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Quantization of canonical isomorphisms and the semiclassical von Neumann theorem

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Abstract. We prove the three mutually related theorems: the theorem on the quantizability of canonical isomorphisms, the theorem on the quantizability of classical canonical commutation relations and a semiclassical version of von Neumann's theorem. Although some similar results can be obtained on the basis of the deformation theory (e.g. [16], [15], [10]), here we present the proofs which involve only elementary methods and notions. Moreover, in our approach we can easily compute the quantum corrections. Our deformation quantizations (semiclassical algebras) are additionally equipped with the deformation involutions and we study here the algebras of entire functions and of polynomials, instead of frequently used algebras of C^{∞} observables.

Key words: semiclassical limit, quantization, canonical commutation relations, canonical isomorphisms, deformation quantization.

Introduction

Canonical isomorphisms, that is the isomorphisms of a phase space preserving its symplectic structure, play an important role in classical mechanics. Their quantum analogues are unitary transformations of Hilbert space, "quantum phase space". One of the interesting problems concerning canonical isomorphisms and unitary transformations is their semiclassical relationship. An important question concerns quantization of canonical isomorphism into a unitary transformation, that is, of finding a unitary transformation which can be treated in some sense as corresponding to a given canonical isomorphism.

Let us note that the meaning of that correspondence cannot be trivial. Namely, consider a phase space X, a canonical automorphism u of X and a procedure of quantization ${}^{\wedge\hbar}$ ($\hbar > 0$) of observables on X into operators acting in a Hilbert space. If the transformation of operators T_u given by $T_u\hat{f}^{\hbar} = (t_u f)^{\wedge\hbar}$ with $t_u f = f\circ u$, had the form $T_u\hat{f}^{\hbar} = U_{\hbar}^{-1}\hat{f}^{\hbar}U_{\hbar}$, where $U_{\hbar}$ is a unitary transformation, then we could call $U_{\hbar}$ a quantization of u (for

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the given procedure $^{\wedge_\hbar}$). However, such $U_\hbar$ usually does not exist, since T_u does not preserve the composition and adjoint of operators (understood in any formal sense). Thus, it seems to be natural from the semiclassical point of view to replace t_u in the definition of T_u by t_u + "quantum corrections", to make T_u preserve the multiplication and adjoint. If this is possible, u should be treated as a quantizable canonical transformation.

This idea can be rigorously formulated, if we use the so-called *deformation model* of quantum mechanics instead of the traditional one. In the deformation model the classical and the quantum algebras of observables are the same set with different algebraic structures. The quantum structure is given by a *deformation quantization* — a family $\{\star_n\}_{n\in\mathbb{N}}$ of bilinear operations in algebra, where $\star_0$ is the classical multiplication of observables and the formal operation $\star_\hbar = \sum_{n=0}^{\infty} \hbar^n \star_n$ is associative and satisfies some conditions of consistence with the classical structure. The deformed multiplication $\star_\hbar$ is an analogue of the multiplication of operators from the traditional model of quantum mechanics, but the subtle difficulties with domains of operators do not exist in the deformation model.

The idea of deformation quantization was first introduced by Moyal (see [19]) and it has been developed by many authors (see e.g. [24], [1], [2] [14], [15], [10], [6], [13], [7], [20]). Deformation quantizations connected with particular procedures of quantization of observables were used for the semiclassical studies in the traditional model of quantum mechanics (see e.g. [4] and [12]).

In our considerations we also follow the ideas of the deformation model. Additionally, to obtain the structure which fully corresponds to the quantum structure based on the Hilbert space, we consider also the deformations of the classical involution. Such deformations, analogues of the adjoint of operator, seem to be a necessary element of the deformation model of quantum mechanics. However, in most of the literature no deformation of involution is considered (see [23] as an exception). Thus we introduce here the notion of *semiclassical algebra*, which includes the deformations of the classical multiplication and of the classical involution. Our studies refer to two kinds of algebras, the *polynomial* and *entire algebras*, which correspond to the algebras of polynomials and of entire functions of $2d$-variables, respectively.

The first of our results, Theorem 1, is a semiclassical version of the well-known von Neumann theorem on unitary equivalence of representations of commutation relations in Hilbert space (see e.g. [21, th. VIII.14]). It states that two selfadjoint systems satisfying the same commutation relations in two formal semiclassical algebras are (under some technical assumptions) equivalent through a uniquely determined formal unitary transformation of these algebras. We compute explicit recurrent formulas for the quantum corrections. The second result concerns the quantization of systems satisfying classical canonical commutation relations. We prove Theorem 2, stating that under some natural assumptions the appropriate quantization exists. Our Theorem 3 asserts that each classical canonical isomorphism is semiclassically quantizable. This result is a direct consequence of the two previous theorems. As an illustration we find simple recurrent formulas for the quantum corrections to the canonical transformation induced by a linear isomorphism of $\mathbb{R}^{2d}$. We also give some examples showing the relationships between the quantization in the deformation model of quantum mechanics and in the traditional one, based on Hilbert space framework.

Some similar results can be obtained by the use of advanced methods of the deformation (and star-product) theory. In particular, the results similar to our Theorem 3 were obtained by Lichnerowicz in [16]. This theorem is also closely related to the problem of equivalence of deformations, which is considered for instance in [15] and [10]. All these papers use Hochschild or Chevalley cohomology groups (see [9] and [11]) and consider the case of arbitrary symplectic manifolds. In the present paper, in contrast, using elementary methods, we study the relationships between the quantization of canonical isomorphisms, the semiclassisal von Neumann theorem and the quantization of classical canonical commutation relations. Therefore, our results can be applied primarily to the case of the linear phase space $\mathbb{R}^{2d}$, which is simple from the deformation theory point of view, but is very important for applications.

There are also some other differences between the results cited and ours. The most important is the use of deformation involution in this paper. Moreover, we do not deal with the commonly used algebra of C^∞ functions, but study the algebras of polynomials and of entire functions. Other important differences concern the assumptions on the formal deformation $\star_\hbar$. Generally authors consider only the so-called *star-products*, which are formal deformations satisfying some symmetry (*parity*) assumptions (see e.g. [14]). They usually also require coefficients $\star_n$ to be bilinear diferential operators. In our paper we consider the general (also "non-symmetric") case of associative formal deformation $\star_\hbar$. The "non-symmetric" deformations play an important role in quantum mechanics, and are related in a natural way to various procedures of quantizations of observables. We also do not make the differentiability assumption on $\star_n$-s in the case of polynomial algebras. (Note that many important operators acting in the space of polynomials are not differential, for instance, the operator of integration.)

In this paper some proofs are omitted or shortened (especially in Section 1). For the details we refer the reader to [17] and [18].

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1. Classical and semiclassical algebras

We introduce here some notions which will be used in this paper, with the main notion of *semiclassical algebra* being the quantization of *classical algebra*.

For multi-indices $\alpha, \beta \in \mathbb{N}^m$ we denote $|\alpha| = \sum_{j=1}^m \alpha_j$, $\alpha! = \prod_{j=1}^m \alpha_j!$, $\binom{\alpha}{\beta} = \prod_{j=1}^m \binom{\alpha_j}{\beta_j}$; $\alpha \leq \beta$ if $\alpha_j \leq \beta_j$ for $j = 1, \ldots, m$; if $a \in \mathbb{C}^m$, then $a^\alpha = \prod_{j=1}^m a_j^{\alpha_j}$; $\mathbf{1}_j$ is the multi-index with 1 in the j-th position and 0 at the remaining ones (here $0 \in \mathbb{N}$). The notation for systems of multi-indices is similar, e.g., for $\boldsymbol{\alpha} \in (\mathbb{N}^m)^k$ $|\boldsymbol{\alpha}| = \sum_{j=1}^k |\boldsymbol{\alpha}_j|$. For $z \in \mathbb{C}^m$ $|z| = \sum_{k=1}^m |z_k|$.

In this paper *algebra* is a complex linear space with bilinear, associative multiplication and with the neutral element $\mathbb{1}$. For an algebra $\mathcal{A}$ characters in boldface, like $\boldsymbol{e}$, $\boldsymbol{f}$, are used to denote elements of $\mathcal{A}^m$ for some $m \in \mathbb{N}$; $\boldsymbol{f}_j$ is the j-th term of $\boldsymbol{f}$, $\boldsymbol{f}^\alpha = \boldsymbol{f}_1^{\alpha_1} \cdots \boldsymbol{f}_m^{\alpha_m}$ for $\alpha \in \mathbb{N}^m$ (with $f^0 = \mathbb{1}$ for any $f \in \mathcal{A}$). We denote also $\boldsymbol{f}' \subset \boldsymbol{f}$ for $\boldsymbol{f} \in \mathcal{A}^m$ and $\boldsymbol{f}' \in \mathcal{A}^{m'}$ with $m' \leq m$, when $\boldsymbol{f}'_j = \boldsymbol{f}_j$ for $j = 1, \ldots, m'$. $\mathcal{M}_m(X)$ is the set of $m \times m$ matrices with elements in X and for $C \in \mathcal{M}_m(X)$ C_{ij} is its element from the i-th row and the j-th column, $C\boldsymbol{f} \in \mathcal{A}^m$ with $(C\boldsymbol{f})_i = \sum_{j=1}^m C_{ij}\boldsymbol{f}_j$. We use the convention that for $F : X \longrightarrow Y$ the same F denotes the product map $F : X^m \longrightarrow Y^m$, $F((x_1, \ldots, x_m)) = (F(x_1), \ldots, F(x_m))$; if F is linear we also omit the brackets, e.g., $t\boldsymbol{e} = (t\boldsymbol{e}_1, \ldots, t\boldsymbol{e}_m)$ for a linear $t : \mathcal{A} \longrightarrow \mathcal{B}$ and $\boldsymbol{e} \in \mathcal{A}^m$.

The typical algebras considered here are $\mathrm{Pol}(\mathbb{C}^m)$ and $\mathrm{Ent}(\mathbb{C}^m)$ — the algebras of polynomials and of entire functions on $\mathbb{C}^m$ and also $\mathrm{Pol}(\mathbb{R}^m)$

and $\mathrm{Ent}(\mathbb{R}^m)$ — the algebra of complex polynomials on $\mathbb{R}^m$ and of complex functions on $\mathbb{R}^m$ having an extension to an entire function on $\mathbb{C}^m$, with the usual pointwise operations. For $\boldsymbol{f} \in \mathcal{A}^m$ $\mathrm{alg}(\boldsymbol{f})$ is the subalgebra of $\mathcal{A}$ generated by $\boldsymbol{f}$ (the smallest one containing all of $\boldsymbol{f}_j$). For $w \in \mathrm{Pol}(\mathbb{C}^m)$, $w(x) = \sum_{\alpha \leq N} w_\alpha x^\alpha$, we set $w(\boldsymbol{f}) = \sum_{\alpha \leq N} w_\alpha \boldsymbol{f}^\alpha$. If all the generators $\boldsymbol{f}_j$ commute, then $\mathrm{alg}(\boldsymbol{f}) = \{w(\boldsymbol{f}) : w \in \mathrm{Pol}(\mathbb{C}^m)\}$ is commutative and then we call $\boldsymbol{f}$ *independent* when $w(\boldsymbol{f}) = 0$ only for the zero $w \in \mathrm{Pol}(\mathbb{C}^m)$. A map * of $\mathcal{A}$ is an *involution* if it is conjugate-linear, $(fg)^* = g^* f^*$ and $(f^*)^* = f$ for $f, g \in \mathcal{A}$. An algebra with an involution will be called **algebra.*

We denote $\mathrm{re}^* f = \frac{1}{2}(f + f^*)$, $\mathrm{im}^* f = \frac{1}{2i}(f - f^*)$, f is *real* if $f^* = f$, $\boldsymbol{f} \in \mathcal{A}^m$ is *real* when all $\boldsymbol{f}_j$ are real. $\mathrm{Pol}(\mathbb{R}^m)$ and $\mathrm{Ent}(\mathbb{R}^m)$ are *algebras with * being the usual conjugation of functions and $\mathrm{Pol}(\mathbb{C}^m)$ and $\mathrm{Ent}(\mathbb{C}^m)$ with the involution * given by the formula $f^*(z) = \overline{f(\bar{z})}$. **Homomorphism* (**isomorphism*) is a homomorphism (isomorphism) of algebras preserving involution.

1.1. Polynomial and entire algebras

Definition 1.1 Let $\mathcal{A}$ be a commutative algebra. $\mathcal{A}$ is a *polynomial algebra* if for some $m \in \mathbb{N}$, $m > 0$ there exists an independent $\boldsymbol{e} \in \mathcal{A}^m$ such that $\mathcal{A} = \mathrm{alg}(\boldsymbol{e})$. Each $\boldsymbol{e}$ satisfying the above conditions is an *algebraic base* of the polynomial algebra $\mathcal{A}$. An algebra $\mathcal{A}$ with a given topology of Fréchet space is an *entire algebra* if the multiplication is joint continuous (i.e. continuous as a map from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$) and

(i) there exists a family of seminorms $\{\| \ \|_j\}_{j \in \mathbb{N}}$ inducing the topology in $\mathcal{A}$ such that $\forall_{f \in \mathcal{A}, j \in \mathbb{N}} \exists_{\varepsilon > 0} \sup_{n \in \mathbb{N}} \varepsilon^n \|f^n\|_j < +\infty$;

(ii) for some $m \in \mathbb{N}$, $m > 0$ there exists $\boldsymbol{e} \in \mathcal{A}^m$ such that each $f \in \mathcal{A}$ can be uniquely expressed as a sum of series $\sum_{\alpha \in \mathbb{N}^m} f_\alpha \boldsymbol{e}^\alpha$ convergent in $\mathcal{A}$, with $f_\alpha \in \mathbb{C}$, and $\forall_{f \in \mathcal{A}, \varepsilon > 0} \exists_{C > 0} \forall_{\alpha \in \mathbb{N}^m} |f_\alpha| < C \varepsilon^{|\alpha|}$.

Each $\boldsymbol{e}$ satisfying (ii) is an *analytic base* of the entire algebra $\mathcal{A}$. The number m is the *dimension* of $\mathcal{A}$ for the both cases of algebras.

The dimension of polynomial and entire algebra is well-defined. $\mathrm{Pol}(\mathbb{R}^m)$ and $\mathrm{Pol}(\mathbb{C}^m)$ are m-dimensional polynomial algebras and we can choose algebraic bases as $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$, where $\mathbf{x}_j(x) = x_j$ for $x \in \mathbb{R}^m$ or $\mathbb{C}^m$ respectively. Moreover, each m-dimensional polynomial algebra $\mathcal{A}$ is isomorphic to $\mathrm{Pol}(\mathbb{C}^m)$ — for any algebraic base $\boldsymbol{e}$ of $\mathcal{A}$ we can define the isomorphism $\phi_{\boldsymbol{e}} : \mathrm{Pol}(\mathbb{C}^m) \longrightarrow \mathcal{A}$ by

$$\phi_{\boldsymbol{e}}(w) = w(\boldsymbol{e}) \tag{1.1}$$

for $w \in \mathrm{Pol}(\mathbb{C}^m)$. Polynomial algebras are in some sense the poorest algebras (by the Baire theorem it is impossibile to define there an interesting topology of a complete space). Note that the first part of (ii) means that $\{\boldsymbol{e}^\alpha\}_{\alpha\in\mathbb{N}^m}$ is a topological base of $\mathcal{A}$ and thus it is also a Schauder base — see e.g. [5]. Note also that each family of seminorms inducing the topology of an entire algebra satisfies the estimate from (i). In $\mathrm{Ent}(\mathbb{C}^m)$ we choose the topology of almost uniform convergence, which is induced by the family $\|f\|_j = \sup_{|z|\leq j} |f(z)|$, $j \in \mathbb{N}$. In $\mathrm{Ent}(\mathbb{R}^m)$ we consider the similar formula for $\|f\|_j$, but in place of $f \in \mathrm{Ent}(\mathbb{R}^m)$ we put $f_{\mathrm{ext}} \in \mathrm{Ent}(\mathbb{C}^m)$, the unique analytic extension of f onto the whole $\mathbb{C}^m$. With these topologies $\mathrm{Ent}(\mathbb{C}^m)$ and $\mathrm{Ent}(\mathbb{R}^m)$ become m-dimensional entire algebras, and for an analytic base we can also choose $\mathbf{x}$. The series from (ii) is then the usual Taylor expansion with the origin in 0. From now on, $\mathrm{Ent}(\mathbb{C}^m)$ and $\mathrm{Ent}(\mathbb{R}^m)$ will designate these algebras with the above defined topologies. Each m-dimensional entire algebra is isomorphically homeomorphic to $\mathrm{Ent}(\mathbb{C}^m)$; we can define $\phi_{\boldsymbol{e}} : \mathrm{Ent}(\mathbb{C}^m) \longrightarrow \mathcal{A}$ by (1.1), where for $w \in \mathrm{Ent}(\mathbb{C}^m)$ with $w(x) = \sum_{\alpha\in\mathbb{N}^m} w_\alpha x^\alpha$ we denote $w(\boldsymbol{e}) = \sum_{\alpha\in\mathbb{N}^m} w_\alpha \boldsymbol{e}^\alpha$ (note that the ambiguous sense of the symbols $\phi_{\boldsymbol{e}}$ and $w(\boldsymbol{e})$ does not lead to confusion). The above series is convergent in $\mathcal{A}$ and by the Banach-Steinhaus theorem it is easily seen that $\phi_{\boldsymbol{e}}$ is an isomorphism and a homeomorphism of $\mathrm{Ent}(\mathbb{C}^m)$ onto $\mathcal{A}$.

