightarrow 0\\
&0 \longrightarrow H^1(S^\bullet) \xrightarrow{L_*^1} \mathbb{C}\left[\pi_1^*\left(\frac{dr}{r}\right)\right] \oplus \mathbb{C}\,[\pi_2^*(\omega)]\\
&\qquad \xrightarrow{c^1} \mathbb{C}\,[\delta_0] \oplus \mathbb{C}\,[\delta_\infty] \xrightarrow{i_*^2} H^2(S^\bullet) \longrightarrow 0.
\end{aligned}$$

Once again, $H^0(S^\bullet) = \mathbb{C}\,[\mathbf{1}]$, but $\dim H^1(S^\bullet) = \dim H^2(S^\bullet) \in \{1, 2\}$. As before, we can show that the rank of the connecting homomorphism c^1 is equal to 1, implying that $\dim H^1(S^\bullet) = \dim H^2(S^\bullet) \in \{1, 2\} = 1$ and ending the proof of the theorem. $\square$

In the following statement, π_1 is again the map $\mathbb{H}^n \longrightarrow \mathbb{R}_+^*$ defined by

$\pi_1(x) = \|x\|$.

Corollary 2.6 *The cohomology of harmonic coclosed automorphic forms with respect to the group Γ generated by a single loxodromy of $\mathbb{H}^n$ satisfies:*

$$H^p(A^\bullet_{hc}(X)) = \begin{cases} \mathbb{C}\,[\mathbf{1}] & \textit{if } p{=}0 \\ \mathbb{C}\left[\pi_1^*\left(\frac{dr}{r}\right)\right] & \textit{if } p=1 \\ 0 & \textit{if } p \geqslant 1 \textit{ and } p \neq \dfrac{n+1}{2}. \end{cases}$$

Proof. When n is even, the Poisson transformation $\Phi^\bullet$ is an isomorphism (up to a multiplicative factor, which depends only on the dimension n of the space and the degree p of the current) from the complex of invariant currents on $\partial\mathbb{H}^n$ "completed" over that of the harmonic coclosed automorphic forms. Here, "completed" means that the augmentation of distributions $\int : T \mapsto \langle T, \mathbf{1}\rangle$ is appended to the complex. The Poisson transformation can thus be written as:

$$\begin{array}{ccccccccccc} 0 \longrightarrow & C^0_\Gamma(\mathbb{S}^{n-1}) & \xrightarrow{d} & C^1_\Gamma(\mathbb{S}^{n-1}) & \xrightarrow{d} \cdots \xrightarrow{d} & C^{n-1}_\Gamma(\mathbb{S}^{n-1}) & \xrightarrow{\int} & \mathbb{C} & \longrightarrow 0 \\ & \downarrow{\scriptstyle \Phi^0} & & \downarrow{\scriptstyle \Phi^1} & \cdots & \downarrow{\scriptstyle \Phi^{n-1}} & & \downarrow{\scriptstyle \Phi^n} & \\ 0 \longrightarrow & A^0_{hc}(X) & \xrightarrow{d} & A^1_{hc}(X) & \xrightarrow{d} \cdots \xrightarrow{d} & A^{n-1}_{hc}(X) & \xrightarrow{d} & A^n_{hc}(X) & \longrightarrow 0. \end{array}$$

More precisely, the Poisson transformation Φ^p (for $p \leqslant n-1$) is given by a *Schwartz kernel* ϕ_p which is a $(p, n-1-p)$-double form (*cf.* [Rh] §7 p.35) on $\mathbb{H}^n \times \partial\mathbb{H}^n$ (for the explicit expression of ϕ_p see [Ga1] §3 Lemmas 1 and 3). Then, if T is a p-current on $\partial\mathbb{H}^n$, we have:

$$\Phi^p(T)(x) = \int_{\{x\}\times\partial\mathbb{H}^n} \phi_p \wedge (\mathbf{1} \otimes T)$$

(where $\int$ still denotes the augmentation of distributions, which extends the integration of $(n-1)$-forms on $\partial\mathbb{H}^n$). Finally, Φ^n is defined as $\Phi^n(1) = \omega$ where ω is the hyperbolic volume form on $\mathbb{H}^n$.

