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A note on value distribution of nonhomogeneous differential polynomials

(Dedicated to Prof. B.K. Lahiri on his 70th birth anniversary)

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Abstract. In the note we prove a result on value distribution of nonhomogeneous differential polynomials which improves a long standing theorem of C.C. Yang.

Key words: differential polynomial, value distribution.

1. Introduction and Definitions

Let f be a transcendental meromorphic function in the open complex plane $\mathbb{C}$. The problem of investigating possible Picard values of the derivative of f leads to the problem of investigating the value distribution of certain polynomials in f and its derivatives which are called differential polynomials generated by f and is explained in *Definition* 2.

Definition 1 A meromorphic function a is said to be a small function of f if $T(r, a) = S(r, f)$.

Definition 2 [1, 3] Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be nonnegative integers. The expression $M_j[f] = b_j(f)^{n_{0j}}(f^{(1)})^{n_{1j}} \cdots (f^{(k)})^{n_{kj}}$ is called a differential monomial generated by f of degree $\gamma_{M_j} = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$, where $T(r, b_j) = S(r, f)$.

The sum of the monomials $P[f] = \sum_{i=1}^{l} M_j[f]$ is called a differential polynomial generated by f of degree $\gamma_P = \max\{\gamma_{M_j} : 1 \le j \le l\}$ and weight $\Gamma_P = \max\{\Gamma_{M_j} : 1 \le j \le l\}$.

The numbers $\underline{\gamma}_P = \min\{\gamma_{M_j} : 1 \le j \le l\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and order of $P[f]$.

$P[f]$ is said to be homogeneous if $\gamma_P = \underline{\gamma}_P$.

Also we denote by γ_P^* the number $\gamma_P^* = \max\{\gamma_{M_j} : \gamma_{M_j} < \gamma_P$ and $1 \le j \le l\}$.

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Definition 3 For a complex number $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N_{1)}(r, a; f)$ the counting function of simple a-points of f.

We do not explain the standard definitions and notations of the value distribution theory because those are available in [6]. Hayman [5] proved the following theorems.

Theorem A *If f is transcendental entire and $n \geq 3$, $a \neq 0$ then $\psi = f' - a(f)^n$ assume all finite values infinitely often.*

Theorem B *If f is transcendental entire and $n \geq 2$ then $f'(f)^n$ assumes all finite values except possibly zero infinitely often.*

Clunie [2] proved *Theorem* B for $n \geq 1$. Later on Sons [7] generalised *Theorem* B and proved the following result.

Theorem C *If f is transcendental entire and $\psi = (f)^{n_0}(f^{(1)})^{n_1} \cdots (f^{(k)})^{n_k}$, where $n_0 \geq 2$, $n_k \geq 1$ and $n_i \geq 0$ for $i \neq 0$, k then $\delta(a; \psi) < 1$ for $a \neq 0, \infty$. Moreover if $N_{1)}(r, 0; f) = o\{T(r, f)\}$ as $r \to \infty$ then for $n_0 \geq 1$ the same conclusion holds.*

For differential polynomials C.C. Yang [8] proved the following theorem.

Theorem D *Let f be transcendental meromorphic with $N(r, f) + N(r, 0; f) = S(r, f)$ and $\psi = \sum a(f)^{p_o}(f^{(1)})^{p_1} \cdots (f^{(k)})^{p_k}$ with no constant term where $T(r, a) = S(r, f)$. If the degree n of ψ is greater than one and $p_0 < n$, $0 \leq p_i \leq n$ for all $i \neq 0$, then $\delta(b, \psi) < 1$ for all $b \neq 0, \infty$.*

Following theorem of Gopalakrishna and Bhoosnurmath [4] shows that for a homogeneous differential polynomial we get a better result.

Theorem E *Let f be meromorphic with $\overline{N}(r, f) + \overline{N}(r, 0; f) = S(r, f)$ and $\psi(f)$ be a nonconstant homogeneous differential polynomial. Then $\Theta(b; \psi) = 0$ for all $b \neq 0, \infty$.*

However for nonhomogeneous differential polynomials *Theorem* E does not hold. For, let $f = \exp(z)$, $\psi = f^2 - 2if'$. Then $\Theta(1; \psi) = \frac{1}{2}$.

For nonhomogeneous differential polynomials C.C. Yang [9] proved the following theorem.

Theorem F *Let f be a transcendental meromorphic function with $N(r, f) + N(r, 0; f) = S(r, f)$. Let $\psi(f)$ be a differential polynomial in f of degree $n \geq 2$ such that all the terms of $\psi(f)$ have degree at least two.*

If $\psi(f)$ is nonhomogeneous then $\delta(b, \psi) \leq 1 - \frac{1}{2n}$ for all $b \neq \infty$.

Now one may naturally ask: ***Is the upper bound $1 - \frac{1}{2n}$ in Theorem* F *sharp ? If not, what is the best possible upper bound ?***

The purpose of the note is to study this problem. We apply a result of H.X. Yi [10] to prove a theorem on the value distribution of nonhomogeneous diferential polynomials which not only gives the best possible upper bound for $\delta(b; P[f])$ in *Theorem* F but also estimate a larger quantity, the ramification index, under weaker hypothesis.

2. Lemmas

In this section we state two lemmas which will be needed in the sequel.

