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Point-extinction and geometric expansion of solutions to a crystalline motion*

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Abstract. We consider the asymptotic behavior of solutions to a generalized crystalline motion which describes evolution of plane curves driven by nonsmooth interfacial energy. Our main results say that solution polygonal curves expand to infinity or shrink to a single point depending on the size of initial data and the sign of the driving force term. In the expanding case, we show that any rescaled solution polygon converges to the boundary of the Wulff shape for the driving force term and hence if the driving force term is a constant, then any solution polygon approaches to an expanding regular polygon even if the motion is anisotropic. We also give lower and upper bounds of the extinction time for the shrinking case. In the appendix, we shall explain the notion of a discrete curvature and crystalline curvature from a numerical point of view.

Key words: crystalline motion, crystalline curvature, discrete curvature, motion by curvature, curve-shortening, point-extinction, geometric expansion, the Wulff shape, estimates of blow-up time, entropy estimate, comparison principle, isoperimetric ratio.

1. Introduction and main results

1.1. The aim of this paper

Let $\mathcal{P}_0$ be a convex closed polygon in the plane $\boldsymbol{R}^2$ with the angle between two adjacent sides of $\mathcal{P}_0$ being $\pi-\Delta\theta$, where $\Delta\theta := 2\pi/n$ and n is the number of sides of the polygon. We consider the evolution problem of finding a family of polygons $\mathcal{P}=\bigcup_{0\leq t<T}(\mathcal{P}_t\times\{t\})$ satisfying

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{x}_j(t)=v_j(t)\boldsymbol{n}_j, \quad 0\leq j<n, \quad 0<t<T, \\ \mathcal{P}\cap\{t=0\}=\mathcal{P}_0, \end{cases} \tag{1.1a}$$

where the vector $\boldsymbol{n}_j$ is the inward normal of the jth side of the polygon $\mathcal{P}_t$ and the vector $\boldsymbol{x}_j(t)$ denotes the point of intersection between the line

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containing the jth side of the polygon $\mathcal{P}_t$ and the line spanned by $\boldsymbol{n}_j$. The function v_j is the inward normal velocity of the jth side which will be specified later. Throughout this paper the interval $[0, T)$, with $T \in (0, \infty]$, will be understood to be the maximal time interval of existence for each solution polygon. We note that the angle between two adjacent sides of $\mathcal{P}_t$ always equals $\pi - \Delta\theta$ as long as the solution polygons exist.

In this paper we consider a generalized crystalline motion of the form

$$v_j(t) = a(\boldsymbol{n}_j)\kappa_j(t) - b(\boldsymbol{n}_j), \quad 0 \leq j < n, \tag{1.1b}$$

where $a > 0$ and b are smooth functions defined on S^1 and κ_j is the crystalline curvature:

$$\kappa_j(t) = \frac{2\tan(\Delta\theta/2)}{d_j(t)}, \quad 0 \leq j < n. \tag{1.1c}$$

Here $d_j(t)$ is the length of the jth side of polygon $\mathcal{P}_t$.

We introduce the set

$$\mathcal{N}_\star := \{\boldsymbol{n}_j = -{}^t(\cos\theta_j, \sin\theta_j) \mid \theta_j = j\Delta\theta,\ \Delta\theta = 2\pi/n,\ 0 \leq j < n\}.$$

This $\mathcal{N}_\star$ is the set of orientations that appear on the Wulff shape (see below) being an regular n-polygon (n-gon in short). A convex polygon $\mathcal{P}$ is called $\mathcal{N}_\star$-*admissible* polygon if the normal vector of each side of $\mathcal{P}$ is the element of $\mathcal{N}_\star$. We can then translate Problem (1.1) into the problem of finding an $\mathcal{N}_\star$-admissible polygon evolved by the crystalline flow (1.1b) with (1.1c). On a general admissibility, we touch upon later.

The aim of this paper is to study the asymptotic behavior of solutions to Problem (1.1). Our main results say that solution polygons shrink to a single point or expand to infinity depending on the size of initial data $\mathcal{P}_0$ and the sign of driving force term b. See Section 1.4. Roughly speaking, if the initial polygon $\mathcal{P}_0$ is sufficiently small, then a solution polygon $\mathcal{P}_t$ shrinks to a single point in a finite time and if $\mathcal{P}_0$ is sufficiently large, then a rescaled solution polygon $\mathcal{P}_t/t$ approaches to the boundary of the Wulff shape $\mathcal{W}_b$ as t tends to infinity. We also give lower and upper bounds of the extinction time for the shrinking case. In the appendix, we shall explain the notion of a discrete curvature and crystalline curvature from a numerical point of view.

1.2. Background

Problem (1.1) is a typical model equation for crystal growth in the plane. In this context the solution polygon represents the boundary curve between two different materials. Such a boundary curve is called the interface or free boundary. The motion of interfaces or free boundaries fascinates many researchers in the fields of applied mathematics, material sciences, physics, biology and so on. The notion of interfacial energy plays an important role in those contexts. As we shall show below, the gradient flow of a total interfacial energy provides a curvature-dependent motion.

Now let us explain how one derives Problem (1.1) in the context of curvature-dependent motion of curves. Let Γ_t be a closed curve parametrized by θ, the angle between the outward normal of Γ_t and the fixed axis. Let f be an interfacial energy defined on Γ_t. If the interfacial energy $f = f(\boldsymbol{n})$ is positively homogeneous of degree one, then the gradient flow of total interfacial energy with respect to the L^2-metric provides the weighted curvature flow $v = \omega := (f(\theta) + f''(\theta))\kappa$. Here we set $f(\theta) = f(\boldsymbol{n}(\theta))$ and $\kappa = \kappa(\theta, t)$ is the curvature of Γ_t. See Elliott [E] and Appendix A.

We note that $f + f''$ is the inverse of the curvature of the boundary of the Wulff shape $\mathcal{W}_f$: a region enclosed by a solution to the problem of finding a closed embedded plane curve Γ that minimizes the total interfacial energy $\int_\Gamma f\, ds$ at fixed enclosed area in the plane. It is not difficult to see that the solution is uniquely determined and the Wulff shape is described by

$$\mathcal{W}_f = \{\boldsymbol{x} \in \boldsymbol{R}^2 \mid \langle \boldsymbol{x}, -\boldsymbol{n}(\theta)\rangle \leq f(\theta) \text{ for all } \theta \in \boldsymbol{R}\}.$$

See, e.g., Gurtin [Gu1] about properties of the Wulff shape.

If the Wulff shape $\mathcal{W}_f$ is a polygon, we call f the crystalline energy (see Angenent-Gurtin [AGu]). Let f be a crystalline energy with $\mathcal{W}_f$ being an n-gon and $\boldsymbol{n}(\theta_j)$ being the normal of the jth side, called facet, of $\partial\mathcal{W}_f$. We can then define the finite set

$$\mathcal{N} := \{\boldsymbol{n}(\theta_j) \mid 0 \leq \theta_0 < \theta_1 < \cdots < \theta_{n-1} < 2\pi\}.$$

For such an energy, Taylor [T2] and Angenent-Gurtin [AGu] restrict the curve Γ_t to the class of $\mathcal{N}$-*admissible* piecewise linear curves $\mathcal{P}_t$, in which (1) each normal vector is the element of $\mathcal{N}$ and (2) normal vectors of two adjacent sides of $\mathcal{P}_t$ are the adjacent in $\mathcal{N}$ (see, e.g., Giga-Gurtin [GGu]). Note that we do not need the condition (2) if $\mathcal{P}_t$ is convex (see the definition of $\mathcal{N}_\star$-admissible above). The evolution equation of $\mathcal{P}_t$ is then reduced to the ordinary differential equations $v_j(t) = \omega_j(t)$. This evolution law is called the crystalline motion or crystalline flow. The function v_j is the velocity of the jth side and ω_j is the jth crystalline curvature defined by $\omega_j(t) = \chi_j l(\boldsymbol{n}_j)/d_j(t)$. Here $l(\boldsymbol{n}_j)$ is the length of the side of $\partial\mathcal{W}_f$ that has orientation $\boldsymbol{n}_j \in \mathcal{N}$, χ_j is the transition number which has the constant value $+1$, -1 or 0 depending on whether the polygon is strictly convex, strictly concave or neither near the jth side of $\mathcal{P}_t$, d_j is the length of the jth side of $\mathcal{P}_t$. In fact, the jth crystalline curvature can be decomposed as follows (see Appendix C):

$$\omega_j(t) = (f + \Delta_\theta f)_j \kappa_j(t), \quad \kappa_j(t) = \chi_j \frac{\gamma_j}{d_j(t)}.$$

Here $\gamma_j := \tan(\Delta\theta_{j+1}/2) + \tan(\Delta\theta_j/2)$ and Δ_θ is a kind of difference operator defined by

$$(\Delta_\theta(\cdot))_j := \frac{(\mathrm{D}_+(\cdot))_j - (\mathrm{D}_+(\cdot))_{j-1}}{\gamma_j}, \quad (\mathrm{D}_+(\cdot))_j := \frac{(\cdot)_{j+1} - (\cdot)_j}{\sin \Delta\theta_{j+1}} \tag{1.2}$$

with $\Delta\theta_j = \theta_j - \theta_{j-1}$. We call κ_j the "discrete curvature," which is an approximation of the real curvature $\kappa(\theta_j)$ if n is sufficiently large (see Appendix B). We note that the discrete curvature and the crystalline curvature are equivalent when the Wulff shape is a regular polygon.

Remark 1.1 In this paper we consider the asymptotic behavior of an $\mathcal{N}_\star$-admissible convex n-gon. Although $\mathcal{N}_\star$ is a special case of $\mathcal{N}$, the set $\mathcal{N}_\star$ is better than $\mathcal{N}$ from a numerical point of view. See Remark 1.8 below and Appendix B.

