Hokkaido Mathematical Journal Vol. 31 (2002) p. 275–281

Finite sums of nilpotent elements in properly infinite C^*-algebras

Nobuhiro KATAOKA

(Received January 24, 2001)

Abstract. We prove that A is the linear span of elements $x \in A$ with $x^2 = 0$ if A is stable or properly infinite. Moreover, we prove the same statement for any closed two-sided ideal I of such C^*-algebras.

Key words: C^*-algebra, commutator, properly infinite, nilpotent elements.

1. Introduction

Recall that a C^*-algebra A is called stable if A is isomorphic to $A \otimes \mathbb{K}$, and a unital C^*-algebra A is called properly infinite if there exist projections $e, f \in A$ such that $e \sim f \sim 1$ and $ef = 0$, where $A \otimes \mathbb{K}$ is the tensor product of A and the C^*-algebra $\mathbb{K}$ of compact operators on a separable infinite dimensional Hilbert space, and $e \sim f$ means that there exists a partial isometry $x \in A$ such that $e = x^*x$, $f = xx^*$. We prove that A is the linear span of elements $x \in A$ with $x^2 = 0$ (or in particular the linear span of nilpotent elements of A) if A is stable or properly infinite. Moreover, we prove the same statement for any closed two-sided ideal I of such C^*-algebras. Denoting by $[A, A]$ the linear span of commutators $[a, b] = ab - ba$, with $a, b \in A$, T. Fack proved in [2] that $[A, A] = A$ if A is stable or properly infinite. We also show the same statement for any closed two-sided ideal I of such C^*-algebras.

The author would like to thank Prof. A. Kishimoto for some helpful comments.

2. Main Results

For each C^*-algebra A, we denote by $N(A)$ the linear span of elements $x \in A$ with $x^2 = 0$. We have the following result;

Theorem 1 *Let A be a properly infinite unital C^*-algebra. Then $I = N(I)$ for any closed two-sided ideal I of A.*

1991 Mathematics Subject Classification : 46L05, 47L05.

When $\{A_k\}_{k=1}^{\infty}$ is a sequence of C^*-algebras, we denote by $\bigoplus_{k=1}^{\infty} A_k$ the direct sum C^*-algebra $\left\{\bigoplus_{k=1}^{\infty} a_k : a_k \in A_k,\ \lim_{k\to\infty} \|a_k\| = 0\right\}$. We also denote by M_n the $n \times n$ matrix algebra, and by I_n the unit of M_n. We begin with the following lemma;

Lemma 2 *Let B be a C^*-algebra, and suppose that $A = B \otimes \left(\bigoplus_{\ell=1}^{\infty} M_{3\ell}\right)$. Define $E_0^\ell \in M_{3\ell}$ for $\ell \in \mathbb{N}$ by*

$$E_0^\ell = \frac{1}{\ell}\begin{pmatrix} I_\ell & 0 & 0 \\ 0 & -\frac{1}{2}I_\ell & 0 \\ 0 & 0 & -\frac{1}{2}I_\ell \end{pmatrix}.$$

Then $x \otimes \left(\bigoplus_{\ell=1}^{\infty} E_0^\ell\right) \in A$ is a element in $N(A)$ for any $x \in B$.

Proof. Define $E_m^\ell \in M_{3\ell}$ for $\ell \in \mathbb{N}$, $m = 1, 2, 3, 4$ by

$$E_1^\ell = \frac{1}{\ell}\begin{pmatrix} I_\ell & I_\ell & I_\ell \\ -\frac{1}{2}I_\ell & -\frac{1}{2}I_\ell & -\frac{1}{2}I_\ell \\ -\frac{1}{2}I_\ell & -\frac{1}{2}I_\ell & -\frac{1}{2}I_\ell \end{pmatrix},$$

$$E_2^\ell = \frac{1}{\ell}\begin{pmatrix} 0 & -I_\ell & -I_\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_3^\ell = \frac{1}{\ell}\begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2}I_\ell & 0 & \frac{1}{2}I_\ell \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_4^\ell = \frac{1}{\ell}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2}I_\ell & \frac{1}{2}I_\ell & 0 \end{pmatrix}.$$

