Hokkaido Mathematical Journal Vol. 31 (2002) p. 261–273

# On the connection of coefficient and structural conditions about Fourier series

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(Received September 29, 2000)

**Abstract.** We extend the validity of some theorems treating the relations of coefficient and structural conditions with respect to Fourier series. The extension means that the conditions given by means of the function $x^\beta$, $\beta > 0$, are replaced by concave or Mulholland-type functions.

*Key words*: Fourier series, coefficient and structural condition, concave function, Mulholland function, best approximation.

## 1. Introduction

Let $f(x)$ be a $2\pi$-periodic Lebesgue integrable to the $p$th power $(p > 1)$ function and let

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

be its Fourier series. Denote

$$\rho_n := (a_n^2 + b_n^2)^{1/2} \quad \text{and} \quad p' := \frac{p}{p-1}.$$

In an old-time paper [4], among others, we proved the following result.

**Theorem A** *Let $w(x)$ $(x \geq 1)$ be a positive and monotone function with the property $w(2n) \leq Aw(n)$ $(A \geq 2,\ n = 1, 2, \ldots)$, moreover let $0 < \beta \leq p'$.*

(i) *If $p \leq 2$ then*

$$\int_0^1 t^{-2} w\left(\frac{1}{t}\right)\left(\int_0^{2\pi} |f(x+2t) + f(x-2t) - 2f(x)|^p dx\right)^{\beta/p} dt < \infty \tag{1.1}$$

2000 *Mathematics Subject Classification* : 42A16, 42A99.

The author was partially supported by the Hungarian National Foundation for Scientific Research under Grant #T029080.

*implies*

$$\sum_{n=1}^{\infty} w(n) \left\{ \sum_{k=n}^{\infty} \rho_k^{p'} \right\}^{\beta/p'} < \infty. \tag{1.2}$$

(ii) *If* $p \geq 2$ *and*

$$\sum_{n=m}^{\infty} n^{-\beta} w(n) \leq K m^{1-\beta} w(m), \tag{1.3}$$

*where* $K := K(\beta, w)$ *is a positive constant, then* (1.2) *implies*

$$\int_0^1 t^{-2} w\left(\frac{1}{t}\right) \left( \int_0^{2\pi} |f(x+t) - f(x-t)|^p dx \right)^{\beta/p} dt < \infty. \tag{1.4}$$

It is easy to see that in the particular case $p = 2$ we can derive an equivalence result from Theorem A, namely assuming the condition (1.3) with $0 < \beta \leq 2$, then the implication chain (1.4)$\Rightarrow$(1.1)$\Rightarrow$(1.2)$\Rightarrow$(1.4) implies the following result.

**Corollary A** *The conditions* (1.1), (1.2) *and* (1.4) *with* $p = 2$ *are equivalent under the assumption* (1.3) *with* $0 < \beta \leq 2$.

This assertion is a part of a theorem proved also in [4].

In a recent paper [5], among others, we generalized the part $\beta \leq 1$ of the assertion (i) of Theorem A such that we replaced the function $x^\beta$ in (1.1) by an arbitrary increasing and concave function $\varphi(x)$ and utilized the quasi $\delta$-power-monotone sequences instead of monotone sequences.

In order to make easy to formulate our result we recall some definitions and notations.

In the sequel we shall assume that $K$, $K_i$ denote positive constants, and may vary from occurance to occurance, $K(\cdot)$ denotes such a constant which depends only those parameters as indicated in the bracket.

We say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is *quasi $\delta$-power-monotone increasing* (*decreasing*) if there exists a constant $K := K(\delta, \gamma) \geq 1$ such that

$$K n^\delta \gamma_n \geq m^\delta \gamma_m \quad (n^\delta \gamma_n \leq K m^\delta \gamma_m)$$

holds for any $n \geq m$, $m = 1, 2, \ldots$

Now we can recall the mentioned result proved in [5], where we shall use the following function linked to a sequence $\omega := \{\omega_n\}$ of positive terms:

$$\omega(x) := \begin{cases} \omega_n, & \text{if } \ x = n, \ \ n \geq 1, \\ \text{linear} & \text{on } \ [n, n+1]. \end{cases}$$

