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Uniqueness of Haar measures for a quasi Woronowicz algebra

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Abstract. It is shown that any two Haar measures for a quasi Woronowicz algebra, a quantum group in the von Neumann algebra framework, are proportional.

Key words: quasi Woronowicz algebras, Haar measures.

Introduction

In [Y1], we introduced a notion of a quasi Woronowicz algebra as a slight generalization of the object studied in [MN]. As shown there, every matched pair of groups gives rise to a concrete quasi Woronowicz algebra. By definition, Woronowicz algebras are quasi ones, so that there are plenty of intriguing examples of quasi Woronowicz algebras. It is shown in [Y2] also that one can perform the double group construction within this "category". Thus, it seems that, as far as the formulation of quantum groups in the language of von Neumann algebras is concerned, the category of quasi Woronowicz algebras provides a nice framework.

The purpose of this note is to offer one more piece of evidence to this statement. To be more precise, we will prove uniqueness of Haar measures for a quasi Woronowicz algebra, which is naturally expected to be true in the "right" framework.

The organization of this note is as follows. Section 1 is concerned with a quick review on quasi Woronowicz algebras. In Section 2, we derive some useful property of a Haar measure for a quasi Woronowicz algebra. Our result heavily relies on the ones obtained in [K]. In Section 3, we illustrate how the result in [DeC, Theorem 2.3] can be extended to the case of quasi Woronowicz algebras. The last section is devoted to proving uniqueness of Haar measures.

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1. Notation

In this section, we give a quick review on quasi Woronowicz algebras, introducing notation that will be used in our later discussion. Quasi Woronowicz algebras are almost like Woronowicz algebras introduced in [MN]. It is not too much to say that what is true for Woronowicz algebras is equally true for quasi Woronowicz algebras. Thus, for the general theory of quasi Woronowicz algebras, we refer the reader to [MN] or [Y1, 2]. Our notation will be mainly adopted from these literatures.

Given a von Neumann algebra $\mathcal{M}$ and a faithful normal semifinite weight ψ on $\mathcal{M}$, we introduce subsets $\mathfrak{n}_\psi$, $\mathfrak{m}_\psi$ and $\mathfrak{m}_\psi^+$ of $\mathcal{M}$ by

$$\mathfrak{n}_\psi = \{x \in \mathcal{M} : \psi(x^*x) < \infty\}, \quad \mathfrak{m}_\psi = \mathfrak{n}_\psi^* \mathfrak{n}_\psi, \quad \mathfrak{m}_\psi^+ = \mathfrak{m}_\psi \cap \mathcal{M}_+.$$

The modular automorphism group of ψ is denoted by σ^ψ.

A *coinvolutive Hopf-von Neumann algebra* is a triple $(\mathcal{M}, \delta, R)$ in which:
(1) $\mathcal{M}$ is a von Neumann algebra;
(2) δ is an injective normal $*$-homomorphism, called a *coproduct* (or a *comultiplication*), from $\mathcal{M}$ into $\mathcal{M} \bar{\otimes} \mathcal{M}$ with the coassociativity condition: $(\delta \otimes id_\mathcal{M}) \circ \delta = (id_\mathcal{M} \otimes \delta) \circ \delta$;
(3) R is a $*$-antiautomorphism of $\mathcal{M}$, called a *coinvolution* or a *unitary antipode*, such that $R^2 = id_\mathcal{M}$ and $\sigma \circ (R \otimes R) \circ \delta = \delta \circ R$, where σ is the usual flip.

A *quasi Woronowicz algebra* is a family $\mathbb{W} = (\mathcal{M}, \delta, R, \tau, h)$ in which:
(1) $(\mathcal{M}, \delta, R)$ is a coinvolutive Hopf-von Neumann algebra;
(2) τ is a continuous one-parameter automorphism group of $\mathcal{M}$, called the *deformation automorphism*, which commutes with the coproduct δ and the antipode R;
(3) h is a τ-invariant faithful normal semifinite weight on $\mathcal{M}$, called a *Haar measure* of $\mathbb{W}$, satisfying the following conditions:
 (a) *Quasi left invariance*: For any ϕ in $\mathcal{M}_*^+$, we have $(\phi \otimes h) \circ \delta(x) = h(x)\phi(1)$ for all $x \in \mathfrak{m}_h^+$;
 (b) *Strong left invariance*: For any $x, y \in \mathfrak{n}_h$ and $\phi \in \mathcal{M}_*$ which is analytic with respect to the adjoint action of the deformation automorphism τ on $\mathcal{M}_*$, the following equality holds:

$$(\phi \otimes h)((1 \otimes y^*)\delta(x)) = (\phi \circ \tau_{-i/2} \circ R \otimes h)(\delta(y^*)(1 \otimes x)).$$

(c) *Commutativity*: $h \circ \sigma_t^{h \circ R} = h$ for all $t \in \mathbf{R}$ (or, equivalently, $h \circ R \circ \sigma_t^h = h \circ R$).