The following simple proposition will be used in the next section.

Proposition 1.1 *If $\mathcal{B}$ is an algebra, $\mathcal{A}$ an m-dimensional polynomial algebra with an algebraic base $\boldsymbol{e}$ and $\boldsymbol{f} \in \mathcal{B}^m$ has commuting elements, then there exists exactly one homomorphism of algebras $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying the condition $\varphi(\boldsymbol{e}) = \boldsymbol{f}$.*

We now consider operators acting in these two kinds of algebra. Let $\mathcal{A}$ be a polynomial or entire algebra and $\boldsymbol{e} \in \mathcal{A}^m$ an algebraic or respectively analytic base. Define $\mathcal{A}' = \mathrm{alg}(\boldsymbol{e})$. We have $\mathcal{A}' = \mathcal{A}$ in the polynomial case. In the analytic case $\mathcal{A}'$ is a dense polynomial subalgebra of $\mathcal{A}$ with an algebraic base $\boldsymbol{e}$. For $\alpha \in \mathbb{N}^m$ the symbol $\partial_{\boldsymbol{e}}^\alpha$ denotes the operator from $\mathcal{A}$ into $\mathcal{A}$ in the polynomial case, and from $\mathcal{A}'$ into $\mathcal{A}'$ or from $\mathcal{A}$ into $\mathcal{A}$ in the analytic case, given by the formula

$$\partial_{\boldsymbol{e}}^\alpha = \phi_{\boldsymbol{e}} \partial^\alpha \phi_{\boldsymbol{e}}^{-1}, \tag{1.2}$$

where $\phi_{\boldsymbol{e}}$ is defined for $\mathcal{A}$ or $\mathcal{A}'$ and $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_m}^{\alpha_m}}$. We also use the symbol ∂^α to denote $\partial_{\boldsymbol{e}}^\alpha$ when the choice of $\boldsymbol{e}$ is clear (e.g. usually $\boldsymbol{e} = \mathbf{x}$ for $\mathrm{Pol}(\mathbb{C}^m)$ and $\mathrm{Ent}(\mathbb{C}^m)$). For $\alpha = \mathbf{1}_j$ we write ∂_{e_j} instead of $\partial_{\boldsymbol{e}}^\alpha$. When $\boldsymbol{\alpha} \in (\mathbb{N}^m)^k$ and $\boldsymbol{f} \in \mathcal{A}^k$ (or $\mathcal{A}'^k$), we denote $\partial_{\boldsymbol{e}}^{\boldsymbol{\alpha}} \boldsymbol{f} = \prod_{j=1}^k \partial_{\boldsymbol{e}}^{\alpha_j} \boldsymbol{f}_j$. Consider a k-linear operator L given by

$$L\boldsymbol{f} = \sum_{\alpha \in (\mathbb{N}^m)^k} l_\alpha \partial_{\boldsymbol{e}}^\alpha \boldsymbol{f} \tag{1.3}$$

for $\boldsymbol{f} \in (\mathcal{A}')^k$, where $l_\alpha \in \mathcal{A}$. This is a well-defined operator $L : (\mathcal{A}')^k \longrightarrow \mathcal{A}$.

Proposition 1.2 *Each k-linear $L : (\mathcal{A}')^k \longrightarrow \mathcal{A}$ has the unique form* (1.3).

Definition 1.2 Let $\mathcal{A}$ be a polynomial or entire algebra. A k-linear $L : \mathcal{A}^k \longrightarrow \mathcal{A}$ is a *differential operator* if

$$L = \sum_{\alpha \in F} l_\alpha \partial_{\boldsymbol{e}}^\alpha, \tag{1.4}$$

for an algebraic or respectively, analytic base $\boldsymbol{e}$ and a finite set $F \subset (\mathbb{N}^m)^k$.

The choice of the base $\boldsymbol{e}$ is not essential in the above definition. When $\mathcal{A}$ is an entire algebra, then any differential operator L is joint continuous.

We introduce the operation $\mathcal{Z}_{\boldsymbol{e}}$ transforming bilinear operators into linear operators. If $S(f,g) = \sum_{\gamma,\gamma' \in \mathbb{N}^m} s_{\gamma,\gamma'} \partial_{\boldsymbol{e}}^\gamma f \cdot \partial_{\boldsymbol{e}}^{\gamma'} g$ for $f, g \in \mathcal{A}'$ ($s_{\gamma,\gamma'} \in \mathcal{A}$), we set

$$\mathcal{Z}_{\boldsymbol{e}}(S) f = \sum_{|\alpha| \geq 2} \left(\frac{1}{2^{|\alpha|} - 2} \sum_{\gamma + \gamma' = \alpha} s_{\gamma,\gamma'} \right) \partial_{\boldsymbol{e}}^\alpha f, \tag{1.5}$$

which is well-defined in two cases. First, when $f \in \mathcal{A}'$, i.e. $\mathcal{Z}_{\boldsymbol{e}}(S) : \mathcal{A}' \longrightarrow \mathcal{A}$. Second, when S is differential — then $\mathcal{Z}_{\boldsymbol{e}}(S) : \mathcal{A} \longrightarrow \mathcal{A}$ (with the same notation) and $\mathcal{Z}_{\boldsymbol{e}}(S)$ is differential.

1.2. Classical algebras

Definition 1.3 A commutative *algebra with a bilinear, antisymmetric operation $\{\ ,\ \} : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ is a *classical algebra* if

(i) (Leibnitz formula) for any $f, g, h \in \mathcal{A}$

$$\{fg, h\} = f\{g,h\} + \{f,h\}g, \tag{1.6}$$

(ii) (Jacobi formula) for any $f, g, h \in \mathcal{A}$

$$\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0, \tag{1.7}$$

(iii) (nondegeneracy) for any $f \in \mathcal{A}$ if $\forall_{g\in\mathcal{A}}\{f,g\} = 0$, then $f = c\mathbb{1}$ for some $c \in \mathbb{C}$,

(iv) (*invariance) for any $f, g \in \mathcal{A}$

$$\{f,g\}^* = \{f^*, g^*\}. \tag{1.8}$$

The operation $\{\ ,\ \}$ in a classical algebra we call the *Poisson bracket.*

In literature the above defined classical algebras may also be referred to as *Poisson algebras with involutions*, but here, for simplicity and because of the semiclasical context, we shall use the former name. The algebras $\mathrm{Pol}(\mathbb{R}^{2d})$, $\mathrm{Ent}(\mathbb{R}^{2d})$, $\mathrm{Pol}(\mathbb{C}^{2d})$ and $\mathrm{Ent}(\mathbb{C}^{2d})$ are classical algebras with the Poisson bracket

$$\{f,g\} = \partial_p f \partial_q g - \partial_p g \partial_q f, \tag{1.9}$$

where ∂_q, ∂_p are the systems of d operators with $(\partial_q)_j = \partial_{q_j} = \frac{\partial}{\partial_{q_j}}$, $(\partial_p)_j = \partial_{p_j} = \frac{\partial}{\partial_{p_j}}$, and the coordinates in $\mathbb{R}^{2d}$ and $\mathbb{C}^{2d}$ are denoted $(q_1, \dots, q_d, p_1, \dots, p_d)$. We also write $\boldsymbol{f}\boldsymbol{g} = \sum_{j=1}^m \boldsymbol{f}_j \boldsymbol{g}_j$ for $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{A}^m$.

Definition 1.4 A system $\boldsymbol{e} \in \mathcal{A}^m$ in a classical algebra $\mathcal{A}$ satisfies *classical canonical commutation relations* (abbreviated to *cccr*) if there exists $D \in \mathcal{M}_m(\mathbb{C})$ such that $\{\boldsymbol{e}_i, \boldsymbol{e}_j\} = D_{ij}\mathbb{1}$ for $i, j = 1, \dots, m$. We define $\mathrm{cr}(\boldsymbol{e}) = D$ then. When $\boldsymbol{e}$ satisfies cccr and $\mathrm{cr}(\boldsymbol{e}) = D$, we say that $\boldsymbol{e}$ satisfies *cccr of D-type.*

If $\boldsymbol{e}$ satisfies cccr of the D-type then $D^\top = -D$. In this paper we consider *polynomial* and *entire classical algebras.* Note that these notions are not simple intersections of the notions of polynomial or entire algebras with the notion of classical algebra.

Definition 1.5 A classical algebra is a *polynomial classical algebra* if it is a polynomial algebra with an algebraic base satisfying cccr; it is an *entire classical algebra* if it is an entire algebra with an analytic base satisfying cccr, the involution is continuous and the Poison bracket is joint continuous. Each algebraic (analytic) base satisfying cccr in a polynomial (entire) classical algebra is called a *canonical base.*

The dimension of any polynomial or entire classical algebra is even. If $\boldsymbol{e} \in \mathcal{A}^m$ is a canonical base, then $\det \operatorname{cr}(\boldsymbol{e}) \neq 0$ and for a nonsingular $C \in \mathcal{M}_m(\mathbb{C})$ the formula $\boldsymbol{f} = C\boldsymbol{e}$ defines also a canonical base, which satisfies

$$\operatorname{cr}(\boldsymbol{f}) = C\operatorname{cr}(\boldsymbol{e})C^{\top}. \tag{1.10}$$

$\operatorname{Pol}(\mathbb{R}^{2d})$, $\operatorname{Pol}(\mathbb{C}^{2d})$ and $\operatorname{Ent}(\mathbb{R}^{2d})$, $\operatorname{Ent}(\mathbb{C}^{2d})$ are our main examples of polynomial and entire classical algebras. The system $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_{2d})$ is a real canonical base for all of them. We denote $(\boldsymbol{q}, \boldsymbol{p}) = \mathbf{x}$, where $\boldsymbol{q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_d)$, $\boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d)$ and

$$\operatorname{cr}((\boldsymbol{q}, \boldsymbol{p})) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

The Poisson bracket is a bilinear differential operator in each polynomial or entire classical algebra $\mathcal{A}$, since for any canonical base $\boldsymbol{e} \in \mathcal{A}^m$ with $\operatorname{cr}(\boldsymbol{e}) = D$ and

$$\{f, g\} = \sum_{i,j=1,\ldots,m} D_{i,j} \partial_{e_i} f \cdot \partial_{e_j} g \tag{1.11}$$

for $f, g \in \mathcal{A}$. A transformation of a phase spaces preserving the symplectic structure is called *canonical.* For canonical u we can define the transformation t_u of algebras of observables by $t_u f = f \circ u$, and t_u preserves the Poisson bracket. In this paper we use the name *canonical* also for transformations of observables.

Definition 1.6 A transformation of classical algebras $t : \mathcal{A} \longrightarrow \mathcal{B}$ is *canonical*, if it is a *homomorphism of *algebras and $t\{f, g\} = \{tf, tg\}$ for $f, g \in \mathcal{A}$. If t is invertible, we call it a *canonical isomorphism.*

Proposition 1.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be both polynomial or both entire classical algebras. If $t : \mathcal{A} \longrightarrow \mathcal{B}$ is continuous in the entire case and $\mathbf{e}$ is a canonical base of $\mathcal{A}$, then*

a) *if t is a canonical isomorphism, then $t\mathbf{e}$ is a canonical base of $\mathcal{B}$ and* $\operatorname{cr}(t\boldsymbol{e}) = \operatorname{cr}(\boldsymbol{e})$;
b) *if $\mathbf{e}$ is real, $t\mathbf{e}$ is a real canonical base of $\mathcal{B}$ with* $\operatorname{cr}(t\boldsymbol{e}) = \operatorname{cr}(\boldsymbol{e})$ *and t is a homomorphism of algebras, then t is a canonical isomorphism.*

The restriction: $tf = f_{|\mathbb{R}^{2d}}$ is a a canonical isomorphism of $\operatorname{Pol}(\mathbb{C}^{2d})$ onto $\operatorname{Pol}(\mathbb{R}^{2d})$ or of $\operatorname{Ent}(\mathbb{C}^{2d})$ onto $\operatorname{Ent}(\mathbb{R}^{2d})$ and $t^{-1}f = f_{\text{ext}}$.

Example 1.1 For $m, n \in \mathbb{N}$ consider a map $u_{(n,m)}$ of $\mathbb{R}^2$ or $\mathbb{C}^2$ given by $u_{(n,m)}(q,p) = (q + (p+q^n)^m, p+q^n)$. This is an analytical diffeomorphism and $u^{-1}_{(n,m)}(q,p) = (q - p^m, p - (q-p^m)^n)$. Define $s_{(n,m)}f = f \circ u_{(n,m)}$ for a function f on $\mathbb{R}^2$ or $\mathbb{C}^2$. By Proposition 1.3 b) $s_{(n,m)}$ can be treated as a canonical isomorphism of $\mathrm{Pol}(\mathbb{R}^2)$, $\mathrm{Pol}(\mathbb{C}^2)$, $\mathrm{Ent}(\mathbb{R}^2)$ or $\mathrm{Ent}(\mathbb{C}^2)$.

1.3. Semiclassical algebras

We now introduce semiclassical notions, which in some sense are the quantizations of the classical notions. We consider *semiclassical algebras* — classical algebras with *deformation product* or *deformation quantization* (see e.g. [1], [7], [14], [20]) and *deformation involution* (see e.g. [23]) being classical multiplication and involution with quantum corrections. Semiclassical algebras are analogues of quantum "algebras" of operators acting in a Hilbert space. We also use the parallel approach with the notion of *formal semiclassical algebra*, an *algebra of the formal power series in the Planck constant $\hbar$ with the coefficients in a classical algebra. The multiplication and the involution correspond there to the operator product and adjoint from the quantum model. The linear space of formal series in $\hbar$ with the coefficients in a linear space $\mathcal{A}$ we denote by $\mathcal{A}[[\hbar]]$ and its elements by $f_\hbar$, $g_\hbar$ etc., the coefficient of $f_\hbar$ of $\hbar^n$ is denoted by $f^{(n)}$ and we write $f_\hbar = \sum_{n=0}^{\infty} \hbar^n f^{(n)} = f^{(0)} + \hbar f^{(1)} + \hbar^2 f^{(2)} + \cdots$. We often identify $f \in \mathcal{A}$ with its image under the embedding: $\mathcal{A} \ni f \longrightarrow f + \hbar 0 + \hbar^2 0 + \cdots \in \mathcal{A}[[\hbar]]$.

We set the hierarchy of some operations: the "strongest" are operations of the type of *, ^{*_n}; the next linear operators like t, t_n, s; the next $\star$, $\star_n$, $\cdot$ and the weakest $+$, $-$. For instance $t_1 f \star_3 t g^{*_7} - t_0 f^* h = [(t_1 f) \star_3 (t(g^{*_7}))] - [(t_0(f^*)) \cdot h]$.