Then $(E_m^\ell)^2 = 0$, $E_0^\ell = \sum_{m=1}^{4} E_m^\ell$ and $\lim_{\ell\to\infty} \|E_m^\ell\| = 0$ for each m. Thus $x \otimes \left(\bigoplus_{\ell=1}^{\infty} E_m^\ell\right) \in A$, $\left(x \otimes \left(\bigoplus_{\ell=1}^{\infty} E_m^\ell\right)\right)^2 = 0$ for each m and

$$x \otimes \left(\bigoplus_{\ell=1}^{\infty} E_0^\ell\right) = x \otimes \left(\bigoplus_{\ell=1}^{\infty} \sum_{m=1}^{4} E_m^\ell\right)$$
$$= \sum_{m=1}^{4} x \otimes \left(\bigoplus_{\ell=1}^{\infty} E_m^\ell\right) \in N(A).$$

□

Note that E_0^ℓ equals $\frac{1}{\ell}\{\sum_{i=1}^{\ell} e_{i,i}^\ell - \frac{1}{2}\sum_{i=\ell+1}^{3\ell} e_{i,i}^\ell\}$ by denoting the matrix units of $M_{3\ell}$ by $\{e_{i,j}^\ell\}$.

Lemma 3 *Let B be a C^*-algebra, and suppose that $A = B \otimes \mathbb{K}$. Denote by $\{e_{i,j}\}$ the matrix units of $\mathbb{K}$. Then $x \otimes e_{1,1} \in N(A)$ for each $x \in B$.*

Proof. Define a sequence $(\lambda_i)_{i=1}^\infty$ by

$$\lambda_i = \begin{cases} \dfrac{1}{4^{k-1}} & (4^{k-1} \leq i \leq 2 \cdot 4^{k-1} - 1) \\ \dfrac{1}{4^{k-1}} \cdot \left(-\dfrac{1}{2}\right) & (2 \cdot 4^{k-1} \leq i \leq 4 \cdot 4^{k-1} - 1), \end{cases}$$

for each $i \in \mathbb{N}$ $\left(\text{i.e. } (\lambda_i)_{i=1}^\infty = (1, -\frac{1}{2}, -\frac{1}{2}, \ldots, \underbrace{\left(-\tfrac{1}{2}\right)^{k-1}, \ldots, \left(-\tfrac{1}{2}\right)^{k-1}}_{2^{k-1} \text{ terms}}, \ldots)\right)$.

Then

$$e_{1,1} = \sum_{i=1}^{\infty} \lambda_i e_{i,i} - \sum_{i=2}^{\infty} \lambda_i e_{i,i}$$
$$= \sum_{k=1}^{\infty} \sum_{i=4^{k-1}}^{4^k-1} \lambda_i e_{i,i} + \sum_{k=1}^{\infty} \sum_{i=2\cdot 4^{k-1}}^{2\cdot 4^k-1} (-\lambda_i) e_{i,i}$$
$$= \sum_{k=1}^{\infty} \frac{1}{4^{k-1}} \left\{ \sum_{i=4^{k-1}}^{2\cdot 4^{k-1}-1} e_{i,i} - \frac{1}{2} \sum_{i=2\cdot 4^{k-1}}^{4\cdot 4^{k-1}-1} e_{i,i} \right\}$$
$$+ \sum_{k=1}^{\infty} \frac{1}{2 \cdot 4^{k-1}} \left\{ \sum_{i=2\cdot 4^{k-1}}^{2\cdot 2\cdot 4^{k-1}-1} e_{i,i} - \frac{1}{2} \sum_{i=2\cdot 2\cdot 4^{k-1}}^{4\cdot 2\cdot 4^{k-1}-1} e_{i,i} \right\}.$$

For each $\ell \in \mathbb{N}$, define $*$-monomorphisms $\imath^\ell : M_{3\ell} \to \mathbb{K}$ by

$$\imath^\ell(e_{i,j}^\ell) = e_{\ell+i-1, \ell+j-1} \qquad 1 \leq i,\ j \leq 3\ell.$$

