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Estimates of spherical derivative of meromorphic functions

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Abstract. The spherical derivative $f^{\#}=|f'|/(1+|f|^2)$ of f meromorphic in $D=\{|z|<1\}$ is estimated from above and below in terms of various geometrical quantities, for example, $\delta^{\#}(z,f)$, $\rho(z,f)$, and $\rho_{au}(z,f)$, in several theorems. A necessary and sufficient condition for $(1-|z|^2)f^{\#}(z)$ to be bounded in D is that there exists r, $0<r\leq 1$, such that $f(w)\neq -1/\overline{f(z)}$ for all z, $w\in D$ satisfying $|w-z|/|1-\bar{z}w|<r$. Also, $(1-|z|^2)f^{\#}(z)$ is bounded in D if and only if $\delta^{\#}(z,f)/\rho_{au}(z,f)$ is bounded in D minus the points z where $f^{\#}(z)=0$. Applications to evaluating the Poincaré density in a plane domain will be considered.

Key words: normal meromorphic function; antipodal point; spherical and Poincaré distances; spherical derivative of meromorphic function; Poincaré density; Bloch function.

1. Introduction

Let a function f be meromorphic in the disk $D=\{|z|<1\}$. The spherical derivative $f^{\#}(z)$ of f at $z\in D$ is defined by $f^{\#}(z)=|f'(z)|/(1+|f(z)|^2)$, if $f(z)\neq\infty$, and $f^{\#}(z)=|(1/f)'(z)|$, if $f(z)=\infty$. Then $f^{\#}=(1/f)^{\#}$ in D, where the constant function ∞ is regarded as a meromorphic function, so that $\infty^{\#}=0$. One can prove that $f^{\#}$ is continuous in D. Actually we shall be mainly concerned with a kind of derivative of f, namely,

$$\Phi_f(z)=(1-|z|^2)f^{\#}(z),\quad z\in D.$$

We call f normal if Φ_f is bounded in D; see [LV] for the details. Let $\rho_a(z,f)$ be the maximum of r, $0<r\leq 1$, such that $f(w)\neq -1/\overline{f(z)}$, the antipodal point of $f(z)$, for all w in the Apollonius disk, or the non-Euclidean disk

$$\Delta(z,r)=\left\{w;\left|\frac{w-z}{1-\bar{z}w}\right|<r\right\}$$

of center z and the non-Euclidean radius $\operatorname{arctanh} r$. Such a $\rho_a(z,f)>0$

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does exist at each point $z \in D$. Let

$$\rho_a(f) = \inf_{z \in D} \rho_a(z, f).$$

Then, $\rho_a(f) > 0$ if and only if there exists r, $0 < r \leq 1$, such that $f(w) \neq -1/\overline{f(z)}$ for each pair $z, w \in D$ with $|w - z|/|1 - \overline{z}w| < r$.

Our beginning and fundamental result is the following

Theorem 1 *A meromorphic function f defined in D is normal if and only if $\rho_a(f) > 0$.*

In fact, the proof depends on the chain (**I**) of inequalities which will appear in Section 3.

Sharp upper and lower estimates of $\Phi_f(z)$ will be given in terms of geometrical quantities, in particular, $\rho(z, f)$ and $\delta^{\#}(z, f)$, together with analytic quantity $(\partial/\partial z) \log \Phi_f(z)$ at z with $f^{\#}(z) \neq 0$. Here, $2\partial/\partial z = (\partial/\partial x) - i(\partial/\partial y)$ and $\rho(z, f)$ is the maximum of r, $0 < r \leq 1$, such that f is univalent in $\Delta(z, r)$; we set $\rho(z, f) = 0$ if $f^{\#}(z) = 0$. Furthermore, $\delta^{\#}(z, f)$ is the maximum of $R > 0$ such that the Riemann image surface of D by f, covering $\mathbf{C}^{\#} = \mathbf{C} \cup \{\infty\}$, contains the one-sheeted spherical cap $\{w; |w - f(z)|/|1 + \overline{f(z)}w| < R\}$ of center $f(z)$, in other words, the single-valued branch F of the inverse of f with $F(f(z)) = z$ can be defined in the cap; again, $\delta^{\#}(z, f) = 0$ if $f^{\#}(z) = 0$. The Liouville theorem applied to the inverse of f then shows that $\delta^{\#}(z, f) < +\infty$.

Set

$$\Lambda_f(z) = (1 - |z|^2) \left| \frac{\partial}{\partial z} \log \Phi_f(z) \right| \tag{1.1}$$

for f at $z \in D$ with $f^{\#}(z) \neq 0$. For example, if $f(z) \neq \infty$ and $f'(z) \neq 0$, then

$$\frac{\partial}{\partial z} \log \Phi_f(z) = \frac{-\overline{z}}{1 - |z|^2} + \frac{1}{2} \cdot \frac{f''(z)}{f'(z)} - \frac{\overline{f(z)} f'(z)}{1 + |f(z)|^2}.$$

Then, at each $z \in D$ with $f^{\#}(z) \neq 0$, one has

$$\text{(A)} \qquad \Phi_f(z) \leq \left(\frac{2}{\rho(z, f)} + \Lambda_f(z) \right) \delta^{\#}(z, f)$$

and

(B) $$\left(1+\frac{\Lambda_f(z)}{2}\right)\delta^{\#}(z,f)\leq\Phi_f(z).$$

Both inequalities (A) and (B) are sharp. In particular, at an arbitrary point $z\in D$ one has

(C) $$\delta^{\#}(z,f)\leq\Phi_f(z).$$

This is observed by C. Pommerenke [Po] without detailed equality condition. Note that (C) is trivial at $z\in D$ with $f^{\#}(z)=0$; in the other case, (C) follows from (B). Our detailed equality condition for (C) will be clarified later in Section 8.

In the specified case where f is univalent in the whole D we shall see the sharp estimate

(D) $$\Lambda_f(z)\leq\rho_a(z,f)+\frac{1}{\rho_a(z,f)}.$$

Indeed, f is then normal, and so the right-hand side of (D) is not greater than $\rho_a(f)+(1/\rho_a(f))<+\infty$. The inequality (D) would be of use for upper and lower estimates of $f^{\#}(z)$ with $f^{\#}(0)=1$; see (5.10) in Section 5.