The cohomology of harmonic coclosed automorphic forms is thus isomorphic to that of the complex $D^\bullet$ defined by $D^p = C^p_\Gamma(\mathbb{S}^{n-1})$ for $p \in \{0, \dots, n-1\}$ and $D^n = \mathbb{C}$:

$$0 \longrightarrow D^0 \xrightarrow{d} D^1 \xrightarrow{d} \cdots \xrightarrow{d} D^{n-1} \xrightarrow{\int} D^n \longrightarrow 0.$$

When n is odd, the isomorphism $H^p\big(A^\bullet_{hc}(X)\big) \simeq H^p(D^\bullet)$ remains true except for $p = \frac{n+1}{2}$ (*cf.* [Gal] Theorem 1).

For $p \in \{0, \dots, n-2\}$, we have $H^p(D^\bullet) = H^p\big(\mathcal{C}^\bullet_\Gamma(\mathbb{S}^{n-1})\big)$. Since $\int \delta_0 = \int \delta_\infty = 1$, the augmentation $\int$ is a non-vanishing linear functionnal so $H^n(D^\bullet) = 0$.

The distribution $\delta_0 + \delta_\infty$ (which generates $H^{n-1}\big(\mathcal{C}^\bullet_\Gamma(\mathbb{S}^{n-1})\big)$ when $n \geqslant 4$) does not belong to $\ker \int$, so $H^{n-1}(D^\bullet) = 0$ for $n \geqslant 4$. If $n = 2$, $H^1\big(\mathcal{C}^\bullet_\Gamma(\mathbb{S}^1)\big)$ is generated by $\delta_0 + \delta_\infty$, which is not closed for the augmentation $\int$, and by Vp which vanishes on constants (as the integral of an odd function on a 0-centered interval). We then have $H^1(D^\bullet) = \mathbb{C}\,[Vp]$.

Now we determine the generators of the cohomology of the complex $A^\bullet_{hc}(X)$. It is clear that the constants are 0-closed and coclosed automorphic forms. Then we have $H^0\big(A_{hc}(X)\big) = \mathbb{C}\,[\mathbf{1}]$. It remains to prove that the 1-form $\pi_1^*\left(\frac{dr}{r}\right)$ (which is Γ-invariant) is automorphic, closed and coclosed. It is trivially closed and a direct computation proves that, where $\langle \mid \rangle$ denotes the euclidian scalar product:

$$\forall x \in \mathbb{H}^n, \quad \forall u \in T_x\mathbb{H}^n, \quad \pi_1^*\left(\frac{dr}{r}\right)_x .u = \frac{\langle x \mid u \rangle}{\|x\|^2}$$

so $\pi_1^*\left(\frac{dr}{r}\right)$ has moderate growth.

Now let's show that its codifferential is zero by using the $*$ Hodge operator defined on 1-forms by $*dx_i = (-1)^{i-1}\frac{1}{x_n^{n-2}} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$. We have:

$$\begin{aligned}
&d * \pi_1^*\left(\frac{dr}{r}\right) \\
&\quad = d * \left[\frac{1}{\|x\|^2} \sum_{i=1}^n x_i dx_i\right] \\
&\quad = d\left[\frac{1}{x_n^{n-2}\|x\|^2} \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\right] \\
&\quad = d\left(\frac{1}{x_n^{n-2}\|x\|^2}\right) \wedge \sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&\qquad + \frac{1}{x_n^{n-2}\|x\|^2} \sum_{i=1}^n (-1)^{i-1} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n
\end{aligned}$$

$$\begin{aligned}
&= \left(-\frac{2}{x_n^{n-2}\|x\|^4}\sum_{i=1}^{n} x_i dx_i\right) \wedge \left(\sum_{i=1}^{n}(-1)^{i-1}x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\right)\\
&\quad - \left(\frac{(n-2)}{x_n^{n-1}\|x\|^2}dx_n\right) \wedge \left(\sum_{i=1}^{n}(-1)^{i-1}x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n\right)\\
&\quad + \frac{n}{x_n^{n-2}\|x\|^2}dx_1 \wedge \cdots \wedge dx_n\\
&= \frac{1}{x_n^{n-2}\|x\|^2}[-2-(n-2)+n]dx_1 \wedge \cdots \wedge dx_n\\
&= 0.
\end{aligned}$$

Consequently, $\pi_1^*\left(\frac{dr}{r}\right) \in A^1_{hc}(X)$. The 1-form $\pi_1^*\left(\frac{dr}{r}\right)$ is not exact because its primitives are the functions $x \mapsto \log\|x\| + K$ with K constant, and that none of those functions is Γ-invariant. □

When n is even, the generators of the cohomology of the complex $A^\bullet_{hc}(X)$ are also the generators of the cohomology of X (which has the homotopy type of a circle), the inclusion $A^\bullet_{hc}(X) \longrightarrow \Omega^\bullet(X)$ is therefore a quasi-isomorphism.