1.3. Generalized crystalline motion and its application

Angenent-Gurtin [AGu] proposed a generalized crystalline motion:

$$\beta(\boldsymbol{n}_j)v_j(t) = \omega_j(t) - U, \tag{1.3}$$

where $\beta(\boldsymbol{n}_j)$ is the kinetic modulus, U is the constant bulk energy. Independently, Taylor [T2] derived the planar crystalline motion under the assumption: $\beta = \text{const.} \times f^{-1}$ and $U \equiv 0$. For the further detail and background of a crystalline flow and a weighted curvature flow, see the papers [AlT, RT, T1, T4], the papers including a survey [T3, TCH, GirK2, GG4, Gu2] and the book [Gu1]. Recently, the three dimensional crystalline flow is analyzed in [GGuM, BNP, Yu]. In [Ry], a Stefan-type problem which has the crystalline interfacial energy is studied. In [IIU], they apply the crystalline motion for the shrinking spiral problem. A numerical simulation is proposed for a curvature-dependent motion with a crystalline type anisotropy in [GP]. Structure and existence of stationary finger of two-dimensional solidification for crystalline energy are investigated in [Al]. See also the very recent work [GG6, GG7].

It is clear that any circle shrinks to a point self-similarly under the isotropic flow $v = \kappa$. In general, we call a solution curve which does not change shape a *self-similar* solution. We can easily check that the boundary of the Wulff shape is a self-similar solution of the weighted curvature flow $v = f\omega = f(f + f'')\kappa$. In [GL], they show the existence of the self-similar solution to the anisotropic flow $v = a(\theta)\kappa$ and obtain the uniqueness under a symmetry assumption. The assumption on $a(\cdot)$ is relaxed to just boundness in [DGM]. Stancu [S1, S2, S3] shows the existence and uniqueness, under a symmetric assumption, of self-similar solution to the crystalline flow $v_j = a(\theta_j)\kappa_j$.

Remark 1.2 Let $\mathcal{P}_t$ be a convex $\mathcal{N}$-admissible polygon with a crystalline energy f. We consider the crystalline motion $v_j = f_j(\omega_j - U)$. Then we can find a self-similar solution $\mathcal{P}_t = \lambda(t)\partial\mathcal{W}_f$ with $\mathcal{P}_0 = \lambda_0\partial\mathcal{W}_f$. Here λ is the solution of

$$\frac{d}{dt}\lambda(t) = -\frac{1}{\lambda(t)} + U, \quad \lambda(0) = \lambda_0.$$

When $U = 0$, it is easy to obtain the exact solution $\lambda(t) = \sqrt{\lambda_0^2 - 2t}$. In general, we have the followings:

- If $U \leq 0$, then the polygon shrinks to a single point;
- If $U > 0$ and $\lambda_0 < U^{-1}$, then the polygon shrinks to a single point;
- If $U > 0$ and $\lambda_0 > U^{-1}$, then the polygon expands into infinity.

Angenent-Gurtin [AGu] extend Remark 1.2 to the following three cases for the evolution equation (1.3) of an $\mathcal{N}$-admissible piecewise linear curve. Let $T > 0$ be a duration of solution polygon of equation (1.3), $\mathcal{L}(t)$ the length and $\mathcal{A}(t)$ the enclosed area. Here and hereafter, we use the term "duration" for the maximal existence time of solution polygons.

- If $U \leq 0$, then $\mathcal{A}(t) \to 0$ as $t \to T < \infty$;
- If $U > 0$ and $\mathcal{L}(0)$ is small enough, then $\mathcal{A}(t) \to 0$ as $t \to T < \infty$;
- If $U > 0$ and $\mathcal{A}(0)$ is large enough, then $\mathcal{A}(t) \to \infty$ as $t \to T = \infty$. Even so, isoperimetric ratio remains bounded: $\limsup_{t\to\infty} \mathcal{L}(t)^2/(4\pi\mathcal{A}(t)) < \infty$. Moreover, they conjecture that (see section 11 in [AGu]), as $t \to \infty$,

$$\text{a solution polygon is asymptotic to the Wulff shape for } \beta^{-1}. \tag{1.4}$$

1.4. Main results

Our goal in this paper is to extend Remark 1.2 and the above results of [AGu] for the motion of convex $\mathcal{N}_\star$-admissible n-gons with general $a > 0$ and b. We assume one of the following:

(A1) $b \leq 0$ is a constant.

(A1)$'$ $b \leq 0$ is not constant and

$$\min_{0\leq j<n} \kappa_j(0) > \frac{\max_{0\leq j<n} b(\boldsymbol{n}_j) - \min_{0\leq j<n} b(\boldsymbol{n}_j)}{\min_{0\leq j<n} a(\boldsymbol{n}_j)}.$$

(A2) $b > 0$ and $\displaystyle\min_{0\leq j<n} \kappa_j(0) \geq \frac{2\max_{0\leq j<n} b(\boldsymbol{n}_j)}{\min_{0\leq j<n} a(\boldsymbol{n}_j)}$.

(A3) $b > 0$ satisfies $(\Delta_\theta b(\boldsymbol{n}) + b(\boldsymbol{n}))_j > \eta$ for a fixed $\eta > 0$,

$(\Delta_\theta a(\boldsymbol{n}) + a(\boldsymbol{n}))_j \geq 0$ and

$$\max_{0\leq j<n} \kappa_j(0) \leq \frac{\min_{0\leq j<n}(\Delta_\theta b(\boldsymbol{n}) + b(\boldsymbol{n}))_j - \eta}{\max_{0\leq j<n}(\Delta_\theta a(\boldsymbol{n}) + a(\boldsymbol{n}))_j}.$$

Assumptions (A1)$'$ and (A2) mean that the initial polygon $\mathcal{P}_0$ is sufficiently small and (A3) means that $\mathcal{P}_0$ is sufficiently large. Note that for (A1)$'$ if $b \leq 0$ is not constant, then $\min_{0\leq j<n} b(\boldsymbol{n}_j) < 0$ and for (A3) there exists b satisfying $(\Delta_\theta b + b)_j > \eta$ since $2\tan(\Delta\theta/2)(\Delta_\theta b + b)_j$ is the length of the jth side of $\partial\mathcal{W}_b$ (see Appendix C).

Our main results are the following.

Theorem A (point-extinction) *Let $n \geq 4$. Assume* (A1) *or* (A1)$'$. *Let $\mathcal{P}_t$ be a solution polygon of Problem* (1.1) *with a duration $T_\star$. Then any solution polygon $\mathcal{P}_t$ shrinks to a single point as $t \to T_\star$ and it holds that*

$$T_\star \leq \frac{1}{2\min_{0\leq j<n} a(\boldsymbol{n}_j)} \left(\frac{\mathcal{L}(0)}{2n\tan(\Delta\theta/2)} \right)^2 .$$

No side of the polygon vanishes before t reaches $T_\star$. Here $\mathcal{L}(0)$ is the length of $\mathcal{P}_0$.

Theorem B (point-extinction) *Let $n \geq 4$. Assume* (A2). *Let $\mathcal{P}_t$ be a solution polygon of Problem* (1.1) *with a duration $T_\star$. Then any solution polygon $\mathcal{P}_t$ shrinks to a single point as $t \to T_\star$. Moreover*

$$T_\star \leq \min\{T_1,\, T_2,\, T_3\}, \quad \textit{where} \quad T_1 = \frac{(\mathcal{L}(0)/2n\tan(\Delta\theta/2))^2}{\min_{0\leq j<n} a(\boldsymbol{n}_j)},$$

$$T_2 = \frac{\mathcal{L}(0)}{2\tan(\Delta\theta/2)\sum_{0\leq j<n} b(\boldsymbol{n}_j)}, \quad T_3 = T_2 - \nu + \sqrt{(\nu - T_2)^2 + \nu T_1},$$

and $\nu = n^2(\sum_{0\leq j<n} b(\boldsymbol{n}_j)\sum_{0\leq j<n} b(\boldsymbol{n}_j)/a(\boldsymbol{n}_j))^{-1}$. No side of the polygon vanishes before t reaches $T_\star$.

Remark 1.3 We call $T_\star$ the "extinction time" or the "blow-up time" (see Section 2.4).

Remark 1.4 If $b \equiv 0$, then the point-extinction holds and the solution is asymptotic self-similar (see [S3]). Let $\mathcal{A}(t)$ be the area of region enclosed by $\mathcal{P}_t$. We can easily check $d\mathcal{A}(t)/dt = -2\tan(\Delta\theta/2)\sum_{0\leq j<n} a(\boldsymbol{n}_j)$, hence we have

$$T_\star = T_{\star\star} = \frac{\mathcal{A}(0)}{2\tan(\Delta\theta/2)\sum_{0\leq j<n} a(\boldsymbol{n}_j)}$$

since point-extinction holds.

For a convex $\mathcal{N}_\star$-admissible polygon $\mathcal{P}_t$, we define the isoperimetric ratio by

$$\mathcal{I}(t) = \frac{\mathcal{L}(t)^2}{4n\tan(\Delta\theta/2)\mathcal{A}(t)}. \tag{1.5}$$

It is not difficult to see that the inequality $\mathcal{I}(t) \geq 1$ holds. The equality $\mathcal{I}(t) = 1$ holds if and only if the polygon $\mathcal{P}_t$ is a regular polygon. See [Y], especially Section 3.

Theorem C (geometric expansion) *Let $n \geq 4$. Assume* (A3). *Let $\mathcal{P}_t$ be a solution polygon of Problem* (1.1). *Then the length $\mathcal{L}(t)$ and the enclosed* *area* $\mathcal{A}(t)$ *of the polygon* $\mathcal{P}_t$ *diverge to infinity as* t *tends to infinity. Every side of the polygon is finite if* t *is finite. Moreover, a rescaled solution polygon* $\mathcal{P}_t/t$ *converges to the boundary of the Wulff shape* $\mathcal{W}_b$ *in the Hausdorff metric as* $t \to \infty$ *and the limit of the isoperimetric ratio* $\mathcal{I}(t)$ *is given as*

$$\lim_{t\to\infty} \mathcal{I}(t) = \frac{\left(\sum_{0\le j<n} b(\boldsymbol{n}_j)\right)^2}{n\sum_{0\le j<n} b(\boldsymbol{n}_j)(\Delta_\theta b(\boldsymbol{n}) + b(\boldsymbol{n}))_j}.$$

Consequently, if $b(\boldsymbol{n}_j)$ *is a positive constant, then any solution polygon* $\mathcal{P}_t$ *expands to infinity approaching an expanding regular polygon in the Hausdorff metric as* $t \to \infty$.