2. The spherical distance and the Poincaré distance

Elementary but necessary facts will be remembered in the present Section. The sphere $\Sigma\subset\mathbf{R}^3$ of diameter one touching the complex plane $\mathbf{C}=\mathbf{R}^2$ at the origin, or the south pole of Σ, from above is identified with $\mathbf{C}^{\#}$ with the aid of the stereographic projection $(x_1,x_2,x_3)\mapsto(x_1+ix_2)/(1-x_3)$ viewed from the north pole $(0,0,1)$ of Σ, which itself is mapped to ∞. The spherical distance $\chi(z,w)$ of z and w of $\mathbf{C}^{\#}$ is then given by

$$\chi(z,w)=\arctan\left|\frac{z-w}{1+\overline{z}w}\right|,$$

where $\chi(z,\infty)=\arctan(1/|z|)$ with $\arctan(+\infty)=\pi/2$, so that $0\leq\chi(z,w)\leq\pi/2$. All the arcs

$$C_\varepsilon(z,z^*)=\left\{\frac{\varepsilon t+z}{1-\overline{z}\varepsilon t};0\leq t\leq+\infty\right\},\quad\varepsilon\in\mathbf{C},\quad|\varepsilon|=1,$$

connect $z\in\mathbf{C}$ and the antipodal point $z^*=-1/\overline{z}\in\mathbf{C}^{\#}$ of z along great circles, whereas, for $z\in\mathbf{C}$ and $w\in\mathbf{C}$ with $z\neq w\neq z^*$, the arc

$$C(z,w)=\left\{\frac{\varepsilon t+z}{1-\overline{z}\varepsilon t};0\le t\le\left|\frac{z-w}{1+\overline{z}w}\right|\right\}$$

with the definite ε,

$$\varepsilon=\frac{w-z}{1+\overline{z}w}\ \Big/\ \left|\frac{z-w}{1+\overline{z}w}\right|,$$

connects z and w on the arc $C_\varepsilon(z,z^*)$. In the case where $z=\infty$ or $w=\infty$ we need obvious change, specifically, $\infty^*=0$ and $0^*=\infty$. We then have

$$\chi(z,w)=\int_{C(z,w)}\frac{|d\zeta|}{1+|\zeta|^2} \tag{2.1}$$

for each pair $z\in\mathbf{C}^\#$, $w\in\mathbf{C}^\#$ with $z\neq w\neq z^*$; evidently,

$$\frac{\pi}{2}=\chi(z,z^*)=\int_{C_\varepsilon(z,z^*)}\frac{|d\zeta|}{1+|\zeta|^2}$$

for all $\varepsilon\in\mathbf{C}$, $|\varepsilon|=1$.

For $a\in\mathbf{C}^\#$ and R, $0<R\le+\infty$, the set

$$\operatorname{Cap}(a,R)=\{z\in\mathbf{C}^\#;\chi(z,a)<\arctan R\},$$

where $0<\arctan R\le\pi/2$, is the spherical cap of center a and of radius $\arctan R$. Hence, $z\in\operatorname{Cap}(a,R)$ if and only if $|(z-a)/(1+\overline{a}z)|<R$. In particular, $\operatorname{Cap}(a,+\infty)=\mathbf{C}^\#\setminus\{a^*\}$.

There are two distances in the open unit disk D; one is the pre-Poincaré distance

$$\tau(z,w)=\left|\frac{z-w}{1-\overline{w}z}\right|$$

and the other is the Poincaré distance

$$\sigma(z,w)=\operatorname{arctanh}\tau(z,w),$$

where $\operatorname{arctanh}x=(1/2)\log\{(1+x)/(1-x)\}$, $0\le x<1$. One needs some device for the proof of the triangle inequality for τ. For $z\neq w$ one has

$$\sigma(z,w)=\int_{\gamma(z,w)}\frac{|d\zeta|}{1-|\zeta|^2},$$

where

$$\gamma(z,w)=\left\{\frac{\varepsilon t+z}{1+\overline{z}\varepsilon t};0\le t\le\tau(z,w)\right\},$$

with $(w-z)/(1-\overline{z}w) = \varepsilon\tau(z,w)$, is the subarc on the circular arc (or, possibly, the diameter) which is orthogonal to the unit circle ∂D at the points $(z+\varepsilon)/(1+\overline{z}\varepsilon)$ and $(z-\varepsilon)/(1-\overline{z}\varepsilon)$.

For f meromorphic in D we have

$$f^{\#}(z) = \lim_{|w-z|\to 0} \frac{\chi(f(w), f(z))}{|w-z|},$$

again the spherical derivative of f at z. One can prove that $f^{\#}(z) \neq 0$ if and only if $f(z) \neq \infty$ and $f'(z) \neq 0$ or f has z as a simple pole. We now have

$$\Phi_f(z) = \lim_{|w-z|\to 0} \frac{\chi(f(w), f(z))}{\sigma(w,z)} = \lim_{|w-z|\to 0} \frac{\chi(f(w), f(z))}{\tau(w,z)}, \quad z \in D.$$

Set

$$\nu(f) = \sup_{z\in D} \Phi_f(z).$$

By definition f is normal if $\nu(f) < +\infty$. This is equivalent to saying that a meromorphic function f is uniformly continuous as a mapping from the metric space (D,σ) into the metric space $(\mathbf{C}^{\#}, \chi)$. We can also replace (D,σ) with (D,τ) in the preceding sentence. For example, if f does not assume the three points of $\mathbf{C}^{\#}$, then f is normal. This is a consequence of the well-known Montel theorem on normal families.

3. Proof of Theorem 1

Let f be meromorphic in D and $z \in D$. Then we always have r, $0 < r \leq 1$, such that $f(w)$ is in the hemisphere $\mathrm{Cap}(f(z), 1)$ for all $w \in \Delta(z,r)$. Hence

$$f(\zeta) \neq f(\eta)^* \quad \text{for all} \quad \zeta,\ \eta \in \Delta(z,r). \tag{3.1}$$

Let $\rho_a^*(z,f)$ be the maximum of r, $0 < r \leq 1$, such that (3.1) holds. Then $\rho_a^*(z,f) > 0$ everywhere in D and

$$\rho_a^*(f) \equiv \inf_{z\in D} \rho_a^*(z,f) \leq \rho_a(f)$$

because $\rho_a^*(z,f) \leq \rho_a(z,f)$ in D. Consequently, $0 \leq \rho_a^*(f) \leq \rho_a(f) \leq 1$.

Theorem 1 immediately follows from the following chain of inequalities for f meromorphic in D;

$$\text{(I)} \qquad \tanh\frac{\pi}{4\nu(f)} \le \rho_a^*(f) \le \rho_a(f) \le 2\rho_a^*(f) \le \frac{2}{\nu(f)},$$

where $+\infty = 1/0$, $0 = 1/+\infty$ and $\tanh(+\infty) = 1$.

THE FIRST INEQUALITY IN (**I**). This is true in both cases $\nu(f) = +\infty$ and $\nu(f) = 0$ because $\rho_a^*(f) = 1$ in the latter case. Suppose that $0 < \nu(f) < +\infty$. Then for each ζ, η in D with $\zeta \ne \eta$ and $f(\zeta) \ne f(\eta)$, one has

$$\begin{aligned}\chi(f(\zeta), f(\eta)) &= \int_\Gamma \frac{|dw|}{1+|w|^2} \le \int_{\mathcal{R}f(\gamma(\zeta,\eta))} \frac{|dw|}{1+|w|^2} \\ &= \int_{\gamma(\zeta,\eta)} f^\#(w)|dw| \le \nu(f) \int_{\gamma(\zeta,\eta)} \frac{|dw|}{1-|w|^2} \\ &= \nu(f)\sigma(\zeta,\eta).\end{aligned}$$

Here, Γ is the geodesic, namely, $\Gamma = C(f(\zeta), f(\eta))$ or $C_\varepsilon(f(\zeta), f(\eta))$ ($|\varepsilon| = 1$) according as $f(\zeta) \ne f(\eta)^*$ or $f(\zeta) = f(\eta)^*$ and further, $\mathcal{R}f(\gamma(\zeta,\eta))$ is the image of $\gamma(\zeta,\eta)$ lying on the Riemann image surface of D by f, which connects $f(\zeta)$ and $f(\eta)$. Hence, for ζ, $\eta \in \Delta(z, \tanh\{\pi/(4\nu(f))\})$, possibly, $\zeta = \eta$, we have

$$\begin{aligned}\chi(f(\zeta), f(\eta)) &\le \chi(f(\zeta), f(z)) + \chi(f(z), f(\eta)) \\ &\le \nu(f)\big(\sigma(\zeta, z) + \sigma(z,\eta)\big) < \frac{\pi}{2}.\end{aligned}$$

Consequently $f(\zeta) \ne f(\eta)^*$, and then $\tanh\{\pi/(4\nu(f))\} \le \rho_a^*(z, f)$ everywhere in D. The first inequality in (**I**) now follows.

