Hokkaido Mathematical Journal Vol. 30 (2001) p. 163–176

Normal extensions and induced characters of 2-groups M_n

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(Received August 27, 1999)

Abstract. Let D_n, Q_n and SD_n be the dihedral group, the generalized quaternion group and the semidihedral group of order 2^{n+1}, respectively. Let C_n be the cyclic 2-group of order 2^n. As is well-known these four kinds of 2-groups play an important role in character theory of 2-groups. Let ϕ be a faithful irreducible character of $H=D_n$, Q_n, SD_n or C_n. In [3] we determined all the 2-groups G such that H is a normal subgroup of G and the induced character ϕ^G is irreducible. There exist other nonabelian 2-groups M_n with a cyclic subgroup of index 2. All the faithful irreducible characters of M_n are algebraically conjugate to each other as in H. The purpose of the paper is to determine all the 2-groups G with a normal subgroup isomorphic to M_n such that ϕ^G is irreducible for a faithful irreducible characters ϕ of the normal subgroup.

Key words: 2-group, induced character, group extension.

1. Introduction

Let D_n, Q_n and SD_n be the dihedral group, the generalized quaternion group and the semidihedral group of order 2^{n+1}, respectively. Let C_n be the cyclic 2-group of order 2^n. We denote the set of complex irreducible characters of G by $\mathrm{Irr}(G)$ and the set of faithful irreducible characters by $\mathrm{FIrr}(G)$. From now on a character means a complex character.

Let $H=D_n$, Q_n or SD_n and $\phi\in\mathrm{FIrr}(H)$. These 2-groups H are known to have many remarkable properties among all 2-groups (cf. [4, Theorem]). We showed the following in [4].

Theorem ([4, Theorem 1]) *Let $H=Q_n$ or D_n or SD_n. Let G be a 2-group which contains H with $[G:H]=2^r$ $(r\geq 1)$. Let ϕ be a faithful irreducible character of H. Suppose that the induced character $\chi=\phi^G$ is irreducible. Then $r\leq n-2$, $\mathbf{Q}(\chi)=\mathbf{Q}(\zeta_{2^{n-r}}+\zeta_{2^{n-r}}^{-1})$, where $\zeta_{2^{n-r}}$ is a primitive 2^{n-r}th root of unity, and $[\mathbf{Q}(\phi):\mathbf{Q}(\chi)]=[G:H]=2^r$. Moreover, the values of ϕ^G depend only on r.*

The original idea of Theorem is due to Yamada [6, Theorem 1]. Accord-

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1991 Mathematics Subject Classification : Primary 20C15; Secondary 20D15.

ing to Theorem, we may say that the induced character ϕ^G of $\phi \in \mathrm{FIrr}(H)$, where $H = D_n$, Q_n or SD_n, has remarkable properties. So in [4] we determined all the 2-groups G such that $G \supset H$, $[G : H] = 2^t$ $(t = 1, 2)$ and $\phi^G \in \mathrm{Irr}(G)$. It follows that a 2-group G is uniquely determined if G contains a normal subgroup H of index 2 or 4, and ϕ^G is irreducible.

There exist other nonabelian 2-groups M_n with a cyclic subgroup of index 2. The 2-group M_n has faithful irreducible characters ϕ, which are algebraically conjugate to each other like D_n, Q_n and SD_n. In [2] in order to compare the results of $H = D_n$, Q_n or SD_n with ones of other 2-groups we determined all the 2-groups G for $H = M_n$ and $\phi \in \mathrm{FIrr}(H)$ such that $G \supset H$, $[G : H] = 2$ and $\phi^G \in \mathrm{Irr}(G)$. The results obtained was in contrast to ones for $H = D_n$, Q_n or SD_n.

Furthermore in [3] we determined all the 2-groups G for $H = D_n$, Q_n, SD_n or C_n and $\phi \in \mathrm{FIrr}(H)$ with $G \rhd H$ and $\phi^G \in \mathrm{Irr}(G)$. It is easily seen that normal extension 2-groups of $H = D_n$, Q_n or SD_n (by which we mean extension 2-groups G with $G \rhd H$) are relative to normal extension 2-groups of $H = C_n$. For $H = D_n$, Q_n or SD_n have characteristic subgroup C_n and $\phi \in \mathrm{FIrr}(H)$ is induced from $\eta \in \mathrm{FIrr}(C_n)$. Indeed we showed the following theorem. The 2-groups in the following theorem are defined in the next section and we use $G_0(M_n)^-$ instead of SD_n according to [3].

Theorem 1 ([3, Theorems 4 and 7]) *Let $H = D_n$, Q_n, $G_0(M_n)^-$ or C_n with $n \geq 3$ and $\phi \in \mathrm{FIrr}(H)$. Let G be a 2-group which contains H as a normal subgroup of index 2^t $(t \geq 1)$. Suppose that the induced character ϕ^G is irreducible. Then $t \leq n-2$ and the following hold:*

(1) *when $H = D_n$, $G \cong G_t(D_n)$,*
(2) *when $H = Q_n$, $G \cong G_t(Q_n)$,*
(3) *when $H = G_0(M_n)^-$, $G \cong G_t(D_n)$ or $G_t(Q_n)$,*
(4) *when $H = C_n$, $G \cong D_n$, Q_n, $G_t(D_n)$, $G_t(Q_n)$, $G_{t-1}(M_n)^+$ or $G_{t-1}(M_n)^-$.*

In particular, there exists a unique 2-group G for $H = D_n$ or Q_n $(n \geq 3)$ and each t $(1 \leq t \leq n-2)$.

For a given group $H = D_n$ or Q_n and $\phi \in \mathrm{FIrr}(H)$, Theorem 1 implies that a 2-group G is uniquely determined such that H is normal in G, $[G : H] = 2^t$ and $\phi^G \in \mathrm{Irr}(G)$ for any t $(1 \leq t \leq n-2)$. This fact is different from results of $G_0(M_n)^-$ $(= SD_n)$ and C_n, and may characterize D_n and Q_n in a sense. For example, let H be a 2-group with faithful irreducible characters which are algebraically conjugate to each other and $\phi \in \mathrm{FIrr}(H)$. Then the following may hold: *If there exists a unique* 2-group G *for any possible integer* t *such that* $G \triangleright H$, $[G : H] = 2^t$ *and* $\phi^G \in \mathrm{Irr}(G)$, *then* G *is isomorphic* D_n *or* Q_n.