Corollary 2.7 *Conjecture* 0.1 *holds for a group of loxodromies of the hyperbolic space of even dimension.*

Now consider an elementary Kleinian group Γ with hyperbolic type. From [Ra] Theorem 5.5.8, it contains a subgroup Γ_0 with finite index generated by a loxodromy. For $p \in \{0, \ldots, n-1\}$, let m^p be the map:

$$\begin{aligned}
m^p : C^p_{\Gamma_0}(\mathbb{S}^{n-1}) &\longrightarrow C^p_{\Gamma}(\mathbb{S}^{n-1})\\
T &\longmapsto \frac{1}{h}\sum_{i=1}^{h}\gamma_i^* T
\end{aligned}$$

where $h = [\Gamma : \Gamma_0]$ and $\{\gamma_1, \ldots \gamma_h\}$ is a system of representatives of right cosets modulo Γ_0.

Lemma 2.8 *The map m^p is well defined, does not depend on the choice of the representatives $\gamma_1, \ldots, \gamma_h$ and defines a retraction of complexes and then a surjection in cohomology.*

Proof. The fact that m^p does not depend on the system of representatives is clear and the Γ-invariance of $m^p(T)$ follows from the fact that the composition of elements of Γ induces a permutation of right cosets modulo Γ_0.

The map $m^\bullet$ is a morphism of complexes because all the γ_i^* commute with the differential d. Finally, it is clear that $m^p(T) = T$ if T is Γ-invariant.

□

Theorem 2.9 *If $n \geqslant 3$, the average $m^\bullet$ is a quasi-isomorphism from $\mathcal{C}^\bullet_{\Gamma_0}(\mathbb{S}^{n-1})$ on $\mathcal{C}^\bullet_\Gamma(\mathbb{S}^{n-1})$. Particularly, the cohomology of Γ-invariant currents on $\mathbb{S}^{n-1}$ is:*

$$H^p(\mathcal{C}^\bullet_\Gamma(\mathbb{S}^{n-1})) = \begin{cases} \mathbb{C}\,[\mathbf{1}] & \text{if } p=0 \\ \mathbb{C}\,[m^1(Vp)] & \text{if } p=1 \\ \mathbb{C}\,[\delta_0+\delta_\infty] & \text{if } p=n-1 \\ 0 & \text{else.} \end{cases}$$

Proof. By using a conjugation in $\mathrm{Iso}^+(\mathbb{H}^n)$ we can suppose that $\Lambda_\Gamma = \{0,\infty\}$. We just apply the average m^p to each of the generators of the cohomology of the complex $\mathcal{C}^\bullet_{\Gamma_0}(\mathbb{S}^{n-1})$ listed in Theorem 2.5. Lemma 2.8 then implies that their images generate the cohomology of the complex $\mathcal{C}^\bullet_\Gamma(\mathbb{S}^{n-1})$.

Since Γ is an elementary group of hyperbolic type whose finite orbit is $\{0,\infty\}$, the distribution $\delta_0+\delta_\infty$ is Γ-invariant, thus equal to its average.