THE THIRD INEQUALITY IN (**I**). We may suppose that $\rho_a(f) > 0$. Then for each $z \in D$ and for all ζ, $\eta \in \Delta(z, \rho_a(f)/2)$, one has

$$\zeta \in \Delta(z, \rho_a(f)/2) \subset \Delta(\eta, \rho_a(f)) \subset \Delta(\eta, \rho_a(\eta, f)),$$

so that $f(\zeta) \ne f(\eta)^*$. Therefore $\rho_a(f)/2 \le \rho_a^*(z, f)$ for all $z \in D$. Hence $\rho_a(f)/2 \le \rho_a^*(f)$.

THE FOURTH INEQUALITY IN (**I**). We propose for the proof that

$$|g'(0)| \le 1 \tag{3.2}$$

for g meromorphic in D with $g(0) = 0$ and $\rho_a^*(0, g) = 1$ (hence, g is pole-free in D). The equality holds in (3.2) if and only if $g(z) \equiv \varepsilon z$, $|\varepsilon| = 1$.

This proposition is attributed to T.-S. Shah [S] in [G, II, p.82, Problem 46]; hereafter our main reference for Univalent Function Theory is [G]. Unfortunately, however, one, together with the present author, might have difficulty in accessing the paper [S]. For a rather easier reference, we recall here the paper [LM] of N.A. Lebedev and I.M. Milin to whom the result appears to be essentially due.

In fact, Lebedev and Milin proved the above proposition for g, furthermore, univalent in D; see [LM, p.397] where they claimed that "Theorem A [LM, p.380] is valid for the class S_Γ." Obviously the equality discussion there should be restricted to the case $n = 1$. If we drop univalency of g in D, then we can find a simply connected, proper subdomain $\mathcal{H}$ of $\mathbf{C}$ such that $g(D) \subset \mathcal{H}$ and $\zeta \neq \eta^*$ for all ζ, $\eta \in \mathcal{H}$. Let h be a conformal mapping from D onto $\mathcal{H}$ with $h(0) = 0$. Then $g = h \circ \phi$ with $\phi = h^{-1} \circ g : D \to D$ holomorphic. The Schwarz inequality $|\phi'(0)| \leq 1$, together with the equality $|\phi'(0)| = 1$ if and only if $\phi(z) \equiv \varepsilon z$, $|\varepsilon| = 1$, now proves the proposition.

The inequality $\rho_a^*(f) \leq 1/\nu(f)$ immediately follows from

$$\textbf{(II)} \qquad \rho_a^*(z, f)\Phi_f(z) \leq 1, \quad z \in D.$$

For the proof we may suppose that $f^\#(z) \neq 0$ and $f(z) \neq \infty$; when $f(z) = \infty$, consider the reciprocal $1/f$. Set $a = \rho_a^*(z, f)$ (> 0) and set

$$g(w) = \frac{f\left(\dfrac{aw + z}{1 + \overline{z}aw}\right) - f(z)}{1 + \overline{f(z)}f\left(\dfrac{aw + z}{1 + \overline{z}aw}\right)}, \quad w \in D.$$

Then $g(0) = 0$ and $\rho_a^*(0, g) = 1$. Hence $a\Phi_f(z) = |g'(0)| \leq 1$, or **(II)**.

The equality holds in **(II)** at z if and only if

$$f(w) = \frac{\lambda \cdot \dfrac{w - z}{1 - \overline{z}w} + \mu}{1 - \overline{\mu}\lambda \cdot \dfrac{w - z}{1 - \overline{z}w}}, \tag{3.3}$$

where $\lambda \in \mathbf{C}$ and $\mu \in \mathbf{C}^\#$ are constants with $|\lambda| \geq 1$. If $\mu = \infty$, then $f(w) = (1 - \overline{z}w)/\{\lambda(w - z)\}$. The function f of (3.3) maps D univalently onto $\mathrm{Cap}(\mu, |\lambda|)$.

THE "ONLY IF" PART. For $g(w) \equiv \varepsilon w$, $|\varepsilon| = 1$, we have (3.3) with $\lambda = \varepsilon/a$ and $\mu = f(z)$.

THE "IF" PART. A calculation shows that $\Phi_f(z) = |\lambda|$ for f of (3.3). To prove $1/|\lambda| \le \rho_a^*(z, f)$ we suppose that there exist ζ, η in $\Delta(z, 1/|\lambda|)$ such that $f(\zeta) = f(\eta)^*$. Setting

$$\zeta' = \frac{\zeta - z}{1 - \bar{z}\zeta} \quad \text{and} \quad \eta' = \frac{\eta - z}{1 - \bar{z}\eta},$$

one then observes that $\lambda\zeta' = (\lambda\eta')^*$ or $|\lambda|^2\zeta'\overline{\eta'} = -1$, together with $|\zeta'| < 1/|\lambda|$ and $|\eta'| < 1/|\lambda|$. We then arrived at a contradiction that $1 = |\lambda|^2|\zeta'||\overline{\eta'}| < 1$. Consequently, $1/\Phi_f(z) = 1/|\lambda| \le \rho_a^*(z, f)$ and hence

$$1 \le \rho_a^*(z, f)\Phi_f(z) \le 1,$$

so that the equality holds in (**II**).

A holomorphic version of a normal meromorphic function is a Bloch function. Later in Section 13 we shall observe the chain (**III**) of inequalities analogous to (**I**). Incidentally, ineterested readers may go directly to Section 13 except for Remark 7.

4. Estimate of $\Lambda_f(z)$

Suppose that $f^\#(z) \ne 0$ at $z \in D$ for f meromorphic in D. Then,

$$\frac{\partial}{\partial z} \log f^\#(z) = \frac{1}{2} \cdot \frac{f''(z)}{f'(z)} - \frac{\overline{f(z)} f'(z)}{1 + |f(z)|^2}$$

if $f(z) \ne \infty$, whereas

$$\frac{\partial}{\partial z} \log f^\#(z) = \frac{\partial}{\partial z} \log |(1/f)'(z)| = \frac{1}{2} \cdot \frac{(1/f)''(z)}{(1/f)'(z)}$$

if $f(z) = \infty$. If f is nonconstant and if $f^\#(z) = 0$, then

$$\lim_{|w-z| \to 0} \frac{\partial}{\partial w} \log f^\#(w) = \infty.$$

Recall here Λ_f of (1.1) for a nonconstant f. Defining $\Lambda_f(z) = +\infty$ at $z \in D$ with $f^\#(z) = 0$ one can then observe that Λ_f is a continuous mapping from D into $[0, +\infty] = C_1(0, \infty)$.