We note that related two results: *1.* For (1.3), Giga-Gurtin [GGu] proves a similar result to Theorem C without the convergence of the isoperimetric ratio. They establish the comparison principle for admissible piecewise linear curves and from which they show the asymptotic shape of the solution curves. *2.* In the case where the interfacial energy f is smooth, Ishii-Pires-Souganidis [IPS] shows that the boundary of a "large" bounded domain in $\boldsymbol{R}^N$ converges to $\partial\mathcal{W}_b$ for the evolution equation $v = a(\boldsymbol{n})H_f - b(\boldsymbol{n})$, where H_f is a weighted mean curvature. Their proof is based on the level set method for a smooth f. Recently, Giga-Giga [GG6] establishes the level set method for a not necessarily smooth f including crystalline. Thus, as pointed out by Giga [G], the result [IPS] is extended for such an energy f in particular for the equation (1.1b).

Remark 1.5 Theorem C gives an answer to the conjecture (1.4) and also asserts that if $b \equiv$ const., then the asymptotic shape is an expanding regular polygon even if $a(\boldsymbol{n}_j)$ is "not" constant, i.e. the motion is anisotropic. We note that the result does not depend on η in Assumption (A3).

Theorem D (lower bound of the blow-up time) Assume $b \not\equiv 0$. Under the same assumption of Theorem A, the blow-up time $T_\star$ is estimated as follows:

$$T_\star \ge \frac{\max_{0\le j<n} a(\boldsymbol{n}_j)}{8\left(\min_{0\le j<n} b(\boldsymbol{n}_j)\right)^2}\left(1 - \sqrt{1 - \frac{8\min_{0\le j<n} b(\boldsymbol{n}_j)\mathcal{A}(0)}{\max_{0\le j<n} a(\boldsymbol{n}_j)\mathcal{L}(0)}}\right)^2.$$

Here $\mathcal{A}(0)$ is the area of the region enclosed by $\mathcal{P}_0$.

Remark 1.6 Let $\mathcal{P}_0$ be a regular polygon. Suppose $a \equiv 1$ and $b \equiv$ const. < 0. We denote the upper bound in Theorem A by T_u and the lower bound in Theorem D by T_ℓ. If we set $b = \mu\kappa(0)$ ($\kappa_j(0) \equiv \kappa(0)$), then we have $\mu < 0$ and $\lim_{\mu \to 0-} T_\ell = T_u = \kappa(0)^{-2}/2 = T_{\star\star}$.

Theorem E (lower bound of the blow-up time) Under the same assumption of Theorem B, the blow-up time $T_\star$ is estimated as follows:

$$T_\star \geq \frac{\mathcal{L}(0)}{8\tan(\Delta\theta/2)\sum_{0 \leq j < n} b(\boldsymbol{n}_j)} \left(\sqrt{1 + \frac{32\tan(\Delta\theta/2)\sum_{0 \leq j < n} b(\boldsymbol{n}_j)\mathcal{A}(0)^2}{\max_{0 \leq j < n} a(\boldsymbol{n}_j)\mathcal{L}(0)^3}} - 1 \right).$$

Remark 1.7 Let $\mathcal{P}_0$ be a regular polygon. Suppose $a \equiv 1$, $b \equiv$ const. > 0 and the Assumption (A2) holds. If we set $b = \mu\kappa(0)$ ($\kappa_j(0) \equiv \kappa(0)$), then we have $\mu \leq 1/2$. We denote the lower bound in Theorem D by T_ℓ. It holds that $T_2 > T_1 > T_3 > T_\ell$ and that $\lim_{\mu \to 0+} T_\ell = \lim_{\mu \to 0+} T_3 = \kappa(0)^{-2}/2 = T_{\star\star}$.

Remark 1.8 (approximation) Many authors have recently studied an approximation of curvature-dependent motions by using crystalline motions. In both [GirK1] and [FG], the convergence results are shown for graph-like curves. In [EGS], the properties of a solution in the sense of [FG] are investigated and several numerical examples are presented in order to visualize their results. The new notion of solutions to a fully nonlinear equation including crystalline motion is introduced and analyzed in [GG1, GG2, GG4]. Its notion is in the realm of viscosity solution theory and so is based on comparison principle which is an extension of [GGu]. The convergence results are discussed in [GG3, GG5] for the solutions in its notion. See also [GG6].

Let the Wulff shape be an $\mathcal{N}_\star$-admissible polygon. Girão [Gir] showed that the crystalline motion $v_j = \omega_j$ approximates the weighted curvature flow $v = \omega$ if the curve is closed and convex. This result was extended by [UY1] for the motion by a power of curvature $v = \kappa^\alpha$ ($\alpha > 0$). Moreover, they constructed a crystalline algorithm to the equation $v = |\kappa|^{\alpha-1}\kappa$ for nonconvex curves in [UY2]. Implicit crystalline algorithm is treated in [UY3] for an area-preserving motion by curvature $v = \kappa - 2\eta\pi/\mathcal{L}$ ($\eta \geq 1$ is a winding number of curve). In [IS], the authors show the approximation of the curve-shortening equation $v = \kappa$ by the crystalline motion $v_j = \kappa_j$ via the level set method. Recently, their results are extended by [GG6, GG7] for general curvature flow equation. See the survey [E] for more general information about an approximation of curvature-dependent motion.

The organization of this paper is as follows: in Section 2, we give several fundamental properties of solutions to Problem (1.1). In Section 3, we present a point-extinction property of solutions via entropy estimates and prove Theorems A and B. In Section 4, we prove Theorem C by the super- and subsolution method or the comparison principle. By using Schwarz inequality twice, we give a lower bound of the extinction time and the proof of Theorems D and E in Section 5. In Appendix A, we give a brief summary on the gradient flow for a total interfacial energy. In Appendices B and C, we explain the notion of the discrete curvature and the crystalline curvature, respectively.

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2. Properties of solutions to Problem (1.1)

In this section we first give an equivalent formulation of Problem (1.1). Secondly, we present comparison principle and evolution of the length and the area. Finally, we show a finite time blow-up of solution.

Throughout this paper we use the notation $\sum_j u_j$, $u_{\max}$, $u_{\min}$ and $\dot{u}(t)$ for $\sum_{0\le j<n} u_j$, $\max_{0\le j<n} u_j$, $\min_{0\le j<n} u_j$ and $du(t)/dt$, respectively. Hereafter we denote $a_j := a(\boldsymbol{n}_j)$ and $b_j := b(\boldsymbol{n}_j)$ for simplicity and assume $n \ge 4$. We note again $\theta_j = j\Delta\theta$.

2.1. A formulation equivalent to Problem (1.1)

Let $\mathcal{P}_t$ be a solution of Problem (1.1). The jth vertex $\boldsymbol{B}_j(t)$ of $\mathcal{P}_t$ is given as the following:

$$\begin{aligned}\boldsymbol{B}_j(t) &= \langle \boldsymbol{x}_{j-1}(t) - \boldsymbol{x}_j(t), \boldsymbol{t}_j + \boldsymbol{n}_j \cot\Delta\theta\rangle\, \boldsymbol{t}_j + \boldsymbol{x}_j(t),\\ &= \boldsymbol{B}_0(t) + \sum_{0\le m<j} d_m(t)\boldsymbol{t}_m, \quad 1 \le j \le n, \quad 0 \le t < T \qquad (2.1)\end{aligned}$$

with $\boldsymbol{B}_0(t) \equiv \boldsymbol{B}_n(t)$, where $\boldsymbol{t}_j = {}^t(-\sin\theta_j, \cos\theta_j)$ is the tangent vector, since the position vector $\boldsymbol{x}_j$ is on the line containing the jth side (n.b. $\boldsymbol{x}_j$ is not necessarily on the jth side) and $\langle\cdot,\cdot\rangle$ is the usual inner product. Then the time evolution of the length of the jth side $d_j(t)$ is given as the following (cf. Figure 10C in [AGu]):

$$\frac{d}{dt}d_j(t) = \frac{d}{dt}|\boldsymbol{B}_{j+1}(t) - \boldsymbol{B}_j(t)| = -2\tan\frac{\Delta\theta}{2}(\Delta_\theta v + v)_j. \tag{2.2}$$

Here the operator Δ_θ is defined by

$$(\Delta_\theta(\cdot))_j := \frac{(\cdot)_{j+1} - 2(\cdot)_j + (\cdot)_{j-1}}{2(1-\cos\Delta\theta)},$$

which is a kind of central difference operator (this is a special version of (1.2)). Then we obtain a discretized version of the equation (2.20) in the book [Gu1]:

$$\frac{d}{dt}\kappa_j(t) = \kappa_j^2(\Delta_\theta v + v)_j, \quad 0 \le j < n, \quad 0 \le t < T.$$

Therefore we can restate Problem (1.1) as follows.

Problem 1 Let $n \geq 4$. Find a function $v(t) = (v_0, v_1, \ldots, v_{n-1}) \in [C[0,T) \cap C^1(0,T)]^n$ and a duration $T \in (0,\infty]$ satisfying

$$\frac{d}{dt}v_j(t) = a_j^{-1}(v_j + b_j)^2(\Delta_\theta v + v)_j, \quad 0 \le j < n, \quad 0 < t < T, \tag{2.3a}$$

$$v_j(0) = a_j\kappa_j(0) - b_j, \quad 0 \le j < n, \tag{2.3b}$$

$$v_{-1}(t) = v_{n-1}(t), \quad v_n(t) = v_0(t), \quad 0 \le t < T, \tag{2.3c}$$

where $\kappa_j(0)$ is the jth initial crystalline curvature of $\mathcal{P}_0$.