**Theorem B** *Let $1 < p \leq 2$, and let $\omega := \{\omega_n\}$ be a quasi $\eta$-power-monotone decreasing sequence of positive numbers with some negative $\eta$, and simultaneously quasi $\rho$-power-monotone increasing with some $\rho < 1$. If $\varphi(u)$ $(u \geq 0,\ \varphi(0) = 0)$ is an increasing and concave function, furthermore*

$$\int_0^1 t^{-2}\omega\left(\frac{1}{t}\right)\varphi\left(\left\{\int_0^{2\pi} |f(x+2t) + f(x-2t) - 2f(x)|^p dx\right\}^{1/p}\right)dt < \infty, \tag{1.5}$$

*then*

$$\sum_{n=1}^{\infty} \omega_n \varphi\left(\left\{\sum_{k=n}^{\infty} \rho_k^{p'}\right\}^{1/p'}\right) < \infty \tag{1.6}$$

*holds.*

Naturally we also wanted to generalize the part $\beta \geq 1$ of the assertion (ii) of Theorem A similarly as that of the assertion (i), that is, to replace the function $x^\beta$ by an increasing concave $\varphi(x)$, but without success.

Analyzing this unsuccessful attempt we have realized that the problem lies in that the increase of a concave function can be very small contrary to that of the function $x^\beta$ with $\beta > 0$. Thus we have started to consider the so-called Mulholland functions, namely their increase can be also prescribed.

As it is well known a *Mulholland class* is determined by two parameters as follows:

Given two numbers $p \geq q \geq 0$, we shall write $\varphi \in \Delta(p,q)$ if $\varphi(t)$ is a nonnegative function defined on $[0,\infty)$ such that $\varphi(0) = 0$, $t^{-p}\varphi(t)$ is nonincreasing, and $t^{-q}\varphi(t)$ is nondecreasing on $(0,\infty)$. Clearly $\varphi(t)$ is nondecreasing on $(0,\infty)$.

It is also plain that the function $\varphi(t) = t^\beta$ belongs to the class $\Delta(\beta,\beta)$. Hence it is also clear that the next step toward the generalizations is to consider a class $\Delta(\beta,\gamma)$ with $0 < \gamma \leq \beta$ and thus we get functions having very similar properties to the function $x^\beta$, or if $0 < \beta \leq 1$ then to any concave function having increase large enough.

On the other hand it is also obvious that every increasing concave function belongs to the class $\Delta(1,0)$, but not conversely. Namely a function from the class $\Delta(1,0)$ is not necessarily concave.

Perceiving this fact and analyzing the proof of Theorem B we have recognized that the proof of Theorem B given in [5] also holds for any function $\varphi \in \Delta(1,0)$ without any change. Thus we emerge to a mild generalization of Theorem B.

In the present paper first we formulate this observation as Theorem 1, and our further results will deal with functions $\varphi$ which belong to a class $\Delta(\beta,\gamma)$ with $0<\gamma\le\beta\le 1$, or $\varphi$ is such a function that the following composite function

$$\varphi_p(x) := \varphi(x^{1-1/p}) \equiv \varphi(x^{1/p'}) \tag{1.7}$$

belongs to $\Delta(\beta,\gamma)$.

We know that an assumption given by (1.7) on the function $\varphi$ is not a beloved one, but useful. Namely our first three theorems generalize only the part $0<\beta\le 1$ of the Theorem A, but it deals with the case $1<\beta\le p'$, too.

It is clear that if $\beta>1$ then the function $\varphi(x)=x^\beta$ is not any more concave, but convex. On the other hand it is clear that $x^\beta$ $(\beta>1)$ also belongs to a Mulholland class, e.g. to the class $\Delta(\beta,1)$, and the functions of this class have similar properties as $x^\beta$.

Then, if $\varphi\in\Delta(\beta,1)$, unfortunately, we are not able to handle effectively the methods of proof used earlier. However we have recognized that by means of the function $\varphi_p(x)$ defined in (1.7) our proofs work successfully, moreover the proofs run word for word as in the proofs given in the case $0<\beta\le 1$.

We remark that the proofs with $\varphi_p(x)$ would be usable for the whole range $0<\beta\le p'$, but the conditions given immediately on $\varphi$ are much easier to survey than if they are formulated via $\varphi_p(x)$, therefore we give and prove the cases $0<\beta\le 1$ and $1<\beta\le p'$ separately. Naturally we do not detail or repeat all of the proofs. We shall detail the proof only at the extension of Theorem 1, namely, as we have mentioned, the proof of Theorem 1 runs as that of Theorem B, and here we do not recall it in the case $0<\beta\le 1$. The proof to be given for this theorem will use the condition given with $\varphi_p$, and it will be effective for the whole range $0<\beta\le p'$.