Lemma 1 [9] *Let $P[f] = \sum_{i=0}^{n} a_i f^i$ where $a_n \not\equiv 0$ and $T(r, a_i) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then $T(r, P[f]) = nT(r, f) + S(r, f)$.*

Lemma 2 [10] *Let $F = f^n + P[f]$, where $n \geq 2$ and $\Gamma_P \leq n - 1$. Then either $P[f] \equiv 0$ or*

$$\begin{aligned}(n - \gamma_P)T(r, f) \leq\ & \overline{N}(r, 0; f) + \overline{N}(r, 0; F) \\ & + (1 + \Gamma_P - \gamma_P)\overline{N}(r, \infty; f) + S(r, f).\end{aligned}$$

3. The Main Result

In this section we prove the main result of the note.

Theorem 1 *Let either $\Gamma_{M_j} = \gamma_{M_j}$ $(j = 1, 2, \ldots, l)$ or $\overline{N}(r, 0; f) + \overline{N}(r, f) = S(r, f)$ and $P[f] = \sum_{j=1}^{l} M_j[f]$ be such that $\gamma_P > \underline{\gamma}_P \geq 1$. Then*

$$\Theta(a; P[f]) \leq \frac{\gamma_P^*}{\gamma_P}$$

for any small function a $(\not\equiv \infty)$ of f.

Proof. Clearly we can write $M_j[f] = c_j f^{\gamma_{M_j}}$, where

$$c_j = b_j \left(\frac{f^{(1)}}{f}\right)^{n_{1j}} \left(\frac{f^{(2)}}{f}\right)^{n_{2j}} \cdots \left(\frac{f^{(k)}}{f}\right)^{n_{kj}}.$$

Now by Milloux theorem {p.55 [6]} we see that

$$m(r, c_j) \le m(r, b_j) + \sum_{i=1}^{k} n_{ij} m\left(r, \frac{f^{(i)}}{f}\right) = S(r, f).$$

Also

$$N(r, c_j) \le N(r, b_j) + \sum_{i=1}^{k} n_{ij} N\left(r, \frac{f^{(i)}}{f}\right).$$

We note that poles of $\frac{f^{(i)}}{f}$ occur only at the poles and zeros of f and a pole or a zero of f is a pole of $\frac{f^{(i)}}{f}$ with multiplicity at most i. So

$$N\left(r, \frac{f^{(i)}}{f}\right) \le i\{\overline{N}(r, f) + \overline{N}(r, 0; f)\}.$$

Therefore

$$\begin{aligned} N(r, c_j) &\le \left\{\sum_{i=1}^{k} i n_{ij}\right\} \{\overline{N}(r, f) + \overline{N}(r, 0; f)\} + S(r, f) \\ &= (\Gamma_{M_j} - \gamma_{M_j})\{\overline{N}(r, f) + \overline{N}(r, 0; f)\} + S(r, f) \\ &= S(r, f), \end{aligned}$$

by the given condition. Hence $T(r, c_j) = S(r, f)$ for $j = 1, 2, \ldots, l$.

Now collecting the same powers of f together and if necessary putting some $\alpha_i \equiv 0$

$$P[f] = \alpha_{\gamma_P} f^{\gamma_P} + \sum_{i=1}^{\gamma_P^*} \alpha_i f^i, \tag{1}$$

where $T(r, \alpha_i) = S(r, f)$ for $i = 1, 2, \ldots, \gamma_P^*, \gamma_P$ and $\alpha_{\gamma_P} \not\equiv 0$.

Now we put

$$\begin{aligned} F &= \frac{P[f]}{\alpha_{\gamma_P}} - \frac{a}{\alpha_{\gamma_P}} \\ &= f^{\gamma_P} + \left\{\sum_{i=1}^{\gamma_P^*} \frac{\alpha_i}{\alpha_{\gamma_P}} f^i - \frac{a}{\alpha_{\gamma_P}}\right\}, \end{aligned}$$

where $T(r, a) = S(r, f)$ and $a \not\equiv \infty$.

Clearly

$$\sum_{i=1}^{\gamma_P^*} \frac{\alpha_i}{\alpha_{\gamma_P}} f^i - \frac{a}{\alpha_{\gamma_P}} \not\equiv 0.$$

For, otherwise by *Lemma* 1 we arrive at a contradiction. Hence by *Lemma* 2 we obtain

$$\begin{aligned}(\gamma_P - \gamma_P^*)T(r,f) &\leq \overline{N}(r,0;F) + S(r,f)\\ &= \overline{N}(r,a;P[f]) + S(r,f).\end{aligned} \tag{2}$$

Also by *Lemma* 1 we get from (1) that

$$T(r,P[f]) = \gamma_P T(r,f) + S(r,f).$$

Therefore it follows from (2) that

$$\left(1 - \frac{\gamma_P^*}{\gamma_P}\right) T(r,P[f]) \leq \overline{N}(r,a;P[f]) + S(r,P[f]),$$

from which the theorem follows. This proves the theorem. □

Remark 1 The condition $\gamma_P \geq 1$ is necessary. For, let $f = \exp(z)$ and $P[f] = (f'')^2 + 2f' - 2f + 1$ {cf. [9]}. Then $\Theta(1;P[f]) = 1$.

Remark 2 The bound γ_P^*/γ_P is sharp. For, let $f = \exp(z)$ and $P[f] = f^3 - f^2$. Then $\Theta(0;P[f]) = 2/3$.

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