Then

$$\imath^{\ell}(E_0^{\ell}) = \frac{1}{\ell}\left\{\sum_{i=\ell}^{2\ell-1} e_{i,i} - \frac{1}{2}\sum_{i=2\ell}^{4\ell-1} e_{i,i}\right\}$$

and $Ran(\imath^{4^{k-1}}) \perp Ran(\imath^{4^{k'-1}})$, $Ran(\imath^{2\cdot 4^{k-1}}) \perp Ran(\imath^{2\cdot 4^{k'-1}})$ for each $k, k' \in \mathbb{N}$, $k \neq k'$, where $Ran(\imath^{\ell})$ is the range of $\imath^{\ell}$ and $\perp$ means the orthogonality relation. Thus the maps

$$\imath_1 = \mathrm{id}_B \otimes \left(\bigoplus_{k=1}^{\infty} \imath^{4^{k-1}}\right) : B \otimes \left(\bigoplus_{k=1}^{\infty} M_{3\cdot 4^{k-1}}\right) \to A$$

$$\imath_2 = \mathrm{id}_B \otimes \left(\bigoplus_{k=1}^{\infty} \imath^{2\cdot 4^{k-1}}\right) : B \otimes \left(\bigoplus_{k=1}^{\infty} M_{3\cdot 2\cdot 4^{k-1}}\right) \to A$$

are well-defined homomorphisms and injective, where id_B is the identity map on B. Since

$$\imath_1\left(N\left(B \otimes \left(\bigoplus_{k=1}^{\infty} M_{3\cdot 4^{k-1}}\right)\right)\right),$$

$$\imath_2\left(N\left(B \otimes \left(\bigoplus_{k=1}^{\infty} M_{3\cdot 2\cdot 4^{k-1}}\right)\right)\right) \subseteq N(A),$$

it follows by Lemma 2 that

$$x \otimes e_{1.1} = \imath_1\left(x \otimes \left(\bigoplus_{k=1}^{\infty} E_0^{4^{k-1}}\right)\right) - \imath_2\left(x \otimes \left(\bigoplus_{k=1}^{\infty} E_0^{2\cdot 4^{k-1}}\right)\right) \in N(A)$$

for each $x \in B$. □

Proof of Theorem 1. Let $e, f \in A$ be projectons such that $e \sim 1 \sim f$, $ef = 0$, and $u, v \in A$ be isometries such that $u^*u = v^*v = 1$, $uu^* = e$ and $vv^* = f$. For each $x \in I$, since $x = exe + ex(1-e) + (1-e)xe + (1-e)x(1-e)$ and $(1-e)xe$, $ex(1-e) \in N(I)$, we only have to prove that exe, $(1-e)x(1-e) \in N(I)$. Set $e_i, f_i \in A$ for each $i \in \mathbb{N}$ by

$$e_i = \begin{cases} v^{i-1}ue & (i \geq 2) \\ e & (i = 1), \end{cases}$$

$$f_i = \begin{cases} u^{i-1}v(1-e) & (i \geq 2) \\ 1-e & (i = 1). \end{cases}$$

Note that e_i's are partial isometries with mutually orthogonal range projection and with the same initial projection e, and that f_i's are partial isometries with mutually orthogonal range projections and with the same initial projection $1-e$. Define $*$-isomorphisms $\varphi : eIe \otimes \mathbb{K} \to I$, $\psi : (1-e)I(1-e) \otimes \mathbb{K} \to I$ by

$$\begin{aligned} \varphi(a \otimes e_{i,j}) &= e_i a e_j^*, \qquad a \in eIe, \qquad i,j \in \mathbb{N}, \\ \psi(b \otimes e_{i,j}) &= f_i b f_j^*, \qquad b \in (1-e)I(1-e), \qquad i,j \in \mathbb{N}. \end{aligned}$$

Then $\varphi(a \otimes e_{1,1}) = a$ for each $a \in eIe$ and $\psi(b \otimes e_{1,1}) = b$ for each $b \in (1-e)I(1-e)$. Thus by Lemma 3,

$$\begin{aligned} exe &= \varphi(exe \otimes e_{1,1}) \in \varphi(N(eIe \otimes \mathbb{K})) \subseteq N(I), \\ (1-e)x(1-e) &= \psi((1-e)x(1-e) \otimes e_{1,1}) \\ &\in \psi(N((1-e)I(1-e) \otimes \mathbb{K})) \subseteq N(I). \end{aligned}$$

□

Corollary 4 *Let B be a C^*-algebra such that the multiplier algebra $M(B)$ is properly infinite, and C be a C^*-algebra. If $A = B \otimes C$ (for instance, if A is a stable algebra, a tensor product with a Cuntz-algebra O_n) then $I = N(I)$ for any closed two-sided ideal I of A, where the tensor product can be taken with respect to any C^*-norm.*

Proof. The multiplier algebra $M(A)$ of A is properly infinite and A is a closed two-sided ideal of $M(A)$. Thus I is a closed two-sided ideal of $M(A)$ and $I = N(I)$ by Theorem 1. □

Recall that a unital C^*-algebra A is called infinite if there exists a projection $e \in A$ such that $e \neq 1$, $e \sim 1$.