A motivation for considering quasi Woronowicz algebras rather than Woronowicz algebras is mentioned in [Y1, 2].

Throughout the remainder of this note, we fix a quasi Woronowicz algebra $\mathbb{W} = (\mathcal{M}, \delta, R, \tau, h)$. We always think of $\mathcal{M}$ as represented on the Hilbert space $L^2(h)$ obtained by the GNS representation from the weight h. By the commutativity of h, there exists a non-singular positive self-adjoint operator Q on $L^2(h)$ affiliated with the centralizer $\mathcal{M}_h = \{x \in \mathcal{M} : \sigma_t^h(x) = x \ (t \in \mathbf{R})\}$ of h such that the Connes' Radon Nikodym derivative $(D(h \circ R) : Dh)_t$ satisfies $(D(h \circ R) : Dh)_t = Q^{it}$ for $t \in \mathbf{R}$. In the notation in [MN], we have $Q = \rho^{-1}$.

By the proof of [MN, Proposition 3.4], the modular automorphism group σ^h of h satisfies

$$(\tau_t \otimes \sigma_t^h) \circ \delta = \delta \circ \sigma_t^h. \qquad (t \in \mathbf{R}) \tag{1.1}$$

From this and the identities $R \circ \sigma_{-t}^h \circ R = \sigma_t^{h \circ R}$, $\sigma \circ (R \otimes R) \circ \delta = \delta \circ R$, one can easily verify that

$$(\sigma_t^{h \circ R} \otimes \tau_{-t}) \circ \delta = \delta \circ \sigma_t^{h \circ R}. \qquad (t \in \mathbf{R}) \tag{1.2}$$

To the given $\mathbb{W}$, there exists another quasi Woronowicz algebra $\widehat{\mathbb{W}} = (\widehat{\mathcal{M}}, \hat{\delta}, \hat{R}, \hat{\tau}, \hat{h})$, called the quasi Woronowicz algebra dual to $\mathbb{W}$. For the construction of $\widehat{\mathbb{W}}$, see [MN].

Finally, the intrinsic group $G(\mathbb{W})$ of the quasi Woronowicz algebra $\mathbb{W}$ is the set of nonzero elements u in $\mathcal{M}$ satisfying the equation

$$\delta(u) = u \otimes u.$$

From [S] and [MN, Section 3], it follows that $G(\mathbb{W})$ is a weakly closed subgroup of the group of unitaries in $\mathcal{M}$.

2. Some property of a Haar measure

Define a subset $\mathcal{D}(h)$ of $\mathcal{M}_+$ by

$$\mathcal{D}(h) = \{x \in \mathcal{M}_+ : (\phi \otimes id)(\delta(x)) \in \mathfrak{m}_h^+ \ (\forall \phi \in \mathcal{M}_*^+)\}.$$

By the definition of quasi left invariance of h, $\mathcal{D}(h)$ contains $\mathfrak{m}_h^+$. By exactly the same argument as in Section 1 of [K], we obtain the next proposition, which is the von Neumann algebra version of Lemma 1.3 of [K].

Proposition 2.1 *For each $x \in \mathcal{D}(h)$, there exists a unique nonnegative number T_x such that*

$$h((\phi \otimes id)(\delta(x))) = T_x \phi(1)$$

for all $\phi \in \mathcal{M}_*^+$.

By using this proposition, we may define, as in Definition 1.4 of [K], a mapping $\tilde{h} : \mathcal{M}_+ \to [0, \infty]$ by

$$\tilde{h}(x) = \begin{cases} T_x, & \text{if } x \in \mathcal{D}(h), \\ \infty, & \text{otherwise.} \end{cases}$$

Thus, if $x \in \mathcal{D}(h)$, the quantity $\tilde{h}(x) = T_x$ is characterized by the equation

$$h((\phi \otimes id)(\delta(x))) = \tilde{h}(x)\phi(1) \qquad (\phi \in \mathcal{M}_*^+).$$

By the same proof as in Proposition 1.5 of [K], one can easily show that $\tilde{h}$ is a faithful normal semifinite weight on $\mathcal{M}$, with $\mathfrak{m}_{\tilde{h}}^+ = \mathcal{D}(h)$, such that $\tilde{h} = h$ on $\mathfrak{m}_h^+$. In particular, we have $\tilde{h} \leq h$. In a moment, we will prove that they actually equal. For this purpose, we state the following proposition, which is Proposition 1.7 of [K].