Our theorems and their extensions in the special case $\varphi(x)=x^\beta$ will reduce to the suitable parts of Theorem A, and they jointly cover the whole range $0 < \beta \le p'$.

In my view, to prove the analogues of our theorems for an arbitrary concave or convex function $\varphi(x)$, it is a hard task if you want that their special cases $\varphi(x) = x^\beta$ should reduce to the suitable parts of Theorem A.

## 2. Theorems

**Theorem 1** *Let $1 < p \le 2$, and let $\omega := \{\omega_n\}$ have the same meaning and properties as in Theorem* B. *If $\varphi \in \Delta(1,0)$ then the condition* (1.5) *implies the inequality* (1.6).

**Theorem 2** *Let us assume that $p \ge 2$, $0 < \gamma \le \beta \le 1$ and that the function $\varphi(u)$ belongs to the class $\Delta(\beta,\gamma)$. Furthermore let us assume that the sequence $\omega := \{\omega_n\}$ is quasi $\eta$-power-monotone decreasing with some $\eta > 1-\gamma$. Then*

$$\sum_{n=1}^{\infty} \omega_n \varphi\left(\left(\sum_{k=n}^{\infty} \rho_k^{p'}\right)^{1/p'}\right) < \infty \tag{2.1}$$

*implies*

$$\int_0^1 t^{-2}\omega\left(\frac{1}{t}\right)\varphi\left(\left(\int_0^{2\pi} |f(x+t)-f(x-t)|^p dx\right)^{1/p}\right) dt < \infty. \tag{2.2}$$

We underline that if $\varphi(u) := u^\beta$, then $\varphi \in \Delta(\beta,\beta)$ clearly holds, and thus with $\omega_n := w(n)$ Theorem 2 reduces to the part (ii) of Theorem A in the special case $\beta \le 1$, namely the assumption (1.3) holds if and only if the sequence $\{w(n)\}$ is quasi $\eta$-power-monotone decreasing with some $\eta > 1-\beta$ ($\gamma = \beta$ if $\varphi \in \Delta(\beta,\beta)$), see e.g. Corollary 2 in [6], here recalled as Lemma 4.

In the case $p = 2$ we can connect the results of the Theorems 1 and 2, and the outcome is the following theorem.

**Theorem 3** *Let $0 < \gamma \le \beta \le 1$. If the function $\varphi(x)$ belongs to the class $\Delta(\beta,\gamma)$ and the sequence $\omega := \{\omega_n\}$ is quasi $\eta$-power-monotone decreasing with some $\eta > 1-\gamma$, and simultaneously quasi $\rho$-power-monotone increasing with some $\rho < 1$, then the following conditions*

$$\int_0^1 t^{-2}\omega\left(\frac{1}{t}\right)\varphi\left(\left(\int_0^{2\pi}|f(x+t)-f(x-t)|^2dx\right)^{1/2}\right)dt<\infty, \tag{2.3}$$

$$\int_0^1 t^{-2}\omega\Big(\frac{1}{t}\Big)\varphi\Big(\Big(\int_0^{2\pi}|f(x+2t)+f(x-2t)-2f(x)|^2dx\Big)^{1/2}\Big)dt<\infty \tag{2.4}$$

*and*

$$\sum_{n=1}^{\infty}\omega_n\varphi\left(\left(\sum_{k=n}^{\infty}\rho_k^2\right)^{1/2}\right)<\infty \tag{2.5}$$

*are equivalent.*

As we have mentioned the function $\varphi(x)=x^\beta$ $(0<\beta\le 1)$ belongs to the class $\Delta(\beta,\beta)$. Thus if the sequence $\omega$ satisfies the conditions given in Theorem 3 on $\omega$, then the sequence $w_n:=\omega_n$ satisfies (1.3). Consequently Theorem 3 for $0<\beta\le 1$, but not for $1<\beta\le 2$, can be considered as an extension of the Corollary A from the class $\Delta(\beta,\beta)$ to the wider class $\Delta(\beta,\gamma)$ with $0<\gamma<\beta\le 1$.