We begin with an upper estimate of $\Lambda_f(z)$ in terms of $\rho(z, f)$ and the smaller of this and $\rho_a(z, f)$, in notation,

$$\rho_{au}(z, f) = \min(\rho(z, f), \rho_a(z, f)), \quad z \in D. \tag{4.1}$$

The function $\rho_{au}(z, f)$ of $z \in D$ is reasonable because the normality criterion for a nonconstant f will be described in terms of this and $\delta^{\#}(z, f)$ after the forthcoming Theorem 5 in Section 8. In short, f is normal if and only if $\delta^{\#}(z, f)/\rho_{au}(z, f)$ is bounded in D minus the points z at which $f^{\#}(z) = 0$.

Theorem 2 *For f meromorphic in D and for $z \in D$ with $f^{\#}(z) \neq 0$ we have*

$$\rho(z, f)\Lambda_f(z) \leq \frac{\rho_{au}(z, f)}{\rho(z, f)} + \frac{\rho(z, f)}{\rho_{au}(z, f)}. \tag{4.2}$$

To describe an equality condition for (4.2) in an "if and only if" form we first let $K(z) = z/(1-z)^2$ be the Koebe function and set

$$K_\varepsilon(z) = \bar{\varepsilon}K(\varepsilon z) = \frac{z}{(1-\varepsilon z)^2}$$

for $\varepsilon \in \mathbf{C}$, $|\varepsilon| = 1$. Then the rational function K_ε of z is, in particular, univalent in D. Set

$$H_{p,\varepsilon} = \frac{K_\varepsilon}{1 + \dfrac{(1+p)^2}{p} \cdot \varepsilon K_\varepsilon}, \quad 0 < p \leq 1, \quad |\varepsilon| = 1.$$

Then $H_{p,\varepsilon}$ is a rational function of z and univalent in D with $H_{1,-\varepsilon} = K_\varepsilon$ and

$$H_{p,-1}(z) = \frac{pz}{(p-z)(1-pz)}, \quad 0 < p \leq 1,$$

so that $H_{p,\varepsilon}(z) = -\bar{\varepsilon}H_{p,-1}(-\varepsilon z)$. Furthermore,

$$\mathbf{C}^{\#} \setminus H_{p,\varepsilon}(D) = \left\{\bar{\varepsilon}t;\ p/(1+p)^2 \leq t \leq p/(1-p)^2\right\};$$

note that $K(p) = p/(1-p)^2$ and $-K(-p) = p/(1+p)^2$ for $0 < p \leq 1$. Calculation shows that

$$H_{p,\varepsilon}(0) = H'_{p,\varepsilon}(0) - 1 = 0, \quad H''_{p,\varepsilon}(0) = -2\varepsilon\left(p + \frac{1}{p}\right).$$

We further have $\rho_a(0, H_{p,\varepsilon}) = p$ for $0 < p \leq 1$ because $H_{p,\varepsilon}(-\bar{\varepsilon}p) = \infty$ in case $0 < p < 1$.

Returning to our inequality (4.2) we now have the equality in (4.2) at

z with $f^{\#}(z) \neq 0$ if and only if

$$f(w) = \frac{\lambda H_{p,\varepsilon}\left(\dfrac{1}{\rho} \cdot \dfrac{w-z}{1-\overline{z}w}\right) + \mu}{1 - \overline{\mu}\lambda H_{p,\varepsilon}\left(\dfrac{1}{\rho} \cdot \dfrac{w-z}{1-\overline{z}w}\right)}, \tag{4.3}$$

where $\lambda \in \mathbf{C} \setminus \{0\}$, $\mu \in \mathbf{C}^{\#}$, $\varepsilon \in \mathbf{C}$, $|\varepsilon| = 1$, p and ρ of $(0,1]$, all are constants. Read

$$1/f(w) = \lambda H_{p,\varepsilon}\left(\frac{1}{\rho} \cdot \frac{w-z}{1-\overline{z}w}\right)$$

in case $\mu = \infty$.

Under the global condition that $f^{\#}$ never vanishes in D, the equality holds in (4.2) if and only if $\rho = 1$ in (4.3), so that f of (4.3) is univalent in the whole D and $\rho_a(z, f) = p$.

For f of (4.3) one can actually observe that $\rho(z, f) = \rho$ and $\rho_{au}(z, f) = \rho_a(z, f) = \rho p$. The latter follows from $H_{p,\varepsilon}(-\overline{\varepsilon}p) = \infty$. Further calculation for f of (4.3) shows that $\rho\Lambda_f(z) = p + (1/p)$.

5. Proof of Theorem 2

Let $S = S(1)$ be the family of f holomorphic and univalent in D satisfying

$$f(0) = f'(0) - 1 = 0. \tag{5.1}$$

For $0 < p < 1$, let $S = S(p)$ be the family of f meromorphic and univalent in D satisfying (5.1) and $f(z_o) = \infty$ at a point z_o depending on f with $|z_o| = p$. In particular, $H_{p,\varepsilon} \in S(p)$ if $0 < p \leq 1$. The Bieberbach and the Komatu results both can be summarized in

Lemma 5.1 *For $f \in S(p)$ with $0 < p \leq 1$, one has*

$$|f''(0)| \leq 2\left(p + \frac{1}{p}\right). \tag{5.2}$$

The equality holds in (5.2) *if and only if $f = H_{p,\varepsilon}$, $|\varepsilon| = 1$.*

The case $p = 1$ is Bieberbach's second-coefficient theorem; see [G, I, p.33, Theorem 1], whereas the case $p < 1$ is attributed to Y. Komatu [K] in [G, I, p.40, Theorem 7].

Proof of Theorem 2. To show (4.2) we may suppose that $f(z) \neq \infty$. We next set $\rho = \rho(z, f) > 0$. Then the function

$$g(w) = \frac{1 + |f(z)|^2}{\rho(1 - |z|^2)f'(z)} \cdot \frac{f\left(\dfrac{\rho w + z}{1 + \overline{z}\rho w}\right) - f(z)}{1 + \overline{f(z)}f\left(\dfrac{\rho w + z}{1 + \overline{z}\rho w}\right)} \tag{5.3}$$

of $w \in D$ is in $S(p)$ with

$$p = \rho_{au}(z, f)/\rho = \rho_{au}(z, f)/\rho(z, f). \tag{5.4}$$

Since a short calculation shows that

$$\rho|\Lambda_f(z)| = \frac{|g''(0)|}{2}, \tag{5.5}$$

the requested inequality (4.2) follows from (5.2) applied to g.

Suppose that the equality holds in (4.2) at z with $f^{\#}(z) \neq 0$, where $f(z) \neq \infty$ without loss of generality. Then $g = H_{p,\varepsilon}$, $|\varepsilon| = 1$, in (5.3) with p of (5.4), so that we have (4.3) with $\mu = f(z)$ and

$$\lambda = \frac{\rho(1 - |z|^2)f'(z)}{1 + |f(z)|^2}.$$

If $f^{\#}$ is zero-free, then ρ in (4.3) must be one. Otherwise, $f^{\#}(\zeta) = 0$ for

$$\zeta = \frac{\rho\overline{\varepsilon} + z}{1 + \overline{z}\rho\overline{\varepsilon}} \in D \tag{5.6}$$

because $H^{\#}_{p,\varepsilon}(\overline{\varepsilon}) = 0$. □

Corollary to Theorem 2 *For f meromorphic in D and for $z \in D$ with $f^{\#}(z) \neq 0$ one has*

$$\rho_{au}(z, f)\Lambda_f(z) \leq 2. \tag{5.7}$$

The equality holds in (5.7) at z with $f^{\#}(z) \neq 0$ if and only if $p = 1$ in (4.3) or $H_{1,\varepsilon} = K_{-\varepsilon}$ there. Furthermore, in case $f^{\#}$ never vanishes in D, the equality holds if and only if f is of (4.3) with $p = 1$ and $\rho = 1$.