Now we have

Problem Let H be a 2-group with faithful irreducible characters which are algebraically conjugate to each other. Let ϕ be any faithful irreducible character of H.

(I) Characterize a 2-group G such that H is a normal subgroup of G and the induced character ϕ^G is irreducible.

(II) Determine all the 2-groups G such that H is a normal subgroup of G and the induced character ϕ^G is irreducible.

The purpose of the paper is to determine completely all the 2-groups G for $H = M_n$ and $\phi \in \mathrm{FIrr}(H)$ with $G \triangleright H$ and $\phi^G \in \mathrm{Irr}(G)$. In fact M_n satisfies the condition about a 2-group H on Problem. The results are very complicated and different from ones of D_n and Q_n (and $G_0(M_n)^-$ $(= SD_n)$). For example in this case G/H is not always cyclic and there exist many such 2-groups G. All the 2-groups G for $H = M_n$ are in Theorems 5, 6 and 8.

Remark 1 From Theorem 6 it follows that [2, Theorem 6] for M_3 is incorrect. $G^{(8-1)}$ and $G^{(8-2)}$ should be removed from [2, Theorem 6].

Remark 2 Let H be a 2-group with $|Z(H)| = 2$ and $\phi \in \mathrm{FIrr}(H)$. Suppose that $\phi(x) = 0$ for all $x \notin Z(H)$. Then from [5, Lemma 2.1] it follows that ϕ is the unique faithful irreducible character of H. It is easy to see that this 2-group H satisfies the condition on Problem and there exists no 2-group G with a normal subgroup H and $\phi^G \in \mathrm{Irr}(G)$. For example, so are extra special 2-groups.

2. Preliminaries

We define notation of some 2-groups which are dealt with in the paper.

C_n is the cyclic 2-group of order 2^n : $C_n = \langle\, a \,\rangle$.

D_n and Q_n are the dihedral group and the generalized quaternion group, respectively, of order 2^{n+1} $(n \geq 2)$:

$$\begin{aligned} D_n &= \langle\, a, b \mid a^{2^n} = 1,\ b^2 = 1,\ bab^{-1} = a^{-1} \,\rangle, \\ Q_n &= \langle\, a, b \mid a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ bab^{-1} = a^{-1} \,\rangle. \end{aligned}$$

We define 2-groups $G_t(D_n)$ $(1 \leq t \leq n-2)$, $G_t(Q_n)$ $(1 \leq t \leq n-2)$, $G_t(M_n)^+$ $(0 \leq t \leq n-3)$ and $G_t(M_n)^-$ $(0 \leq t \leq n-3)$ as follows:

$$G_t(D_n) = \left\langle a,\, b,\, x \;\middle|\; \begin{array}{l} a^{2^n} = 1,\ b^2 = 1,\ x^{2^t} = 1, \\ bab^{-1} = a^{-1},\ xax^{-1} = a^{1+2^{n-t}},\ xbx^{-1} = b \end{array} \right\rangle,$$

$$G_t(Q_n) = \left\langle a,\, b,\, x \;\middle|\; \begin{array}{l} a^{2^n} = 1,\ b^2 = a^{2^{n-1}},\ x^{2^t} = 1, \\ bab^{-1} = a^{-1},\ xax^{-1} = a^{1+2^{n-t}},\ xbx^{-1} = b \end{array} \right\rangle,$$

$$G_t(M_n)^+ = \langle\, a,\, b \mid a^{2^n} = 1,\ b^{2^{t+1}} = 1,\ bab^{-1} = a^{1+2^{n-t-1}} \,\rangle,$$

$$G_t(M_n)^- = \langle\, a,\, b \mid a^{2^n} = 1,\ b^{2^{t+1}} = 1,\ bab^{-1} = a^{-1+2^{n-t-1}} \,\rangle.$$

We note that $G_0(M_n)^-$ is the semidihedral 2-group SD_n and $G_0(M_n)^+$ is M_n. Namely if a nonabelian 2-group G of order 2^{n+1} has a cyclic subgroup of index 2, then G is isomorphic to D_n, Q_n, $G_0(M_n)^-$ or $G_0(M_n)^+$. After Theorem 8 we show that 2-groups discussed in the paper are not isomorphic to each other.

Here we recall $\mathrm{FIrr}(M_n)$. The 2-group M_n has 2^{n-2} faithful irreducible characters ϕ_ν $(1 \leq \nu \leq 2^{n-1}$ and $2 \nmid \nu)$:

$$\phi_\nu(a^{2i}) = 2\zeta^{2\nu i} \quad (1 \leq i \leq 2^n), \qquad \phi_\nu(x) = 0 \quad (x \notin \langle\, a^2 \,\rangle),$$

where ζ is a primitive 2^nth root of unity. Each faithful irreducible character ϕ_ν is induced from the faithful linear character η_ν of the cyclic subgroup $\langle\, a \,\rangle : \eta_\nu(a^i) = \zeta^{\nu i}$ $(1 \leq i \leq 2^n)$. Thus it follows that the faithful irreducible characters ϕ_ν are algebraically conjugate to each other.

We have the following criterion for irreducibility of induced characters.

Lemma 2 *Let G be a 2-group containing a normal subgroup M_n and $\phi \in \mathrm{FIrr}(M_n)$. Then the following statements are equivalent.*
(1) *ϕ^G is irreducible.*
(2) *$\phi^g(a^2) \neq \phi(a^2)$ for all $g \notin M_n$.*
(3) *$C_G(a^2) \subset M_n$.*

Proof. We have already seen that $\phi(x) = 0$ for any $x \notin \langle\, a^2 \,\rangle$. Because the subgroup $\langle\, a^2 \,\rangle$ is the center of M_n, it follows that $G \rhd \langle\, a^2 \,\rangle$. The lemma is easily shown from Clifford's Theorem (cf. [1, p.329]). □

Now we have the following useful lemma.

Lemma 3 *Let G be a 2-group such that $G \rhd M_n$ and $\phi \in \mathrm{FIrr}(M_n)$. Suppose that ϕ^G is irreducible. Then there exists an embedding*

$$\alpha : G/M_n \longrightarrow \mathrm{Aut}\langle a^2 \rangle.$$

In particular, G/M_n is abelian. If $n = 3$, then G/M_n is cyclic.