Now let us consider the following quadratic moduli of continuity:

$$\omega^{(2)}(f,\delta):=\sup_{0\le t\le\delta}\left\{\int_0^{2\pi}[f(x+t)-f(x-t)]^2dx\right\}^{1/2},$$

$$\omega_2^{(2)}(f,\delta):=\sup_{0\le t\le\delta}\left\{\int_0^{2\pi}[f(x+2t)+f(x-2t)-2f(x)]^2dx\right\}^{1/2},$$

$$w^{(2)}(f,\delta):=\left\{\frac{1}{\delta}\int_0^{\delta}\left(\int_0^{2\pi}[f(x+t)-f(x-t)]^2dx\right)dt\right\}^{1/2},$$

$$w_2^{(2)}(f,\delta):=\left\{\frac{1}{\delta}\int_0^{\delta}\left(\int_0^{2\pi}[f(x+2t)+f(x-2t)-2f(x)]^2dx\right)dt\right\}^{1/2}.$$

If $\Omega(f,\delta)$ denotes one of the moduli of continuity defined above, then we can enlarge the equivalence chain given in Theorem 3 as follows:

**Corollary 1** *Under the assumptions of Theorem* 3 *the conditions* (2.3), (2.4), (2.5),

$$\sum_{n=1}^{\infty} \omega_n \varphi\left(\Omega\left(f, \frac{1}{n}\right)\right) < \infty \tag{2.6}$$

*and*

$$\sum_{n=1}^{\infty} \omega_n \varphi(E_n(f,2)) < \infty \tag{2.7}$$

*are equivalent, where* $E_n(f,2)$ *denotes the best trigonometric approximation of* $f$ *by trigonometric polynomials of order at most* $(n-1)$ *in the space* $L^2$.

**Extensions** *The assertions of the Theorems* 1, 2, 3 *and Corollary* 1 *will uphold if among their assumptions we replace the condition given on the function* $\varphi(x)$ *by the same condition on the composite function* $\varphi_p(x)$ *defined in* (1.7).

As an example we formulate the extension of Theorem 1 as follows.

**Theorem 1*** *Let* $0 < p \leq 2$, *and let* $\omega := \{\omega_n\}$ *have the same meaning and properties as in Theorem* B. *If* $\varphi_p \in \Delta(1,0)$ *then the condition* (1.5) *implies* (1.6).

## 3. Lemmas

We shall apply the following lemmas

**Lemma 1** ([7, p.314]) *If* $\varphi \in \Delta(p,q)$ *for some* $0 \leq q \leq p$ *and* $t_m \geq 0$ *for all* $m$, *then*

$$\varepsilon^p \varphi(t) \leq \varphi(\varepsilon t) \leq \varepsilon^q \varphi(t) \quad \text{for} \quad 0 \leq \varepsilon \leq 1 \quad \text{and} \quad t \geq 0, \tag{3.1}$$

*and*

$$\varphi\left(\sum_{m=0}^{\infty} t_m\right) \leq \sum_{m=0}^{\infty} \varphi(t_m) \quad \text{for} \quad 0 < p \leq 1. \tag{3.2}$$

**Lemma 2** ([1, p.348]) *If* $f \in L^2$ *then*

$$\left[\omega^{(2)}\left(f, \frac{1}{n}\right)\right]^2 \leq \frac{8\pi}{n^2} \sum_{k=1}^{n} k \sum_{\nu=k}^{\infty} \rho_\nu^2.$$

**Lemma 3** ([3. p.241]) *If $f \in L^2$ then*

$$w_2^{(2)}\left(f, \frac{1}{n}\right) \geq E_n(f, 2).$$

**Lemma 4** ([6, p.11]) *A positive sequence $\{\gamma_n\}$ bounded by blocks, that is, if the inequalities*

$$\alpha_1 \min(\gamma_{2^k}, \gamma_{2^{k+1}}) \leq \gamma_n \leq \alpha_2 \max(\gamma_{2^k}, \gamma_{2^{k+1}})$$

*hold for any $2^k \leq n \leq 2^{k+1}$, $k = 1, 2, \dots$ with $0 < \alpha_1 \leq \alpha_2 < \infty$, is quasi $\delta$-power-monotone increasing (decreasing) with a certain negative (positive) exponent $\delta$ if and only if the inequality*

$$\sum_{n=1}^{m} \gamma_n n^{-1} \leq K\gamma_m \qquad \left(\sum_{n=m}^{\infty} \gamma_n n^{-1} \leq K\gamma_m\right)$$