Suppose now $p=1$. Because $\widetilde{\omega}$ is the Euclidean volume form on $\mathbb{S}^{n-2}$, $\pi_2^*(\widetilde{\omega})$ is Γ_0-invariant. Indeed, we have, for $\gamma\in\Gamma_0$:

$$\gamma^*\left(\pi_2^*(\widetilde{\omega})\right) = (\pi_2\circ\gamma)^*(\widetilde{\omega})$$

with $\pi_2\circ\gamma=\pi_2$. Since R is an euclidian isometry, $\pi_2^*(\widetilde{\omega})$ is also R-invariant. We can then consider its average $\omega = m^{n-2}\left(\pi_2^*(\widetilde{\omega})\right)$. Then we get

$$\begin{aligned} \langle m^1(Vp),\omega\rangle &= \langle Vp, m^{n-2}(\omega)\rangle = \langle Vp,\omega\rangle \\ &= \frac{1}{h}\sum_{\gamma\in\Gamma_0\backslash\Gamma}\int_{\mathbb{R}^{n-1}-\{0\}} \pi_1^*\left(\frac{dr}{r}\right)\wedge\gamma^*\left(\pi_2^*(\widetilde{\omega})\right). \end{aligned}$$

Since every $\gamma\in\Gamma$ preserves the orientation, each of these integrals is that of a positive function. We can then conclude that $\langle Vp,\omega\rangle\neq 0$, thus m_* is an isomorphism. □

The result does not hold for $n=2$, as we shall show. We begin with the characterization of elementary discrete subgroups of $\mathrm{Conf}^+(\mathbb{S}^1)$ of hyperbolic type.

Proposition 2.10 *Every elementary discrete subgoup of* $\mathrm{Iso}^+(\mathbb{H}^2)$ *of hyperbolic type is conjugated to one of the following groups:*
- *an infinite cyclic group* $\Gamma = \langle \gamma_a \rangle$ *generated by* $z \mapsto az$ *with* $a \in]0,1[$;
- *a group generated by such a map and the inversion* $\sigma : z \mapsto -\frac{1}{z}$.

Proof. This proposition follows from the fact that, among the elements of $\mathrm{PSL}(2,\mathbb{R})$, the only homographies preserving $\{0,\infty\}$ of finite order are identity and σ. □

Proposition 2.11 *Let* $\Gamma = \langle \gamma_a, \sigma \rangle$ *the elementary Kleinian group of hyperbolic type generated by* $\gamma_a : z \mapsto az$ *and the inversion* $\sigma : z \mapsto -\frac{1}{z}$. *The cohomology of* Γ*-invariant currents on* $\mathbb{S}^1$ *is*

$$H^p\big(\mathcal{C}^\bullet_\Gamma(\mathbb{S}^1)\big) = \begin{cases} \mathbb{C}\,[\mathbf{1}] & \text{if } p = 0 \\ \mathbb{C}\,[\delta_0 + \delta_\infty] & \text{if } p = 1 \\ 0 & \text{else.} \end{cases}$$

Proof. From Lemma 2.8, this cohomology is generated by the average of each of the generators of the cohomology of γ_a-invariant currents listed in Theorem 2.5. In the same manner, $\mathbf{1}$ and $\delta_0 + \delta_\infty$ are Γ-invariant thus equal to their image under $m^\bullet$. But an elementary computation shows that $\sigma^* Vp = -Vp$, implying that $m^1(Vp) = 0$. □

As before, the average map on Γ_0-invariant forms on $\mathbb{H}^n$ $m^\bullet : \Omega^\bullet(X_0) \longrightarrow \Omega^\bullet(X)$ is defined by

$$m^p(\omega) = \frac{1}{h} \sum_{\overline{\gamma} \in \Gamma_0 \backslash \Gamma} \gamma^*(\omega)$$

and it can be proved likewise that it is well defined and is a retraction of complexes.

Corollary 2.12 *If* $n \geqslant 3$, *the complex of harmonic coclosed automorphic forms with respect to an elementary Kleinian group of hyperbolic type has the same* p*-cohomology as a circle for* $p \neq \frac{n+1}{2}$. *More precisely, such a group is conjugated to a group* Γ *satisfying*

$$H^p\big(A^\bullet_{hc}(X)\big) = \begin{cases} \mathbb{C}\,[\mathbf{1}] & \textit{if } p=0 \\ \mathbb{C}\left[m^1\left(\pi_1^*\left(\frac{dr}{r}\right)\right)\right] & \textit{if } p=1 \\ 0 & \textit{if } p \geqslant 1 \textit{ and } p \neq \dfrac{n+1}{2}. \end{cases}$$

Proof. We can show in the same manner as in Corollary 2.6 that the p-cohomology space ($p \neq \frac{n+1}{2}$) of the complex $A^\bullet_{hc}(X)$ is that of $\mathbb{S}^1$. To show that a generator of $H^1\big(A^\bullet_{hc}(X)\big)$ is the one we announced, we only need to see that it is a closed and coclosed automorphic form that is not exact; this follows from the fact that the Poisson transformation is linear and commutes to hyperbolic isometries and from the same properties of the 1-current $m^1(Vp)$. □

The same method applied to the result of Proposition 2.11 yields the following result.