Definition 1.7 A classical algebra $\mathcal{A}$ with a family $\{\star_n\}_{n\in\mathbb{N}}$ of bilinear operations and a family $\{^{*_n}\}_{n\in\mathbb{N}}$ of conjugate-linear operations in $\mathcal{A}$ is a *semiclassical algebra* if

(i) for any $f, g, h \in \mathcal{A}$, $n \in \mathbb{N}$

$$\sum_{j+k=n} (f \star_j g) \star_k h = \sum_{j+k=n} f \star_j (g \star_k h); \tag{1.12}$$

(ii) for any $f \in \mathcal{A}$ and $n > 0$

$$\mathbb{1} \star_n f = f \star_n \mathbb{1} = 0; \tag{1.13}$$

(iii)

$$\star_0 = \cdot, \quad \{\ ,\ \}_1 = \{\ ,\ \}, \tag{1.14}$$

where for $f, g \in \mathcal{A}$

$$\{f, g\}_n = i(f \star_n g - g \star_n f); \tag{1.15}$$

(iv) for any $f, g \in \mathcal{A}$, $n \in \mathbb{N}$

$$\sum_{j+k=n} (f \star_j g)^{*_k} = \sum_{j+k+l=n} g^{*_k} \star_j f^{*_l}; \tag{1.16}$$

(v) for any $f \in \mathcal{A}$ i $n \in \mathbb{N}$

$$\sum_{j+k=n} (f^{*_j})^{*_k} = \begin{cases} f & \text{for } n = 0 \\ 0 & \text{for } n > 0; \end{cases} \tag{1.17}$$

(vi)

$$^{*_0} = {}^*. \tag{1.18}$$

The family $\{\star_n\}_{n\in\mathbb{N}}$ is called a *deformation product* or *deformation quantization in* $\mathcal{A}$ and $\{^{*_n}\}_{n\in\mathbb{N}}$ — a *deformation involution in* $\mathcal{A}$. A family $\{f^{(n)}\}_{n\in\mathbb{N}}$ of elements of $\mathcal{A}$ is *selfadjoint* when for any $n \in \mathbb{N}$

$$\sum_{k+l=n} (f^{(k)})^{*_l} = f^{(n)}; \tag{1.19}$$

a family $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ of elements $(\boldsymbol{f}_1^{(n)}, \dots, \boldsymbol{f}_m^{(n)}) \in \mathcal{A}^m$ is *selfadjoint* when all $\{\boldsymbol{f}_i^{(n)}\}_{n\in\mathbb{N}}$ are selfadjoint.

To simplify the notation we shall usually denote a semiclassical algebra by a single letter. Let $\mathcal{A}$ be an algebra, $\{\star_n\}_{n\in\mathbb{N}}$ a family of bilinear operations and $\{^{*_n}\}_{n\in\mathbb{N}}$ a family of conjugate-linear operations in $\mathcal{A}$. For $f_\hbar, g_\hbar \in \mathcal{A}[[\hbar]]$ define

$$f_\hbar \star_\hbar g_\hbar = \sum_{n=0}^{\infty} \hbar^n \sum_{j+k+l=n} f^{(k)} \star_j g^{(l)}, \tag{1.20}$$

$$f_\hbar^{*_\hbar} = \sum_{n=0}^{\infty} \hbar^n \sum_{k+l=n} (f^{(k)})^{*_l}. \tag{1.21}$$

Proposition 1.4 *If $\mathcal{A}$ is a classical algebra, $\star_0$, $\star_1$ and ^{*_0} satisfy* (1.14) *and* (1.18), *then $\mathcal{A}$ with the families $\{\star_n\}_{n\in\mathbb{N}}$ and $\{{}^{*_n}\}_{n\in\mathbb{N}}$ is a semiclassical algebra iff $\mathcal{A}[[\hbar]]$ is an *algebra with the multiplication $\star_\hbar$, the unity $\mathbb{1}\in\mathcal{A}$ and the involution ${}^{*_\hbar}$. Moreover, an element $f_\hbar$ of this *algebra is real iff $\{f^{(n)}\}_{n\in\mathbb{N}}$ is selfadjoint in $\mathcal{A}$.*

We define a *formal semiclassical algebra* as an *algebra $\mathcal{A}[[\hbar]]$ related to a semiclassical algebra $\mathcal{A}$ by the above proposition.

Example 1.2 Consider a matrix $M \in \mathcal{M}_d(\mathbb{R})$ and let $\mathcal{A}$ be one of the classical algebras $\mathrm{Pol}(\mathbb{R}^{2d})$, $\mathrm{Pol}(\mathbb{C}^{2d})$, $\mathrm{Ent}(\mathbb{R}^{2d})$, $\mathrm{Ent}(\mathbb{C}^{2d})$. Define

$$\begin{aligned} f\star_n^M g &= \sum_{|\alpha+\beta|=n} \frac{i^n}{\alpha!\beta!}\left((E\partial_q)^\alpha\partial_p^\beta f\right)\left((M\partial_q)^\beta\partial_p^\alpha g\right), \\ f^{*_n^M} &= \sum_{|\alpha|=n}\frac{i^n}{\alpha!}(G\partial_q)^\alpha\partial_p^\alpha f^* \end{aligned}$$

for $f,g\in\mathcal{A}$, where $E = I+M$, $G = I+2M$. $\{\mathcal{A},\{\star_n^M\}_{n\in\mathbb{N}},\{{}^{*_n^M}\}_{n\in\mathbb{N}}\}$ is a semiclassical algebra (see [17]). We denote it respectively $\mathrm{Pol}_M(\mathbb{R}^{2d})$, $\mathrm{Pol}_M(\mathbb{C}^{2d})$, $\mathrm{Ent}_M(\mathbb{R}^{2d})$, $\mathrm{Ent}_M(\mathbb{C}^{2d})$ and we call them *M-semiclassical algebras.* The best known semiclassical algebras which can be obtained this way are the following (see e.g. [3])

a) $\mathrm{Pol}_{q\text{-}p}(\mathbb{R}^{2d})$, etc. — *$q$-$p$ semiclassical algebras*, when $M=-I$:

$$f\star_n^{q\text{-}p}g = \sum_{|\alpha|=n}\frac{(-i)^n}{\alpha!}\partial_p^\alpha f\partial_q^\alpha g, \quad f^{*_n^{q\text{-}p}} = \sum_{|\alpha|=n}\frac{(-i)^n}{\alpha!}\partial_q^\alpha\partial_p^\alpha f^*;$$

b) $\mathrm{Pol}_{p\text{-}q}(\mathbb{R}^{2d})$, etc. — *$p$-$q$ semiclassical algebras*, when $M=0$:

$$f\star_n^{p\text{-}q}g = \sum_{|\alpha|=n}\frac{i^n}{\alpha!}\partial_p^\alpha f\partial_q^\alpha g, \quad f^{*_n^{p\text{-}q}} = \sum_{|\alpha|=n}\frac{i^n}{\alpha!}\partial_q^\alpha\partial_p^\alpha f^*;$$

c) $\mathrm{Pol}_W(\mathbb{R}^{2d})$, etc. — *Weyl semiclassical algebras*, when $M=-\frac{1}{2}I$:

$$\begin{aligned} f\star_n^W g &= \sum_{|\alpha+\beta|=n}\frac{(-1)^{|\beta|}i^n}{2^n\alpha!\beta!}(\partial_q^\alpha\partial_p^\beta f)(\partial_q^\beta\partial_p^\alpha g), \\ f^{*_n^W} &= \begin{cases} f^* & \text{for } n=0 \\ 0 & \text{for } n>0. \end{cases} \end{aligned}$$

Example 1.3 Let $\mathrm{Pol}_{\eta\xi}(\mathbb{C}^{2d})$ and $\mathrm{Ent}_{\eta\xi}(\mathbb{C}^{2d})$ be classical algebras given as follows. As algebras they simply are $\mathrm{Pol}(\mathbb{C}^{2d})$ and $\mathrm{Ent}(\mathbb{C}^{2d})$ respectively, for $\mathrm{Ent}_{\eta\xi}(\mathbb{C}^{2d})$ the Fréchet space topology is the same as in $\mathrm{Ent}(\mathbb{C}^{2d})$. The Poisson bracket is given by the formula $\{f,g\} = i(\partial_\eta f \partial_\xi g - \partial_\eta g \partial_\xi f)$, where the coordinates in $\mathbb{C}^{2d}$ are denoted here by (η, ξ). We completely change the formula for involution: $f^*(\eta,\xi) = \overline{f(\bar{\xi},\bar{\eta})}$ for $(\eta,\xi) \in \mathbb{C}^{2d}$. $\mathrm{Pol}_{\eta\xi}(\mathbb{C}^{2d})$ is a polynomial and $\mathrm{Ent}_{\eta\xi}(\mathbb{C}^{2d})$ an entire classical algebra. As a canonical base we can take the system $\mathbf{x}$ denoted here by $(\boldsymbol{\eta}, \boldsymbol{\xi})$. This canonical base is not real, but $\boldsymbol{\eta}^* = \boldsymbol{\xi}$, $\boldsymbol{\xi}^* = \boldsymbol{\eta}$. The *$\eta\xi$-Weyl semiclassical algebras* $\mathrm{Pol}_{\eta\xi\text{-}W}(\mathbb{C}^{2d})$, $\mathrm{Ent}_{\eta\xi\text{-}W}(\mathbb{C}^{2d})$ are given by

$$
\begin{aligned}
f \star_n^{\eta\xi\text{-}W} g &= \sum_{|\alpha+\beta|=n} \frac{(-1)^{|\beta|}}{2^n \alpha! \beta!} (\partial_\xi^\alpha \partial_\eta^\beta f)(\partial_\xi^\beta \partial_\eta^\alpha g), \\
f^{*_n^{\eta\xi\text{-}W}} &= \begin{cases} f^* & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases}
\end{aligned}
$$

The above examples are related to some procedures of quantizations of observables: q-p, p-q and Weyl semiclassical algebras to q-p, p-q and Weyl quantizations (see [3]); $\eta\xi$-Weyl semiclassical algebra to Weyl quantization in Bargmann space (see [4]) and M-semiclassical algebras to the quantization given by the so called T-symbol (see [22], [17]), where $T = \begin{pmatrix} A & A+B \\ A & B \end{pmatrix} \in \mathcal{M}_{2d}(\mathbb{R})$ for some $A, B \in \mathcal{M}_d(\mathbb{R})$ with $|\det A| = 1$, and $M = (A^{-1}B)^\top$.

Definition 1.8 A semiclassical algebra $\mathcal{A}$ is a *polynomial semiclassical algebra* if $\mathcal{A}$ is a polynomial classical algebra; it is an *entire semiclassical algebra* if $\mathcal{A}$ is an entire classical algebra, all ^{*_n} are continuous and $\star_n$ are joint continuous. A polynomial or entire semiclassical algebra is *differential* if all $\star_n$ are differential.

All the semiclassical algebras from the examples 1.2 and 1.3 are differential. However, it is easy to construct non-differential examples.

The *quantum commutator* $\frac{i}{\hbar}[\ ,\ ]$ in $\mathcal{A}[[\hbar]]$ is a counterpart of the Poisson bracket:

$$
\frac{i}{\hbar}[f_\hbar, g_\hbar] = \sum_{n=0}^{\infty} \hbar^n \sum_{j+k+l=n} \{f^{(k)}, g^{(l)}\}_{1+j} \tag{1.22}
$$

for $f_\hbar, g_\hbar \in \mathcal{A}[[\hbar]]$ (where $\{\ ,\ \}_n$ is given by (1.15)). Thus, informally, $\frac{i}{\hbar}[f_\hbar, g_\hbar] = \frac{i}{\hbar}(f_\hbar \star_\hbar g_\hbar - g_\hbar \star_\hbar f_\hbar)$. The well-known Moyal bracket (see e.g. [1], [19]) is the quantum commutator for Weyl semiclassical algebras.

Proposition 1.5 *Let $\mathcal{A}$ be a semiclassical algebra. We have*
(i) (*Jacobi formula in $\mathcal{A}[[\hbar]]$*) *for $f_\hbar, g_\hbar, h_\hbar \in \mathcal{A}[[\hbar]]$*

$$\frac{i}{\hbar}\left[\frac{i}{\hbar}[f_\hbar, g_\hbar], h_\hbar\right] + \frac{i}{\hbar}\left[\frac{i}{\hbar}[h_\hbar, f_\hbar], g_\hbar\right] + \frac{i}{\hbar}\left[\frac{i}{\hbar}[g_\hbar, h_\hbar], f_\hbar\right] = 0; \tag{1.23}$$

(ii) *for any $f, g, h \in \mathcal{A}$, $n \in \mathbb{N}$*

$$\sum_{k+l=n} \{\{f,g\}_{1+k}, h\}_{1+l} + \{\{h,f\}_{1+k}, g\}_{1+l} + \{\{g,h\}_{1+k}, f\}_{1+l} = 0; \tag{1.24}$$

(iii)

$$\mathbb{1}^{*_\hbar} = \mathbb{1}; \tag{1.25}$$

(iv) *for any $f_\hbar, g_\hbar \in \mathcal{A}[[\hbar]]$*

$$\left(\frac{i}{\hbar}[f_\hbar, g_\hbar]\right)^{*_\hbar} = \frac{i}{\hbar}[f_\hbar^{*_\hbar}, g_\hbar^{*_\hbar}]. \tag{1.26}$$

Definition 1.9 Let $\mathcal{A}$ be a semiclassical algebra. A family $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ of elements of $\mathcal{A}^m$ satisfies *quantum canonical commutation relations with corrections* (abbreviated to *qccr+c*) if for some $D \in \mathcal{M}_m(\mathbb{C})$ and for all $i, j = 1, \ldots, m$

$$\sum_{k+l+s=n} \{\boldsymbol{f}_i^{(k)}, \boldsymbol{f}_j^{(l)}\}_{1+s} = \begin{cases} D_{ij}\mathbb{1} & \text{for } n = 0 \\ 0 & \text{for } n > 0. \end{cases} \tag{1.27}$$

If (1.27) holds we say that $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ satisfies *qccr+c of D-type*. A system $\boldsymbol{f} \in \mathcal{A}^m$ satisfies *quantum canonical commutation relations* (abbreviated to *qccr*) if for some $D \in \mathcal{M}_m(\mathbb{C})$ and for all $i, j = 1, \ldots, m$

$$\{\boldsymbol{f}_i, \boldsymbol{f}_j\}_n = \begin{cases} D_{ij}\mathbb{1} & \text{for } n = 1 \\ 0 & \text{for } n > 1. \end{cases} \tag{1.28}$$

Analogically, when (1.28) holds we say that $\boldsymbol{f}$ satisfies *qccr of D-type*. A system $\boldsymbol{f} \in \mathcal{A}^m$ *is quantizable* if there exists $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ satisfying qccr+c with $\boldsymbol{f}^{(0)} = \boldsymbol{f}$; a family $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ is then a *quantization* of $\boldsymbol{f}$ and $\boldsymbol{f}^{(n)}$ for $n \geq 1$ are *quantum corrections to* $\boldsymbol{f}$. We say that $\boldsymbol{f}$ is *selfadjoint quantizable* if it has a quantization which is a selfadjoint family (which we call a *selfadjoint quantization*).

For a family $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ consider a system $(f_{\hbar,1}, \ldots, f_{\hbar,m}) \in (\mathcal{A}[[\hbar]])^m$ given by

$$f_{\hbar,i} = \sum_{n=0}^{\infty} \hbar^n \boldsymbol{f}_i^{(n)}. \tag{1.29}$$

Proposition 1.6 *If $\mathcal{A}$ is a semiclassical algebra, then $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ satisfies qccr+c of D-type iff $\frac{i}{\hbar}[f_{\hbar,i}, f_{\hbar,j}] = D_{ij}\mathbb{1}$ for $i, j = 1, \ldots, m$.*

Thus qccr+c are analogue of operator canonical commutation relations in a Hilbert space. Note that $\boldsymbol{f}$ satisfies qccr of D-type iff the family $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ given by $\boldsymbol{f}^{(0)} = \boldsymbol{f}$ and $\boldsymbol{f}^{(n)} = 0$ for $n > 0$ satisfies qccr+c of D-type. Roughly speaking, qccr is qccr+c with zero corrections. Hence, if $\boldsymbol{f}$ satisfies qccr, then $\boldsymbol{f}$ is quantizable. If $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ satisfies qccr+c of D-type, then $\boldsymbol{f}^{(0)}$ satisfies cccr of D-type. The inverse fact on quantizability of systems satisfying cccr is one of the main problems considered in this paper (see Theorem 2). The system $(\boldsymbol{q}, \boldsymbol{p})$ in M-semiclassical algebras and $(\boldsymbol{\eta}, \boldsymbol{\xi})$ in $\eta\xi$-Weyl semiclassical algebra are quantizable since they satisfy qccr.

We define now *semiclassical unitary transformations*, which are in some sense quantizations of canonical transformations. In the quantum case, for a unitary operator U acting in a Hilbert space, we can consider the transformation T_U, $T_U A = U^{-1}AU$, acting on quantum observables (operators) A. Thus T_U preserves the algebraic structure of "observables algebra". A similar property defines semiclassical unitary transformations.