Corollary 5 *If A is a simple unital infinite C^*-algebra then $A = N(A)$*

Proof. By [1], A is properly infinite. Thus $A = N(A)$ by Theorem 1. □

If we do not assume that A is simple in Corollary 5 then the conclusion does not follow in general. For instance, the Toeplitz algebra $\mathfrak{T}$ is a unital infinite C^*-algebra with a closed two-sided ideal $\mathbb{K}$, and the quotient C^*-algebra $\mathfrak{T}/\mathbb{K}$ is isomorphic to $C(\mathbb{S})$, where $C(\mathbb{S})$ is the C^*-algebra of complex continuous functions on $\mathbb{S}$ and $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$. Then $N(\mathfrak{T}) = \mathbb{K} \neq \mathfrak{T}$. For $N(\mathbb{K}) = \mathbb{K}$ by Corollary 4, and $N(\mathfrak{T}/\mathbb{K}) = N(C(\mathbb{S})) = \{0\}$. Thus $N(\mathfrak{T}) \subseteq \mathrm{Ker}(\pi) = \mathbb{K} = N(\mathbb{K}) \subseteq N(\mathfrak{T})$ since $\pi(N(\mathfrak{T})) \subseteq N(\mathfrak{T}/\mathbb{K}) = \{0\}$, where π is the quotient map from $\mathfrak{T}$ onto $\mathfrak{T}/\mathbb{K}$.

Finally we consider the relation between $[A, A]$ and $N(A)$.

Proposition 6 *For any C^*-algebra A, $N(A) \subseteq [A, A]$.*

Proof. For each $x \in A$ with $x^2 = 0$, set $x = u|x|$, where $|x| = (x^*x)^{\frac{1}{2}}$ and u in the double dual A^{**} of A is the partial isometry of the polar-decomposition of x. Then since $u|x|^{\frac{1}{2}} \in A$,

$$x = \left[u|x|^{\frac{1}{2}}, |x|^{\frac{1}{2}}\right] \in [A, A].$$

$\square$

Corollary 7 *Let A be a properly infinite C^*-algebra. Then $I = [I, I]$ for any closed two-sided ideal I of A.*

Proof. By Theorem 1 and Proposition 6, $I = N(I) \subseteq [I, I] \subseteq I$. $\square$

Corollary 8 *Let B be a C^*-algebra such that the multiplier algebra $M(B)$ is properly infinite, and C be a C^*-algebra. If $A = B \otimes C$ then $I = [I, I]$ for any closed two-sided ideal I of A, where the tensor product can be taken with respect to any C^*-norm.*

Proof. The multiplier algebra $M(A)$ is properly infinite and A is a closed two-sided ideal of $M(A)$. Thus I is a closed two-sided ideal of $M(A)$ and $I = [I, I]$ by Corollary 7. $\square$

References

[1] Cuntz J., *The structure of multiplication and addition in simple C^*-algebras*. Math. Scand. **40** (1977), 215–233.

[2] Fack T., *Finite sums of commutators in C^*-algebras*. Ann. Inst. Fourier, Grenoble **32** (1982), 129–137.

[3] Marcoux L.W., *On the closed Lie ideals of certain C^*-algebras*. J. Integr. Equat. Oper. Th. **22** (1995), 463–475.

[4] Marcoux L.W. and Murphy G.J., *Unitarily-invariant linear spaces in C^*-algebaras*. Proc. Amer. Math. Soc. **126** (1998), 3597–3605.

[5] Pedersen G., *C^*-algebras and their Automorphism Group*. Academic Press, London/ New York/ San Francisco (1979).

[6] Thomsen K., *Finite sums and products of commutators in inductive limit C^*-algebras*. Ann. Inst. Fourier. Grenoble **43** (1993), 225–249.

[7] Wegge-Olsen N.E., *K-Theory and C^*-Algebras*. Oxford University Press, Oxford/ New York/ Tokyo (1993).

Department of Mathematics
Graduate School of Science
Hokkaido University
Sapporo 060-0810, Japan
E-mail: kataoka@math.sci.hokudai.ac.jp