Proof. From Lemma 2 it follows that $C_G(\langle a^2 \rangle) = M_n$. Because M_n is a normal subgroup of G and $\langle a^2 \rangle$ is the center of M_n, we have the embedding. □

We also recall the automorphism groups of $\langle a^2 \rangle$ and M_n. The cyclic group of order n is denoted by Z_n:

$$\mathrm{Aut}\langle a^2 \rangle = \langle \theta \rangle \times \langle \psi \rangle \cong Z_2 \times Z_{2^{n-3}},$$

where $\theta(a^{2i}) = a^{-2i}$ and $\psi(a^{2i}) = a^{2\times 5i}$ $(1 \leq i \leq 2^{n-1})$.

$$\mathrm{Aut}\, M_n = \{f_{i,j,k} \mid 1 \leq i \leq 2^n,\ 2 \nmid i,\ j \in \{0,\, 1\},\ k \in \{0,\, 1\}\}$$

where $f_{i,j,k}(a) = a^i b^j$, $f_{i,j,k}(b) = a^{2^{n-1}k} b$.

In order to determine groups we use the following (cf. [7, III, §7]):

Proposition 4 *Let H be a finite group. Let G be a finite group such that H is a normal subgroup of G and $G/H = \langle xH \rangle$ is a cyclic group of order $t \geq 2 : x^t = r \in H$. Let θ be a map defined as $\theta(h) = xhx^{-1}$ for any $h \in H$. Then the following statements hold:*
(1) $\theta \in \mathrm{Aut}\, H$,
(2) $\theta^t(h) = rhr^{-1}$ *for any* $h \in H$,
(3) $\theta(r) = r$.
Conversely, if $\theta \in \mathrm{Aut}\, H$ and $r \in H$ satisfy (2) and (3), then there exists a unique extension group G of H such that $G/H = \langle vH \rangle$ is a cyclic group of order t, $\theta(h) = vhv^{-1}$ for any $h \in H$ and $v^t = r$.

3. Determination of cyclic extensions of M_n

It is easy to show the following from Theorem 1:

Theorem 5 *Let G be a 2-group which contains $M_n = \langle a, b \rangle$ $(n \geq 3)$ as a normal subgroup of index 2^t $(t \geq 1)$ and $\phi \in \mathrm{FIrr}(M_n)$. Suppose that $G \rhd \langle a \rangle$ and the induced character ϕ^G is irreducible. Then $G \cong G_t(D_n)$ $(t \leq n-2)$, $G_t(Q_n)$ $(t \leq n-2)$, $G_t(M_n)^+$ $(t \leq n-3)$ or $G_t(M_n)^-$ $(t \leq n-3)$.*

So in the rest of the paper we assume that $\langle a \rangle \ntriangleleft G$.

In this section we consider the case that G/M_n is cyclic. Lemma 3 implies that all extensions of M_3 are discussed here. We set $G = \langle a, b, x \rangle$ and $x^{2^t} \in M_n = \langle a, b \rangle$ $(t \geq 1)$. Considering $\operatorname{Aut} M_n$, we have $xbx^{-1} = b$ or $a^{2^{n-1}}b$. When $xbx^{-1} = a^{2^{n-1}}b$, because $G = \langle a, b, ax \rangle$ and $(ax)b(ax)^{-1} = aa^{2^{n-1}}ba^{-1} = a^{1+2^{n-1}}a^{-(1+2^{n-1})}b = b$, we may assume that $xbx^{-1} = b$ in $G = \langle a, b, x \rangle$.

We separate the proof into two cases depending on the action of x on a^2. We note if $n = 3$, then we have only Case I.

Case I: $xa^2x^{-1} = a^{-2}$. Since $x^2a^2x^{-2} = a^2$ and $\phi^G \in \operatorname{Irr}(G)$, we have $x^2 \in M_n$ and $t = 1$. This case is treated in [2]. We demonstrate the proof again for the completeness.

We set $xax^{-1} = a^ib$. Then $xa^2x^{-1} = (a^ib)^2 = a^{2i(1+2^{n-2})} = a^{-2}$ and $i \equiv -1 + 2^{n-2} \pmod{2^{n-1}}$. So we have $xax^{-1} = a^{-1+2^{n-2}}b$ or $a^{-1+3\cdot 2^{n-2}}b$. If $xax^{-1} = a^{-1+3\cdot 2^{n-2}}b$, we have $(bx)a(bx)^{-1} = a^{(-1+3\cdot 2^{n-2})(1+2^{n-1})}b = a^{-1+2^{n-2}}b$ by $n \geq 3$. Since $G = \langle a, b, bx \rangle$, we may assume that $xax^{-1} = a^{-1+2^{n-2}}b$ in $G = \langle a, b, x \rangle$. Now because $(a^{-1+2^{n-2}}b)^2 = a^{-2}$ we have

$$\begin{aligned} x^2ax^{-2} &= xa^{-1+2^{n-2}}bx^{-1} = (a^{-1+2^{n-2}}b)^{-1+2^{n-2}}b \\ &= b^{-1}a^{1-2^{n-2}}a^{-2^{n-2}}b = a \end{aligned}$$

and so $x^2 = a^k$ for some integer k. From $x^2bx^{-2} = a^kba^{-k} = b$ we have $2 \mid k$. Let $x^2 = a^{2k}$ for some integer k. Then $a^{2k} = xa^{2k}x^{-1} = (xa^2x^{-1})^k = a^{-2k}$ and $2k \equiv 0 \pmod{2^{n-1}}$. Hence we have $x^2 = 1$ or $x^2 = a^{2^{n-1}}$. Consequently $G = \langle a, b, x \rangle$ $(\triangleright M_n)$ in this case has one of the following relations:

$$\begin{aligned} &xax^{-1} = a^{-1+2^{n-2}}b, \quad xbx^{-1} = b, \quad x^2 = 1, \\ &xax^{-1} = a^{-1+2^{n-2}}b, \quad xbx^{-1} = b, \quad x^2 = a^{2^{n-1}}. \end{aligned}$$