Definition 1.10 A family $\{t_n\}_{n\in\mathbb{N}}$ of linear transformations between semiclassical algebras $\mathcal{A}$ and $\mathcal{B}$ is a *semiclassical unitary transformation* of $\mathcal{A}$ into $\mathcal{B}$ if

(i) for any $f, g \in \mathcal{A}$, $n \in \mathbb{N}$

$$\sum_{j+k=n} t_k(f \star_j g) = \sum_{j+k+l=n} t_k f \star_j t_l g; \tag{1.30}$$

(ii)

$$t_n(\mathbb{1}) = \begin{cases} \mathbb{1} & \text{for } n = 0 \\ 0 & \text{for } n > 0, \end{cases} \tag{1.31}$$

(iii) for any $f \in \mathcal{A}$, $n \in \mathbb{N}$

$$\sum_{j+k=n} t_k f^{*_j} = \sum_{j+k=n} (t_k f)^{*_j}. \tag{1.32}$$

A transformation $t : \mathcal{A} \longrightarrow \mathcal{B}$ is *quantizable* if there exists a semiclassical unitary transformation $\{t_n\}_{n\in\mathbb{N}}$ of $\mathcal{A}$ into $\mathcal{B}$ such that $t_0 = t$; $\{t_n\}_{n\in\mathbb{N}}$ is then a *quantization* of t. The transformations t_n for $n \in \mathbb{N}$ are *coefficients* of the semiclassical unitary transformation or, for $n > 0$, the *quantum corrections to* t.

Let $\{t_n\}_{n\in\mathbb{N}}$ be a family of linear transformations of $\mathcal{A}$ into $\mathcal{B}$. We consider the transformation $t_\hbar : \mathcal{A}[[\hbar]] \longrightarrow \mathcal{B}[[\hbar]]$ given by

$$t_\hbar f_\hbar = \sum_{n=0}^{\infty} \hbar^n \sum_{j+k=n} t_j f^{(k)}. \tag{1.33}$$

Proposition 1.7 *If $\mathcal{A}$ and $\mathcal{B}$ are semiclassical algebras, then $\{t_n\}_{n\in\mathbb{N}}$ is a semiclassical unitary transformation of $\mathcal{A}$ into $\mathcal{B}$ iff $t_\hbar$ is a *homomorphism of the formal semiclassical algebra $\mathcal{A}[[\hbar]]$ into $\mathcal{B}[[\hbar]]$.*

If $\{t_n\}_{n\in\mathbb{N}}$ is a semiclassical unitary transformation, then we call $t_\hbar$ a *formal unitary transformation.* By Proposition 1.7

$$t_\hbar\left(\frac{i}{\hbar}[f_\hbar, g_\hbar]\right) = \frac{i}{\hbar}[t_\hbar f_\hbar, t_\hbar g_\hbar] \tag{1.34}$$

for $f_\hbar, g_\hbar \in \mathcal{A}[[\hbar]]$ or, equivalently, for $f, g \in \mathcal{A}$, $n \in \mathbb{N}$

$$\sum_{j+k=n} t_k\{f, g\}_{1+j} = \sum_{j+k+l=n} \{t_k f, t_l g\}_{1+j}. \tag{1.35}$$

In particular we obtain canonicity of t_0. The inverse fact on quantizability of canonical transformations is one of the main subjects of this paper (see Theorem 3).

Consider semiclassical algebras $\mathcal{A}$, $\mathcal{B}$. Let $\{e^{(n)}\}_{n\in\mathbb{N}}$ and $\{f^{(n)}\}_{n\in\mathbb{N}}$ be families of elements of $\mathcal{A}^m$ and $\mathcal{B}^m$ respectively and let $\{t_n\}_{n\in\mathbb{N}}$ be a family of linear transformations of $\mathcal{A}$ into $\mathcal{B}$. Then $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ *is an image of* $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$ *by* $\{t_n\}_{n\in\mathbb{N}}$ if

$$\sum_{j+k=n} t_k \boldsymbol{e}_i^{(j)} = \boldsymbol{f}_i^{(n)} \tag{1.36}$$

for $i = 1, \dots, m$, $n \in \mathbb{N}$. The above can be also written in the form $t_\hbar e_{\hbar,i} = f_{\hbar,i}$, where $e_{\hbar,i}$, $f_{\hbar,i}$ are given by (1.29). Hence, if $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$ satisfies qccr+c of D-type, $\{t_n\}_{n\in\mathbb{N}}$ is a semiclassical unitary transformation and $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ is an image of $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$ by $\{t_n\}_{n\in\mathbb{N}}$, then by (1.34) $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ also satisfies qccr+c of D-type.

Consider semiclassical algebras $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ with a family $\{t_n\}_{n\in\mathbb{N}}$ of linear transformations of $\mathcal{A}$ into $\mathcal{B}$ and $\{s_n\}_{n\in\mathbb{N}}$ of $\mathcal{B}$ into $\mathcal{C}$. The family $\{u_n\}_{n\in\mathbb{N}} = \{\sum_{j+k=n} s_j \circ t_k\}_{n\in\mathbb{N}}$ is the *superposition* of $\{s_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$; we denote it by $\{s_n\}_{n\in\mathbb{N}} \circ \{t_n\}_{n\in\mathbb{N}}$. On the level of formal semiclassical algebras the above simply means that $u_\hbar = s_\hbar \circ t_\hbar$, hence superposition is associative. When $\mathcal{A} = \mathcal{C}$, then the family $\{s_n\}_{n\in\mathbb{N}}$ is the *inverse* of $\{t_n\}_{n\in\mathbb{N}}$, if $\{s_n\}_{n\in\mathbb{N}} \circ \{t_n\}_{n\in\mathbb{N}} = \{Id_{\mathcal{A},n}\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}} \circ \{s_n\}_{n\in\mathbb{N}} = \{Id_{\mathcal{B},n}\}_{n\in\mathbb{N}}$, where $Id_{\mathcal{A},n} = Id_{\mathcal{A}}$ for $n = 0$ and $Id_{\mathcal{A},n} = 0$ for $n > 0$ and similarly for $\mathcal{B}$. If the inverse of $\{t_n\}_{n\in\mathbb{N}}$ exists, then we denote it by $\{t_n\}_{n\in\mathbb{N}}^{-1}$ and we say that $\{t_n\}_{n\in\mathbb{N}}$ is *invertible*. Obviously, $\{s_n\}_{n\in\mathbb{N}} = \{t_n\}_{n\in\mathbb{N}}^{-1}$ iff $s_\hbar = t_\hbar^{-1}$. By Proposition 1.7, superpositions and the inverses of semiclassical unitary transformations are also semiclassical unitary transformations.

Definition 1.11 An invertible semiclassical unitary transformation is a *semiclassical unitary isomorphism.* The formal unitary transformation corresponding to a semiclassical unitary isomorphism is a *formal unitary isomorphism.*

If $\{t_n\}_{n\in\mathbb{N}}$ is a semiclassical unitary isomorphism, then t_0 is a canonical isomorphism. Moreover we have:

Proposition 1.8 *If* $\{t_n\}_{n\in\mathbb{N}}$ *is a quantization of a canonical isomorphism* t_0*, then* $\{t_n\}_{n\in\mathbb{N}}$ *is a semiclassical unitary isomorphism and* $\{t_n\}_{n\in\mathbb{N}}^{-1}$ *is a quantization of* t_0^{-1}.

2. The semiclassical von Neumann theorem

In this section we prove a semislassical version of von Neumann theorem on unitary equivalence of quantum operator commutation relations in

Hilbert space (see [21, th. VIII.14]), Theorem 1 below.

In formal semiclassical algebras terms the idea of the semiclassical von Neumann theorem may be expressed as follows: *under some "technical assumptions", if* $(e_{\hbar,1},\dots,e_{\hbar,m})$, $(f_{\hbar,1},\dots,f_{\hbar,m})$ *are real systems of elements of formal semiclassical algebras* $\mathcal{A}[[\hbar]]$, $\mathcal{B}[[\hbar]]$ *respectively and the commutation relations*

$$\frac{i}{\hbar}[e_{\hbar,i},e_{\hbar,j}] = D_{i,j}\mathbb{1}, \quad \frac{i}{\hbar}[f_{\hbar,i},f_{\hbar,j}] = D_{i,j}\mathbb{1}$$

hold for $i,j=1,\dots,m$ *and some* $D\in\mathcal{M}_m(\mathbb{C})$, *then there exists a formal unitary transformation* $t_\hbar:\mathcal{A}[[\hbar]]\longrightarrow\mathcal{B}[[\hbar]]$, *such that* $t_\hbar e_{\hbar,i}=f_{\hbar,i}$ *for* $i=1,\dots,m$. We thus need a tool to construct a proper semiclassical unitary transformation.

2.1. The inductive lemma

The following lemma is our main technical tool. We shall say that a condition with a natural parameter n is satisfied *on the level* s, if it is satisfied for $n=s$.

Lemma 2.1 (The inductive lemma) *Suppose that* $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$, $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ *satisfy qccr+c of the same type in semiclassical algebras* $\mathcal{A}$, $\mathcal{B}$ *respectively and that* $\boldsymbol{e}^{(0)}\in\mathcal{A}^m$ *is independent. If* $n\geq 1$ *and for any* $k=0,\dots,n-1$ *there exist linear transformations* $t_k:\mathcal{A}\longrightarrow\mathcal{B}$ *satisfying* (1.30), (1.31) *and* (1.36) *on the levels* $0,\dots,n-1$, *then there exists the exactly one linear transformation* $t'_n:\mathcal{A}'\longrightarrow\mathcal{B}$, *where* $\mathcal{A}'=\mathrm{alg}(\boldsymbol{e}^{(0)})$, *satisfying*

(i) $t'_n(fg)=t'_nf\cdot t_0g+t_0f\cdot t'_ng-P_n(f,g)$ *for* $f,g\in\mathcal{A}'$, *where* $P_n:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{B}$,

$$P_n(f,g)=\sum_{\substack{j+k=n\\ k\neq n}} t_k(f\star_j g)-\sum_{\substack{j+k+l=n\\ k,l\neq n}} t_kf\star_j t_lg; \tag{2.1}$$

(ii) $t'_n\mathbb{1}=0$;

(iii) $t'_n\boldsymbol{e}^{(0)}=\widetilde{\boldsymbol{f}}^{(n)}$, *where* $\widetilde{\boldsymbol{f}}^{(n)}\in\mathcal{B}^m$, $\widetilde{\boldsymbol{f}}^{(n)}=\boldsymbol{f}^{(n)}-\sum_{\substack{j+k=n\\ k\neq n}} t_k\boldsymbol{e}^{(j)}$.

Proof. The uniqueness of the choice of t'_n is immediate, since $\mathrm{alg}(\boldsymbol{e}^{(0)})=\mathcal{A}'$. We prove the existence of t'_n employing the idea from the theory of ordinary differential equations, where the extended phase space is constructed to replace the problem of solving of a non-autonomous equation by the problem of solving of the corresponding autonomous equation. Therefore we first find the linear transformation $\varphi : \mathcal{A}' \longrightarrow \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}} = \mathcal{A} \times \mathcal{B}$, satisfying

(i′) $\varphi(fg) = \varphi(f) \odot \varphi(g)$ for $f, g \in \mathcal{A}'$, where the operation $\odot : \widetilde{\mathcal{A}} \times \widetilde{\mathcal{A}} \longrightarrow \widetilde{\mathcal{A}}$ is defined for $f, g \in \mathcal{A}$, $f', g' \in \mathcal{B}$ by

$$(f, f') \odot (g, g') = (fg,\ f' \cdot t_0 g + t_0 f \cdot g' - P_n(f, g)); \tag{2.2}$$

(ii′) $\varphi(\mathbb{1}) = (\mathbb{1}, 0)$;

(iii′) $\varphi(\boldsymbol{e}_i^{(0)}) = (\boldsymbol{e}_i^{(0)}, \widetilde{\boldsymbol{f}}_i^{(n)})$ for any $i = 1, \ldots, m$.

If φ is as above, then it has the form $\varphi = (\varphi_1, \varphi_2)$, and by (i′) and (ii′) $\varphi_1 : \mathcal{A}' \longrightarrow \mathcal{A}$ is a homomorphism of the algebras. Since $\mathcal{A}' = \operatorname{alg}(\boldsymbol{e}^{(0)})$ and (iii′) holds, $\varphi_1(f) = f$ for $f \in \mathcal{A}'$. We can thus define $t_n' = \varphi_2$ and by (i′), (ii′), (iii′) it satisfies the conditions (i), (ii), (iii).

We have to prove the existence of φ. Suppose that $\widetilde{\mathcal{A}}$ is the algebra with the multiplication $\odot$ and with the unity $(\mathbb{1}, 0)$, in which $(\boldsymbol{e}_i^{(0)}, \widetilde{\boldsymbol{f}}_i^{(n)})$ commute for $i = 1, \ldots, m$. Then the above conditions on φ exactly mean that φ is a homomorphism of the algebras with values on $\boldsymbol{e}^{(0)}$ fixed by (iii′). Observe that by the independence of $\boldsymbol{e}^{(0)}$ the algebra $\mathcal{A}'$ is a polynomial algebra and $\boldsymbol{e}^{(0)}$ is its algebraic base. The existence of φ follows then from Proposition 1.1, provided our suppositions on $\widetilde{\mathcal{A}}$ are true.

We first check that $(\boldsymbol{e}_r^{(0)}, \widetilde{\boldsymbol{f}}_r^{(n)})$ and $(\boldsymbol{e}_s^{(0)}, \widetilde{\boldsymbol{f}}_s^{(n)})$ commutes under $\odot$. By (2.2) we can write this as $P_n(\boldsymbol{e}_r^{(0)}, \boldsymbol{e}_s^{(0)}) = P_n(\boldsymbol{e}_s^{(0)}, \boldsymbol{e}_r^{(0)})$, and this is, by (2.1) and (1.15), equivalent to

$$\sum_{i+j=n-1} t_i \{\boldsymbol{e}_r^{(0)}, \boldsymbol{e}_s^{(0)}\}_{1+j} = \sum_{i+k+z=n-1} \{t_i \boldsymbol{e}_r^{(0)}, t_k \boldsymbol{e}_s^{(0)}\}_{1+z}. \tag{2.3}$$

The above condition corresponds to (1.35) on the level $n-1$ for $\boldsymbol{e}_r^{(0)}$, $\boldsymbol{e}_s^{(0)}$ (thus it would follow from (1.30) on the level n, but we assumed (1.30) only on the levels $0, \ldots, n-1$). To prove (2.3), consider first the case $n > 1$ and transform the LHS of (2.3) using the fact that $\{\boldsymbol{e}^{(n)}\}_{n \in \mathbb{N}}$ satisfies qccr+c and that $t_j \mathbb{1} = 0$ for $j > 0$, and next using (1.35) on the levels $0, \ldots, n-2$ (which holds by (1.30) on the levels $1, \ldots, n-1$):

$$\sum_{i+j=n-1} t_i \{\boldsymbol{e}_r^{(0)}, \boldsymbol{e}_s^{(0)}\}_{1+j} = - \sum_{i+j=n-1} t_i \sum_{\substack{u+w+z=j \\ z \neq j}} \{\boldsymbol{e}_r^{(u)}, \boldsymbol{e}_s^{(w)}\}_{1+z}$$

$$= - \sum_{\substack{i+u+w+z=n-1\\ u+w\neq 0}} t_i\{\boldsymbol{e}_r^{(u)}, \boldsymbol{e}_s^{(w)}\}_{1+z}$$

$$= - \sum_{0<u+w\leq n-1} \sum_{i+z=n-1-(u+w)} t_i\{\boldsymbol{e}_r^{(u)}, \boldsymbol{e}_s^{(w)}\}_{1+z}$$

$$= - \sum_{\substack{i+k+z+u+w=n-1\\ u+w\neq 0}} \{t_i\boldsymbol{e}_r^{(u)}, t_k\boldsymbol{e}_s^{(w)}\}_{1+z}.$$

Using (1.36) on the levels $0, \ldots, n-1$ and the fact that $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ satisfies qccr+c, we also have

$$0 = \sum_{j+z+l=n-1} \{\boldsymbol{f}_r^{(j)}, \boldsymbol{f}_s^{(l)}\}_{1+z} = \sum_{j+z+l=n-1} \sum_{\substack{k+w=l\\ i+u=j}} \{t_i\boldsymbol{e}_r^{(u)}, t_k\boldsymbol{e}_s^{(w)}\}_{1+z}$$

$$= \sum_{\substack{i+k+z+u+w=n-1\\ u+w\neq 0}} \{t_i\boldsymbol{e}_r^{(u)}, t_k\boldsymbol{e}_s^{(w)}\}_{1+z} + \sum_{i+k+z=n-1} \{t_i\boldsymbol{e}_r^{(0)}, t_k\boldsymbol{e}_s^{(0)}\}_{1+z},$$

and this equality together with the previous one implies (2.3) for $n > 1$. When $n = 1$ it is sufficient to use the condition $t_0\boldsymbol{e}^{(0)} = \boldsymbol{f}^{(0)}$ and the fact that $\boldsymbol{e}^{(0)}$, $\boldsymbol{f}^{(0)}$ satisfy cccr of the same type.