Remark 2.1 (equivalence) Problem (1.1) and Problem 1 are equivalent except the indefiniteness of position of the polygon. Indeed, suppose v is a solution of Problem 1, then we have

$$\begin{aligned}\frac{1}{2\tan(\Delta\theta/2)}\frac{d}{dt}&\sum_j \frac{2a_j\tan(\Delta\theta/2)}{v_j(t)+b_j}\boldsymbol{t}_j \\ &= -\sum_j(\Delta_\theta v + v)_j\boldsymbol{t}_j = -\sum_j(\Delta_\theta \boldsymbol{t} + \boldsymbol{t})_j v_j = \boldsymbol{0}.\end{aligned}$$

Here we have used the relation of summation by parts:

$$\sum_j f_j(\Delta_\theta g)_j = -\sum_j(\mathrm{D}_+ f)_j(\mathrm{D}_+ g)_j = \sum_j g_j(\Delta_\theta f)_j, \tag{2.4}$$

and the relation $(\Delta_\theta \boldsymbol{t})_j = -\boldsymbol{t}_j$. Here and hereafter, we define the forward difference such as

$$(\mathrm{D}_+ f)_j := \frac{f_{j+1} - f_j}{2\sin(\Delta\theta/2)}.$$

Hence by equation (2.1), we can construct a closed convex n-gon whose length of the jth side is $2a_j \tan(\Delta\theta/2)/(v_j(t) + b_j) =: d_j(t)$ and the jth normal vector is $\boldsymbol{n}_j$, as long as v is a solution of Problem 1. This n-gon is the very solution polygon of Problem (1.1).

2.2. Comparison principle

The following comparison principle plays an important role in this paper.

Lemma 2.2 *Fix $T > 0$. Let $(p_j(t))_{0\le j<n} > 0$ and $(q_j(t))_{0\le j<n}$ be defined on $t \in [0,T]$. If $u = (u_j(t))_{0\le j<n} \in [C[0,T] \cap C^1(0,T)]^n$ is a solution of*

$$\begin{cases} \dfrac{d}{dt}u_j \ge p_j(\Delta_\theta u)_j + q_j u_j, & 0 \le j < n, \quad 0 < t < T, \\ u_{-1}(t) = u_{n-1}(t), \quad u_n(t) = u_0(t), & 0 \le t \le T, \\ u_j(0) \ge 0, \quad 0 \le j < n, & \end{cases}$$

then $u_j(t) \ge 0$ holds for $0 \le j < n$ and $0 \le t \le T$.

See, e.g., [Y] for the proof of this lemma.

As an application of the above lemma, we obtain the next:

Lemma 2.3 *For a solution v of Problem 1 and fixed $T \in (0, T_\star)$, we have the followings.*

(1) *For a constant $c \ge 0$, if $v_j(0) \ge c$, then $v_j(t) \ge c$ for all $t \in [0,T]$.*

(2) *For a constant $c \le 0$, if $v_j(0) \le c$, then $v_j(t) \le c$ for all $t \in [0,T]$.*

(3) *If v_j^u is a supersolution of Problem 1, i.e. a solution of*

$$\dot{v}_j^u \ge a_j^{-1}(v_j^u + b_j)^2(\Delta_\theta v_j^u + v_j^u)_j, \quad 0 \le j < n, \quad 0 < t < T,$$

with $v_j^u(0) \ge v_j(0)$ and periodic boundary condition (2.3c), then $v_j^u(t) \ge v_j(t)$ holds for all $0 \le t \le T$ and $0 \le j < n$.

(4) *If v_j^l is a subsolution of Problem 1, i.e. a solution of*

$$\dot{v}_j^l \le a_j^{-1}(v_j^l + b_j)^2(\Delta_\theta v_j^l + v_j^l)_j, \quad 0 \le j < n, \quad 0 < t < T,$$

with $v_j^l(0) \le v_j(0)$ and periodic boundary condition (2.3c), then $v_j^l(t) \le$

$v_j(t)$ holds for all $0 \le t \le T$ and $0 \le j < n$.

Proof. For each proposition, put (1) $u_j = v_j - c$; (2) $u_j = c - v_j$; (3) $u_j = v_j^u - v_j$; (4) $u_j = v_j - v_j^l$; and apply Lemma 2.2. □

2.3. The length and the area

The (total) length of the polygon is

$$\mathcal{L}(t) := \sum_j d_j = 2\tan\frac{\Delta\theta}{2}\sum_j \kappa_j^{-1} = 2\tan\frac{\Delta\theta}{2}\sum_j \frac{a_j}{v_j + b_j}, \tag{2.5}$$

and the rate of change of $\mathcal{L}(t)$ can be computed by

$$\dot{\mathcal{L}}(t) = -2\tan\frac{\Delta\theta}{2}\sum_j v_j(t). \tag{2.6}$$

If $v_j(0) \ge 0$ (resp., "≤ 0"), then $v_j(t) \ge 0$ (resp., "≤ 0") by Lemma 2.3 and so $\dot{\mathcal{L}}(t) \le 0$ (resp., "≥ 0"), i.e. the motion of solution polygons is a discretized curve-shortening (resp., curve-lengthening). The area enclosed by the polygon is

$$\mathcal{A}(t) := -\frac{1}{2}\sum_j \langle \boldsymbol{x}_j(t), \boldsymbol{n}_j\rangle d_j(t), \tag{2.7}$$

and the rate of change of $\mathcal{A}(t)$ can be computed by

$$\dot{\mathcal{A}}(t) = -2\tan\frac{\Delta\theta}{2}\sum_j \frac{a_j v_j}{v_j + b_j}.$$

Here we have used equations (2.2) and (2.4), definition $\langle \dot{\boldsymbol{x}}_j, \boldsymbol{n}_j\rangle = v_j$ and geometric relation $d_j = -2\tan(\Delta\theta/2)(\Delta_\theta\langle \boldsymbol{x}, \boldsymbol{n}\rangle + \langle \boldsymbol{x}, \boldsymbol{n}\rangle)_j$.

2.4. Finite time blow-up

In this subsection, we give a partial proof of Theorems A and B, namely the statement concerning finite time blow-up.

Lemma 2.4 (finite time blow-up) *Suppose v is a solution of Problem 1. Under the same assumption of Theorem A, there exists a finite time $T_\star > 0$ such that the maximum of $\{\kappa_j = (v_j + b_j)/a_j\}$ blows up to infinity as $t \nearrow T_\star$:*

$$T_\star \le \frac{1}{2\min_{0\le j<n} a(\boldsymbol{n}_j)}\left(\frac{\mathcal{L}(0)}{2n\tan(\Delta\theta/2)}\right)^2.$$

Proof. Since $n^2 = (\sum_j 1)^2 = (\sum_j \kappa_j^{1/2}\kappa_j^{-1/2})^2$, Schwarz inequality and the assumption $b \le 0$ yields

$$\begin{aligned}\left(2n\tan\frac{\Delta\theta}{2}\right)^2 &\le -\frac{1}{2a_{\min}}\frac{d}{dt}\mathcal{L}(t)^2 + 2\tan\frac{\Delta\theta}{2}\mathcal{L}(t)\sum_j \frac{b_j}{a_j} \\ &\le -\frac{1}{2a_{\min}}\frac{d}{dt}\mathcal{L}(t)^2.\end{aligned} \tag{2.8}$$

By the general argument for ordinary differential equation, a solution v of Problem 1 exists uniquely and locally in time. Put $T_\star > 0$ such as maximal existing time. Take $0 < t < T_\star$. Integration of the above inequality over $(0, t)$ yields

$$\mathcal{L}(t) \le \sqrt{\mathcal{L}(0)^2 - 2a_{\min}\left(2n\tan\frac{\Delta\theta}{2}\right)^2 t}.$$

Since $\mathcal{L}(t) \ge 2n\tan(\Delta\theta/2)/\kappa_{\max}$, we have

$$\kappa_{\max} \ge 2n\tan\frac{\Delta\theta}{2}\left(\mathcal{L}(0)^2 - 2a_{\min}\left(2n\tan\frac{\Delta\theta}{2}\right)^2 t\right)^{-1/2},$$

and the assertion is concluded. □

Lemma 2.5 (finite time blow-up) *Suppose v is a solution of Problem* 1. *Under the same assumption of Theorem* B, *there exists a finite time $T_\star > 0$ such that the maximum of $\{\kappa_j = (v_j + b_j)/a_j\}$ blows up to infinity as $t \nearrow T_\star$:*

$$T_\star \le \min\{T_1,\ T_2,\ T_3\}.$$

Here T_1, T_2 and T_3 have been defined in Theorem B.

Proof. The Assumption (A2) implies $v_j(0) \ge b_{\max}$. Then Lemma 2.3 provides $v_j(t) \ge b_{\max} \ge b_j$. Hence, we get T_1 by a similar proof of Lemma 2.4.

Integration of $-\dot{\mathcal{L}}(t) = 2\tan(\Delta\theta/2)\sum_j v_j \ge 2\tan(\Delta\theta/2)\sum_j b_j$ over $(0, t)$ yields

$$\mathcal{L}(t) \le \mathcal{L}(0) - 2\tan\frac{\Delta\theta}{2}\sum_j b_j t, \tag{2.9}$$

and then we obtain T_2.

Substitute the inequality (2.9) to (2.8), integrate it over $(0,t)$ and solve it. Then we get $t \le T_3$. □

3. Point-extinction (proof of Theorems A and B)

Before we give the proof of Theorems A and B, we present the following theorem.