Corollary 2.13 *The cohomology of harmonic coclosed automorphic forms with respect to an elementary Kleinian group with hyperbolic type, which is not infinite cyclic, is the cohomology of a point*:

$$H^p\big(A^\bullet_{hc}(X)\big) = \begin{cases} \mathbb{C}\,[\mathbf{1}] & \textit{if } p=0 \\ 0 & \textit{else.} \end{cases}$$

Corollary 2.14 *If n is even, Conjecture* 0.1 *holds for an elementary Kleinian group $\Gamma \subset \mathrm{Iso}^+(\mathbb{H}^n)$ with hyperbolic type.*

Proof. The morphism of complexes $m^\bullet$ induces a surjection from $H^p(X_0)$ to $H^p(X)$, with $X_0 = \Gamma_0\backslash\mathbb{H}^n$. Since X_0 has the homotopy type of a circle, we have $H^p(X) = 0$ for $p \geqslant 2$ and $\dim H^p(X) \leqslant 1$ for $p \in \{0,1\}$. Of course $H^0(X) = \mathbb{C}\,[\mathbf{1}]$ and we only need to examine $p = 1$. Suppose once again that Γ_0 is a group of loxodromies fixing 0 and ∞, conjugating Γ in $\mathrm{Iso}^+(\mathbb{H}^n)$ if needed. Then $H^1(X_0) = \mathbb{C}\,\left[\pi_1^*\left(\frac{dr}{r}\right)\right]$ so $H^1(X)$ is generated by $m^1\left(\pi_1^*\left(\frac{dr}{r}\right)\right)$. It is also a generator of the cohomology of $A^\bullet_{hc}(\mathbb{H}^n)$ for $n \geqslant 4$ as seen at Corollary 2.12.

Suppose $n = 2$. Since the Poisson transformation is linear and commutes with hyperbolic isometries, it also commutes with the average map m^1, so we have

$$m^1\left(\pi_1^*\left(\frac{dr}{r}\right)\right) = m^1(\Phi^1(Vp)) = \Phi^1(m^1(Vp)) = 0$$

from Proposition 2.11.

In both cases, the inclusion $A^\bullet_{hc}(X) \longrightarrow \Omega^\bullet(X)$ is a quasi-isomorphism. $\square$

3. The case of a translation in $\mathbb{H}^2$

Now let Γ be the Kleinian group generated by the translation $t : z \longmapsto z + b$ in the upper half space $\mathbb{H}^n$, with $b = (b_1, \dots, b_{n-1}, 0) \in \mathbb{R}^{n-1} \times \{0\}$ fixed. The limit set of Γ is $\Lambda_\Gamma = \{\infty\}$ and its domain of discontinuity is $D_\Gamma = \mathbb{R}^{n-1}$. A fundamental domain for the action of Γ on $\mathbb{H}^n$ is the region

$$B = \{(x_1, \dots, x_n) \in \mathbb{H}^n,\ \forall i \in \{1, \dots, n-1\},\ 0 \leqslant x_i \leqslant b_i\}$$

The quotient $X = \Gamma \backslash \mathbb{H}^n$ has infinite volume and the homotopy type of a circle.

It is well known that every elementary Kleinian group $\Gamma \subset \mathrm{Iso}^+(\mathbb{H}^2)$ with parabolic type is conjugated to such a group (*cf.* [Fre] Lemma 1.9 p.12).

The following lemma holds in every dimension and can be proved by direct computation.

Lemma 3.1 *The critical exponent of the group Γ is*

$$\delta(\Gamma) = \frac{1}{2}.$$

Definition 3.2 Let $s = (s_1, \dots, s_{n-1}) \in \mathbb{N}^{n-1}$ be a multi-index whose length is denoted by $|s|$ and let ξ be a $(n-1-p)$-tangent vector at 0 to $\mathbb{R}^{n-1}$. We define

$$D^s \delta^\infty_\xi := \sigma^*(D^s \delta_\xi)$$

where σ is the inversion $x \longmapsto \frac{x}{\|x\|^2}$ and D^s the partial derivative operator $\frac{\partial^{|s|}}{\partial x_1^{s_1} \cdots \partial x_{n-1}^{s_{n-1}}}$.