*holds for any natural number $m$.*

**Lemma 5** *Let us assume that the sequence $\omega := \{\omega_n\}$ of positive numbers is quasi $\eta$-power-monotone decreasing with some negative $\eta$ and simultaneously quasi $\rho$-power-monotone increasing with some $\rho < 1$. Then there exist constants $K(\omega) \geq 1$, $A = A(\omega) \geq 2$ and an increasing sequence $\{p_m\}$ of integers such that $p_0 = 0$ and for all $m \geq 0$ the inequalities*

$$A^m \leq \sum_{n=p_m+1}^{p_{m+1}} \omega_n \leq A^{m+1}, \tag{3.3}$$

*and for $m \geq 1$*

$$p_{m+1} \leq K(\omega) p_m \tag{3.4}$$

*hold.*

The assertions of Lemma 5 can be found in the proof of Theorem 3 of [5] implicitely.

## 4. Proof of the theorems

*Proof of Theorem* 2. By the Hausdorff-Young theorem (see [8, p.101]) we obtain that

$$\left\{\int_0^{2\pi} |f(x+t)-f(x-t)|^p dx\right\}^{1/p} \le K\left\{\sum_{k=1}^{\infty} \rho_k^{p'} |\sin kt|^{p'}\right\}^{1/p'}.$$

Hence, since the function $\varphi(u) \in \Delta(\beta,\gamma)$ and $u^{1/p'}$ is concave, using the inequality (3.2), we obtain that

$$\begin{aligned}
&\int_0^1 t^{-2}\omega\left(\frac{1}{t}\right)\varphi\left(\left\{\int_0^{2\pi} |f(x+t)-f(x-t)|^p dx\right\}^{1/p}\right)dt \\
&\le K_1 \sum_{m=0}^{\infty} \int_{2^{-m-1}}^{2^{-m}} t^{-2}\omega\left(\frac{1}{t}\right)\varphi\left(\left\{\sum_{k=1}^{\infty} \rho_k^{p'} |\sin kt|^{p'}\right\}^{1/p'}\right)dt \\
&\le K_2 \sum_{m=0}^{\infty} 2^m \omega_{2^m} \varphi\left(\left\{\sum_{k=1}^{2^m-1} \rho_k^{p'} k^{p'} 2^{-mp'}\right\}^{1/p'} + \left\{\sum_{k=2^m}^{\infty} \rho_k^{p'}\right\}^{1/p'}\right) \\
&\le K_3 \sum_{m=0}^{\infty} 2^m \omega_{2^m} \varphi\left(\sum_{\nu=0}^{m-1} 2^{\nu-m}\left\{\sum_{k=2^\nu}^{2^{\nu+1}-1} \rho_k^{p'}\right\}^{1/p'}\right) \\
&\quad + K_3 \sum_{m=0}^{\infty} 2^m \omega_{2^m} \varphi\left(\left\{\sum_{k=2^m}^{\infty} \rho_k^{p'}\right\}^{1/p'}\right).
\end{aligned} \tag{4.1}$$

Here the second sum is finite by (2.1). Next we estimate the first sum utilizing again (3.2) and (3.1) with $q=\gamma$.

$$\begin{aligned}
&\sum_{m=0}^{\infty} 2^m \omega_{2^m} \varphi\left(\sum_{\nu=0}^{m-1} 2^{\nu-m}\left\{\sum_{k=2^\nu}^{2^{\nu+1}-1} \rho_k^{p'}\right\}^{1/p'}\right) \\
&\le \sum_{m=0}^{\infty} 2^m \omega_{2^m} \sum_{\nu=0}^{m-1} 2^{(\nu-m)\gamma} \varphi\left(\left\{\sum_{k=2^\nu}^{2^{\nu+1}-1} \rho_k^{p'}\right\}^{1/p'}\right) \\
&\le \sum_{\nu=0}^{\infty} 2^{\nu\gamma} \varphi\left(\left\{\sum_{k=2^\nu}^{2^{\nu+1}-1} \rho_k^{p'}\right\}^{1/p'}\right) \sum_{m=\nu}^{\infty} 2^{m(1-\gamma)}\omega_{2^m}.
\end{aligned} \tag{4.2}$$