The bilinearity of $\odot$ and the neutrality of $(\mathbb{1}, 0) \in \widetilde{\mathcal{A}}$ is evident, so it remains to prove the associativity of $\odot$. By the equality $t_0(fg) = t_0 f \cdot t_0 g$ ((1.30) on the level 0) it suffices to show that for $f, g, h \in \mathcal{A}$

$$P_n(fg, h) - P_n(f, gh) - t_0 f \cdot P_n(g, h) + P_n(f, g) \cdot t_0 h = 0. \qquad (2.4)$$

Thus, by (2.1) we must prove that $L_I + L_{II} + R_I + R_{II} = 0$, where

$$L_I = \sum_{\substack{i+j=n\\ i\neq n}} t_i[(fg)\star_j h - f\star_j(gh)];$$

$$L_{II} = \sum_{\substack{i+j=n\\ i\neq n}} t_i(f\star_j g) \cdot t_0 h - t_0 f \cdot t_i(g\star_j h);$$

$$R_I = \sum_{\substack{i+j+k=n\\ i,k\neq n}} t_i f\star_j t_k(gh) - t_k(fg)\star_j t_i h;$$

$$R_{II} = \sum_{\substack{i+j+k=n\\ i,k\neq n}} t_0 f \cdot (t_i g\star_j t_k h) - (t_k f\star_j t_i g) \cdot t_0 h.$$

Using (1.12) for f, g, h we have

$$
\begin{aligned}
0=\sum_{i=0}^{n-1}0 &= \sum_{i=0}^{n-1} t_i[(fg)\star_{(n-i)} h - f\star_{(n-i)}(gh)] \\
&\quad + \sum_{k=1}^{n}\sum_{i=0}^{n-k} t_i[(f\star_k g)\star_{(n-i-k)} h - f\star_{(n-i-k)}(g\star_k h)] \\
&= L_I + \sum_{k=1}^{n}\sum_{r+u+s=n-k} t_r(f\star_k g)\star_u t_s h - t_s f\star_u t_r(g\star_k h),
\end{aligned}
$$

where the last equality follows from (1.30) on the levels $0,\dots,n-1$ for $f\star_k g$, h and for f, $g\star_k h$. Thus we have

$$
\sum_{(r,u,s,k)\in A} t_r(f\star_k g)\star_u t_s h - t_s f\star_u t_r(g\star_k h) = -L_I, \tag{2.5}
$$

where $A=\{(r,u,s,k)\in\mathbb{N}^4 : r+u+s+k=n,\ k>0\}$. By (1.12) for $t_k f$, $t_l g$, $t_m h$ we also have

$$
\begin{aligned}
0=\sum_{i=0}^{n}\sum_{\substack{r+l+s=i\\ r,l,s\neq n}} 0 &= \sum_{i=0}^{n}\sum_{\substack{r+l+s=i\\ r,l,s\neq n}}\sum_{k+u=n-i} (t_r f\star_k t_l g)\star_u t_s h - t_r f\star_u(t_l g\star_k t_s h) \\
&= \sum_{(r,u,s,k,l)\in C\cup D} (t_r f\star_k t_l g)\star_u t_s h - t_s f\star_u(t_r g\star_k t_l h),
\end{aligned} \tag{2.6}
$$

with

$$
\begin{aligned}
C &= \{(r,u,s,k,l)\in\mathbb{N}^5 : r+u+s+k+l=n,\ r+k+l\neq n,\ s\neq n\},\\
D &= \{(r,u,s,k,l)\in\mathbb{N}^5 : r+k+l=n,\ s=0,\ k=0,\ r\neq n,\ l\neq n\},
\end{aligned}
$$

where in the last equality we have used the symmetry in r, l, s and the formula

$$
\begin{aligned}
&C\cup D\\
&\quad=\{(r,u,s,k,l)\in\mathbb{N}^5 : r+u+s+k+l=n,\ r\neq n,\ l\neq n,\ s\neq n\}.
\end{aligned}
$$

By (1.30) on the levels $0,\dots,n-1$ for f, g and for g, h we have respectively

$$
0=\sum_{\substack{j+u+s=n\\ j\neq n, s\neq n}} 0 = \sum_{\substack{j+u+s=n\\ j\neq n, s\neq n}} \left[\sum_{r+k=j} t_r(f\star_k g) - \sum_{r+k+l=j} t_r f\star_k t_l g\right]\star_u t_s h;
$$

$$0=\sum_{\substack{j+u+s=n\\ j\neq n, s\neq n}} 0=\sum_{\substack{j+u+s=n\\ j\neq n, s\neq n}} t_s f \star_u \left[\sum_{r+k=j} t_r(g\star_k h) - \sum_{r+k+l=j} t_r g\star_k t_l h\right].$$

Since $C\cap D=\emptyset$, substracting the above equalities and using (2.6), we get

$$\begin{aligned}\sum_{(r,u,s,k)\in B} & t_r(f\star_k g)\star_u t_s h - t_s f\star_u t_r(g\star_k h)\\ &= \sum_{(r,u,s,k,l)\in C} (t_r f\star_k t_l g)\star_u t_s h - t_s f\star_u (t_r g\star_k t_l h)\\ &= -\sum_{(r,u,s,k,l)\in D} (t_r f\star_k t_l g)\star_u t_s h - t_s f\star_u (t_r g\star_k t_l h) = R_{II}, \qquad (2.7)\end{aligned}$$

where $B=\{(r,u,s,k)\in\mathbb{N}^4 : r+u+s+k=n,\ r+k\neq n,\ s\neq n\}$. Let

$$\begin{aligned}A_1 &= A\setminus B = \{(r,u,s,k)\in\mathbb{N}^4 : r+k=n,\ k>0,\ u=s=0\},\\ A_2 &= B\setminus A = \{(r,u,s,k)\in\mathbb{N}^4 : r+u+s=n,\ r\neq n,\ s\neq n,\ k=0\}.\end{aligned}$$

As $A\setminus A_1 = A\cap B = B\setminus A_2$, we can replace the sum "$\sum_{(r,u,s,k)\in A}$" in the LHS of (2.5) by "$\sum_{(r,u,s,k)\in B}+\sum_{(r,u,s,k)\in A_1}-\sum_{(r,u,s,k)\in A_2}$" and then, by (2.7), this sum is equal to $R_{II}+L_{II}+R_I$, which establishes (2.4). □

Note, that if $\mathcal{A}=\mathcal{A}'$ and t_0 is invertible, then (2.4) from the above proof means, that $t_0^{-1}P_n$ is a Hochschild 2-cocycle and (i) of Lemma 2.1 means, that this 2-cocycle is exact (see e.g. [14])

We can prove now the corollary, a "simple version" of the semiclassical von Neumann theorem.

Corollary 2.1 *Consider a semiclassical algebra $\mathcal{B}$ and a polynomial semiclassical algebra $\mathcal{A}$. If $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$, $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ are selfadjoint families satisfying qccr+c of the same type in $\mathcal{A}$ and $\mathcal{B}$ respectively and $\boldsymbol{e}^{(0)}$ is a canonical base of $\mathcal{A}$, then there exists a unique semiclassical unitary transformation $\{t_n\}_{n\in\mathbb{N}}$ such that $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ is an image of $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$ by $\{t_n\}_{n\in\mathbb{N}}$.*

Proof. Since we have $\mathcal{A}'=\mathcal{A}$, using Proposition 1.1 and the inductive lemma we can obtain the existence of t_0 and then of t_n-s, such that the corresponding $t_\hbar$ is the unique homomorphism of $\mathcal{A}[[\hbar]]$ into $\mathcal{B}[[\hbar]]$ of the form (1.33), for which $t_\hbar e_{\hbar,i}=f_{\hbar,i}$ (where $e_{\hbar,i}$, $f_{\hbar,i}$ are connected with $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$, $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ by (1.29)). Moreover, by Proposition 1.4 all $e_{\hbar,i}$ and $f_{\hbar,i}$ are real. By Proposition 1.7, it suffices to prove that $t_\hbar f_\hbar^{*\hbar}=(t_\hbar f_\hbar)^{*\hbar}$

for $f_\hbar \in \mathcal{A}[[\hbar]]$. Consider $s_\hbar : \mathcal{A}[[\hbar]] \longrightarrow \mathcal{B}[[\hbar]]$ given by $s_\hbar f_\hbar = (t_\hbar f_\hbar^{*\hbar})^{*\hbar}$ for $f_\hbar \in \mathcal{A}[[\hbar]]$. From the properties of an ivolution we see that $s_\hbar$ is also a homomorphism of $\mathcal{A}[[\hbar]]$ into $\mathcal{B}[[\hbar]]$ satisfying $s_\hbar e_{\hbar,i} = f_{\hbar,i}$ and it has the form (1.33). Thus, by the unicity, we have $s_\hbar = t_\hbar$, which finishes the proof. □

This corollary doesn't give the explicit formulas for the coefficients t_n of the semiclassical unitary transformation. We shall obtain such formulas in the "main" semiclassical von Neumann theorem.

2.2. The explicit formulas for polynomial and entire algebras

We formulate here the main result of this section.

Theorem 1 (The semiclassical von Neumann theorem) *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are both polynomial semiclassical algebras or both entire differential semiclassical algebras, $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$ and $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ are selfadjoint families satisfying qccr+c of the same type in $\mathcal{A}$ and $\mathcal{B}$ respectively and $\boldsymbol{e}^{(0)}$, $\boldsymbol{f}^{(0)}$ are canonical bases of $\mathcal{A}$ and $\mathcal{B}$ respectively. Then there exists a unique semiclassical unitary transformation $\{t_n\}_{n\in\mathbb{N}}$ with continuous coefficients in the entire differential case such that $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ is an image of $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$ by $\{t_n\}_{n\in\mathbb{N}}$. Moreover, t_0 is a canonical isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ given by the formula*

$$t_0 = \phi_{\boldsymbol{f}^{(0)}}\phi^{-1}_{\boldsymbol{e}^{(0)}} \tag{2.8}$$

and t_n for $n \in \mathbb{N}$ have the form $t_n = t_0 D_n$, where $D_n : \mathcal{A} \longrightarrow \mathcal{A}$ are recurrently defined by

$$D_0 = Id_{\mathcal{A}}, \quad D_n = -\mathcal{Z}_{\boldsymbol{e}^{(0)}}(R_n) + \sum_{j=1}^{m} \tilde{e}_j^{(n)} \partial_{e_j^{(0)}} \tag{2.9}$$

for $n \geq 1$, where $R_n : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$,

$$R_n(f,g) = \sum_{\substack{j+k=n \\ k\neq n}} D_k(f\star_j g) - \sum_{\substack{j+k+l=n \\ k,l\neq n}} (D_k f)\tilde{\star}_j(D_l g), \tag{2.10}$$

with $\tilde{\star}_j : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ given for $f, g \in \mathcal{A}$, $j \in \mathbb{N}$ by

$$f\tilde{\star}_j g = t_0^{-1}(t_0 f \star_j t_0 g) \tag{2.11}$$

and where $\widetilde{\boldsymbol{e}}^{(n)} \in \mathcal{A}^m$ *are given by*

$$\widetilde{\boldsymbol{e}}^{(n)} = t_0^{-1}\boldsymbol{f}^{(n)} - \sum_{\substack{j+k=n \\ k\neq n}} D_k \boldsymbol{e}^{(j)}. \tag{2.12}$$

In the entire differential case D_n *are differential operators.*

Proof. Observe first that $\boldsymbol{e}^{(0)}$ and $\boldsymbol{f}^{(0)}$ are real and thus, by Proposition 1.3, t_0 defined by (2.8) is a canonical isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ and by definition is a homeomorphism in the entire differential case. We prove by induction that $\{t_n\}_{n\in\mathbb{N}}$ is well-defined, that it satisfies (1.30), (1.31), (1.36), and that D_n are differential in the entire differential case. The zero step is already done; suppose that the above is true on the levels $0,\ldots,n-1$. By Lemma 2.1 there exists $t'_n : \mathcal{A}' = \mathrm{alg}(\boldsymbol{e}^{(0)}) \longrightarrow \mathcal{B}$ satisfying (i), (ii), (iii). By (2.1), (2.10) and by our inductive assumptions we have $R_n = t_0^{-1}P_n$. Let us define $D'_n : \mathcal{A}' \longrightarrow \mathcal{A}$ and $R'_n : \mathcal{A}' \times \mathcal{A}' \longrightarrow \mathcal{A}$ by the formulas

$$D'_n = t_0^{-1}t'_n, \qquad R'_n = R_{n|\mathcal{A}'\times\mathcal{A}'}.$$

By (i) we have

$$D'_n(fg) - D'_n f\cdot g - f\cdot D'_n g = -R'_n(f,g) \tag{2.13}$$

for $f,g\in\mathcal{A}'$. By Proposition 1.2 the operators D'_n and R'_n can be written in the form

$$D'_n f = \sum_{\alpha\in\mathbb{N}^m} d_\alpha \partial^\alpha_{\boldsymbol{e}^{(0)}} f, \quad R'_n(f,g) = \sum_{\gamma,\gamma'\in\mathbb{N}^m} r_{\gamma,\gamma'}\partial^\gamma_{\boldsymbol{e}^{(0)}} f\cdot \partial^{\gamma'}_{\boldsymbol{e}^{(0)}} g,$$

with $d_\alpha, r_{\gamma,\gamma'}\in\mathcal{A}$ for $f,g\in\mathcal{A}'$. Thus we can rewrite (2.13) as

$$\sum_{\gamma,\gamma'\neq 0} d_{\gamma+\gamma'}\binom{\gamma+\gamma'}{\gamma}\partial^\gamma_{\boldsymbol{e}^{(0)}} f\cdot\partial^{\gamma'}_{\boldsymbol{e}^{(0)}} g = -\sum_{\gamma,\gamma'\in\mathbb{N}^m} r_{\gamma,\gamma'}\partial^\gamma_{\boldsymbol{e}^{(0)}} f\cdot \partial^{\gamma'}_{\boldsymbol{e}^{(0)}} g$$

and hence, by the uniqueness of the form (1.3) of an operator, we have $d_{\gamma+\gamma'}\binom{\gamma+\gamma'}{\gamma} = -r_{\gamma,\gamma'}$ for $\gamma,\gamma'\in\mathbb{N}^m\setminus\{0\}$ and $r_{\gamma,\gamma'}=0$ if $\gamma=0$ or $\gamma'=0$. Therefore, for any $\alpha\in\mathbb{N}^m\setminus\{0\}$

$$-\sum_{\gamma+\gamma'=\alpha} r_{\gamma,\gamma'} = -\sum_{\substack{\gamma+\gamma'=\alpha\\ \gamma,\gamma'\neq 0}} r_{\gamma,\gamma'} = d_\alpha \sum_{\substack{\gamma+\gamma'=\alpha\\ \gamma,\gamma'\neq 0}}\binom{\alpha}{\gamma}$$

$$= d_\alpha \sum_{\substack{\gamma\neq 0,\alpha \\ \gamma\leq\alpha}} \binom{\alpha}{\gamma} = d_\alpha(2^{|\alpha|}-2).$$

By the above, we can express D'_n by R'_n and by a first-order differential operator using the operation $\mathcal{Z}_{\boldsymbol{e}^{(0)}}$ (see (1.5)). We have

$$D'_n = -\mathcal{Z}_{\boldsymbol{e}^{(0)}}(R'_n) + \sum_{|\alpha|\leq 1} d_\alpha \partial^\alpha_{\boldsymbol{e}^{(0)}}.$$

The coefficients d_α for $|\alpha| \leq 1$ are determined by the conditions (ii) and (iii) of Lemma 2.1, which imply that $D'_n \mathbb{1} = 0$ and $D'_n \boldsymbol{e}^{(0)} = t_0^{-1}\widetilde{\boldsymbol{f}}^{(n)}$ with $\widetilde{\boldsymbol{f}}^{(n)}$ defined in (iii) of Lemma 2.1. Thus $d_0 = 0$ and by (2.12) $d_{\boldsymbol{1}_j} = \widetilde{\boldsymbol{e}}^{(n)}_j$. Finally we obtain

$$D'_n = -\mathcal{Z}_{\boldsymbol{e}^{(0)}}(R'_n) + \sum_{j=1}^{m} \widetilde{\boldsymbol{e}}^{(n)}_j \partial_{\boldsymbol{e}^{(0)}_j}. \tag{2.14}$$

Observe that in the polynomial case the above means that $D'_n = D_n$ (since $\mathcal{A}' = \mathcal{A}$ and $R'_n = R_n$ then) and that $t'_n = t_n$. Therefore, we obtain (1.30), (1.31) and (1.36) which completes the induction in this case. In the entire differential case we have to prove first that R_n is a differential operator. By (2.10) and by the inductive assumption it is enough to prove that for any $j \in \mathbb{N}$ $\tilde{\star}_j$ is a differential operator, which is easy to check by remarks after Definition 1.4. By (2.9), the operator D_n and then also t_n is well-defined, since R_n is differential and we can apply $\mathcal{Z}_{\boldsymbol{e}^{(0)}}$ to it. Moreover, D_n is also differential. Since differential operators are continuous we may complete the induction in this case by the standard continuity arguments. To prove (1.32) we proceed analogously to the proof of Corollary 2.1. The uniqueness immediatelly follows from Lemma 2.1. □

Remarks

(i) Theorem 1 and Corollary 2.1 can be somewhat generalized. We can assume *σ-selfadjointness* of $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$ and $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ for any permutation σ of $\{1,\ldots,m\}$ (the same for $\mathcal{A}$ and $\mathcal{B}$) instead of selfadjointnes, where we call a system $\boldsymbol{f} \in \mathcal{A}^m$ *σ-real* for a *algebra $\mathcal{A}$ if $\boldsymbol{f}_i^* = \boldsymbol{f}_{\sigma(i)}$ for $i = 1,\ldots,m$ and we call a family $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ *σ-selfadjoint* in a semiclassical algebra $\mathcal{A}$ if the corresponding system $(f_{\hbar,1},\ldots,f_{\hbar,m}) \in (\mathcal{A}[[\hbar]])^m$ is σ-real. Note that $\boldsymbol{f}_{\sigma^2(i)} = \boldsymbol{f}_i$ for $i = 1,\ldots,m$ and for a σ-real $\boldsymbol{f}$. The canonical base $(\boldsymbol{\eta}, \boldsymbol{\xi})$ in $\mathrm{Ent}_{\eta\xi}(\mathbb{C}^{2d})$ is an example of a σ-real system for σ given by $\sigma(j) = (j + d) \bmod 2d$ for $j \in \{1, \dots, 2d\}$.