First, (5.7) follows from (4.2) because

$$\Lambda_f(z) \leq \frac{1}{\rho_{au}(z, f)}\left(\frac{\rho_{au}(z, f)^2}{\rho(z, f)^2} + 1\right) \leq \frac{2}{\rho_{au}(z, f)}. \tag{5.8}$$

Suppose that the equality holds in (5.7). Then $\rho_{au}(z,f)=\rho(z,f)\equiv\rho$, and p of (5.4) must be one. Hence f is of (4.3) with $p=1$. The converse is not difficult to prove. The remaining part of the proof is now obvious.

Remark 1 Set

$$\rho(f)=\inf_{z\in D}\rho(z,f)$$

for f meromorphic in D and $z\in D$. For $f(z)=z^2$, one has $\nu(f)<+\infty$ and $\rho(f)=\rho(0,f)=0$. On the other hand, P. Lappan [L] found an f pole-free in D with $\nu(f)=+\infty$ and $\rho(f)>0$. Hence there is no implication relation between the inequalities $\nu(f)<+\infty$ and $\rho(f)>0$ even for holomorphic f.

We set, for f meromorphic with nonvanishing $f^{\#}$ in D,

$$B(f)\equiv\sup_{z\in D}\frac{1}{\rho(z,f)}\left(\frac{\rho_{au}(z,f)}{\rho(z,f)}+\frac{\rho(z,f)}{\rho_{au}(z,f)}\right)\le\sup_{z\in D}\frac{2}{\rho_{au}(z,f)}.$$

When $\nu(f)<+\infty$ and $\rho(f)>0$ at the same time, we have

$$B(f)\le\frac{2}{\min\big(\rho(f),\rho_a(f)\big)},$$

so that $B(f)<+\infty$. It then follows from (4.2) that

$$\Lambda_f(z)\le B(f),\qquad z\in D \tag{5.9}$$

for f with $\nu(f)<+\infty$ and $\rho(f)>0$.

Remark 2 Returning to (D) in Section 1 we now have this as a consequence of (4.2). The equality holds there if and only if f is of (4.3) with $\rho=1$. Let $\mathcal{M}$ be the family of f meromorphic and univalent in D with the normalization $f^{\#}(0)=1$. Note that $B(f)=\rho_a(f)+(1/\rho_a(f))\ge 2$ for $f\in\mathcal{M}$. A problem is whether or not

$$\sup_{f\in\mathcal{M}}B(f)<+\infty,\quad\text{or equivalently,}\quad\inf_{f\in\mathcal{M}}\rho_a(f)>0,$$

is valid. The answer is in the negative. Indeed, for $H_{p,\varepsilon}\in\mathcal{M}$ we have

$$\rho_a(H_{p,\varepsilon})\le\rho_a(0,H_{p,\varepsilon})=p,\quad 0<p<1,$$

so that

$$\inf_{f\in\mathcal{M}}\rho_a(f)=0.$$

It further follows from (**I**) that $\sup_{f\in\mathcal{M}} \nu(f) = +\infty$. See the forthcoming Section 14.

We now have estimates of $f^{\#}$ for $f \in \mathcal{M}$ with the aid of (5.9). Namely,

$$\frac{(1-|z|)^{B(f)-1}}{(1+|z|)^{B(f)+1}} \le f^{\#}(z) \le \frac{(1+|z|)^{B(f)-1}}{(1-|z|)^{B(f)+1}}, \quad z \in D, \tag{5.10}$$

for all $f \in \mathcal{M}$. The right-hand side of (5.10) should be compared with $f^{\#}(z) \le \nu(f)(1+|z|)^{-1}/(1-|z|)$, $z \in D$, with the multiplier $\nu(f)$. Note that $\nu(K) > 2$; see Section 14. To achieve (5.10) we note that $\Phi_f(0) = 1$ and

$$(1-|\zeta|^2)|\operatorname{grad}\log\Phi_f(z)| = 2\Lambda_f(\zeta) \le 2B(f), \quad \zeta \in D.$$

Hence for $z = |z|e^{i\theta} \in D\setminus\{0\}$ and $\zeta = u + iv = re^{i\theta}$, $0 < r < |z|$, we have

$$\begin{aligned}|\log\Phi_f(z)| &= \left|\int_0^{|z|}\left(\cos\theta\cdot\frac{\partial}{\partial u}\log\Phi_f(\zeta) + \sin\theta\cdot\frac{\partial}{\partial v}\log\Phi_f(\zeta)\right)dr\right| \\ &\le \int_0^{|z|}|\operatorname{grad}\log\Phi_f(\zeta)|dr \le B(f)\log\frac{1+|z|}{1-|z|},\end{aligned} \tag{5.11}$$

whence (5.10) for $z \ne 0$.

One can then prove that if the first or the second equality holds in (5.10) at $z \ne 0$, then f is "similar" to $H_{p,\varepsilon}$ for $0 < p \le 1$ and $|\varepsilon| = 1$. More precisely,

$$\Lambda_f(\zeta) = \rho_a(\zeta, f) + \frac{1}{\rho_a(\zeta, f)} = B(f)$$

for all $\zeta = re^{i\theta}$, $0 < r < |z|$, where $z = |z|e^{i\theta}$. On letting $\zeta \to 0$ one has $\Lambda_f(0) = B(f)$. Hence

$$B(f) = \Lambda_f(0) \le \rho_a(0, f) + \frac{1}{\rho_a(0, f)} \le B(f),$$

so that $\Lambda_f(0) = \rho_a(0,f) + (1/\rho_a(0,f))$. Since $f \in \mathcal{M}$, one has $f = \big(\lambda H_{p,\varepsilon} + \mu\big)/\big(1 - \overline{\mu}\lambda H_{p,\varepsilon}\big)$, $|\lambda| = 1$, $\mu \in \mathbf{C}^{\#}$, in view of (4.3) on setting $z = 0$. In particular, under the assumption that the equality in the left or right in (5.10) holds at $z \ne 0$ for f, we have $B(f) = \rho_a(0,f) +$ $(1/\rho_a(0,f))$.

Unfortunately, we cannot proceed further since determination of $B(f)$ even for the specified extremal functions $f = H_{p,\varepsilon}$ is interesting but difficult to obtain. Moreover, in view of the above one might conjecture that $B(H_{p,\varepsilon}) = p + (1/p)$. In fact, this is false for $H_{1,-1} = K$. Actually, $\rho_a(z, K) < 1$ for $z \in D$ except for the real interval $[0, K^{-1}(4)]$, where $K^{-1}(4) = (9 - \sqrt{17})/8$. Hence $\rho_a(K) < 1$, so that $B(K) > 2$.