Case II: $xa^2x^{-1} = a^{2(\pm 1+2^{n-t-1})}$. In this case we have $n \geq 4$ and $1 \leq t \leq n-3$. We set $xax^{-1} = a^ib$. Then $xa^2x^{-1} = (a^ib)^2 = a^{2i(1+2^{n-2})} = a^{2(\pm 1+2^{n-t-1})}$ and $i \equiv (1+2^{n-2})(\pm 1 + 2^{n-t-1}) \pmod{2^{n-1}}$. So we have

$$xax^{-1} = a^{(1+2^{n-2})(\pm 1+2^{n-t-1})}b \quad \text{or} \quad a^{(1+2^{n-2})(\pm 1+2^{n-t-1})+2^{n-1}}b.$$

If $xax^{-1} = a^{(1+2^{n-2})(\pm 1+2^{n-t-1})+2^{n-1}}b$, we have

$$\begin{aligned} (bx)a(bx)^{-1} &= ba^{(1+2^{n-2})(\pm 1+2^{n-t-1})+2^{n-1}}bb^{-1} \\ &= a^{(1+2^{n-2}+2^{n-1})(\pm 1+2^{n-t-1})+2^{n-1}}b \end{aligned}$$

$$= a^{(1+2^{n-2})(\pm 1+2^{n-t-1})}b.$$

Since $G = \langle a, b, bx \rangle$, we may assume that $xax^{-1} = a^{(1+2^{n-2})(\pm 1+2^{n-t-1})}b$ in $G = \langle a, b, x \rangle$.

When $xax^{-1} = a^{(1+2^{n-2})(1+2^{n-t-1})}b$, we have

$$\begin{aligned} x^2ax^{-2} &= (a^{(1+2^{n-2})(1+2^{n-t-1})}b)^{(1+2^{n-2})(1+2^{n-t-1})}b \\ &= a^{(1+2^{n-t-1})\{(1+2^{n-2})(1+2^{n-t-1})-1\}}a^{(1+2^{n-2})(1+2^{n-t-1})}bb \\ &= a^{(1+2^{n-t-1})(1+2^{n-t-1}+2^{n-1})} \\ &= a^{(1+2^{n-t-1})^2+2^{n-1}} \end{aligned}$$

because $(a^{(1+2^{n-2})(1+2^{n-t-1})}b)^2 = a^{2(1+2^{n-t-1})}$.

If $t = 1$, we have $x^2ax^{-2} = xax^{-1} = a$. So $x^2 = a^k$ for some integer k and $x^2bx^{-2} = (a^k)b(a^k)^{-1} = a^{2^{n-1}k}b = b$. Hence we have $2 \mid k$. Let $x^2 = a^{2k}$ for some integer k. Since $a^{2k} = xa^{2k}x^{-1} = a^{2k(1+2^{n-2})}$ and $2 \mid k$. So we get $x^2 = a^{4k}$. If $x^2 = a^{4k}$, then we have

$$\begin{aligned} &(a^{2k(-1+2^{n-3})}x)a(a^{2k(-1+2^{n-3})}x)^{-1} = ab, \\ &(a^{2k(-1+2^{n-3})}x)b(a^{2k(-1+2^{n-3})}x)^{-1} = b \\ &(a^{2k(-1+2^{n-3})}x)^2 = a^{4k(-1+2^{n-3})(1+2^{n-3})+4k} = 1. \end{aligned}$$

Since $G = \langle a, b, a^{2k(-1+2^{n-3})}x \rangle$, we get $x^2 = 1$ in $G = \langle a, b, x \rangle$. Consequently $G = \langle a, b, x \rangle$ ($\triangleright M_n$) has the following relations:

$$xax^{-1} = a^{1+2^{n-1}}b, \quad xbx^{-1} = b, \quad x^2 = 1.$$

Remark 3 We may set $xax^{-1} = ab$ in $G = \langle a, b, x \rangle$ as in [2, Theorem 5].

If $t \geq 2$, we have

$$\{(1 + 2^{n-t-1})^2 + 2^{n-1}\}^{2^{t-1}} \equiv (1 + 2^{n-t-1})^{2^t} \equiv 1 + 2^{n-1} \pmod{2^n}.$$

Hence $x^{2^t}ax^{-2^t} = a^{1+2^{n-1}}$. So $x^{2^t} = a^k b$ for some integer k and $x^{2^t}bx^{-2^t} = a^kba^{-k} = a^ka^{-k(1+2^{n-1})}b = a^{2^{n-1}k}b = b$. Hence we have $2 \mid k$. Let $x^{2^t} = a^{2k}b$ for some integer k. Since $a^{2k}b = xa^{2k}bx^{-1} = a^{2(1+2^{n-t-1})k}b$, we have $k \equiv (1 + 2^{n-t-1})k \pmod{2^{n-1}}$ and $2^t \mid k$. So we get $x^{2^t} = a^{2^{t+1}k}b$.

Set $r = 1 + 2^{n-t-1}$. There exists a solution ν satisfying

$$2\nu\frac{r^{2^t} - 1}{r - 1} + 2^{t+1}k \equiv 0 \pmod{2^n},$$

because $2^t \left\| \frac{r^{2^t}-1}{r-1}\right.$ So we have

$$(a^{2\nu}x)a(a^{2\nu}x)^{-1} = a^{(1+2^{n-2})(1+2^{n-t-1})}b,$$
$$(a^{2\nu}x)b(a^{2\nu}x)^{-1} = b$$
$$(a^{2\nu}x)^{2^t} = a^{2\nu\frac{r^{2^t}-1}{r-1}+2^{t+1}k}b = b.$$