(ii) By Proposition 1.8, the family $\{t_n\}_{n\in\mathbb{N}}$ from Theorem 1 is a semiclassical unitary **isomorphism** and in the entire differential case $\{t_n\}^{-1}_{n\in\mathbb{N}}$ has continuous coefficients.

We use the formulas from Theorem 1 to compute the operator D_1 in a simple but nontrivial case.

Example 2.1 Let $\mathcal{A} = \mathcal{B}$ be a two-dimmensional q-p or p-q or Weyl semiclassical algebra (polynomial or entire) and consider $\boldsymbol{e}^{(0)} = \boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{p})$, $\boldsymbol{f}^{(0)} = (\boldsymbol{q}, \boldsymbol{p} + \boldsymbol{q}^2)$. Both systems satisfy qccr of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$-type in $\mathcal{A}$, hence taking $\boldsymbol{e}^{(n)} = \boldsymbol{f}^{(n)} = 0$ for $n \geq 1$ we obtain $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$ and $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ satisfying qccr+c. Since $\boldsymbol{e}^{(0)}$ is a canonical base, $\boldsymbol{f}^{(0)} = s_{(2,0)}\boldsymbol{e}^{(0)}$ (see Example 1.1) is also a canonical base by Proposition 1.3. We also have $t_0 = s_{(2,0)}$ and, moreover, $\widetilde{\boldsymbol{e}}^{(1)} = 0$ and $R_1(f,g) = f \star_1 g - f \tilde{\star}_1 g$ for $f, g \in \mathcal{A}$. Therefore we compute

$$D_1 = \mathcal{Z}_{\mathbf{x}}(\tilde{\star}_1 - \star_1).$$

Let us denote D_1 for the considered cases by $D_1^{(q\text{-}p)}$, $D_1^{(p\text{-}q)}$, $D_1^{(W)}$ respectively. In q-p case we have

$$f \star_1 g = -i\partial_p f \partial_q g$$

for $f, g \in \mathcal{A}$, hence

$$f \tilde{\star}_1 g = -i\partial_p f(2\boldsymbol{q}\partial_p g + \partial_q g),$$

which follows that

$$D_1^{(q\text{-}p)} = -i\boldsymbol{q}\partial_q\partial_p.$$

Analogously, for p-q and Weyl cases, we find

$$D_1^{(p\text{-}q)} = i\boldsymbol{q}\partial_q\partial_p, \quad D_1^{(W)} = 0.$$

The following result is an important (and immediate) consequence of Theorem 1.

Corollary 2.2 *Suppose that two structures of polynomial or of entire differential semiclassical algebras are defined in a polynomial or, respectively,* *in an entire classical algebra and that there exists a real canonical base of this classical algebra being selfadjoint quantizable in the both semiclassical algebras. Then these semiclassical algebras are semiclassically unitary isomorphic.*

This proves the semiclassical "equivalence" of all M-semiclassical algebra structures (see Example 1.2) defined in one of $\mathrm{Pol}(\mathbb{R}^{2d})$, $\mathrm{Pol}(\mathbb{C}^{2d})$, $\mathrm{Ent}(\mathbb{R}^{2d})$ and $\mathrm{Ent}(\mathbb{C}^{2d})$, since the canonical base $\mathbf{x} = (\boldsymbol{q}, \boldsymbol{p})$ is selfadjoint quantizable in all these semiclassical algebras. However, as we shall see soon, the quantizability assumption in the above corollary is not necessary!

3. Quantization of classical canonical commutation relations

We consider here the problem of quantization of systems satisfying classical canonical commutation relations.

Let $\mathcal{A}$ be an m-dimmensional polynomial or entire classical algebra and $\boldsymbol{e}$ a canonical base of $\mathcal{A}$. For $s = 1 \dots, m$ we denote

$$\nabla_{\boldsymbol{e},s} g = (\{g, \boldsymbol{e}_1\}, \dots, \{g, \boldsymbol{e}_s\}) \in \mathcal{A}^s,$$

$$\mathfrak{D}_{\boldsymbol{e},s} \boldsymbol{g} = (\{\boldsymbol{g}_i, \boldsymbol{e}_j\})_{i,j=1,\dots,s} = \begin{pmatrix} \nabla_{\boldsymbol{e},s} \boldsymbol{g}_1 \\ \vdots \\ \nabla_{\boldsymbol{e},s} \boldsymbol{g}_s \end{pmatrix} \in \mathcal{M}_s(\mathcal{A})$$

for $g \in \mathcal{A}$ and for $\boldsymbol{g} \in \mathcal{A}^s$. When $s = m$, we shall also write $\nabla_{\boldsymbol{e}}$ and $\mathfrak{D}_{\boldsymbol{e}}$ instead of $\nabla_{\boldsymbol{e},s}$ and $\mathfrak{D}_{\boldsymbol{e},s}$. For $C \in \mathcal{M}_s(\mathcal{A})$ and $s' = 1, \dots, s$ we denote by $C_{[s']}$ the $s' \times s'$ matrix with $(C_{[s']})_{i,j} = C_{i,j}$, $i, j = 1, \dots s'$.

3.1. Equations $\mathfrak{D}_{\boldsymbol{e}} \boldsymbol{g} = C$ and $\mathfrak{D}_{\boldsymbol{e}} \boldsymbol{g} - (\mathfrak{D}_{\boldsymbol{e}} \boldsymbol{g})^{\top} = C$

In the present subsection we study solvability of some equations arising in the quantization of cccr.

Proposition 3.1 *Consider an m-dimmensional polynomial or entire classical algebra with a canonical base $\boldsymbol{e}$ and matrices $C \in \mathcal{M}_m(\mathcal{A})$ and $C' \in \mathcal{M}_{s+1}(\mathcal{A})$ for some $s \in \{1, \dots, m-1\}$.*

a) *The equation $\mathfrak{D}_{\boldsymbol{e}} \boldsymbol{g} = C$ has a solution $\boldsymbol{g} \in \mathcal{A}^m$ iff $\{C_{i,j}, \boldsymbol{e}_k\} = \{C_{i,k}, \boldsymbol{e}_j\}$ for $i, j, k = 1, \dots, m$.*

b) *If $C'^{\top} = -C'$ and*

$$\{C'_{i,j}, \boldsymbol{e}_k\} + \{C'_{k,i}, \boldsymbol{e}_j\} + \{C'_{j,k}, \boldsymbol{e}_i\} = 0 \tag{3.1}$$

for $i,j,k=1,\dots,s+1$ *and* $\breve{\boldsymbol{g}} \in \mathcal{A}^s$ *satisfies* $(\mathcal{D}_{\boldsymbol{e},s}\breve{\boldsymbol{g}}) - (\mathcal{D}_{\boldsymbol{e},s}\breve{\boldsymbol{g}})^{\top} = C'_{[s]}$, *then there exists* $\boldsymbol{g} \in \mathcal{A}^{s+1}$ *such that* $\breve{\boldsymbol{g}} \subset \boldsymbol{g}$ *and*

$$(\mathcal{D}_{\boldsymbol{e},s+1}\boldsymbol{g}) - (\mathcal{D}_{\boldsymbol{e},s+1}\boldsymbol{g})^{\top} = C'. \tag{3.2}$$

c) *The equation*

$$(\mathcal{D}_{\boldsymbol{e}}\boldsymbol{g}) - (\mathcal{D}_{\boldsymbol{e}}\boldsymbol{g})^{\top} = C \tag{3.3}$$

has a solution $\boldsymbol{g} \in \mathcal{A}^m$ *iff* $C^{\top} = -C$ *and for* $i,j,k=1,\dots,m$

$$\{C_{i,j}, \boldsymbol{e}_k\} + \{C_{k,i}, \boldsymbol{e}_j\} + \{C_{j,k}, \boldsymbol{e}_i\} = 0.$$

This proposition can be easily proved using the following lemma (we omit here the details and also the proof of the lemma — see [17] or [18]).

Lemma 3.1 *Suppose that* $\boldsymbol{g} \in \mathcal{A}^j$ *for some* $j=1,\dots,m$. *Then the equation*

$$\nabla_{\boldsymbol{e},j} f = \boldsymbol{g} \tag{3.4}$$

has a solution $f \in \mathcal{A}$ *iff*

$$\mathcal{D}_{\boldsymbol{e},j}\boldsymbol{g} = (\mathcal{D}_{\boldsymbol{e},j}\boldsymbol{g})^{\top}. \tag{3.5}$$

3.2. Quantization of canonical bases

We prove here the following theorem on the quantization of canonical bases:

Theorem 2 *Each canonical base of a polynomial or entire semiclassical algebra is quantizable and moreover it is selfadjoint quantizable if it is real.*

To prove this theorem we need a recursive lemma.

Lemma 3.2 *Suppose that* $\mathcal{A}$ *is an* m*-dimmensional polynomial or entire semiclassical algebra with a canonical base* $\boldsymbol{e}$ *and that* $\breve{\breve{\boldsymbol{e}}} \subset \breve{\boldsymbol{e}} \subset \boldsymbol{e}$, *where* $\breve{\boldsymbol{e}} \in \mathcal{A}^{s+1}$, $\breve{\breve{\boldsymbol{e}}} \in \mathcal{A}^s$ *for some* $s=1,\dots,m-1$. *If* $\{\breve{\breve{\boldsymbol{e}}}^{(n)}\}_{n\in\mathbb{N}}$ *is a quantization of* $\breve{\breve{\boldsymbol{e}}}$, *then there exists* $\{\breve{\boldsymbol{e}}^{(n)}\}_{n\in\mathbb{N}}$ *being a quantization of* $\breve{\boldsymbol{e}}$ *and satisfying* $\breve{\breve{\boldsymbol{e}}}^{(n)} \subset \breve{\boldsymbol{e}}^{(n)}$ *for* $n \in \mathbb{N}$. *Moreover, if* $\breve{\boldsymbol{e}}$ *is real and* $\{\breve{\breve{\boldsymbol{e}}}^{(n)}\}_{n\in\mathbb{N}}$ *is selfadjoint, then the above* $\{\breve{\boldsymbol{e}}^{(n)}\}_{n\in\mathbb{N}}$ *can be also choosen selfadjoint.*

Sketch of the proof. We have to prove that for any $n \in \mathbb{N}$ there exists

$\breve{\boldsymbol{e}}^{(n)} \in \mathcal{A}^{s+1}$ satisfying $\breve{\boldsymbol{e}}^{(0)} = \breve{\boldsymbol{e}}$ and for $n \geq 1$ $\breve{\boldsymbol{e}}^{(n)} \supset \breve{\breve{\boldsymbol{e}}}^{(n)} \in \mathcal{A}^s$ and

$$\sum_{r+t+v=n} \{\breve{e}_i^{(r)}, \breve{e}_j^{(t)}\}_{1+v} = 0$$

for any $i, j = 1, \ldots, s+1$. We recurrently construct $\breve{\boldsymbol{e}}^{(n)}$. Suppose that $n \geq 1$ and that $\breve{\boldsymbol{e}}^{(k)}$ for $k = 0, \ldots, n-1$ satisfying the above conditions are already constructed. Thus the needed $\breve{\boldsymbol{e}}^{(n)}$ should satisfy $\breve{\boldsymbol{e}}^{(n)} \supset \breve{\breve{\boldsymbol{e}}}^{(n)}$ and the matrix equation

$$(\mathcal{D}_{\boldsymbol{e},s+1}\breve{\boldsymbol{e}}^{(n)}) - (\mathcal{D}_{\boldsymbol{e},s+1}\breve{\boldsymbol{e}}^{(n)})^{\top} = C^{(n)},$$

with $C^{(n)} \in \mathcal{M}_{s+1}(\mathcal{A})$,

$$C_{ij}^{(n)} = - \sum_{\substack{r+t+v=n \\ r,t\neq n}} \{\breve{e}_i^{(r)}, \breve{e}_j^{(t)}\}_{1+v}.$$

As $\{\breve{\breve{\boldsymbol{e}}}^{(n)}\}_{n\in\mathbb{N}}$ satisfies qccr+c, we have

$$(\mathcal{D}_{\boldsymbol{e},s}\breve{\breve{\boldsymbol{e}}}^{(n)}) - (\mathcal{D}_{\boldsymbol{e},s}\breve{\breve{\boldsymbol{e}}}^{(n)})^{\top} = C_{[s]}^{(n)}.$$

From (1.15) it follows that $(C^{(n)})^{\top} = -C^{(n)}$. Using the Jacobi identity (1.24) and the inductive assumption we can check that (3.1) holds for $C = C^{(n)}$ (see [17] or [18]). Therefore the existence of $\breve{\boldsymbol{e}}^{(n)}$ follows from Proposition 3.1 b). Suppose now that $\breve{\boldsymbol{e}}$ is real and $\{\breve{\breve{\boldsymbol{e}}}^{(n)}\}_{n\in\mathbb{N}}$, is selfadjoint, and choose an arbitrary $\{\breve{\boldsymbol{e}}^{(n)}\}_{n\in\mathbb{N}}$ satisfying the conditions of the already proved first part of the lemma. Let $\breve{e}_{\hbar,i} = \sum_{n=0}^{\infty} \hbar^n \breve{e}_i^{(n)}$ for $i = 1, \ldots, s+1$ and $\breve{\breve{e}}_{\hbar,i} = \sum_{n=0}^{\infty} \hbar^n \breve{\breve{e}}_i^{(n)}$ for $i = 1, \ldots, s$ ($\breve{e}_{\hbar,i}, \breve{\breve{e}}_{\hbar,i} \in \mathcal{A}[[\hbar]]$). Thus we have $\breve{e}_{\hbar,i} = \breve{\breve{e}}_{\hbar,i}$ for $i = 1, \ldots, s$. Moreover $(\breve{\breve{e}}_{\hbar,i})^{*_\hbar} = \breve{\breve{e}}_{\hbar,i}$ for $i = 1, \ldots, s$ and

$$\frac{i}{\hbar}[\breve{e}_{\hbar,i},\ \breve{e}_{\hbar,j}] = (\mathrm{cr}(\breve{\boldsymbol{e}}))_{ij}\mathbb{1} \tag{3.6}$$

for $i, j = 1, \ldots, s+1$. By (1.25), (1.26) and since $(\mathrm{cr}(\breve{\boldsymbol{e}}))_{ij} \in \mathbb{R}$ (which holds by reality of $\breve{\boldsymbol{e}}$ and by (1.8)) we have $\frac{i}{\hbar}[\breve{e}_{\hbar,i},\ \breve{e}_{\hbar,s+1}^{*_\hbar}] = (\mathrm{cr}(\breve{\boldsymbol{e}}))_{i,s+1}\mathbb{1}$ for $i = 1, \ldots, s$. Hence by (3.6) $\frac{i}{\hbar}[\breve{e}_{\hbar,i},\ \mathrm{re}^{*_\hbar}\breve{e}_{\hbar,s+1}] = (\mathrm{cr}(\breve{\boldsymbol{e}}))_{i,s+1}\mathbb{1}$. By antisymmetricity of $\frac{i}{\hbar}[\ ,\ ]$ and of $\mathrm{cr}(\breve{\boldsymbol{e}})$, the family $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ of systems from $\mathcal{A}^{s+1}$ given by the conditions $\boldsymbol{f}_i^{(n)} = \breve{e}_i^{(n)}$ for $i = 1, \ldots, s$ and by $\sum_{n=0}^{\infty} \hbar^n \boldsymbol{f}_{s+1}^{(n)} = \mathrm{re}^{*_\hbar}\breve{e}_{\hbar,s+1}$, is selfadjoint, it satisfies $\breve{\breve{\boldsymbol{e}}}^{(n)} \subset \boldsymbol{f}^{(n)}$ for $n \in \mathbb{N}$ and it is a quantization of $\breve{\boldsymbol{e}}$. $\square$