Theorem 3.1 *Assume* (A1) *or* (A1)$'$ *or* (A2). *If the area* $\mathcal{A}(t)$ *is bounded away from zero, then a solution* v *of Problem* 1 *is uniformly bounded for* $t \in [0, T_\star)$, *where the blow-up time* $T_\star$ *attains* $\mathcal{A}(T_\star) = 0$.

Remark 3.2 This theorem does not claim that the polygon shrinks to a single point.

We use the analogue of several estimates by Gage-Hamilton [GH] for the curvature and by Girão [Gir] for the weighted curvature. For reader's convenience, we do not omit the proofs except completely the same one.

Lemma 3.3 *There exists a constant* $C_1 = C_1(v(0), \Delta\theta) \ge 0$ *such that*

$$2\tan\frac{\Delta\theta}{2}\sum_{0\le j<n}(\mathrm{D}_+v)_j^2 \le 2\tan\frac{\Delta\theta}{2}\sum_{0\le j<n}v_j^2 + C_1.$$

Proof. It can be shown that the next estimate:

$$\begin{aligned}
&2\tan\frac{\Delta\theta}{2}\frac{d}{dt}\sum_j(v^2 - (\mathrm{D}_+v)^2)_j \\
&\quad = 4\tan\frac{\Delta\theta}{2}\sum_j a_j^{-1}(v_j+b_j)^2(\Delta_\theta v + v)_j^2 \ge 0.
\end{aligned}$$

By the integration of this inequality over $(0,t)$ and putting

$$C_1 \ge \max\left\{-2\tan\frac{\Delta\theta}{2}\sum_j(v(0)^2 - (\mathrm{D}_+v(0))^2)_j,\ 0\right\},$$

we get the assertion. □

One can easily get: $\sum_{j=1}^{[n/2]}\sin\theta_j \le 2\cot(\Delta\theta/2)$, where $[n/2]$ is $n/2$ for n even and $(n-1)/2$ for n odd, since the left-hand side equals to $\cot(\Delta\theta/2)$ for n even and $(1+\sec(\Delta\theta/2))\cot(\Delta\theta/2)/2$ for n odd.

We introduce the median normal velocity which is a similar to the median curvature in [GH] and the median discrete weighted curvature in [Gir].

Definition 3.4 (median normal velocity)
$v_*(t) := \max_{0\le j<n} \min_{j+1\le i\le j+[n/2]} v_i(t)$.

Lemma 3.5 *Assume* (A1) *or* (A1)$'$ *or* (A2). *Fix* $t \in [0, T_\star)$. *If* $\mathcal{A}(t)$ *is bounded away from zero, then* $v_*(t)$ *is bounded.*

Proof. We note that if we assume (A1)$'$, then we have $v_{\min}(0) + b_{\min} > 0$, from which and the lower bound $v_*(t) \ge v_{\min}(0)$ by Lemma 2.3, it follows that $v_* + b_{\min}$ is positive for all $t \ge 0$ and, under the assumption (A1) or (A2), it is always positive.

Now let j_0 be a value of j which attains the maximum of v. A polygon lies between parallel lines whose distance is less than

$$\sum_{j=j_0+1}^{j_0+[n/2]} \sin(\theta_j - \theta_{j_0}) d_j = 2\tan\frac{\Delta\theta}{2} \sum_{j=1}^{[n/2]} \frac{a_{j+j_0}\sin\theta_j}{v_{j+j_0} + b_{j+j_0}}$$
$$\le \frac{2\tan(\Delta\theta/2)a_{\max}}{v_* + b_{\min}} \sum_{j=1}^{[n/2]} \sin\theta_j \le \frac{4a_{\max}}{v_* + b_{\min}}.$$

The diameter is bounded by $\mathcal{L}/2$ and the area is bounded by the width times the diameter:

$$\mathcal{A}(t) \le \frac{2a_{\max}\mathcal{L}(t)}{v_*(t) + b_{\min}}.$$

Hence $v_*(t) \le 2a_{\max}\mathcal{L}(0)/\mathcal{A}(t) - b_{\min}$.

The assertion is proved in a similar way if we assume (A1) or (A2). □

Definition 3.6 Let the entropy be:

$$\mathcal{E}(t) := 2\tan\frac{\Delta\theta}{2} \sum_{0\le j<n} \left(a_j \log\kappa_j(t) + \frac{b_j}{\kappa_j(t)} \right).$$

Lemma 3.7 *Assume* (A1) *or* (A1)$'$ *or* (A2). *Fix* $t \in [0, T_\star)$. *It there exists a constant* $C_* > 0$ *such that* $v_*(\tau) \le C_*$ *for* $0 \le \tau \le t$, *then* $\mathcal{E}(t)$ *is bounded.*

Proof. By using the summation by parts (2.4), one has

$$\dot{\mathcal{E}}(t) = 2\tan\frac{\Delta\theta}{2}\sum_j (v^2 - (\mathrm{D}_+ v)^2)_j.$$

We use the same estimates as in the proof of Girão [Gir] (Section 2, *Fourth*) and have the next estimate:

$$2\tan\frac{\Delta\theta}{2}\sum_j (v^2 - (\mathrm{D}_+ v)^2)_j \le 2n\tan\frac{\Delta\theta}{2}v_*^2 - 2v_*\dot{\mathcal{L}}(t).$$

Hence, $\mathcal{E}(t) \le \mathcal{E}(0) + 2n\tan(\Delta\theta/2)C_*^2 T_\star + 2C_*\mathcal{L}(0)$ holds. □

Lemma 3.8 *Assume* (A1) *or* (A1)$'$ *or* (A2). *If $\mathcal{E}(t)$ is bounded, then for any $\delta > 0$ there exists a constant $C_2 > \max\{1, a_{\max}\} - b_{\min}$ if $b \le 0$ and $C_2 > a_{\max}$ if $b > 0$ such that $v_j(t) \le C_2$ except for θ_j in intervals of length less than δ for $t \in [0, T_\star)$.*

Proof. If $v_j \ge C_2$ for m values of j and $m\Delta\theta \ge \delta$, then

$$\begin{aligned}\mathcal{E}(t) &\ge 2\tan\frac{\Delta\theta}{2}\Bigg(m a_{\min}\log\frac{C_2 + b_{\min}}{a_{\max}} \\ &\qquad\qquad - a_{\max}(n-m)\left|\log\frac{2\tan(\Delta\theta/2)}{\mathcal{L}(0)}\right|\Bigg) + B \\ &\ge \frac{2}{\Delta\theta}\tan\frac{\Delta\theta}{2}\Bigg(\delta a_{\min}\log\frac{C_2 + b_{\min}}{a_{\max}} \\ &\qquad\qquad - a_{\max}(2\pi - \delta)\left|\log\frac{2\tan(\Delta\theta/2)}{\mathcal{L}(0)}\right|\Bigg) + B\end{aligned}$$

where $B = b_{\min}\mathcal{L}(0)$ when $b \le 0$ and

$$\begin{aligned}\mathcal{E}(t) &\ge 2\tan\frac{\Delta\theta}{2}\left(m a_{\min}\log\frac{C_2}{a_{\max}} - (n-m)a_{\max}\left|\log\frac{2\tan(\Delta\theta/2)}{\mathcal{L}(0)}\right|\right) \\ &\ge \frac{2}{\Delta\theta}\tan\frac{\Delta\theta}{2}\left(\delta a_{\min}\log\frac{C_2}{a_{\max}} - (2\pi - \delta)a_{\max}\left|\log\frac{2\tan(\Delta\theta/2)}{\mathcal{L}(0)}\right|\right)\end{aligned}$$

when $b > 0$. This gives a contradiction when C_2 is large. □

Lemma 3.9 *Assume* (A1) *or* (A1)$'$ *or* (A2). *For $t \in [0, T_\star)$, if $v_j(t) \le C_2$ for some constant $C_2 \gg 1$ except for θ_j in intervals of length less than δ and $\delta > 0$ is small enough, then $v_{\max}(t)$ is bounded.*

Proof. As in the proof of Girão [Gir] (Section 2, *Sixth*), we have the next estimate:

$$
\begin{aligned}
v_j &= v_i + \sum_{i \le m < j} (v_{m+1} - v_m) \\
&\le C_2 + \left(\sum_{i \le m < j} \frac{2(1-\cos\Delta\theta)}{2\tan(\Delta\theta/2)} \right)^{1/2} \left(2\tan\frac{\Delta\theta}{2} \sum_{i \le m < j} (\mathrm{D}_+ v)_m^2 \right)^{1/2} \\
&\le C_2 + \sqrt{(j-i)\sin\Delta\theta} \left(2\tan\frac{\Delta\theta}{2} \sum_{0 \le m < n} v_m^2 + C_1 \right)^{1/2} \\
&\le C_2 + \sqrt{\delta} \left(2n\tan\frac{\Delta\theta}{2} v_{\max}^2 + C_1 \right)^{1/2} \\
&\le C_2 + \sqrt{\delta} \left(\sqrt{2\sqrt{2}\pi} v_{\max} + \sqrt{C_1} \right)
\end{aligned}
$$

since $v_j \le C_2$ and $\theta_i - \theta_j \le \delta$. Here we have used Lemma 3.3.

Hence $(1 - \sqrt{2\sqrt{2}\pi\delta}) v_{\max} \le C_2 + \sqrt{C_1 \delta}$ holds and we get $v_{\max} \le (C_2 + \sqrt{C_1\delta})/(1 - \sqrt{2\sqrt{2}\pi\delta})$ for small δ. □

Proof of Theorem 3.1. Suppose that a side of $\mathcal{P}_t$ disappears for $t < T_\star$ where $T_\star$ attains $\mathcal{A}(T_\star) = 0$. Put t_0 as the first time that happens (n.b. $t_0 > 0$ is clear). Then $\mathcal{A}(t) > 0$ for $0 \le t \le t_0$ and the estimates above imply that $\sup_{0 \le t \le t_0} v_{\max}(t)$ is bounded, so $d_{\min}(t_0) > 0$. This is a contradiction. Hence the assertion holds. □

We are now ready to present of the proof of Theorems A and B.