Proposition 2.2 *Let $x \in \mathcal{M}_+$. Then x belongs to $\mathfrak{m}_{\tilde{h}}^+$ if and only if $(\phi \otimes id)(\delta(x))$ belongs to $\mathfrak{m}_{\tilde{h}}^+$ for all $\phi \in \mathcal{M}_*^+$.*

Theorem 2.3 *The weight $\tilde{h}$ equals the Haar measure h.*

Proof. We first recall (see (1.1)) that the modular automorphism group σ^h of h satisfies

$$(\tau_t \otimes \sigma_t^h) \circ \delta = \delta \circ \sigma_t^h \qquad (t \in \mathbf{R}). \tag{2.3.1}$$

Let $x \in \mathfrak{m}_{\tilde{h}}^+$. Thanks to (2.3.1), for any $\phi \in \mathcal{M}_*^+$ and $t \in \mathbf{R}$, we have

$$\begin{aligned} h((\phi \otimes id)(\delta(\sigma_t^h(x)))) &= h((\phi \circ \tau_t \otimes id)(\delta(x))) \\ &= \phi \circ \tau_t(1)\tilde{h}(x) = \phi(1)\tilde{h}(x) < \infty. \end{aligned}$$

This shows that $\sigma_t^h(x) \in \mathcal{D}(h) = \mathfrak{m}_{\tilde{h}}^+$. Hence we find that $\sigma_t^h(\mathfrak{m}_{\tilde{h}}^+) = \mathfrak{m}_{\tilde{h}}^+$ for all $t \in \mathbf{R}$. From this, it follows that $\tilde{h} \circ \sigma_t^h = \tilde{h}$ for all $t \in \mathbf{R}$. As noted just before this theorem, $\tilde{h}$ equals h on $\mathfrak{m}_h^+$, hence on $\mathfrak{m}_h$. By [PT, Proposition 5.9], we have $\tilde{h} = h$. □

Corollary 2.4 *Let $x \in \mathcal{M}_+$. Then the following are equivalent;*
(1) *x belongs to $\mathfrak{m}_h^+$.*
(2) *$(\phi \otimes id)(\delta(x))$ belongs to $\mathfrak{m}_h^+$ for any $\phi \in \mathcal{M}_*^+$.*

Proof. The assertion immediately follows from Proposition 2.2 and Theorem 2.3. □

3. The intrinsic group of a quasi Woronowicz algebra

Let $G(\widehat{\mathbb{W}})$ be the intrinsic group of the dual Woronowicz algebra $\widehat{\mathbb{W}}$. By the same argument as in Section I of [S] (see also the discussion following [MN, Corollary 3.11.1]), it is shown that each element $v \in G(\widehat{\mathbb{W}})$ defines an automorphism β_v of $\mathcal{M}$ given by

$$\beta_v(x) = vxv^* \qquad (x \in \mathcal{M}).$$

With this notation, we can show the next theorem:

Theorem 3.1 *Let v be a unitary on $L^2(h)$. Then v is in $G(\widehat{\mathbb{W}})$ if and only if the two conditions below are satisfied:*
(1) *$\beta_v := \operatorname{Ad} v|_{\mathcal{M}}$ is an automorphism of $\mathcal{M}$ which satisfies*

$$(\beta_v \otimes id) \circ \delta = \delta \circ \beta_v,$$

(2) *v is the canonical implementation of β_v.*

A proof of the above theorem goes exactly parallel to that of [DeC, Theorem 2.3] with the aid of both Proposition 2.1 and Lemma 2.2 of [DeC] as generalized to our setting. Thus we leave the verification to the reader. However some remarks are in order. Proposition 2.1 of [DeC] is true for quasi Woronowicz algebras, and is shown in the same manner as in [DeC]. Lemma 2.2 of [DeC] is also true for quasi Woronowicz algebras. But its proof given in [DeC] can not be adopted in our setting, becasue we only have quasi left invariance of h. Hence we present its proof in the following which utilizes the results obtained in the preceding section.

Lemma 3.2 *If β is an automorphism of $\mathcal{M}$ such that $(\beta \otimes id) \circ \delta = \delta \circ \beta$, then we have $h \circ \beta = h$.*

Proof. It suffices to prove that $\beta(\mathfrak{m}_h^+) = \mathfrak{m}_h^+$.