Since $G = \langle a, b, a^{2\nu}x \rangle$, we have $x^{2^t} = 1$ in $G = \langle a, b, x \rangle$. Consequently $G = \langle a, b, x \rangle$ ($\triangleright M_n$) has the following relations:

$$xax^{-1} = a^{(1+2^{n-2})(1+2^{n-t-1})}b, \quad xbx^{-1} = b, \ x^{2^t} = b.$$

When $xax^{-1} = a^{(1+2^{n-2})(-1+2^{n-t-1})}b$, we have

$$\begin{aligned} x^2ax^{-2} &= (a^{(1+2^{n-2})(-1+2^{n-t-1})}b)^{(1+2^{n-2})(-1+2^{n-t-1})}b \\ &= a^{(-1+2^{n-t-1})\{(1+2^{n-2})(-1+2^{n-t-1})-1\}}a^{(1+2^{n-2})(-1+2^{n-t-1})}bb \\ &= a^{(1+2^{n-t-1})(1+2^{n-t-1})} \\ &= a^{(1+2^{n-t-1})^2} \end{aligned}$$

because $(a^{(1+2^{n-2})(1+2^{n-t-1})}b)^2 = a^{2(1+2^{n-t-1})}$. We have

$$x^{2^t}ax^{-2^t} = a^{(-1+2^{n-t-1})^{2^t}} = a^{1+2^{n-1}}$$

by $t \geq 1$. So $x^{2^t} = a^k b$ for some integer k and

$$x^{2^t}bx^{-2^t} = a^k ba^{-k} = a^k a^{-k(1+2^{n-1})}b = a^{2^{n-1}k}b = b.$$

Hence we have $2 \,|\, k$. Let $x^{2^t} = a^{2k}b$ for some integer k. Since $a^{2k}b = xa^{2k}bx^{-1} = a^{2(-1+2^{n-t-1})k}b$, we have $k \equiv k(-1+2^{n-t-1}) \pmod{2^{n-1}}$ and $2^{n-2} \mid k$. So we get $x^{2^t} = a^{2^{n-1}k}b$.

Set $r = -1 + 2^{n-t-1}$. If $x^{2^t} = a^{2^{n-1}}b$, then we have

$$(a^2x)a(a^2x)^{-1} = a^{(1+2^{n-2})(-1+2^{n-t-1})}b,$$
$$(a^2x)b(a^2x)^{-1} = b,$$
$$(a^2x)^{2^t} = a^{2\frac{r^{2^t}-1}{r-1}+2^{n-1}}b = b$$

because $2^{n-2} \left\| \frac{r^{2^t}-1}{r-1}\right.$. Since $G = \langle a, b, a^2x \rangle$, we get $x^{2^t} = b$ in $G = \langle a, b, x \rangle$. Consequently $G = \langle a, b, x \rangle$ ($\triangleright M_n$) has the following relations:

$$xax^{-1} = a^{(1+2^{n-2})(-1+2^{n-t-1})}b, \ xbx^{-1} = b, \ x^{2^t} = b.$$

Summarizing, we have

Theorem 6 *Let $M_n = \langle a, b \mid a^{2^n} = 1, b^2 = 1, bab^{-1} = a^{1+2^{n-1}} \rangle$ $(n \geq 3)$ and $\phi \in \mathrm{FIrr}(M_n)$. Let G be a 2-group containing a normal subgroup M_n of index 2^t $(t \geq 1)$. Suppose that G/M_n is cyclic, $G \not\triangleright \langle a \rangle$ and $\phi^G \in \mathrm{Irr}(G)$. Then G is isomorphic to one of the following:*

(1) $G_1^D = \langle M_n, x \mid xax^{-1} = a^{-1+2^{n-2}}b,\ xbx^{-1} = b,\ x^2 = 1 \rangle$,

(2) $G_1^Q = \langle M_n, x \mid xax^{-1} = a^{-1+2^{n-2}}b,\ xbx^{-1} = b,\ x^2 = a^{2^{n-1}} \rangle$,

(3) $G_1^+ = \langle M_n, x \mid xax^{-1} = a^{1+2^{n-1}}b,\ xbx^{-1} = b,\ x^2 = 1 \rangle$ $(n \geq 4)$,

(4) $G_t^+ = \langle M_n, x \mid xax^{-1} = a^{(1+2^{n-2})(1+2^{n-t-1})}b,\ xbx^{-1} = b,\ x^{2^t} = b \rangle$ $(2 \leq t \leq n-3)$,

(5) $G_t^- = \langle M_n, x \mid xax^{-1} = a^{(1+2^{n-2})(-1+2^{n-t-1})}b,\ xbx^{-1} = b,\ x^{2^t} = b \rangle$ $(1 \leq t \leq n-3)$.

4. Determination of noncyclic extensions of M_n $(n \geq 4)$

We consider noncyclic extensions of M_n of index 2^{t+1} $(t \geq 1)$. In this case it follows from Lemma 3 that

$$G/M_n \cong \langle \theta \rangle \times \langle \psi^{2^{n-t-3}} \rangle \cong \langle -1 \rangle \times \langle 1 + 2^{n-t-1} \rangle,$$

where $n \geq 4$ and $1 \leq t \leq n-3$. So we may set $G = \langle M_n, x, y \rangle = \langle a, b, x, y \rangle$ with

$$xa^2x^{-1} = a^{2(1+2^{n-t-1})}, \quad ya^2y^{-1} = a^{-2} \quad \text{and} \quad yxy^{-1} \in M_n x.$$

Furthermore we suppose that $G \not\triangleright \langle a \rangle$. Then we have the useful lemma, which is due to K. Sekiguchi.

Lemma 7 *Let $M_n = \langle a, b \mid a^{2^n} = 1,\ b^2 = 1,\ bab^{-1} = a^{1+2^{n-1}} \rangle$ $(n \geq 4)$ and $\phi \in \mathrm{FIrr}(M_n)$. Let G be a 2-group containing a normal subgroup M_n of index 2^t $(t \geq 1)$. Suppose that $G \not\triangleright \langle a \rangle$ and $\phi^G \in \mathrm{Irr}(G)$. Then the normalizer $N_G(\langle a \rangle)$ of $\langle a \rangle$ is a subgroup of G of index 2.*

Proof. Let $x, y \in G \setminus N_G(\langle a \rangle)$. Let $xax^{-1} = a^i b$ $(2 \nmid i)$ and $yay^{-1} = a^j b$ $(2 \nmid j)$. Then because $xbx^{-1} = a^{2^{n-1}k}b$ $(k = 0$ or $1)$ and j is odd,

$$(xy)a(xy)^{-1} = xa^jbx^{-1} = (a^ib)^j a^{2^{n-1}k} b \in \langle a \rangle.$$

So we have $xy \in N_G(\langle a \rangle)$. This lemma is completely proved. □

First we consider $H = \langle M_n, x \rangle = \langle a, b, x \rangle$. This group H satisfies that $H \rhd M_n$, $\phi^H \in \mathrm{Irr}(H)$ and H/M_n is cyclic of order 2^t $(t \geq 1)$. From Theorem 5 and Theorem 6 it follows that $H \cong G_t(M_n)^+$, G_1^+ or G_t^+, because $xa^2x^{-1} = a^{2(1+2^{n-t-1})}$.