Proof of Theorem A *and* B. By Theorem 3.1, we have $\mathcal{A}(T_\star) = 0$. If n is odd, then $\mathcal{L}(T_\star) = 0$ since the angle between two adjacent sides of polygon is always $\pi - \Delta\theta$ and we have no two sides which are parallel to each other. Suppose that n is even. Then the jth side and the $(j + n/2)$th side are parallel. Let w_j be the distance between the jth and the $(j + n/2)$th side and we have

$$
w_m = \sum_{j=m+1}^{m+n/2} \sin(\theta_j - \theta_m)\, d_j = \sum_{j=1}^{n/2} \sin\theta_j\, d_{j+m}, \quad \text{or}
$$

$$
w_m = - \sum_{j=n/2+1}^{n} \sin\theta_j\, d_{j+m}.
$$

Therefore,

$$2w_m = \sum_j |\sin\theta_j|\, d_{j+m} = 2\tan\frac{\Delta\theta}{2}\sum_j |\sin\theta_j|\, \frac{a_{j+m}}{v_{j+m}+b_{j+m}}.$$

Then we have

$$\begin{aligned}
\dot{w}_m &= -\tan\frac{\Delta\theta}{2}\sum_j |\sin\theta_j|\,(\Delta_\theta v + v)_{j+m} \\
&= -\tan\frac{\Delta\theta}{2}\sum_j v_j\,(\Delta_\theta|\sin\theta| + |\sin\theta|)_{j-m} \\
&= -(v_m + v_{m+n/2})
\end{aligned}$$

since

$$(\Delta_\theta|\sin\theta| + |\sin\theta|)_i = \begin{cases} \cot(\Delta\theta/2) & \text{if } i = 0,\ n/2; \\ 0 & \text{if otherwise.} \end{cases}$$

Proof of Theorem A. Put $C > 0$ such as $\dot{\mathcal{A}}(t) \geq -2\tan(\Delta\theta/2)\sum_j a_j + b_{\min}\mathcal{L}(0) =: -C$. By Theorem 3.1, we have $\mathcal{A}(t) \leq C(T_\star - t)$. Then $\dot{w}_m \leq -v_m \leq -2\tan(\Delta\theta/2)a_m d_m^{-1}$ and $\mathcal{A}(t) \geq w_m d_m/2$ yield

$$\frac{\dot{w}_m}{w_m} \leq -a_m\frac{\tan(\Delta\theta/2)}{\mathcal{A}(t)} \leq -a_m\frac{\tan(\Delta\theta/2)}{C(T_\star - t)}.$$

Hence, by integration over $(0,t)$, we have

$$w_m(t) \leq w_m(0)\left(\frac{T_\star - t}{T_\star}\right)^{a_m\tan(\Delta\theta/2)/C}$$

and $w_m(T_\star) = 0$ for all m. Then $\mathcal{L}(T_\star) = 0$ is concluded. □

Proof of Theorem B. Since $b > 0$, it holds that $\mathcal{A}(t) \leq C(T_\star - t)$ for a positive constant $C > 0$. By the condition (A2), we have $v_j(t) \geq b_{\max}$ and

$$-v_m = -a_m\frac{2\tan(\Delta\theta/2)}{d_m} + b_{\max} \leq -a_m\frac{2\tan(\Delta\theta/2)}{d_m} + v_{m+n/2}.$$

Then $\dot{w}_m \leq -2\tan(\Delta\theta/2)a_m d_m^{-1}$ and $\mathcal{A}(t) \geq w_m d_m/2$ provide the point-extinction in a way similar to the proof of Theorem A. □□

4. Geometric expansion (proof of Theorem C)

We shall prove Theorem C by the super- and subsolution method or the comparison principle.

Lemma 4.1 *Let v be a solution of Problem 1. Under the Assumption (A3), the functions*

$$v_j^u(t) = \frac{a_j}{\eta t + \kappa_{\max}(0)^{-1}} - b_j \quad and \quad v_j^l(t) = \frac{a_j}{\mu t + \kappa_{\min}(0)^{-1}} - b_j$$

are super- and subsolutions of Problem 1, respectively. Here μ is a positive constant satisfying $\mu \geq (\Delta_\theta b + b)_{\max}$ and η is defined in (A3).

Proof. By Assumption (A3): $\kappa_{\max}(0)(\Delta_\theta a + a)_{\max} - (\Delta_\theta b + b)_{\min} \leq -\eta$, we have

$$\dot{v}_j^u = -\eta a_j^{-1}(v_j^u + b_j)^2 \geq a_j^{-1}(v_j^u + b_j)^2(\Delta_\theta v^u + v^u)_j.$$

Here v_j^u is a supersolution of Problem 1 since $v_j^u(0) \geq v_j(0)$ holds.

In the same way, one can prove that v_j^l is a subsolution of Problem 1 by Assumption (A3): $(\Delta_\theta a + a)_j \geq 0$ and the assumption $\mu \geq (\Delta_\theta b + b)_{\max}$. □

We are now ready to present the proof of Theorem C.

Proof of Theorem C. By Lemma 4.1, we can estimate $\dot{\mathcal{L}}(t)$ such as

$$\frac{\sum_j a_j}{\mu t + \kappa_{\min}(0)^{-1}} - \sum_j b_j \leq \frac{\dot{\mathcal{L}}(t)}{-2\tan(\Delta\theta/2)} \leq \frac{\sum_j a_j}{\eta t + \kappa_{\max}(0)^{-1}} - \sum_j b_j.$$

Integration over $(0, t)$ yields

$$\frac{\mathcal{L}(t)}{2\tan(\Delta\theta/2)} \leq \frac{\mathcal{L}(0)}{2\tan(\Delta\theta/2)} - \frac{\sum_j a_j}{\mu} \log(\mu\kappa_{\min}(0)t + 1) + \sum_j b_j t$$

and

$$\frac{\mathcal{L}(t)}{2\tan(\Delta\theta/2)} \geq \frac{\mathcal{L}(0)}{2\tan(\Delta\theta/2)} - \frac{\sum_j a_j}{\eta} \log(\eta\kappa_{\max}(0)t + 1) + \sum_j b_j t.$$

Let the jth support function be

$$h_j(t) := \langle \boldsymbol{x}_j, -\boldsymbol{n}_j \rangle.$$

Then $v_j(t) = -\dot{h}_j(t)$. By Lemma 2.3 (3) (4) and Lemma 4.1, we have $v_j^l \le v_j \le v_j^u$, i.e.

$$\frac{a_j}{\mu t + \kappa_{\min}(0)^{-1}} - b_j \le -\dot{h}_j(t) \le \frac{a_j}{\eta t + \kappa_{\max}(0)^{-1}} - b_j.$$

Integration of this inequality over $(0, t)$ yields

$$\begin{aligned} -\frac{a_j}{\eta}\log(\eta\kappa_{\max}(0)t + 1) &\le h_j(t) - b_j t - h_j(0) \\ &\le -\frac{a_j}{\mu}\log(\mu\kappa_{\min}(0)t + 1) \le 0. \end{aligned} \tag{4.1}$$

Therefore one has

$$\lim_{t\to\infty}\frac{h_j(t)}{t} = b_j, \quad 0 \le j < n. \tag{4.2}$$

By using (2.7) and the geometric relation $d_j(t) = 2\tan(\Delta\theta/2)(\Delta_\theta h(t) + h(t))_j$, we have the upper bound of the area $\mathcal{A}(t)$ as follows:

$$\begin{aligned} \mathcal{A}(t) &= \frac{1}{2}\sum_j d_j(t)h_j(t) \\ &\le \frac{1}{2}\sum_j d_j(t)(b_j t + h_j(0)) \\ &= \tan\frac{\Delta\theta}{2}t\sum_j h_j(t)(\Delta_\theta b + b)_j + \tan\frac{\Delta\theta}{2}\sum_j h_j(t)(\Delta_\theta h(0) + h(0))_j \\ &\le \tan\frac{\Delta\theta}{2}t\sum_j(\Delta_\theta b + b)_j(b_j t + h_j(0)) + \frac{1}{2}\sum_j d_j(0)(b_j t + h_j(0)) \\ &= \tan\frac{\Delta\theta}{2}t^2\sum_j b_j(\Delta_\theta b + b)_j + \tan\frac{\Delta\theta}{2}t\sum_j b_j(\Delta_\theta h(0) + h(0))_j \\ &\qquad + \frac{t}{2}\sum_j d_j(0)b_j + \frac{1}{2}\sum_j d_j(0)h_j(0) \\ &\le \tan\frac{\Delta\theta}{2}t^2\sum_j b_j(\Delta_\theta b + b)_j + b_{\max}\mathcal{L}(0)t + \mathcal{A}(0). \end{aligned}$$

Here we have used (4.1) for the upper bound of $h_j(t)$ twice, the summation by parts (2.4) several times and the assumption (A3). In a similar way, we obtain the lower bound of the area $\mathcal{A}(t)$:

$$\begin{aligned}\mathcal{A}(t) \geq \tan\frac{\Delta\theta}{2}t^2\sum_j b_j(\Delta_\theta b+b)_j \\ -2\tan\frac{\Delta\theta}{2}t\frac{\log(\eta\kappa_{\max}(0)t+1)}{\eta}\sum_j a_j(\Delta_\theta b+b)_j \\ +\frac{\mathcal{L}(0)}{2}\left(b_{\min}t-\frac{a_{\max}}{\eta}\log(\eta\kappa_{\max}(0)t+1)\right).\end{aligned}$$

Therefore it holds that the limits: $\mathcal{L}(t)$, $\mathcal{A}(t)\to\infty$ as $t\to\infty$. Moreover, one can easily calculate the limit of isoperimetric ratio $\mathcal{I}(t)=\mathcal{L}(t)^2/(4n\tan(\Delta\theta/2)\mathcal{A}(t))$ such as

$$\lim_{t\to\infty}\mathcal{I}(t)=\frac{(\sum_j b_j)^2}{n\sum_j b_j(\Delta_\theta b+b)_j}.$$

This limit and (4.2) assert that a rescaled solution polygon $\mathcal{P}_t/t$ converges to the boundary of the Wulff shape $\partial\mathcal{W}_b$ in the Hausdorff metric as $t\to\infty$.