These are the partial derivatives of Dirac p-currents at ∞. Of course, the linear combinations of these currents are the only currents on $\mathbb{S}^{n-1}$ with support in $\{\infty\}$.

From now on, we assume that $n = 2$ and, without loss of generality, we can also suppose $b = 1$.

Lemma 3.3 *The Γ-invariant currents on $\mathbb{S}^1 = \partial\mathbb{H}^2$ with support in the limit set $\Lambda_\Gamma = \{\infty\}$ are:*

$$\mathcal{C}^p_\Gamma(\mathbb{S}^1, \Lambda_\Gamma) = \begin{cases} \mathbb{C}\,\delta_\infty \oplus \mathbb{C}\,\delta'_\infty & \text{if } p = 1 \\ \mathbb{C}\,\delta^\infty_{\frac{\partial}{\partial x}} & \text{if } p = 0. \end{cases}$$

Proof. We shall find out which of the currents introduced at Definition 3.2 are invariant by the translation t, or, equivalently, which of their images by σ^* are invariant by $\tau = \sigma \circ t \circ \sigma$.

First let's assume $p = 1$: we are looking for derivatives of δ_0 which are τ-invariant. We use a result about the derivatives of a composition of functions, for which the proof is immediate by induction: for $m \in \mathbb{N}^*$,

$$(\varphi \circ \tau)^{(m)} = \sum_{p=1}^{m} a_p^m \varphi^{(p)} \circ \tau$$

where each coefficient a_p^m is a homogeneous polynomial of degree p in $\tau', \tau'', \ldots, \tau^{(m-p+1)}$. More precisely, we have, for $m \geqslant 1$:

$$\begin{cases} a_1^m = \tau^{(m)} \qquad a_m^m = (\tau')^m \\ \forall p \in \{2, \ldots, m\}, \ \ a_p^{m+1} = (a_p^m)' + a_{p-1}^m \tau' \\ a_{m-1}^m = \dfrac{m(m-1)}{2}(\tau')^{m-2}\tau'' \text{ si } m \geqslant 2. \end{cases}$$

The computation of $\tau^*\delta_0^{(m)}$ gives, for $m \geqslant 1$, $\tau^*\delta_0^{(m)} = \sum_{p=1}^m (-1)^{m-p} a_p^m(0)\delta_0^{(p)}$. In particular, the vector subspace $V = \bigoplus_{p=1}^m \mathbb{C}\,\delta_0^{(p)}$ is stable by τ^*.

Since $\tau'(0) = 1$ and $\tau''(0) = -2$, we get $a_m^m(0) = 1$ and $a_{m-1}^m(0) = -m(m-1)$. Hence the matrix of the endomorphism of V induced by τ with respect to the basis $(\delta_0^{(p)})_{p=1}^m$ has the following form:

$$M = \begin{pmatrix} 1 & -2 & * & \cdots & \cdots & * \\ 0 & 1 & -6 & \cdots & \cdots & * \\ 0 & 0 & 1 & \ddots & * & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & -m(m-1) \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

The matrix $M - \mathrm{id}_V$ has rank $m-1$ so the subspace associated to the eigenvalue 1 is generated by δ'_∞. This ends the proof of $\mathcal{C}^1_\Gamma(\mathbb{S}^1, \Lambda_\Gamma) = \mathbb{C}\,\delta_\infty \oplus \mathbb{C}\,\delta'_\infty$.

Now assume $p = 0$. Let $m \in \mathbb{N}$ and $T = \left(\delta^\infty_{\frac{\partial}{\partial x}}\right)^{(m)}$. An elementary computation shows that $dT = -\delta_\infty^{(m+1)}$. Since the spaces $\mathcal{C}^p_\Gamma(\mathbb{S}^1, \Lambda_\Gamma)$ form a differential complex, for T to be Γ-invariant, it is necessary for dT to be invariant too. From the case $p = 1$ studied above we have $m + 1 \in \{0, 1\}$, implying $m = 0$.