6. Riemann image surface; an upper estimate of Φ_f

The quantity $\delta^{\#}(z, f)$ defined in Section 1 appears in

Theorem 3 *Let f be meromorphic in D and suppose that $f^{\#}(z) \neq 0$ at a point $z \in D$. Then*

$$\Phi_f(z) \leq \left(\frac{2}{\rho(z, f)} + \Lambda_f(z)\right) \delta^{\#}(z, f). \tag{6.1}$$

Set

$$F_{R,\varepsilon} = \frac{K_\varepsilon}{1 + R\varepsilon K_\varepsilon}, \quad 0 \leq R < +\infty, \quad \varepsilon \in \mathbf{C}, \quad |\varepsilon| = 1.$$

This is rational and univalent in D, together with

$$F_{R,\varepsilon}(D) = \mathbf{C}^{\#} \setminus \left\{\frac{\overline{\varepsilon}}{t}; R - 4 \leq t \leq R\right\}.$$

One can especially observe that

$$H_{p,\varepsilon} = F_{R,\varepsilon} \quad \text{for} \quad R = \frac{(1 + p)^2}{p} \geq 4, \quad 0 < p \leq 1.$$

The equality holds in (6.1) at $z \in D$ with $f^{\#}(z) \neq 0$ if and only if there exist four parameters $\rho \in (0, 1]$, $R \in [0, +\infty)$; $\varepsilon \in \mathbf{C}$, $|\varepsilon| = 1$, $\lambda \in \mathbf{C} \setminus \{0\}$, and $\mu \in \mathbf{C}^{\#}$, such that

$$f(w) = \frac{\lambda F_{R,\varepsilon}\left(\dfrac{1}{\rho} \cdot \dfrac{w - z}{1 - \overline{z}w}\right) + \mu}{1 - \overline{\mu}\lambda F_{R,\varepsilon}\left(\dfrac{1}{\rho} \cdot \dfrac{w - z}{1 - \overline{z}w}\right)}. \tag{6.2}$$

Suppose that $f^{\#}$ never vanishes in D in Theorem 3. Then the equality holds in (6.1) at $z \in D$ if and only if $\rho = 1$ further in (6.2).

Remark 3 Combining (4.2) and (6.1) one has

$$\Phi_f(z) \leq \frac{\left(\rho(z, f) + \rho_{au}(z, f)\right)^2}{\rho(z, f)^2 \rho_{au}(z, f)} \delta^{\#}(z, f) \tag{6.3}$$

at $z \in D$ with $f^{\#}(z) \neq 0$. Moreover, one has $\Phi_f(z) \leq 4\delta^{\#}(z, f)/\rho_{au}(z, f)$ if $f^{\#}(z) \neq 0$. The equality holds in (6.3) at $z \in D$ with $f^{\#}(z) \neq 0$ if and only if f is of (4.3). If, furthermore, $f^{\#}$ never vanishes in the whole D, then $\rho = 1$ in (4.3) in addition.

7. Proof of Theorem 3

The following lemma occupies a central position in the proof of Theorem 3.

Lemma 7.1 *For $f \in S(p)$ with $0 < p \leq 1$, one has*

$$\frac{2}{4 + |f''(0)|} \leq \delta^{\#}(0, f). \tag{7.1}$$

The equality holds in (7.1) if and only if $f = F_{R,\varepsilon}$, $0 \leq R \leq 4$, $|\varepsilon| = 1$, in case $p = 1$, whereas $f = H_{p,\varepsilon}$ in case $p < 1$.

Lemma 7.1 for $p < 1$ is essentially due to W. Fenchel [F]; see [G, II, p.245, Theorem 33]; one needs little technique for the proof of the equality condition, and so we include here the proof of Lemma 7.1 for $p < 1$. For the case $p = 1$, see, for example, [Y2, p.106, Lemma 3.1]. The proof of the equality condition is considerably delicate in both cases.

For the proof of Lemma 7.1 in case $p < 1$ we let $c \in \mathbf{C}^{\#} \setminus f(D)$ be arbitrary. Then $0 \neq c \neq \infty$ and the function $g = cf/(c - f)$ is in S with

$$g''(0) = f''(0) + \frac{2}{c}. \tag{7.2}$$

Hence

$$\frac{1}{|c|} \leq \frac{|g''(0)|}{2} + \frac{|f''(0)|}{2} \leq 2 + \frac{|f''(0)|}{2} \tag{7.3}$$

by the Bieberbach inequality $|g''(0)| \leq 4$. Consequently,

$$|c| \geq \frac{2}{4 + |f''(0)|}, \tag{7.4}$$

from which follows (7.1).

Suppose that the equality holds in (7.1) and choose $c \in \mathbf{C}$ on the boundary of $f(D)$ such that $|c| = \delta^{\#}(0, f)$. Then for $g = cf/(c - f)$ for the present c the equality holds in (7.4), so that the right-most in (7.3) is $1/|c|$.

Consequently, $g = K_\varepsilon$, whence

$$f = \frac{K_\varepsilon}{1+(1/c)K_\varepsilon}. \tag{7.5}$$

Since $f''(0) = 4\varepsilon - (2/c)$, it follows from

$$\frac{1}{|c|} = 2 + \frac{|f''(0)|}{2} = 2 + \left|2\varepsilon - \frac{1}{c}\right|$$

that $c = |c|\bar{\varepsilon}$. On the other hand, since $f(\varepsilon' p) = \infty$ for some ε', $|\varepsilon'| = 1$, it follows that $K_\varepsilon(\varepsilon' p) = -c$. Hence $K(\varepsilon\varepsilon' p) = -|c| < 0$, so that $\varepsilon\varepsilon' p$ must be negative, or, $\varepsilon\varepsilon' = -1$. We thus have $|c| = -K(-p)$, or,

$$\frac{1}{c} = \frac{(1+p)^2}{p}\varepsilon. \tag{7.6}$$

We can now conclude from (7.6) that $f = H_{p,\varepsilon}$.

Conversely, for $f = H_{p,\varepsilon}$, we have

$$\delta^{\#}(0,f) = \frac{p}{(1+p)^2} \quad \text{and} \quad |f''(0)| = 2\left(p + \frac{1}{p}\right).$$

We thus observe the equality in (7.1).

Proof of Theorem 3. There is no loss of generality in supposing that $f(z) \neq \infty$. Set $\rho = \rho(z,f)$ and recall g of (5.3) for which $|g''(0)| = 2\rho\Lambda_f(z)$ and

$$\frac{2}{4+|g''(0)|} \leq \delta^{\#}(0,g) \leq \frac{\delta^{\#}(z,f)}{\rho\Phi_f(z)}$$

by Lemma 7.1 for the first inequality. Hence (6.1). Suppose that the equality holds in (6.1). Then $g = F_{R,\varepsilon}$, $0 \leq R \leq 4$, or $g = H_{p,\varepsilon}$ according as $p = \rho_{au}(z,f)/\rho(z,f)$ of (5.4) is one or less than one. If $f^{\#}$ never vanishes in D, then $\rho = 1$; otherwise $f^{\#}(\zeta) = 0$ for ζ of (5.6). Conversely if f is of (6.2), then one has $g = F_{R,\varepsilon}$, $0 \leq R \leq 4$, or $g = H_{p,\varepsilon}$ for g of (5.3) for the present f. One can now easily prove the equality in (6.1). □

Combining (5.7) and (6.1) one has the following

Corollary to Theorem 3 *Let f be meromorphic in D and let $f^{\#}(z) \neq 0$ at a point $z \in D$. Then*

$$\Phi_f(z) \leq 2\left(\frac{1}{\rho(z,f)} + \frac{1}{\rho_{au}(z,f)}\right)\delta^{\#}(z,f). \tag{7.7}$$

The equality condition is now easily obtained.