Case I: $H = G_t(M_n)^+ = \langle M_n, x \mid xax^{-1} = a^{1+2^{n-t-1}}, xbx^{-1} = b, x^{2^t} = b \rangle$ $(1 \leq t \leq n-3)$.

By Lemma 7 we have $N_G(\langle a \rangle) \not\ni y$ and $yay^{-1} = a^i b$ $(2 \nmid i)$. Because $a^{-2} = ya^2y^{-1} = (a^ib)^2 = a^{2i(1+2^{n-2})}$ we have $i \equiv -(1+2^{n-2}) \pmod{2^{n-1}}$. So $yay^{-1} = a^{-1\pm 2^{n-2}}b$. If $yay^{-1} = a^{-1-2^{n-2}}b$, then $(by)a(by)^{-1} = a^{(-1-2^{n-2})(1+2^{n-1})}b = a^{-1+2^{n-2}}b$. Hence we may assume that $yay^{-1} = a^{-1+2^{n-2}}b$ in $G = \langle a, b, x, y \rangle$. We have already $yby^{-1} = a^{2^{n-1}}b$ or b. If $yby^{-1} = a^{2^{n-1}}b$, then

$$\begin{aligned}(aby)b(aby)^{-1} &= aa^{2^{n-1}}ba^{-1} = b,\\ (aby)a(aby)^{-1} &= aba^{-1+2^{n-2}}a^{-1} = a^{-1+2^{n-2}}b\end{aligned}$$

and $G = \langle M_n, x, aby \rangle$. So we may assume that $yby^{-1} = b$. Because $y^2 \in M_n$ and $y^2ay^{-2} = (a^{-1+2^{n-2}}b)^{-1+2^{n-2}}b = a^{-2^{n-2}}b^{-1}a^{1-2^{n-2}}b = a^{-2^{n-1}+(1+2^{n-1})} = a$, we have $y^2 \in \langle a \rangle$. We set $y^2 = a^k$. Because $a^k = ya^ky^{-1} = (a^{-1+2^{n-2}}b)^k$, we have $2 \mid k$ and $a^k = a^{-k}$. So $k \equiv 0 \pmod{2^{n-1}}$. We have $y^2 = 1$ or $a^{2^{n-1}}$.

Next we determine the action of y on x. From $yxy^{-1} \in M_nx$ we have two cases, i.e., $yxy^{-1} = a^ix$ or a^ibx. If $yxy^{-1} = a^ibx$, then because $xbx^{-1} = b$ we have $(a^ibx)b(a^ibx)^{-1} = a^iba^{-i} = a^{2^{n-1}i}b = b$ and $2 \mid i$. Then because $xax^{-1} = a^{1+2^{n-t}}$,

$$\begin{aligned}(a^ibx)a^{-1+2^{n-2}}b(a^ibx)^{-1} &= a^ia^{(-1+2^{n-2})(1+2^{n-t})(1+2^{n-1})}a^{-i(1+2^{n-1})}b\\ &= a^{-1-2^{n-2}-2^{n-t}}b\\ &\neq (a^{-1+2^{n-2}}b)^{1+2^{n-t}} = a^{-1+2^{n-2}-2^{n-t}}b.\end{aligned}$$

Consequently we have $yxy^{-1} = a^ix$. Because $y^2 \in Z(G)$, which is the center of G, we have $x = y^2xy^{-2} = (a^{-1+2^{n-2}}b)^ia^ix$ and $2 \mid i$. Hence $b = yby^{-1} = yx^{2^t}y^{-1} = (a^ix)^{2^t} = a^{i\frac{r^{2^t}-1}{r-1}}x^{2^t} = a^{i\frac{r^{2^t}-1}{r-1}}b$, where $r = 1 + 2^{n-t-1}$. Because $2^t \parallel \frac{r^{2^t}-1}{r-1}$, we have $2^{n-t} \mid i$. Then $(a^ix)^{2^t} = a^{i\frac{r^{2^t}-1}{r-1}}x^{2^t} = b$. Now from $2^{n-t} \mid i$ there exists a solution λ satisfying

$$i - 2^{n-t-1}\lambda \equiv 0 \pmod{2^n} \quad \text{and} \quad 2 \mid \lambda.$$

Then we have

$$(a^\lambda y)x(a^\lambda y)^{-1} = a^\lambda a^i x a^{-\lambda} = a^{\lambda+i-\lambda(1+2^{n-t-1})}x = x,$$
$$(a^\lambda y)a(a^\lambda y)^{-1} = a^\lambda a^{-1+2^{n-2}}ba^{-\lambda} = a^{-1+2^{n-2}}b,$$

$(a^\lambda y)^2 = y^2$, $(a^\lambda y)b(a^\lambda y)^{-1} = b$ and $G = \langle M_n, x, a^\lambda y\rangle$. So in this case we have the following relation in $G = \langle M_n, x, y\rangle$:

$$yay^{-1} = a^{-1+2^{n-2}}b, \quad yby^{-1} = b, \quad yxy^{-1} = x, \quad y^2 = 1 \text{ or } a^{2^{n-1}}.$$

Case II: $H = G_1^+ = \langle M_n, x \mid xax^{-1} = a^{1+2^{n-1}}b,\ xbx^{-1} = b,\ x^2 = 1\rangle$ $(n \geq 4)$.

By Lemma 7 we have $N_G(\langle a\rangle) \ni y$ and $yay^{-1} = a^i$ $(2 \nmid i)$. Because $a^{-2} = ya^2y^{-1} = a^{2i}$ we have $i \equiv -1 \pmod{2^{n-1}}$. So $yay^{-1} = a^{-1}$ or $a^{-1+2^{n-1}}$. If $yay^{-1} = a^{-1+2^{n-1}}$, then $(by)a(by)^{-1} = a^{-1}$. Hence we may assume that $yay^{-1} = a^{-1}$ in $G = \langle a, b, x, y\rangle$. We have already $yby^{-1} = a^{2^{n-1}}b$ or b. If $yby^{-1} = a^{2^{n-1}}b$, then $(ay)b(ay)^{-1} = aa^{2^{n-1}}ba^{-1} = b$, $(ay)a(ay)^{-1} = a^{-1}$ and $G = \langle M_n, x, ay\rangle$. So we may assume that $yby^{-1} = b$. Because $y^2 \in M_n$ and $y^2ay^{-2} = a$, we have $y^2 \in \langle a\rangle$. We set $y^2 = a^k$. Because $a^k = ya^ky^{-1} = a^{-k}$, we have $2^{n-1} \mid k$. We have $y^2 = 1$ or $a^{2^{n-1}}$.