Conversely, if $\omega \in \Omega^1(\mathbb{S}^1)$ is locally expressed as $\omega = \varphi dx$, we get

$$\begin{aligned}\left\langle \delta_{\frac{\partial}{\partial x}}, \tau^*\omega \right\rangle &= \left\langle \delta_{\frac{\partial}{\partial x}}, \varphi \circ \tau\tau' dx \right\rangle \\ &= \varphi \circ \tau(0)\tau'(0) = \varphi(0) = \langle \delta_0, \varphi \rangle = \left\langle \delta_{\frac{\partial}{\partial x}}, \omega \right\rangle\end{aligned}$$

which shows that $\delta_{\frac{\partial}{\partial x}}$ is τ-invariant. □

Lemma 3.4 *The cohomology of Γ-invariant currents on $\mathbb{S}^1$ with support in $\Lambda_\Gamma = \{\infty\}$ is:*

$$H^p(\mathcal{C}^\bullet_\Gamma(\mathbb{S}^1, \Lambda_\Gamma)) = \begin{cases} \mathbb{C}\,[\delta_\infty] & \textit{if } p = 1 \\ 0 & \textit{else.} \end{cases}$$

Proof. The distribution δ_∞ is not exact because it does not vanish on constant functions. But $d(\delta^\infty_{\frac{\partial}{\partial x}}) = -\delta'_\infty$ as seen in the previous lemma. This implies that $[\delta'_\infty] = 0$ and that $\delta^\infty_{\frac{\partial}{\partial x}}$ is not closed. □

Definition 3.5 The finite part of dx, denoted by $Pf(dx)$, is defined for every $\varphi \in \Omega^0(\mathbb{S}^1)$ by

$$\langle Pf(dx), \varphi \rangle = \lim_{A \to \infty} \int_{-A}^{A} [\varphi(x) - \varphi(\infty)] dx.$$

Lemma 3.6 *$Pf(dx)$ is a Γ-invariant distribution on $\mathbb{S}^1$.*

Proof. By using an asymptotic expansion of a function $\varphi \in \Omega^0(\mathbb{S}^1)$ near ∞, we can show that the limit of the integral exists as $A \to \infty$. The continuity of $Pf(dx)$ for the Schwartz topology is not a problem either.

This expansion also proves that the integrals

$$\int_A^{A+1} [\varphi(x) - \varphi(\infty)]\, dx \quad \text{and} \quad \int_{-A+1}^{-A} [\varphi(x) - \varphi(\infty)]\, dx$$

tend to 0 as $A \to \infty$, implying the invariance of $Pf(dx)$ by t^*. $\square$

Proposition 3.7 *The localization map of Γ-invariant distributions $L^1 : \mathcal{C}^1_\Gamma(\mathbb{S}^1) \longrightarrow \mathcal{C}^1_\Gamma(D_\Gamma)$ is surjective.*

Proof. The group Γ acts freely on $D_\Gamma = \mathbb{R}$ and the quotient $\Gamma \backslash D_\Gamma \simeq \mathbb{S}^1$ is compact and connected so, by Lemma 1.2, we have $\ker \widehat{\theta} \subset \operatorname{Im} L^1$ where $\widehat{\theta} : T \mapsto \langle T, f \rangle$. Moreover, we have

$$\mathcal{C}^1_\Gamma(\mathbb{R}) = \ker \widehat{\theta} \oplus \mathbb{C}\, dx.$$

Indeed, f is continuous, positive so $\widehat{\theta}(dx) > 0$. Because the subspace $\ker \widehat{\theta}$ is an hyperplane of $\mathcal{C}^1_\Gamma(\mathbb{R})$, dx generates a supplementary.