In particular, if b is a constant, then $\lim_{t\to\infty}\mathcal{I}(t)=1$ and the Bonnesen's type inequality (see [Eg]) provides that $\mathcal{P}_t/t$ converges to a regular polygon in the Hausdorff metric. This completes the proof of Theorem C. □

5. Lower bound of the blow-up time (proof of Theorems D and E)

We will use Schwarz inequality twice to obtain a lower bound of blow-up time. A similar idea was used in Giga-Yama-uchi [GY] to give a bound for the mean curvature flow in higher dimension.

Proof of Theorem D *and* E. By Schwarz inequality, we have

$$\begin{aligned}-\dot{\mathcal{A}}(t) &= 2\tan\frac{\Delta\theta}{2}\sum_j a_j-2\tan\frac{\Delta\theta}{2}\sum_j b_j\kappa_j^{-1} \\ &= 2\tan\frac{\Delta\theta}{2}\sum_j a_j^{1/2}\kappa_j^{1/2}a_j^{1/2}\kappa_j^{-1/2}-2\tan\frac{\Delta\theta}{2}\sum_j b_j\kappa_j^{-1}\end{aligned}$$

$$\leq \sqrt{a_{\max}}\Big(2\tan\frac{\Delta\theta}{2}\mathcal{L}(t)\sum_j b_j - \frac{1}{2}\frac{d}{dt}\mathcal{L}(t)^2\Big)^{1/2} \\ - 2\tan\frac{\Delta\theta}{2}\sum_j b_j\kappa_j^{-1}.$$

Integration the above inequality over $(0,T_\star)$, the point-extinction and Schwarz inequality yield

$$\begin{aligned}\mathcal{A}(0) &\leq \sqrt{a_{\max}}\int_0^{T_\star}\left(2\tan\frac{\Delta\theta}{2}\mathcal{L}(t)\sum_j b_j - \frac{d}{dt}\frac{\mathcal{L}(t)^2}{2}\right)^{1/2} dt - B \\ &\leq \sqrt{a_{\max}}\left(\int_0^{T_\star} dt\right)^{1/2} \\ &\qquad \left(\int_0^{T_\star}\left(2\tan\frac{\Delta\theta}{2}\mathcal{L}(t)\sum_j b_j - \frac{d}{dt}\frac{\mathcal{L}(t)^2}{2}\right)dt\right)^{1/2} - B \\ &= \sqrt{a_{\max}}\sqrt{T_\star}\left(\frac{\mathcal{L}(0)^2}{2} + 2\tan\frac{\Delta\theta}{2}\sum_j b_j\int_0^{T_\star}\mathcal{L}(t)\,dt\right)^{1/2} - B\end{aligned}$$

where $B = 2\tan(\Delta\theta/2)\int_0^{T_\star}\sum_j b_j/\kappa_j\,dt$.

Proof of Theorem D. Since $b \leq 0$, we get

$$\mathcal{A}(0) \leq \sqrt{\frac{a_{\max}}{2}}\mathcal{L}(0)\sqrt{T_\star} - b_{\min}\mathcal{L}(0)T_\star.$$

Assumption $b \not\equiv 0$ means $b_{\min} < 0$. Hence the solution of this inequality provides the lower bound of $T_\star$. □

Proof of Theorem E. Since $b > 0$, we get

$$\mathcal{A}(0)^2 \leq a_{\max}T_\star\left(2\tan\frac{\Delta\theta}{2}\sum_j b_j\mathcal{L}(0)T_\star + \frac{1}{2}\mathcal{L}(0)^2\right).$$

The solution of this inequality provides the lower bound of $T_\star$. □□

Appendices

A. Gradient flow of a total interfacial energy

If the interfacial energy on the curve Γ is distributed uniformly as constant 1, then the total interfacial energy of Γ is given by

$$E[\Gamma] = \int_\Gamma 1 ds = \int_{\boldsymbol{T}} |\boldsymbol{x}_\theta| d\theta,$$
$$(ds = |\boldsymbol{x}_\theta| d\theta : \text{ the arc-length parameter}),$$

where $\boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z}$ is the flat torus. The first variation of E has the form:

$$\frac{\delta E[\Gamma^\varepsilon_{\boldsymbol{z}}]}{\delta \boldsymbol{z}} := \frac{d}{d\varepsilon} E[\Gamma^\varepsilon_{\boldsymbol{z}}]\bigg|_{\varepsilon=0} = \int_\Gamma \langle -\boldsymbol{t}_s, \boldsymbol{z}\rangle ds$$

where $\Gamma^\varepsilon_{\boldsymbol{z}} = \{\vec{x} \in \boldsymbol{R}^2 \mid \vec{x} = \boldsymbol{x} + \varepsilon \boldsymbol{z}(\theta),\ \boldsymbol{x} \in \Gamma_t,\ \theta \in \boldsymbol{T}\}$. Hence the gradient of E with L^2-metric is $\operatorname{grad} E[\Gamma] = -\boldsymbol{t}_s$. Then Frenet-Serret formula $\boldsymbol{t}_s = \kappa \boldsymbol{n}$ yields $\boldsymbol{x}_t = -\operatorname{grad} E[\Gamma] = \kappa \boldsymbol{n}$, i.e. $v = \langle \boldsymbol{x}_t, \boldsymbol{n}\rangle = \kappa$. This equation is called the classical curve-shortening equation, and is investigated by many authors (see [GH, Gry, AV, And] and references therein).

If the interfacial energy $f = f(\boldsymbol{n})$ is a positively homogeneous of degree one in $C^2(\boldsymbol{R}^2 \backslash \{\boldsymbol{0}\})$, then the gradient flow of total interfacial energy $E[\Gamma] = \int_\Gamma f(\boldsymbol{n}) ds$ is computed as $\boldsymbol{x}_t = {}^t(\kappa \operatorname{Hess} f(\boldsymbol{n})\boldsymbol{t})^\perp$ (see Elliott [E]). Here ${}^t(x_1, x_2)^\perp = {}^t(-x_2, x_1)$. Then we obtain $v = \langle \boldsymbol{x}_t, \boldsymbol{n}\rangle = \langle (\kappa \operatorname{Hess} f(\boldsymbol{n})\boldsymbol{t})^\perp, \boldsymbol{n}\rangle = \kappa \langle \operatorname{Hess} f(\boldsymbol{n})\boldsymbol{t}, \boldsymbol{t}\rangle$. Moreover, if we put $f(\theta) = f(\boldsymbol{n}(\theta))$, then we get the weighted curvature flow $v = \omega = (f + f'')\kappa$ since $\langle \operatorname{Hess} f(\boldsymbol{n})\boldsymbol{t}, \boldsymbol{t}\rangle = f + f''$ holds. The function $f + f''$ is the inverse of curvature on the boundary $\partial \mathcal{W}_f$ of the Wulff shape $\mathcal{W}_f$. Indeed, the locus of the boundary of the Wulff shape $\partial \mathcal{W}_f$ is

$$\partial \mathcal{W}_f = \{\vec{x} \in \boldsymbol{R}^2 \mid \vec{x} = \boldsymbol{y}(\theta) = -f(\theta)\boldsymbol{n}(\theta) + f'(\theta)\boldsymbol{t}(\theta),\ \ \theta \in \boldsymbol{T}\},$$

and then its curvature is $\kappa_{\mathcal{W}} = -\langle \boldsymbol{y}_\theta, \boldsymbol{y}^\perp_{\theta\theta}\rangle |\boldsymbol{y}_\theta|^{-3} = (f + f'')^{-1}$.

B. Discrete curvature

B.1. Characterization

Let $\mathcal{P}$ be an $\mathcal{N}_\star$-admissible piecewise linear curve. Each side of $\mathcal{P}$ has zero curvature, if the curvature is defined in the standard way based on the Frenet-Serret's formula. However, the curvature of smooth curves can alternatively be defined as follows: *1.* the negative of the gradient of length (see Appendix A); *2.* the negative of derivative of the length w.r.t. signed area for smooth deformations of the curve. In these sense, the analogous quantity for $\mathcal{P}$ can be defined. These are the two of characterizations of κ_j. See, e.g., Rybka [Ry].

Now we present the third characterization of the curvature as the following. Since $\mathcal{P}$ is an $\mathcal{N}_\star$-admissible piecewise linear curve, the inverse of the discrete curvature is given by $1/\kappa_j = \chi_j \times d_j/2\tan(\Delta\theta/2)$. In other words, we have the next relation:

$$1/\text{discrete curvature} = \chi_j \times \text{radius of the largest}$$
$$\text{(inscribed circle of) inscribed regular polygon.}$$

This relation is a discretized version of the inverse of the usual curvature:

$$1/\text{curvature} = \text{sign} \times \text{radius of the largest inscribed circle.}$$

In this sense each side of $\mathcal{P}$ does have nonzero curvature κ_j. See Figure 2.

B.2. Discrete curvature κ_j vs. curvature $\kappa(\theta_j)$

Suppose a subarc of a curve Γ, say Γ_{sub}, is Gauss-parametrized and strictly convex as follows:

$$\Gamma_{\text{sub}} = \{\vec{x} \in \boldsymbol{R}^2 \mid \vec{x} = \boldsymbol{x}(\theta),\ \ \theta \in [\theta_{j-1}, \theta_{j+1}],\ \ \theta_{j-1} < \theta_j < \theta_{j+1}\}.$$

We define a part of circumscribed piecewise linear curve, say $\mathcal{P}_{\text{sub}}$, of Γ_{sub} such as

$$\Gamma_{\text{sub}} \cap \mathcal{P}_{\text{sub}} = \{\boldsymbol{x}(\theta_{j-1}), \boldsymbol{x}(\theta_j), \boldsymbol{x}(\theta_{j+1})\}.$$

See Figure 1.