Next we determine the action of y on x. From $yxy^{-1} \in M_nx$ we have $yxy^{-1} = a^ix$ or a^ibx. If $yxy^{-1} = a^ix$, then because $xbx^{-1} = b$ we have $(a^ix)b(a^ix)^{-1} = a^iba^{-i} = a^{2^{n-1}i}b = b$ and $2 \mid i$. Then because $xax^{-1} = a^{1+2^{n-1}}b$, $(a^ix)a^{-1}(a^ix)^{-1} = a^i(a^{1+2^{n-1}}b)^{-1}a^{-i} = ba^{-1+2^{n-1}} = a^{-1}b \neq a^{-1+2^{n-1}}b$. Consequently we have $yxy^{-1} = a^ibx$. Because $xbx^{-1} = b$, we have $(a^ibx)b(a^ibx)^{-1} = a^iba^{-i} = a^{2^{n-1}i}b = b$ and $2 \mid i$. Then $(a^ibx)^2 = a^iba^{i(1+2^{n-2})}b = a^{2i(1+2^{n-3})} = 1$. Consequently we have $2^{n-1} \mid i$. Namely $yxy^{-1} = bx$ or $a^{2^{n-1}}bx$. When $yxy^{-1} = a^{2^{n-1}}bx$,

$$(a^2y)x(a^2y)^{-1} = a^2a^{2^{n-1}}bxa^{-2} = a^{2+2^{n-1}}ba^{-2(1+2^{n-2})}x = bx,$$

$(a^2y)a(a^2y)^{-1} = a^{-1}$, $(a^2y)b(a^2y)^{-1} = b$, $(a^2y)^2 = y^2$ and $G = \langle M_n, x, a^2y\rangle$. So in this case we have the following relation in $G = \langle M_n, x, y\rangle$:

$$yay^{-1} = a^{-1}, \quad yby^{-1} = b, \quad yxy^{-1} = bx, \quad y^2 = 1 \text{ or } a^{2^{n-1}}.$$

Case III: $H = G_t^+ = \langle M_n, x \mid xax^{-1} = a^{(1+2^{n-2})(1+2^{n-t-1})}b,\ xbx^{-1} = b,\ x^{2^t} = b\rangle$ $(2 \leq t \leq n-3)$.

As well as Case II it follows that we may assume that in $G = \langle M_n, x, y \rangle$

$$yay^{-1} = a^{-1}, \ \ yby^{-1} = b \ \text{ and } \ y^2 = 1 \ \text{ or } \ a^{2^{n-1}}.$$

We determine the action of y on x. From $yxy^{-1} \in M_n x$ we have $yxy^{-1} = a^i x$ or $a^i bx$. If $yxy^{-1} = a^i x$, then because $xbx^{-1} = b$ we have $(a^i x)b(a^i x)^{-1} = a^i ba^{-i} = a^{2^{n-1}i}b = b$ and $2 \mid i$. So because $xax^{-1} = a^{(1+2^{n-2})(1+2^{n-t-1})}b$,

$$\begin{aligned}(a^i x)a^{-1}(a^i x)^{-1} &= a^i(a^{(1+2^{n-2})(1+2^{n-t-1})}b)^{-1}a^{-i} \\ &= (a^{(1+2^{n-2})(1+2^{n-t-1})}b)^{-1} \\ &= a^{-(1-2^{n-2})(1+2^{n-t-1})}b \neq a^{-(1+2^{n-2})(1+2^{n-t-1})}b.\end{aligned}$$

Consequently $yxy^{-1} = a^i bx$. Because $xbx^{-1} = b$, we have $(a^i bx)b(a^i bx)^{-1} = a^i ba^{-i} = a^{2^{n-1}i}b = b$ and $2 \mid i$. Then it follows that

$$(a^i bx)^{2^t} = (a^{2i(1+2^{n-t-2})}x^2)^{2^{t-1}} = a^{2i(1+2^{n-t-2})\frac{r^{2^t}-1}{r^2-1}}b = b.$$

From $2^{t-1} \parallel \frac{r^{2^t}-1}{r^2-1}$ we have $2^{n-t} \mid i$. Then there exists a solution λ such that

$$i - 2^{n-t-1}\lambda \equiv 0 \pmod{2^n} \quad \text{and} \quad 2 \mid \lambda.$$

Then we have

$$\begin{aligned}(a^\lambda y)x(a^\lambda y)^{-1} &= a^\lambda a^i bxa^{-\lambda} = a^{\lambda+i}a^{-\lambda(1+2^{n-t-1})}bx \\ &= a^{i-2^{n-t-1}\lambda}bx = bx.\end{aligned}$$

$(a^\lambda y)a(a^\lambda y)^{-1} = a^{-1}$, $(a^\lambda y)b(a^\lambda y)^{-1} = b$, $(a^\lambda y)^2 = y^2$ and $G = \langle a, b, x, a^\lambda y \rangle$. So in this case we have the following relation in $G = \langle M_n, x, y \rangle$:

$$yay^{-1} = a^{-1}, \quad yby^{-1} = b, \quad yxy^{-1} = bx, \quad y^2 = 1 \quad \text{or} \quad a^{2^{n-1}}.$$

Summarizing, we have

Theorem 8 *Let $M_n = \langle a, b \mid a^{2^n} = 1, b^2 = 1, bab^{-1} = a^{1+2^{n-1}} \rangle$ $(n \geq 3)$ and $\phi \in \mathrm{FIrr}(M_n)$. Let G be a 2-group with a normal subgroup M_n of index 2^t $(t \geq 1)$. Suppose that G/M_n is noncyclic, $G \not\triangleright \langle a \rangle$ and $\phi^G \in \mathrm{Irr}(G)$. Then G is isomorphic to one of the following*:

(1) $G_t(M_n)^{+D} = \langle M_n, x, y \mid xax^{-1} = a^{1+2^{n-t-1}}, xbx^{-1} = b, x^{2^t} = b, yay^{-1} = a^{-1+2^{n-2}}b, yby^{-1} = b, yxy^{-1} = x, y^2 = 1 \rangle$ $(1 \leq t \leq n-3)$,

(2) $G_t(M_n)^{+Q} = \langle M_n, x, y \mid xax^{-1} = a^{1+2^{n-t-1}}, xbx^{-1} = b, x^{2^t} = b, yay^{-1} = a^{-1+2^{n-2}}b, yby^{-1} = b, yxy^{-1} = x, y^2 = a^{2^{n-1}} \rangle$ $(1 \leq t \leq n-3)$,

(3) $G_1^{+D} = \langle M_n, x, y \mid xax^{-1} = a^{1+2^{n-1}}b, xbx^{-1} = b, x^2 = 1, yay^{-1} = a^{-1}, yby^{-1} = b, yxy^{-1} = bx, y^2 = 1 \rangle$ $(n \geq 4)$,

(4) $G_1^{+Q} = \langle M_n, x, y \mid xax^{-1} = a^{1+2^{n-1}}b, xbx^{-1} = b, x^2 = 1, yay^{-1} = a^{-1}, yby^{-1} = b, yxy^{-1} = bx, y^2 = a^{2^{n-1}} \rangle$ $(n \geq 4)$,

(5) $G_t^{+D} = \langle M_n, x, y \mid xax^{-1} = a^{(1+2^{n-2})(1+2^{n-t-1})}b, xbx^{-1} = b, x^{2^t} = b, yay^{-1} = a^{-1}, yby^{-1} = b, yxy^{-1} = bx, y^2 = 1 \rangle$ $(2 \leq t \leq n-3)$,

(6) $G_t^{+Q} = \langle M_n, x, y \mid xax^{-1} = a^{(1+2^{n-2})(1+2^{n-t-1})}b, xbx^{-1} = b, x^{2^t} = b, yay^{-1} = a^{-1}, yby^{-1} = b, yxy^{-1} = bx, y^2 = a^{2^{n-1}} \rangle$ $(2 \leq t \leq n-3)$.

G		Order	Involutions	$Z(G)$	G'	G/G'
$G_t(D_n)$	$1 \leq t \leq n-2$	2^{n+t+1}	$3 \times 2^{n-1}$	2	2^t	$2^{n-t} \times 2^t \times 2$
$G_t(Q_n)$	$1 \leq t \leq n-2$	2^{n+t+1}	2^{n-1}	2	2^t	$2^{n-t} \times 2^t \times 2$
$G_t(M_n)^+$	$1 \leq t \leq n-3$	2^{n+t+1}	0	2^{n-t-1}	2^{t+1}	$2^{n-t-1} \times 2^{t+1}$
$G_t(M_n)^-$	$1 \leq t \leq n-3$	2^{n+t+1}	0	2	2^{n-1}	$2^{t+1} \times 2$
G_1^D	$n = 3$	2^{3+1+1}	2^3	2	2×2	4×2
	$n \geq 4$	2^{n+1+1}	2^n	2	2^{n-1}	4×2
G_1^Q	$n = 3$	2^{3+1+1}	0	2	2×2	4×2
	$n \geq 4$	2^{n+1+1}	0	2	2^{n-1}	4×2
G_1^+	$n \geq 4$	2^{n+1+1}	4	2^{n-1}	2×2	$2^{n-1} \times 2$
G_t^+	$2 \leq t \leq n-3$	2^{n+t+1}	0	2^{n-t}	2^{t+1}	$2^{n-t} \times 2^t$
G_1^-	$n \geq 4$	2^{n+1+1}	2^{n-1}	2	2^{n-1}	4×2
G_t^-	$2 \leq t \leq n-3$	2^{n+t+1}	0	2	2^{n-1}	4×2^t
$G_t(M_n)^{+D}$	$1 \leq t \leq n-3$	$2^{n+t+1+1}$	$3 \times 2^{n-1}$	2	2^{n-1}	$4 \times 2^t \times 2$
$G_t(M_n)^{+Q}$	$1 \leq t \leq n-3$	$2^{n+t+1+1}$	2^{n-1}	2	2^{n-1}	$4 \times 2^t \times 2$
G_1^{+D}	$n \geq 4$	$2^{n+1+1+1}$	$4 + 2^{n+1}$	2	$2^{n-1} \times 2$	$2 \times 2 \times 2$
G_1^{+Q}	$n \geq 4$	$2^{n+1+1+1}$	$4 + 2^n$	2	$2^{n-1} \times 2$	$2 \times 2 \times 2$
G_t^{+D}	$2 \leq t \leq n-3$	$2^{n+t+1+1}$	$3 \times 2^{n-1}$	2	$2^{n-1} \times 2$	$2 \times 2^t \times 2$
G_t^{+Q}	$2 \leq t \leq n-3$	$2^{n+t+1+1}$	2^{n-1}	2	$2^{n-1} \times 2$	$2 \times 2^t \times 2$

We note that 2-groups G in Theorems 5, 6 and 8 are groups by Proposition 4. In order to show these 2-groups G that contains M_n of order 2^n are not isomorphic to each other, we have the above table and a fact. In the table Order is the order of G and Involutions is the number of involutions outside M_n. Furthermore in center $Z(G)$, commutator subgroups G' and G/G' we have the form as the direct product of cyclic 2-groups. For example, $2^{n-t} \times 2^t$ means the direct product of two cyclic 2-groups of order 2^{n-t} and 2^t.

Finally we need to show that $G_1(M_n)^- \not\cong G_1^Q$ for $n \geq 4$. This is shown from a fact that $G_1(M_n)^-$ has a normal subgroup of order 2^n and G_1^Q has no such subgroup (cf. [2, p.343]).

Acknowledgment The author would like to express his sincere gratitude to Professor Toshihiko Yamada and Professor Katsusuke Sekiguchi for their useful suggestions. In particular, Problem in introduction is based on an idea of Professor Yamada. The idea of Lemma 7 is due to Professor Sekiguchi.

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