8. Lower estimtes of Φ_f or upper estimates of $\delta^\#$

We begin with

Theorem 4 *Let f be meromorphic in D. Then at each $z \in D$ with $f^\#(z) \neq 0$,*

$$\delta^\#(z,f) \leq \frac{\Phi_f(z)}{1+(1/2)\Lambda_f(z)}. \tag{8.1}$$

In particular, one always has (C) in Section 1. A consequence is that if f is normal, then $\delta^\#(z,f)$ is bounded. The converse is, however, false [Po, p.6].

The equality holds in (8.1) at z with $f^\#(z) \neq 0$ if and only if

$$f(w) = \frac{\lambda \cdot \dfrac{w-z}{1-\overline{z}w} + \mu}{1-\overline{\mu}\lambda \cdot \dfrac{w-z}{1-\overline{z}w}}, \tag{8.2}$$

where $\lambda \in \mathbf{C} \setminus \{0\}$ and $\mu \in \mathbf{C}^\#$ are constants. For f of (8.2) one has $\delta^\#(z,f) = \Phi_f(z) = |\lambda|$, $\Lambda_f(z) = 0$, and $\rho_{au}(z,f) = 1$.

An equality condition for (C) can be given in "if and only if" form. If $f^\#(z) \neq 0$, then the equality holds in (C) if and only if (8.2) holds. The proof is now obvious.

Our next result is

Theorem 5 *Let f be meromorphic in D. Then at each $z \in D$,*

$$\delta^\#(z,f) \leq \frac{4\rho_{au}(z,f)}{\left(1+\rho_{au}(z,f)\right)^2}\,\Phi_f(z). \tag{8.3}$$

The equality holds in (8.3) at z with $f^\#(z) \neq 0$ if and only if f is of (8.2). Again (C) is a consequence of (8.3).

It follows from (7.7) and (8.3) that

$$\frac{1}{4} \cdot \frac{\delta^\#(z,f)}{\rho_{au}(z,f)} \leq \Phi_f(z) \leq 4 \cdot \frac{\delta^\#(z,f)}{\rho_{au}(z,f)} \tag{8.4}$$

for $z \in D$ with $f^\#(z) \neq 0$. The right-hand side of (8.4) is observed also in Remark 3. By the continuity of Φ_f in the whole D one now has a criterion that a nonconstant and meromorphic function f is normal if and only if there exists a constant $c(f) > 0$ such that

$$\delta^{\#}(z, f) \leq c(f)\rho_{au}(z, f)$$

for all $z \in D$.

Remark 4 It seems reasonable to suspect that one of the right-hand sides of (8.1) and (8.3) is always less than the other, so that (8.1) is a consequence of (8.3) or the converse.

To observe that this is actually not the case we set

$$W(z) = \frac{1}{2}\Lambda_f(z), \quad U(z) = \frac{1}{1 + W(z)}, \quad \text{and}$$

$$Y(z) = \frac{4\rho_{au}(z, f)}{\left(1 + \rho_{au}(z, f)\right)^2}$$

for $z \in D$ with $f^{\#}(z) \neq 0$. Then for the specified function $f(z) \equiv z^2$ and for $0 < x < 1$, we have

$$W(x) = \frac{|\phi(x)|}{4x(1 + x^4)}, \quad \phi(x) = (x^2 + 1)(x^4 - 4x^2 + 1),$$

$$\rho_{au}(x, f) = \rho(x, f) = x, \quad \text{and} \quad Y(x) = \frac{4x}{(1 + x)^2} < 1.$$

Since $\phi(x_o) = 0$ for $x_o = (\sqrt{6} - \sqrt{2})/2 \in (0, 1)$, one immediately has $Y(x_o) < 1 = U(x_o)$. On the other hand, $U(x) \to 2/3$ and $Y(x) \to 1$ as $x \to 1 - 0$. Hence there exists x_1, $x_o \leq x_1 < 1$, such that $Y(x) > U(x)$ for all x, $x_1 < x < 1$.

9. Proofs of Theorems 4 and 5

The bounded Koebe function κ_M for M, $1 \leq M < +\infty$ is defined by

$$\kappa_M(z) = MK^{-1}\left(\frac{K(z)}{M}\right), \quad z \in D,$$

where K^{-1} is the inverse of K in $K(D)$. As a specified case, we have $\kappa_1(z) \equiv z$. Suppose that $M > 1$. At each ζ of the unit circle ∂D one then has the limit $\kappa_M(\zeta) = \lim_{z \to \zeta} \kappa_M(z)$ lying on the circle $\{\eta;\ |\eta| = M\}$ or in the real, left-open interval

$$\left(-M, -M\left(2M-1-2\sqrt{M(M-1)}\right)\right].$$

We actually have $\chi \in \partial D$ with $\operatorname{Im}\chi > 0$ such that $\kappa_M(\chi) = \kappa_M(\overline{\chi}) = -M$. Let I be the closed subarc of ∂D bisected by -1, whose end-points are χ and $\overline{\chi}$. Then κ_M maps $\partial D \setminus I$ one to one and onto $\{\eta;\, |\eta| = M\} \setminus \{-M\}$. Let Υ be the open subarc of I connecting χ and -1. Then for each $\zeta \in \Upsilon$ one observes that $\kappa_M(\zeta) = \kappa_M(\overline{\zeta}) \in \left(-M, \kappa_M(-1)\right)$.

We shall make use of Lemmata due to G. Pick [Pi]. Extremal functions will be

$$\kappa_{\varepsilon,M}(z) = \overline{\varepsilon}\kappa_M(\varepsilon z), \quad |\varepsilon| = 1,$$

so that $\kappa_{\varepsilon,1}(z) \equiv z$.

Lemma 9.1 *For $f \in S$ with $|f| < M$ in D one has*

$$|f''(0)| \leq 4\left(1 - \frac{1}{M}\right). \tag{9.1}$$

Note that $M \geq 1$ in the above because $1 = f'(0) \leq M$ by the Schwarz lemma for f/M. The equality holds in (9.1) if and only if $f = \kappa_{\varepsilon,M}$. See [G, I, p.38, Theorem 4].

Lemma 9.2 *For $f \in S$ with $|f| < M$ in D one has*

$$-\kappa_M(-1) \leq \delta^{\#}(0, f). \tag{9.2}$$

The equality holds in (9.2) if and only if $f = \kappa_{\varepsilon,M}$. See [Y2, p.111, Lemma 5.1] for the details.

Since

$$-\kappa_M(-1) = M\left(2M-1-2\sqrt{M(M-1)}\right),$$

we observe that

$$\frac{-\kappa_M(-1)}{M} = 2M-1-2\sqrt{M(M-1)} \leq W \iff \frac{(W+1)^2}{4W} \leq M, \tag{9.3}$$

where $0 < W \leq 1 \leq M < +\infty$.