Now if we use the decreasing part of Lemma 4 with $\gamma_n := n^{1-\gamma}\omega_n$, then we get that

$$\sum_{m=\nu}^{\infty} 2^{m(1-\gamma)}\omega_{2^m} \le K 2^{\nu(1-\gamma)}\omega_{2^\nu}. \tag{4.3}$$

Utilizing this estimation we get that the last sum in (4.2) is less than

$$K\sum_{\nu=0}^{\infty} 2^{\nu}\omega_{2^\nu}\varphi\left(\left\{\sum_{k=2^\nu}^{2^{\nu+1}} \rho_k^{p'}\right\}^{1/p'}\right),$$

and this is obviously finite by (2.1).

The estimations (4.1) and (4.2) thus imply the implication (2.1) $\Rightarrow$ (2.2); and this was to be proved. □

*Proof of Corollary* 1. By the classical theorem of Gram [2]

$$E_n^2(f,2) = \pi\sum_{k=n}^{\infty}\rho_k^2, \tag{4.4}$$

thus (2.5) and (2.7) are obviously equivalent. It is also clear that

$$\frac{1}{2}w_2^{(2)}\left(f,\frac{1}{n}\right) \le \Omega\left(f,\frac{1}{n}\right) \le 2\omega^{(2)}\left(f,\frac{1}{n}\right) \tag{4.5}$$

holds. In order to verify the implication (2.7) $\Rightarrow$ (2.6), by (4.5) it is sufficient to show that (2.7) implies

$$\sigma_1 := \sum_{n=1}^{\infty}\omega_n\varphi\left(\omega^{(2)}\left(f,\frac{1}{n}\right)\right) < \infty. \tag{4.6}$$

Using the abbreviation $E_n := E_n\left(f,\frac{1}{n}\right)$, (3.1), (3.2), (4.3), (4.4) and the Lemma 2 we obtain that

$$\begin{aligned}
\sigma_1 &\le K\sum_{n=1}^{\infty}\omega_n\varphi\left(\left\{\frac{1}{n^2}\sum_{k=1}^{n} kE_k^2\right\}^{1/2}\right)\\
&\le K_1\sum_{m=0}^{\infty}\sum_{n=2^m}^{2^{m+1}}\omega_n\varphi\left(\left\{\sum_{\nu=0}^{m}\sum_{k=2^\nu}^{2^{\nu+1}}\frac{k}{n^2}E_k^2\right\}^{1/2}\right)\\
&\le K_2\sum_{m=0}^{\infty}2^m\omega_{2^m}\sum_{\nu=0}^{m}2^{(\nu-m)\gamma}\varphi(E_{2^\nu})\\
&= K_2\sum_{\nu=0}^{\infty}2^{\nu\gamma}\varphi(E_{2^\nu})\sum_{m=\nu}^{\infty}2^{m(1-\gamma)}\omega_{2^m}\\
&\le K_3\sum_{\nu=0}^{\infty}2^{\nu}\omega_{2^\nu}\varphi(E_{2^\nu}) \le K_4\sum_{n=1}^{\infty}\omega_n\varphi(E_n),
\end{aligned}$$

thus, by (2.7), we verified (4.6), and herewith the implication (2.7) $\Rightarrow$ (2.6), too.

To prove the implication (2.6) $\Rightarrow$ (2.7) it is enough to show (2.7) follows from

$$\sum_{n=1}^{\infty} \omega_n w_2^{(2)}\left(f, \frac{1}{n}\right) < \infty,$$

but this, by Lemma 3 and (2.6), clearly holds.

Let us associate the equivalence (2.6) $\Leftrightarrow$ (2.7) with the equivalence chain (2.3) $\Leftrightarrow$ (2.4) $\Leftrightarrow$ (2.5) proved in the Theorem 3, and considering the obvious equivalence (2.5) $\Leftrightarrow$ (2.6) (see (4.4)), the proof of Corollary 1 is complete.