Proof of Theorem 2. Let $\mathcal{A}$ be a polynomial or entire m-dimmensional semiclassical algebra and $\boldsymbol{e}$ its canonical base. We recurrently construct quantizations of all $\breve{\boldsymbol{e}} \subset \boldsymbol{e}$ having the dimmension between 1 and m. The recursion is possible by Lemma 3.2, provided that $\breve{\boldsymbol{e}} = \boldsymbol{e}_1 \in \mathcal{A}^1$ is quantizable (or selfadjoint quantizable for real $\boldsymbol{e}$). To obtain a quantization only, we can take arbitrary elements of $\mathcal{A}$ as $\boldsymbol{e}_1^{(n)}$ for $n \geq 1$, since (1.27) always holds when $i = j$ (e.g. by antisymmetricity of $\{\ \}_n$). By (1.19) the family $\{\boldsymbol{e}_1^{(n)}\}_{n\in\mathbb{N}}$ will be selfadjoint if for any $n \in \mathbb{N}$

$$2i \operatorname{im}^{*_0}(\boldsymbol{e}_1^{(n)}) = \boldsymbol{e}_1^{(n)} - (\boldsymbol{e}_1^{(n)})^{*_0} = \sum_{\substack{k+l=n \\ k\neq n}} (\boldsymbol{e}_1^{(k)})^{*_l}.$$

If $\boldsymbol{e}$ is real, the above condition holds for $n = 0$ (since $^{*_0} = {}^*$). We shall obtain it also for $n \geq 1$ defining recurrently

$$\boldsymbol{e}_1^{(n)} = f^{(n)} + \frac{1}{2} \sum_{\substack{k+l=n \\ k\neq n}} (\boldsymbol{e}_1^{(k)})^{*_l},$$

where $f^{(n)}$ are arbitrary real elements of $\mathcal{A}$. □

Remarks

(i) As can be seen from the proof of Theorem 2 and of Lemma 3.2, quantization (and salfadjoint quantization) of canonical bases is non-unique. This non-uniqueness is a consequence of the fact that solutions of the equation (3.3) are defined up to the term of the form $\nabla_{\boldsymbol{e}} f$ for $f \in \mathcal{A}$ (see Lemma 3.1).

(ii) From Theorem 2 we immediately obtain quantizability (or salfadjoint quantizability) of “subsystems” of canonical bases, that is, of such systems $\boldsymbol{f} \in \mathcal{A}^s$ that $\boldsymbol{f}_j \in \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_m\}$ for $j = 1, \ldots, s$, where $\boldsymbol{e} \in \mathcal{A}^m$ is a canonical base of $\mathcal{A}$ (real for salfadjoint quantizability).

We now present an example of a family satisfying qccr+c with non-trivial quantum corrections.

Example 3.1 Consider a two-dimmensional Weyl semiclassical algebra with the canonical base

$$\boldsymbol{e} = (\boldsymbol{q} + (\boldsymbol{p} + \boldsymbol{q}^k)^l,\ \boldsymbol{p} + \boldsymbol{q}^k) = s_{(k,l)}(\boldsymbol{q}, \boldsymbol{p}),$$

$k, l \in \mathbb{N}$, ($s_{(k,l)}$ is a canonical isomorphism from Example 1.1 and $\boldsymbol{p} = \boldsymbol{p}_1$,

$\boldsymbol{q} = \boldsymbol{q}_1$, since $d = 1$). Proceeding as in the proof of Theorem 2 we construct the quantum corrections $\boldsymbol{e}^{(n)}$ to $\boldsymbol{e}$. These corrections cannot be all equal to zero, since $\boldsymbol{e}$ does not satisfy qccr, e.g. for $k = l = 3$

$$\{\boldsymbol{e}_1, \boldsymbol{e}_2\}_3 = \{(\boldsymbol{p} + \boldsymbol{q}^3)^3, \boldsymbol{q}^3\}_3 = \frac{3}{2} \neq 0.$$

We can quantize $\boldsymbol{e}_2$ taking zero corrections, that is,

$$\boldsymbol{e}_2^{(n)} = 0 \tag{3.7}$$

for $n \geq 1$ (it is a selfadjoint quantization, since $*_n = 0$ for $n \geq 1$). A family $\{\boldsymbol{e}^{(n)}\}_{n \in \mathbb{N}}$ will be a quantization of $\boldsymbol{e}$, if for $n \geq 1$

$$\{\boldsymbol{e}_2^{(n)}, \boldsymbol{e}_1\} = \{\boldsymbol{e}_1^{(n)}, \boldsymbol{e}_2\} - \sum_{\substack{r+t+v=n \\ r,t \neq n}} \{\boldsymbol{e}_2^{(r)}, \boldsymbol{e}_1^{(t)}\}_{1+v}. \tag{3.8}$$

Using the canonical isomorphism $s_{(k,l)}^{-1}$, by (3.7) we obtain

$$\{\boldsymbol{p},\ s_{(k,l)}^{-1}\boldsymbol{e}_1^{(n)}\} = s_{(k,l)}^{-1} \sum_{t=0}^{n-1} \{\boldsymbol{e}_1^{(t)},\ \boldsymbol{p} + \boldsymbol{q}^k\}_{1+(n-t)},$$

and since $\{\boldsymbol{p}, \cdot\} = \partial_q$ and $\{\cdot, \boldsymbol{p}\}_j = 0$ for $j > 1$, we can choose

$$\boldsymbol{e}_1^{(n)} = s_{(k,l)} \int_{\boldsymbol{q}} s_{(k,l)}^{-1} \sum_{t=0}^{n-1} \{\boldsymbol{e}_1^{(t)},\ \boldsymbol{q}^k\}_{1+(n-t)} \tag{3.9}$$

for $n \geq 1$, where the operetor $\int_{\boldsymbol{q}} : \mathrm{Ent}(\mathbb{C}^2) \longrightarrow \mathrm{Ent}(\mathbb{C}^2)$ (or $\mathrm{Ent}(\mathbb{R}^2) \longrightarrow \mathrm{Ent}(\mathbb{R}^2)$) is given for $f = \sum_{i,j \in \mathbb{N}} f_{ij} \boldsymbol{q}^i \boldsymbol{p}^j$, $f_{ij} \in \mathbb{C}$ by the formula

$$\int_{\boldsymbol{q}} f = \sum_{i,j \in \mathbb{N}} \frac{1}{j+1} f_{ij} \boldsymbol{q}^{i+1} \boldsymbol{p}^j.$$

Finally $\{\boldsymbol{e}^{(n)}\}_{n \in \mathbb{N}}$ with $\boldsymbol{e}^{(0)} = \boldsymbol{e}$ and with $\boldsymbol{e}^{(n)}$ given recurrently for $n \geq 1$ by (3.9) and (3.7) is a selfadjoint quantization of $\boldsymbol{e}$, since $* = *_0$ and $*_n = 0$ for $n \geq 1$ and $\{\ \}_r$, $s_{k,l}$ and $\int_{\boldsymbol{q}}$ commute with $*$ (being the usual conjugation of function here).

4. Quantization of canonical isomorphisms

We prove here our result concerning quantizability of canonical isomorphisms of polynomial and entire algebras. As an illustration we find simple recurrent formulas for the quantum corrections in the case of linear canonical isomorphisms of the phase space for M-semiclassical algebras. We also find corresponding unitary transformations acting in Hilbert space.

4.1. The quantizability theorem

We can combine the results of Theorem 1 and of Theorem 2 to obtain the following result.

Theorem 3 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are both polynomial semiclassical algebras or both entire differential semiclassical algebras and that $\mathcal{A}$ possesses a real canonical base. If $t : \mathcal{A} \longrightarrow \mathcal{B}$ is a canonical isomorphism, continuous in the entire differential case, then t is quantizable and in the entire differential case the quantum corrections to t can be choosen continuous.*

Proof. Let $\boldsymbol{e}$ be a real canonical base of $\mathcal{A}$. By proposition 1.3 $t\boldsymbol{e}$ is a real canonical base of $\mathcal{B}$ with $\mathrm{cr}(t\boldsymbol{e}) = \mathrm{cr}(\boldsymbol{e})$ and thus by Theorem 2 $\boldsymbol{e}$ and $t\boldsymbol{e}$ are selfadjoint quantizable. Therefore there exist selfadjoint families $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$, $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ satisfying qccr+c of the same type in $\mathcal{A}$ and $\mathcal{B}$ respectively such that $\boldsymbol{e}^{(0)} = \boldsymbol{e}$, $\boldsymbol{f}^{(0)} = t\boldsymbol{e}$. By Theorem 1 there exists a semiclassical unitary transformation $\{t_n\}_{n\in\mathbb{N}}$ (with continuous coefficients in the entire differential case) such that $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ is an image of $\{\boldsymbol{e}^{(n)}\}_{n\in\mathbb{N}}$ by $\{t_n\}_{n\in\mathbb{N}}$. In particular $t_0\boldsymbol{e} = t_0\boldsymbol{e}^{(0)} = \boldsymbol{f}^{(0)} = t\boldsymbol{e}$ and since t_0 and t are homomorphisms of algebras, we have $t_0 = t$. Thus $\{t_n\}_{n\in\mathbb{N}}$ is a quantization of t. □

Remarks

(i) Quantum corrections for t are not uniquely determined (see also the remark (i) after the proof of Theorem 2).

(ii) If a canonical base $\boldsymbol{e}$ and some selfadjoint quantizations of $\boldsymbol{e}$ and $t\boldsymbol{e}$ are choosen, then there is a unique choice of quantum corrections t_n. They are given by the formulas from Theorem 1, that is, $t_n = tD_n$ for $n \geq 1$ and D_n are recurrently defined by (2.9), (2.10), (2.11), (2.12) with $t_0 = t$ and $\boldsymbol{e}^{(0)} = \boldsymbol{e}$. The remark (ii) after the proof of Theorem 1 is also valid here.

(iii) The existence of a real canonical base of $\mathcal{A}$ is an important assumption of Theorem 3. It can be proved (see [17] or [18]) that if $\mathcal{A}$ possesses σ-real canonical base for some permutation σ (see remark (i) after the proof of Theorem 1), then $\mathcal{A}$ possesses also a real canonical base.

4.2. The case of linear isomorphisms of $\mathbb{R}^{2d}$

Consider an m-dimmensional polynomial or entire classical algebra with a real canonical base $\boldsymbol{e}$. Let $C \in \mathcal{M}_m(\mathbb{R})$ satisfy

$$\mathrm{cr}(\boldsymbol{e}) = C\mathrm{cr}(\boldsymbol{e})C^{\top} \tag{4.1}$$

and denote $t_C : \mathcal{A} \longrightarrow \mathcal{A}$

$$t_C = \phi_{\boldsymbol{f}}\, \phi_{\boldsymbol{e}}^{-1}, \tag{4.2}$$

where $\boldsymbol{f} = C\boldsymbol{e}$. Since C has real coefficients, $\boldsymbol{f}$ is a real system and thus by (4.1) and Proposition 1.3 $\boldsymbol{f}$ is a real canonical base. Therefore, by Proposition 1.3, t_C is a canonical isomorphism of $\mathcal{A}$ (and a homeomorphism in the entire case).

We define now some classes of multilinear differential operators for $\mathcal{A}$. By $\mathcal{O}^{(n)}_{\boldsymbol{e},k}$ we denote the set of all k-linear operators $P : \mathcal{A}^k \longrightarrow \mathcal{A}$ of the form

$$P = \sum_{\substack{\alpha\in(\mathbb{N}^m)^k \\ |\alpha|=n}} p_\alpha \partial^\alpha_{\boldsymbol{e}},$$

where $p_\alpha \in \mathbb{C}$ (in particular these operators have "constant coefficients relatively to $\boldsymbol{e}$"). For instance, the multiplication $\cdot$ in $\mathcal{A}$ is in $\mathcal{O}^{(0)}_{\boldsymbol{e},2}$, and $\{\ ,\ \} \in \mathcal{O}^{(2)}_{\boldsymbol{e},2}$ by (1.11). Note that all operators from $\bigcup_{n\in\mathbb{N}} \mathcal{O}^{(n)}_{\boldsymbol{e},1}$ commute.

We shall define also the operation $\tilde{\mathcal{Z}}_{\boldsymbol{e}}$, which transforms bilinear differential operators into linear operators in $\mathcal{A}$. It is defined by the formula which is similar to (1.5):

$$\tilde{\mathcal{Z}}_{\boldsymbol{e}}(S)f = \sum_{\alpha\in\mathbb{N}^m} \sum_{\gamma+\gamma'=\alpha} s_{\gamma,\gamma'} \partial^\alpha_{\boldsymbol{e}} f$$

for $f \in \mathcal{A}$, where S is a bilinear differential operator,

$$S(f,g) = \sum_{\gamma,\gamma'\in\mathbb{N}^m} s_{\gamma,\gamma'} \partial^\gamma_{\boldsymbol{e}} f \cdot \partial^{\gamma'}_{\boldsymbol{e}} g$$

for $f, g \in \mathcal{A}$. For instance

$$\tilde{\mathcal{Z}}_{\boldsymbol{e}}(\cdot) = Id. \tag{4.3}$$

Note that if $S \in \mathcal{O}^{(n)}_{\boldsymbol{e},2}$ for $n \geq 2$, then $\tilde{\mathcal{Z}}_{\boldsymbol{e}}(S) \in \mathcal{O}^{(n)}_{\boldsymbol{e},1}$ and

$$\mathcal{Z}(S) = \frac{1}{2^n - 2} \tilde{\mathcal{Z}}_{\boldsymbol{e}}(S). \tag{4.4}$$

For linear operators D, D' and a bilinear P, we define bilinear operators P_C, $P(D,D')$ and a linear D_C by the formulas

$$\begin{aligned}&P_C(f,g)=t_C^{-1}P(t_Cf,t_Cg), \quad D_C=t_C^{-1}D\,t_C,\\&P(D,D')(f,g)=P(Df,D'g),\end{aligned}$$

$f,g\in\mathcal{A}$.

Lemma 4.1 *Under the previous assumptions on $\mathcal{A}$, $\mathbf{e}$ and C, if $P\in\mathcal{O}_{\mathbf{e},2}^{(k)}$, $D\in\mathcal{O}_{\mathbf{e},1}^{(l)}$ and $D'\in\mathcal{O}_{\mathbf{e},1}^{(l')}$, then*

a) $DP\in\mathcal{O}_{\mathbf{e},2}^{(k+l)}$ *and* $\tilde{\mathcal{Z}}_{\mathbf{e}}(DP)=2^lD\tilde{\mathcal{Z}}_{\mathbf{e}}(P)$;

b) $P(D,D')\in\mathcal{O}_{\mathbf{e},2}^{(k+l+l')}$ *and* $\tilde{\mathcal{Z}}_{\mathbf{e}}(P(D,D'))=DD'\tilde{\mathcal{Z}}_{\mathbf{e}}(P)$;

c) $P_C\in\mathcal{O}_{\mathbf{e},2}^{(k)}$ *and* $\tilde{\mathcal{Z}}_{\mathbf{e}}(P_C)=(\tilde{\mathcal{Z}}_{\mathbf{e}}(P))_C$.

We can now formulate a result concerning quantization of canonical isomorphism t_C.

Corollary 4.1 *Let $\mathcal{A}$ be an m-dimmensional polynomial or entire semiclassical algebra with a real canonical base $\mathbf{e}$ and suppose that* (i) *$\mathbf{e}$ satisfies qccr;* (ii) *$\mathbf{e}^{\star_n}=0$ for $n>0$;* (iii) *$\star_n\in\mathcal{O}_{\mathbf{e},2}^{(2n)}$ for $n\in\mathbb{N}$. If $C\in\mathcal{M}_m(\mathbb{R})$ satisfies* (4.1) *and the operators $D_n:\mathcal{A}\longrightarrow\mathcal{A}$ are given by the recurrent formula*

$$D_0=Id_{\mathcal{A}}, \quad D_n=\frac{-1}{4^n-2}\left[\sum_{\substack{j+k=n\\k\neq n}}4^kD_kL_j-\sum_{\substack{j+k+l=n\\k,l\neq n}}D_kD_l(L_j)_C\right] \tag{4.5}$$

for $n\geq1$, where

$$L_j=\tilde{\mathcal{Z}}_{\mathbf{e}}(\star_j),$$

then $D_n\in\mathcal{O}_{\mathbf{e},1}^{(2n)}$ and the family $\{t_n\}_{n\in\mathbb{N}}$ with $t_n=t_CD_n$ is a quantization of t_C (with continuous coefficients in the entire case).