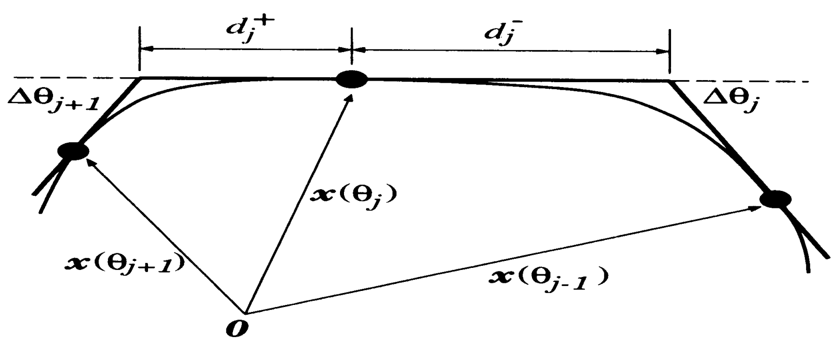

Fig. 1. $\mathcal{P}_{\text{sub}}$ (outside: the part of circumscribed piecewise linear curve of Γ_{sub}), and Γ_{sub} (inside: the subarc of Γ).

We call the side including $\boldsymbol{x}(\theta_j)$ of $\mathcal{P}_{\text{sub}}$ the jth side. The length of the jth side is denoted by d_j. The jth side is a part of tangent line which has the orientation $\boldsymbol{t}(\theta_j) = {}^t(-\sin\theta_j, \cos\theta_j)$ since the inward normal at $\boldsymbol{x}(\theta_j)$ is $\boldsymbol{n}(\theta_j)$. We note that the transition number is $\chi_j = +1$.

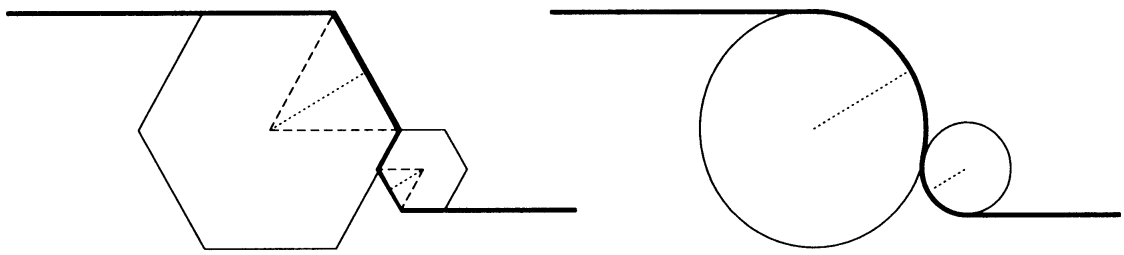

Fig. 2. Symbolic figure to compare the discrete curvature and the usual curvature. Thick solid = piecewise linear curve $\mathcal{P}$ (left) and curve Γ (right), Solid = the largest inscribed polygon (left), and the largest inscribed circle (right), Dashed = radius (both), Long dashed = half of diagonal (left).

Let $\kappa(\theta_j)$ be the curvature at $\boldsymbol{x}(\theta_j) \in \Gamma_{\mathrm{sub}}$ and $\kappa_j = \gamma_j / d_j$ the discrete curvature defined on the jth side of $\mathcal{P}_{\mathrm{sub}}$.

The relation between κ_j and $\kappa(\theta_j)$ is calculated as follows (cf. Section 3 in [Gir]). First, we decompose the length of the jth side as $d_j = d_j^+ + d_j^-$ (see Figure 1). Next, we obtain

$$d_j^+ = \frac{1}{\kappa(\theta_j)} \left(\frac{\Delta\theta_{j+1}}{2} - \frac{(\Delta\theta_{j+1})^2}{6} \frac{\kappa'(\theta_j)}{\kappa(\theta_j)} + \mathrm{O}((\Delta\theta_{j+1})^3) \right)$$

by the Taylor expansion of

$$\boldsymbol{x}(\theta_{j+1}) - \boldsymbol{x}(\theta_j) = \int_{\theta_j}^{\theta_{j+1}} \frac{\boldsymbol{t}(\theta)}{\kappa(\theta)} d\theta = \int_0^{\Delta\theta_{j+1}} \frac{\boldsymbol{t}(\theta_j + \mu)}{\kappa(\theta_j + \mu)} d\mu$$

around θ_j and the decomposition:

$$d_j^+ = \langle \boldsymbol{x}(\theta_{j+1}) - \boldsymbol{x}(\theta_j), \boldsymbol{t}_j - \cot \Delta\theta_{j+1} \boldsymbol{n}_j \rangle.$$

In the same way, we obtain

$$d_j^- = \frac{1}{\kappa(\theta_j)} \left(\frac{\Delta\theta_j}{2} + \frac{(\Delta\theta_j)^2}{6} \frac{\kappa'(\theta_j)}{\kappa(\theta_j)} + \mathrm{O}((\Delta\theta_j)^3) \right).$$

Therefore we have

$$\kappa_j = \frac{\gamma_j}{d_j^+ + d_j^-} = \kappa(\theta_j) + \frac{\kappa'(\theta_j)}{3} (\Delta\theta_{j+1} - \Delta\theta_j) + \mathrm{O}((\Delta\theta_{\max})^2) \tag{B.1}$$

since

$$\begin{aligned}\gamma_j &= \tan\frac{\Delta\theta_{j+1}}{2} + \tan\frac{\Delta\theta_j}{2}\\ &= \frac{\Delta\theta_{j+1}+\Delta\theta_j}{2} + \frac{(\Delta\theta_{j+1})^3+(\Delta\theta_j)^3}{24} + \mathrm{O}((\Delta\theta_{j+1})^5+(\Delta\theta_j)^5)\end{aligned}$$

holds. Here $\Delta\theta_{\max} = \max\{\Delta\theta_{j+1}, \Delta\theta_j\}$ and $\mathrm{O}(\cdot)$ in equation (B.1) depends on

$$\sum_{1\le\ell\le2}\max_{\theta\in[\theta_{j-1},\theta_{j+1}]}\left|\frac{d^\ell}{d\theta^\ell}\kappa(\theta)\right| \quad\text{and}\quad \min_{\theta\in[\theta_{j-1},\theta_{j+1}]}\kappa(\theta).$$

Hence, it is reasonable to treat $\mathcal{N}_\star$-admissible piecewise linear curves from a numerical point of view.

C. Crystalline curvature

Let f be a crystalline energy and $\mathcal{P}_t$ an $\mathcal{N}$-admissible piecewise linear curve. Then the Wulff shape $\mathcal{W}_f$ is a polygon and the distance between the origin and the jth side (which has the orientation $\boldsymbol{n}_j \in \mathcal{N}$) is f_j. See Figure 3.

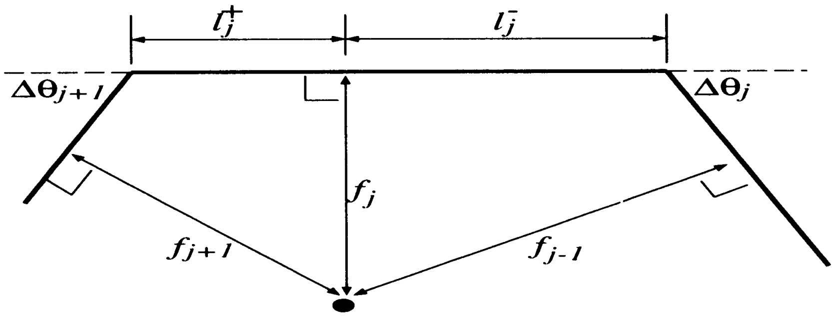

Fig. 3. The jth side of the Wulff shape $\mathcal{W}_f$ if f is a crystalline energy.

If we decompose the length of the jth side such as $l(\boldsymbol{n}_j) = l_j^+ + l_j^-$ (see Figure 3 again), then we get $l(\boldsymbol{n}_j) = \gamma_j(f+\Delta_\theta f)_j$ since

$$\begin{aligned}l_j^+ &= f_{j+1}\sin\Delta\theta_{j+1} - \frac{f_j - f_{j+1}\cos\Delta\theta_{j+1}}{\tan\Delta\theta_{j+1}}, \quad\text{and}\\ l_j^- &= f_{j-1}\sin\Delta\theta_j - \frac{f_j - f_{j-1}\cos\Delta\theta_j}{\tan\Delta\theta_j}\end{aligned}$$

hold (awake around equation (1.2) again).

The discrete curvature of the polygon $\partial\mathcal{W}_f$ is given as $\gamma_j/l(\boldsymbol{n}_j) = (f + \Delta_\theta f)_j^{-1}$. Hence the crystalline curvature $\omega_j(t)$ is

$$\begin{aligned}\omega_j(t) &= \frac{\text{discrete curvature of } \mathcal{P}_t}{\text{discrete curvature of polygon } \partial\mathcal{W}_f} \\ &= \frac{\kappa_j(t)}{(f + \Delta_\theta f)_j^{-1}} = (f + \Delta_\theta f)_j \kappa_j(t).\end{aligned}$$

This is a discrete version of weighted curvature $\omega(\theta, t)$:

$$\begin{aligned}\omega(\theta, t) &= \frac{\text{curvature of } \Gamma_t}{\text{curvature of } \partial\mathcal{W}_f} = \frac{\kappa(\theta, t)}{(f(\theta) + f''(\theta))^{-1}} \\ &= (f(\theta) + f''(\theta))\kappa(\theta, t)\end{aligned}$$

at the point (θ_j, t) if f is smooth. Namely, the crystalline curvature is a discrete weighted curvature.

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