□

*Proof of Theorem* 1*. Using the property (3.2) of $\varphi_p(x)$, and (3.3) we obtain that

$$\begin{aligned}
\sum_{n=1}^{\infty} \omega_n \varphi\left(\left\{\sum_{k=n}^{\infty} \rho_k^{p'}\right\}^{1/p'}\right) &\equiv \sum_{n=1}^{\infty} \omega_n \varphi_p\left(\sum_{k=n}^{\infty} \rho_k^{p'}\right) \\
&\leq \sum_{m=0}^{\infty} \sum_{n=p_m+1}^{p_{m+1}} \omega_n \sum_{\nu=m}^{\infty} \varphi_p\left(\sum_{k=p_\nu+1}^{p_{\nu+1}} \rho_k^{p'}\right) \\
&\leq \sum_{\nu=0}^{\infty} \varphi_p\left(\sum_{k=p_\nu+1}^{p_{\nu+1}} \rho_k^{p'}\right) \sum_{m=\nu}^{\infty} A^{m+1} \\
&\leq A^2 \sum_{\nu=0}^{\infty} A^{\nu} \varphi\left(\left\{\sum_{k=p_\nu+1}^{p_{\nu+1}} \rho_k^{p'}\right\}^{1/p'}\right). \qquad (4.7)
\end{aligned}$$

Next the Hausdorff-Young theorem gives that

$$\begin{aligned}
&\left\{\sum_{k=1}^{\infty} \rho_k^{p'} |\sin kt|^{2p'}\right\}^{1/p'} \\
&\qquad \leq \left\{\int_0^{2\pi} |f(x+2t) + f(x-2t) - 2f(x)|^p dx\right\}^{1/p}.
\end{aligned}$$

This and (1.5) imply that

$$I := \int_0^1 t^{-2}\omega\left(\frac{1}{t}\right)\varphi\left(\left\{\sum_{k=1}^{\infty}\rho_k^{p'}|\sin kt|^{2p'}\right\}^{1/p'}\right)dt < \infty. \tag{4.8}$$

On the other hand

$$I \geq \sum_{m=1}^{\infty}\int_{1/p_{m+1}}^{1/p_m} t^{-2}\omega\left(\frac{1}{t}\right)\varphi\left(\left\{\sum_{k=p_{m-1}+1}^{p_m}\rho_k^{p'}|\sin kt|^{2p'}\right\}^{1/p'}\right)dt. \tag{4.9}$$

By (3.4), the products $kt$, appearing in the integrals above, satisfy

$$0 < c \leq \frac{p_{m-1}}{p_{m+1}} \leq kt \leq \frac{p_m}{p_m} = 1,$$

thus (4.8) and (4.9) yield that

$$\sum_{m=1}^{\infty}\varphi\left(\left\{\sum_{k=p_{m-1}+1}^{p_m}\rho_k^{p'}\right\}^{1/p'}\right)\int_{1/p_{m+1}}^{1/p_m} t^{-2}\omega\left(\frac{1}{t}\right)dt < \infty. \tag{4.10}$$

Since, by (3.3),

$$\int_{1/p_{m+1}}^{1/p_m} t^{-2}\omega\left(\frac{1}{t}\right)dt \geq A^m,$$

thus, (4.7) and (4.10) prove the implication (1.5) $\Rightarrow$ (1.6).

This completes the proof. □

## References

[1] Bari N.K., *Trigonometric series.* (in Russian), Moscow: Gosudarstv. Izdat. Fiz.-Mat. Lit., 1961.

[2] Gram J.P., *Über die Entwicklung reeller Funktionen in Reihen mittels der Methode der kleinsten Quadrate.* J. reine angew. Math. **94** (1883), 41–73.

[3] Leindler L., *Über verschiedene Konvergenzarten trigonometrischer Reihen.* Acta Sci. Math. Szeged **25** (1964), 233–249.

[4] Leindler L., *Über Strukturbedingungen für Fourierreihen.* Math. Zeitschr. **88** (1965), 418–431.

[5] Leindler L., *On the generalized absolute convergence of Fourier series.* Hokkaido Math. J. **30** (2001), 241–251.

[6] Leindler L. and Németh J., *On the connection between quasi power-monotone and quasi geometrical sequences with application to integrability theorems for power series.* Acta Math. Hungar. **68** (1995), 7–19.

[7] Mateljević M. and Pavlović M., *$L^p$-behavior of power series with positive coefficients and Hardy spaces*. Proc. Amer. Math. Soc. **87** (1983), 309–316.

[8] Zygmund A., *Trigonometric series* I. Cambridge University Press, 1959.

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