Proof. By (iii) $L_j\in\mathcal{O}_{\mathbf{e},1}^{(2j)}$ and thus D_n given by (4.5) are in $\mathcal{O}_{\mathbf{e},1}^{(2n)}$ for any $n\in\mathbb{N}$. On the other hand, by the remark (ii) after the prof of Theorem 3, the recurrent formulas (2.9), (2.10), (2.11) and (2.12) with $t_0=t_C$ are valid for quantization of t_C. Thus define $\mathbf{e}^{(n)}=\boldsymbol{f}^{(n)}=0$ for $n>0$ and $\mathbf{e}^{(0)}=\mathbf{e}$, $\boldsymbol{f}^{(0)}=t_C\mathbf{e}$. Obviously, by (i) and (ii) $\{\mathbf{e}^{(n)}\}_{n\in\mathbb{N}}$ is a selfadjoint quantization of $\boldsymbol{e}$. We have

$$\{\boldsymbol{f}_k, \boldsymbol{f}_l\}_n = \sum_{i,j=1,\dots,m} C_{ki}C_{lj}\{\boldsymbol{e}_k, \boldsymbol{e}_l\}_n = 0$$

for $n > 0$, thus analogously $\{\boldsymbol{f}^{(n)}\}_{n\in\mathbb{N}}$ is a selfadjoint quantization of the canonical base $\boldsymbol{f}$. By (2.12), the RHS of the recurrent formula (2.9) reduces to $-\mathcal{Z}_{\boldsymbol{e}}(R_n)$. Hence by (4.4) and Lemma 4.1, the two definitions of operators D_n are equivalent. □

Example 4.1 Consider the classical algebra $\mathcal{A} = \mathrm{Pol}(\mathbb{R}^{2d})$ or $\mathrm{Ent}(\mathbb{R}^{2d})$ and an arbitrary linear canonical transformation C of the phase space $\mathbb{R}^{2d}$ (that is, a linear transformation preserving the standard symplectic form $dq \wedge dp$ in $\mathbb{R}^{2d}$). If we identify C with the element of $\mathcal{M}_{2d}(\mathbb{R})$, then this property of C is just defined by (4.1) with $\boldsymbol{e} = (\boldsymbol{q}, \boldsymbol{p}) = \mathbf{x}$ (see e.g. [8]). We can rewrite (4.1) in the form of the system of matrix equations

$$KL^\top = LK^\top, \quad MN^\top = NM^\top, \quad KN^\top - LM^\top = I,$$

where C has the block form

$$C = \begin{pmatrix} K & L \\ M & N \end{pmatrix}, \tag{4.6}$$

with $K, L, M, N \in \mathcal{M}_d(\mathbb{R})$. By (4.2), the canonical isomorphism t_C (acting on the algebra $\mathcal{A}$ level) satisfies $t_C f = f \circ C$ for $f \in \mathcal{A}$.

We can compute now a family of quantum corrections for t_C in M-semiclassical algebras $\mathrm{Pol}_M(\mathbb{R}^{2d})$ or $\mathrm{Ent}_M(\mathbb{R}^{2d})$ (see Example 1.2) using the above corollary. Note that the only terms in (4.5) depending on the choice of semiclassical algebra are operators L_j, hence it is enough to compute these operators. Denote the operator L_j by L_j^M for a given M-semiclassical algebra. By (4.3) and Lemma 4.1 we have

$$\begin{aligned} L_n^M &= \sum_{|\alpha+\beta|=n} \frac{i^n}{\alpha!\beta!}(E\partial_q)^\alpha (M\partial_q)^\beta \partial_p^{\alpha+\beta} \\ &= \sum_{|\gamma|=n} \frac{i^n}{\gamma!}\left[\sum_{\alpha\le\gamma}\binom{\gamma}{\alpha}(E\partial_q)^\alpha (M\partial_q)^{\gamma-\alpha}\right]\partial_p^\gamma \\ &= \sum_{|\gamma|=n} \frac{i^n}{\gamma!}(E\partial_q + M\partial_q)^\gamma \partial_p^\gamma = \sum_{|\gamma|=n} \frac{i^n}{\gamma!}(G\partial_q)^\gamma \partial_p^\gamma. \end{aligned}$$

Usually the corrections t_n for t_C are compositions of t_C and of some quite complicated differential operators with constant coefficients depending on C and M. Hovever, when we deal with Weyl semiclassical algebras, then $G = 0$ and thus all the corrections t_n are zero for $n \geq 1$. This is the well-known result and the invariance of all $\star_n^W$ from Weyl deformation quantization for linear canonical transformations of the phace space is an immediate consequence of it (see [3] V §1.4). When $G \neq 0$, then quantum corrections are usually nonzero. For instance, in q-p semiclassical algebra (for $G = -I$) for $C = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ we have $t_1 = it_C \partial_p \partial_q$.

Observe also that by the above formula for L_n^M and by the formulas for deformations in M-semiclassical algebras, in each algebra of this type $\star_n^M$ and $*_n^M$ are closely related by the equality

$$f^{*_n^M} = L_n^M(f^*) = \tilde{\mathcal{Z}}_{\mathbf{x}}(\star_n)(f^*).$$

4.3. Connections with unitary transformations of Hilbert space

It would be interesting to find a relationship between semiclassical unitary isomorphisms considered in this paper and unitary transformations of Hilbert space. To do this we need some map between semiclassical algebra and "algebra" of quantum observables (operators) in the appropriate Hilbert space. We consider here the simple case of the semiclassical unitary isomorphism being the quantization of canonical transformation t_C (from the previous subsection) in the semiclassical algebra $\mathcal{A} = \mathrm{Pol}_{q\text{-}p}(\mathbb{R}^{2d})$ or $\mathrm{Pol}_{p\text{-}q}(\mathbb{R}^{2d})$ or $\mathrm{Pol}_W(\mathbb{R}^{2d})$. The natural Hilbert space corresponding to $\mathcal{A}$ is $L^2(\mathbb{R}^d)$ and the natural map between $\mathcal{A}$ and operators in $L^2(\mathbb{R}^d)$ is the procedure ${}^{\wedge_\hbar}$ of quantization of observables from $\mathcal{A}$ — q-p, p-q or Weyl quantization respectively (see e.g. [3]). In particular we have ${}^{\wedge_\hbar} : \mathcal{A} \longrightarrow \mathcal{D}\mathit{iff}(\mathbb{R}^d)$ for $\hbar > 0$ (note that $\hbar$ is no longer a formal parameter here), where $\mathcal{D}\mathit{iff}(\mathbb{R}^d)$ is a set of differential operators with polynomial coefficients in $L^2(\mathbb{R}^d)$ with the (invariant) domain $\mathcal{S}(\mathbb{R}^d)$ — the space of Schwartz functions. Note that $\mathcal{D}\mathit{iff}(\mathbb{R}^d)$ with the composition of operators, with the identity operator in $\mathcal{S}(\mathbb{R}^d)$ and with the adjoint being the restriction of the usual Hilbert adjoint of operators to $\mathcal{S}(\mathbb{R}^d)$ forms a *algebra. Using Proposition 1.4 it is easy to prove that $\{\mathrm{Pol}(\mathbb{R}^{2d}), \star_{(\hbar)}, \mathbb{1}, {}^{*(\hbar)}\}$ with $\star_{(\hbar)} : \mathrm{Pol}(\mathbb{R}^{2d}) \times \mathrm{Pol}(\mathbb{R}^{2d}) \longrightarrow \mathrm{Pol}(\mathbb{R}^{2d})$ and ${}^{*(\hbar)} : \mathrm{Pol}(\mathbb{R}^{2d}) \longrightarrow \mathrm{Pol}(\mathbb{R}^{2d})$ given by

$$f \star_{(\hbar)} g = \sum_{n=0}^{\infty} \hbar^n f \star_n g, \quad f^{*(\hbar)} = \sum_{n=0}^{\infty} \hbar^n f^{*n}$$

is also a *algebra for any $\hbar > 0$ (note that by the formulas from Example 1.2 the above sums are finite for fixed $f, g \in \mathcal{A}$). Moreover, from the construction of the semicalssical structure in $\mathcal{A}$ (see [3]) it follows that ${}^{\wedge\hbar}$ is a *isomorphism of this *algebra onto $\mathcal{D}\mathit{iff}(\mathbb{R}^d)$ for any $\hbar > 0$ and

$$\hat{\boldsymbol{q}}^{\hbar} = Q_{\hbar}, \quad \hat{\boldsymbol{p}}^{\hbar} = P_{\hbar}, \tag{4.7}$$

where $(Q_{\hbar,j}\varphi)(x) = x_j\varphi(x)$, $(P_{\hbar,j}\varphi)(x) = \frac{\hbar}{i}\frac{\partial}{\partial x_j}\varphi(x)$ for $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$. We denote by ${}^{\vee\hbar}$ the inverse *isomorphism. Let $\{t_n\}_{n\in\mathbb{N}}$ be the quantization of t_C constructed in Example 4.1. It is easily seen from Propositipon 1.7 that for $\hbar > 0$ the transformation $t_{(\hbar)} : \mathrm{Pol}(\mathbb{R}^{2d}) \longrightarrow \mathrm{Pol}(\mathbb{R}^{2d})$ given for $f \in \mathcal{A}$ by

$$t_{(\hbar)} f = \sum_{n=0}^{\infty} \hbar^n t_n f$$

is a *isomorphism of $\{\mathrm{Pol}(\mathbb{R}^{2d}), \star_{(\hbar)}, \mathbb{1}, {}^{*(\hbar)}\}$ (the above sum is finite for any f by Corollary 4.1) and

$$t_{(\hbar)}(\boldsymbol{q}, \boldsymbol{p}) = t_C(\boldsymbol{q}, \boldsymbol{p}) = C(\boldsymbol{q}, \boldsymbol{p}) \tag{4.8}$$

(since $D_n \boldsymbol{q}_i = D_n \boldsymbol{p}_i = 0$ for $n \geq 1$, $i = 1, \ldots, d$). For $\hbar > 0$ consider now the transformation $\mathcal{T}_{C,\hbar} : \mathcal{D}\mathit{iff}(\mathbb{R}^d) \longrightarrow \mathcal{D}\mathit{iff}(\mathbb{R}^d)$, $\mathcal{T}_{C,\hbar} = {}^{\wedge\hbar} \circ t_{(\hbar)} \circ {}^{\vee\hbar}$, which is a *isomorphism by the above considerations. By (4.7) and (4.8) we have

$$\mathcal{T}_{C,\hbar}(Q_{\hbar}, P_{\hbar}) = C(Q_{\hbar}, P_{\hbar}). \tag{4.9}$$

We can now precisely formulate our problem as the question about the exisistence of such a unitary transformation $\mathcal{U}_{C,\hbar}$ of $L^2(\mathbb{R}^d)$ preserving $\mathcal{S}(\mathbb{R}^d)$ that for any $X \in \mathcal{D}\mathit{iff}(\mathbb{R}^d)$ and $\hbar > 0$ $\mathcal{T}_{C,\hbar} X = \mathcal{U}_{C,\hbar}^{-1} X\, \mathcal{U}_{C,\hbar}$. The ansver is positive for all C. To find the appropriate $\mathcal{U}_{C,\hbar}$ we shall use the fact that any linear canonical transformation C of the phase space R^{2d} has a decomposition of the form

$$\begin{pmatrix} D^{-1} & 0 \\ 0 & D^{\mathsf{T}} \end{pmatrix} \begin{pmatrix} I & 0 \\ R & I \end{pmatrix} \begin{pmatrix} P_r & P'_r \\ -P'_r & P_r \end{pmatrix} \begin{pmatrix} I & \widetilde{R} \\ 0 & I \end{pmatrix} \begin{pmatrix} \widetilde{D}^{-1} & 0 \\ 0 & \widetilde{D}^{\mathsf{T}} \end{pmatrix}, \tag{4.10}$$

where $r \in \{0,\dots,d\}$, $D,\widetilde{D},R,\widetilde{R} \in \mathcal{M}_d(\mathbb{R})$ satisfy $\det D, \det \widetilde{D} \neq 0$, $R^\top = R$, $\widetilde{R}^\top = \widetilde{R}$ and P_r, P'_r are projections in $\mathbb{R}^d$ given by $P_r x = (x_1,\dots,x_r, 0,\dots,0)$ and $P'_r x = (0,\dots,0,x_{r+1},\dots,x_d)$ (in particular $P_0 = P'_d = 0$) (see [17] for the proof). We first restrict our considerations to the cases of C being one of the terms of the above decomposition. Since the condition (4.9) uniquely determines the *isomorphism of $\mathcal{D}\mathit{iff}(\mathbb{R}^d)$, we can easily find $\mathcal{U}_{C,\hbar}$ in these cases:

(i) if $C = \begin{pmatrix} P_r & P'_r \\ -P'_r & P_r \end{pmatrix}$ with $r \in \{0,\dots,d\}$, then $\mathcal{U}_{C,\hbar} = \mathcal{F}_{r,\hbar}$, where $\mathcal{F}_{r,\hbar}$ is the quantum Fourier transform in $L^2(\mathbb{R}^d)$ in the last $d-r$ coordinates, that is, $\mathcal{F}_{r,\hbar} = I$ when $r = d$ and for $r < d$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$

$$(\mathcal{F}_{r,\hbar}\varphi)(x) = (2\pi\hbar)^{-(d-r)/2} \int_{\mathbb{R}^{(d-r)}} \exp\left(-\frac{i}{\hbar}(x_{r+1},\dots,x_d)s\right)\varphi(x_1,\dots,x_r,s)\,ds$$

(and it is uniquely extended to the unitary transformation of the whole $L^2(\mathbb{R}^d)$);

(ii) if $C = \begin{pmatrix} I & 0 \\ R & I \end{pmatrix}$ with $R = R^\top$, then $\mathcal{U}_{C,\hbar} = \mathcal{M}_{R,\hbar}$, where $\mathcal{M}_{R,\hbar}$ is the operator of multiplication by the function $g_{R,\hbar} : \mathbb{R}^d \longrightarrow \mathbb{C}$, $g_{R,\hbar}(x) = \exp\left(\frac{i}{2\hbar}(Rx)x\right)$ for $x \in \mathbb{R}^d$;

(iii) if $C = \begin{pmatrix} I & R \\ 0 & I \end{pmatrix}$ with $R = R^\top$, then $\mathcal{U}_{C,\hbar} = \mathcal{V}_{R,\hbar}$, where $\mathcal{V}_{R,\hbar} = \mathcal{F}_{0,\hbar}^{-1}\mathcal{M}_{(-R),\hbar}\mathcal{F}_{0,\hbar}$;

(iv) if $C = \begin{pmatrix} D^{-1} & 0 \\ 0 & D^\top \end{pmatrix}$ with $\det D \neq 0$, then $\mathcal{U}_{C,\hbar} = \mathcal{S}^\diamond_D$, where $\mathcal{S}^\diamond_D$ is the normalized change of variables connected with D, that is, for $\varphi \in L^2(\mathbb{R}^d)$ $\mathcal{S}^\diamond_D\varphi = |\det D|^{\frac{1}{2}}\varphi \circ D$.

Using these special cases and the decomposition (4.10) we can define $\mathcal{U}_{C,\hbar}$ for $\hbar > 0$ for an arbitrary C by the formula

$$\mathcal{U}_{C,\hbar} = \mathcal{S}^\diamond_{\widetilde{D}}\mathcal{V}_{\widetilde{R},\hbar}\mathcal{F}_{r,\hbar}\mathcal{M}_{R,\hbar}\mathcal{S}^\diamond_D.$$

It seems natural to treat the obtained family of unitary transformation $\mathcal{U}_{C,\hbar}$ as a quantization of the canonical map C. Similar construction can be made when we take an arbitrary semiclassical algebra $\mathrm{Pol}_M(\mathbb{R}^{2d})$ as $\mathcal{A}$. In this case the quantization $^{\wedge\hbar}$ should be defined by T-symbol (see [17] for the details).

Unfortunatelly, the case of quantizations of nonlinear canonical isomorphisms of the phase space is much more difficult. It remains an open problem to rigorously construct in like manner a family of unitary transformations corresponding to a given semiclassical unitary transformation in a more general case.

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