Let $x \in \mathcal{M}_+$. Then, by Corollary 1.4, we have

$$\begin{aligned} x \in \mathfrak{m}_h^+ &\iff (\phi \otimes id)(\delta(x)) \in \mathfrak{m}_h^+ \qquad (\forall \phi \in \mathcal{M}_*^+) \\ &\iff (\phi \circ \beta \otimes id)(\delta(x)) \in \mathfrak{m}_h^+ \quad (\forall \phi \in \mathcal{M}_*^+) \\ &\iff (\phi \otimes id)(\delta(\beta(x))) \in \mathfrak{m}_h^+ \quad (\forall \phi \in \mathcal{M}_*^+) \\ &\iff \beta(x) \in \mathfrak{m}_h^+. \end{aligned}$$

Thus we are done. □

4. Uniqueness of Haar measures

In this section, we prove the main theorem of this note which asserts uniqueness of Haar measures for a quasi Woronowicz algebra.

Let ψ be another Haar measure for a quasi Woronowicz algebra $\mathbb{W} = (\mathcal{M}, \delta, R, \tau, h)$.

Lemma 4.1 *The weight $\psi \circ R$ commutes with h.*

Proof. Set $\beta_t := \sigma_{-t}^{\psi} \circ \sigma_t^h$ $(t \in \mathbf{R})$. From (1.1), each β_t satisfies

$$(\beta_t \otimes id) \circ \delta^\circ = \delta^\circ \circ \beta_t,$$

where $\delta^\circ = \sigma \circ \delta$, which is the coproduct of the coopposite $\mathbb{W}^\circ$ of $\mathbb{W}$. Note that $\psi \circ R$ is a Haar measure for $\mathbb{W}^\circ$. Hence, by Lemma 3.2, we have

$$\psi \circ R \circ \beta_t = \psi \circ R.$$

From this, it follows that

$$\begin{aligned} \psi \circ R &= \psi \circ R \circ \beta_t = \psi \circ R \circ \sigma_{-t}^{\psi} \circ \sigma_t^h = \psi \circ \sigma_t^{\psi \circ R} \circ R \circ \sigma_t^h \\ &= \psi \circ R \circ \sigma_t^h. \end{aligned}$$

Thus we are done. □

By the preceding lemma and [PT], there exists a nonsingular positive self-adjoint operator S, affiliated with the centralizer $\mathcal{M}_h$ of h, such that

$$(D\psi \circ R : Dh)_t = S^{it} \qquad (t \in \mathbf{R}).$$

Next we consider the automorphism γ_t $(t \in \mathbf{R})$ of $\mathcal{M}$ given by $\gamma_t := \sigma_t^{\psi \circ R} \circ \sigma_{-t}^{h \circ R}$. From (1.2), we easily find that each γ_t satisfies

$$(\gamma_t \otimes id) \circ \delta = \delta \circ \gamma_t.$$

Hence, by Theorem 3.1, there exists a unique group-like element $w_t \in G(\widehat{\mathbb{W}})$ that is the canonical implementation of $\gamma_t : \gamma_t = \mathrm{Ad}\, w_t$.

Lemma 4.2 *For each $t \in \mathbf{R}$, $\gamma_t = id_{\mathcal{M}}$. In particular, we have $\sigma^\psi = \sigma^h$.*

Proof. Let $t \in \mathbf{R}$. We have

$$\sigma_t^{\psi \circ R} = \gamma_t \circ \sigma_t^{h \circ R} = \mathrm{Ad}(w_t Q^{it}) \circ \sigma_t^h.$$

On the other hand, we have $\sigma_t^{\psi \circ R} = \mathrm{Ad}\, S^{it} \circ \sigma_t^h$. It follows that $\mathrm{Ad}(w_t Q^{it}) = \mathrm{Ad}\, S^{it}$ as automorphisms of $\mathcal{M}$. Hence $S^{-it} w_t Q^{it}$ belongs to the center $\mathcal{Z}(\mathcal{M})$ of $\mathcal{M}$. In particular, w_t lies in $\mathcal{M}$. Since $\mathcal{M} \cap \widehat{\mathcal{M}} = \mathbf{C}$ by [MN, Proposition 3.11], w_t must be a scalar, which in turn entails that γ_t is the identity map. Therefore, $\sigma^{\psi \circ R} = \sigma^{h \circ R}$. In particular, we have $\sigma^\psi = \sigma^h$.

□

Corollary 4.3 *The weight ψ commutes with h, i.e., we have $\psi \circ \sigma_t^h = \psi$ for any $t \in \mathbf{R}$.*

Theorem 4.4 *If ψ be another Haar measure for a quasi Woronowicz algebra $\mathbb{W} = (\mathcal{M}, \delta, R, \tau, h)$, then ψ is proportional to h.*

Proof. Since we now have Corollary 4.3 at hand, we can follow the proof of [S, Théorème III.3] to obtain the result. The details are left to the reader.

□

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