But $Pf(dx)$ is an element of $\mathcal{C}^1_\Gamma(\mathbb{S}^1)$ satisfying $L^1(Pf(dx)) = dx$ then this implies that $\mathbb{C}\, dx \subset \operatorname{Im} L^1$, and $\operatorname{Im} L^1 = \mathcal{C}^1_\Gamma(\mathbb{R})$. $\square$

Theorem 3.8 *The cohomology of Γ-invariant currents on $\partial\mathbb{H}^2$ is*

$$H^p(\mathcal{C}^\bullet_\Gamma(\mathbb{S}^1)) = \begin{cases} \mathbb{C}\,[\mathbf{1}] & \text{if } p = 0 \\ \mathbb{C}\,[\delta_\infty] \oplus \mathbb{C}\,[Pf(dx)] & \text{if } p = 1 \\ 0 & \text{else.} \end{cases}$$

Proof. By Lemma 3.1, $\delta(\Gamma) = \frac{1}{2}$ so the localization map of p-currents with $p < 2 - 1 - \delta(\Gamma) = \frac{1}{2}$, *i.e.* $p = 0$, is surjective. By Proposition 3.7, the localization of distributions is also surjective. The situation is that of the following commutative diagram, where the rows are exact sequences:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C^0 \arrow[r, hook, "i^0"] \arrow[d, "d"] & S^0 \arrow[r, "L^0"] \arrow[d, "d"] & Q^0 \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & C^1 \arrow[r, hook, "i^1"] \arrow[d] & S^1 \arrow[r, "L^1"] \arrow[d] & Q^1 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The complexes $C^\bullet$, $S^\bullet$ and $Q^\bullet$ respectively denote $\mathcal{C}^\bullet_\Gamma(\mathbb{S}^1, \Lambda_\Gamma)$, $\mathcal{C}^\bullet_\Gamma(\mathbb{S}^1)$ and $\mathcal{C}^\bullet_\Gamma(D_\Gamma)$. This induces an exact sequence in cohomology:

$$0 \longrightarrow H^0(C^\bullet) \xrightarrow{i^0_*} H^0(S^\bullet) \xrightarrow{L^0_*} H^0(Q^\bullet) \xrightarrow{c^0} H^1(C^\bullet) \xrightarrow{i^1_*} H^1(S^\bullet) \xrightarrow{L^1_*} H^1(Q^\bullet) \longrightarrow 0.$$

Lemma 3.4 gives the cohomology of the complex $C^\bullet$ and the cohomology of $Q^\bullet$ is isomorphic to the de Rham cohomology of the manifold $\Gamma\backslash D_\Gamma$. It can then easily be shown that $H^0(Q^\bullet) = \mathbb{C}\,[\mathbf{1}]$ and $H^1(Q^\bullet) = \mathbb{C}\,[dx]$ (by noticing that dx is not exact in $Q^\bullet$). The exact sequence then reduces to

$$0 \longrightarrow H^0(S^\bullet) \xrightarrow{L^0_*} \mathbb{C}\,[\mathbf{1}] \xrightarrow{c^0} \mathbb{C}\,[\delta_\infty] \xrightarrow{i^1_*} H^1(S^\bullet) \xrightarrow{L^1_*} \mathbb{C}\,[dx] \longrightarrow 0$$

The constants are the only closed 0-currents on $\mathbb{S}^1$ and are Γ-invariant, so $H^0(S^\bullet) = \mathbb{C}\,[\mathbf{1}]$. The map L^0_* is thus an isomorphism. The connecting homomorphism is therefore zero by exactness of the sequence and the fact that i^1_* is injective. Since $L^1_*([Pf(dx)]) = [dx]$, we have $H^1(S^\bullet) = \mathbb{C}\,[\delta_\infty] \oplus \mathbb{C}\,[Pf(dx)]$. $\square$

Corollary 3.9 *The cohomology of harmonic coclosed automorphic forms with respect to the group* Γ *generated by a translation of* $\mathbb{H}^2$ *is*:

$$H^p\big(A^\bullet_{hc}(X)\big) = \begin{cases} \mathbb{C}\,[\mathbf{1}] & \textit{if } p = 0 \\ \mathbb{C}\,[dx] & \textit{if } p = 1 \\ 0 & \textit{else.} \end{cases}$$

In particular, Conjecture 0.1 *holds in this case.*

The proof is similar to that of Corollary 2.6.

Final note:

by a method of average very similar to the one detailed in paragraph 2, it is easy to compute the cohomologies when Γ is elementary of elliptic type (*i.e.* finite) and show that Conjecture 0.1 also holds in this case.

P.-Y. Gaillard gave me a proof of Conjecture 0.1 for the quotient of a simple non compact Lie group G with rank 1 and trivial center by a maximal compact subgroup and for a finite subgroup Γ of G. He uses the theory of representations and the theory of Casselman-Wallach.

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