Proof of Theorem 4. We may suppose that $f(z) \neq \infty$. Let ϕ be the inverse function of f in $\mathrm{Cap}(f(z),\delta)$, $\delta=\delta^{\#}(z,f)$, with $\phi(f(z))=z$. Set

$$g(w)=\frac{(1-|z|^2)f'(z)}{\delta(1+|f(z)|^2)}\cdot\frac{\phi\left(\dfrac{\delta w+f(z)}{1-\overline{f(z)}\delta w}\right)-z}{1-\bar{z}\phi\left(\dfrac{\delta w+f(z)}{1-\overline{f(z)}\delta w}\right)},\quad w\in D. \tag{9.4}$$

Then $g\in S$ and $|g|<M\equiv\Phi_f(z)/\delta$. A laborious but simple calculation with the aid of

$$\frac{\phi''(f(z))}{\phi'(f(z))}=-\frac{f''(z)}{f'(z)^2},$$

yields that

$$\frac{g''(0)}{2}=\frac{-\delta(1+|f(z)|^2)}{(1-|z|^2)f'(z)}\cdot(1-|z|^2)\frac{\partial}{\partial z}\log\Phi_f(z).$$

An appeal to Lemma 9.1 for g now immediately produces that

$$\frac{\Lambda_f(z)}{M}\le 2\left(1-\frac{1}{M}\right),$$

whence (8.1).

Suppose that the equality holds in (8.1) at z with $f^{\#}(z)\neq 0$. Then $g=\kappa_{\varepsilon,M}$ in (9.4) shows that

$$\frac{\delta w+f(z)}{1-\overline{f(z)}\delta w}=f\left(\frac{A\kappa_{\varepsilon,M}(w)+z}{1+\bar{z}A\kappa_{\varepsilon,M}(w)}\right),\quad w\in D, \tag{9.5}$$

where

$$A=\frac{\delta(1+|f(z)|^2)}{(1-|z|^2)f'(z)}$$

with $|A|=1/M$. Suppose that $M>1$. For $t\in\big(-M,\kappa_M(-1)\big)$ we have $\zeta\in\Upsilon$ with $\kappa_M(\zeta)=\kappa_M(\bar{\zeta})=t$, so that $\kappa_{\varepsilon,M}(\bar{\varepsilon}\zeta)=\kappa_{\varepsilon,M}(\bar{\varepsilon}\bar{\zeta})=\bar{\varepsilon}t$. Since $|A\bar{\varepsilon}t|=-t/M<1$, the identity (9.5) shows that f has different values

$$\frac{\delta\bar{\varepsilon}\zeta+f(z)}{1-\overline{f(z)}\delta\bar{\varepsilon}\zeta}\quad\text{and}\quad\frac{\delta\bar{\varepsilon}\bar{\zeta}+f(z)}{1-\overline{f(z)}\delta\bar{\varepsilon}\bar{\zeta}}$$

at the same point

$$\frac{A\bar{\varepsilon}t+z}{1+\bar{z}A\bar{\varepsilon}t}\in D.$$

This contradiction shows that M must be one, so that $\kappa_{\varepsilon,M}(w) \equiv w$. Since $|A| = 1$, we now have (8.2) with $\lambda = \overline{A}\delta$, $|\lambda| = \delta$, and $\mu = f(z)$. Conversely, for f of (8.2) the equality holds in (8.1). $\square$

Proof of Theorem 5. Without any loss of generality we may suppose that $f^{\#}(z) \neq 0$ and $f(z) \neq \infty$. To the function g of (9.4) one may apply, this time, Lemma 9.2 to conclude that

$$-\kappa_M(-1) \leq \delta^{\#}(0, g) \leq M\rho_{au}(z, f),$$

where $M = \Phi_f(z)/\delta$; note that $f(z)^* \notin \mathrm{Cap}(f(z), \delta)$. On the basis of (9.3) one has that

$$\frac{\left(1 + \rho_{au}(z, f)\right)^2}{4\rho_{au}(z, f)} \leq M.$$

Hence (8.3). Suppose that the equality holds in (8.3) at z with $f^{\#}(z) \neq 0$. Then $g = \kappa_{\varepsilon,M}$. The same argument as in the proof of Theorem 4 yields that $M = 1$ and so f is of (8.2). The converse is now obvious. $\square$

10. Schwarzian derivative

For f nonconstant and meromorphic in a domain $\mathcal{D}$ in $\mathbf{C}^{\#}$, the Schwarzian derivative $Sw(f)$ of f is a meromorphic function defined by

$$Sw(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

in $\mathcal{D}\setminus\{\infty\}$. Consequently, $Sw(f)$ has a point $z \in \mathcal{D}\setminus\{\infty\}$ as a pole of order exactly 2 if and only if $f^{\#}(z) = 0$, whereas, $Sw(f)(z) \neq \infty$ at $z \in \mathcal{D}\setminus\{\infty\}$ if and only if $f^{\#}(z) \neq 0$. One should be careful enough in case z is a simple pole of f. If $\infty \in \mathcal{D}$, then we define

$$Sw(f)(\infty) = \lim_{z \to \infty} Sw(f)(z).$$

More exactly,

$$Sw(f)(\infty) = \lim_{z \to 0} 2z^4 Sw(g)(z) = 0,$$

where $g(z) = f(1/z)$ for z near 0, so that $Sw(f)$ is meromorphic in the whole $\mathcal{D}$. Furthermore, $Sw(f)$ has ∞ as a zero of order 2 if and only if $g^{\#}(0) = 0$.

A well known theorem of Z. Nehari [N] reads that if f is meromorphic in D and if

$$\sup_{z\in D}(1-|z|^2)^2|Sw(f)(z)| \le 2, \tag{10.1}$$

then f is univalent in D. Furthermore, the constant 2 in the right of (10.1) is the best possible [H].

Theorem 6 *Suppose that $f^{\#}(z) \neq 0$ at a point $z \in D$ for f meromorphic in D. Then*

$$(1-|z|^2)^2|Sw(f)(z)| \le 6\left\{\left(\frac{\Phi_f(z)}{\delta^{\#}(z,f)}\right)^2 - 1\right\}. \tag{10.2}$$

Again (C) in Section 1 follows. The equality holds in (10.2) at z with $f^{\#}(z) \neq 0$ if and only if f is of (8.2). For f of (8.2) both sides of (10.2) are zero.

A generalized function $\kappa_{\varepsilon,M,R}$ of $\kappa_{\varepsilon,M}$ with a nonnegative parameter R appears in the proof of Theorem 6 as in the equality argument in the proofs of Theorems 4 and 5. For

$$\varepsilon \in \mathbf{C},\ |\varepsilon| = 1; \quad 1 \le M < +\infty; \quad 0 \le R \le 4\left(1 - \frac{1}{M}\right)\ (< 4),$$

set

$$\kappa_{\varepsilon,M,R}(z) = \bar{\varepsilon} M K^{-1}\left(\frac{K(\varepsilon z)}{M(1 + RK(\varepsilon z))}\right), \quad z \in D.$$

Note that the present parameter R has the upper bound. Then $\kappa_{\varepsilon,M} = \kappa_{\varepsilon,M,0}$. Furthermore, $\kappa_{\varepsilon,1,0}(z) = \kappa_{\varepsilon,1}(z) \equiv z$. One can prove that $\kappa_{\varepsilon,M,R} \in S$ and moreover, the function $\kappa_{\varepsilon,M,R}$ maps D univalently onto the disk $\{|z| < M\}$ minus the union $J_A \cup J_B$ of the sets

$$J_A = \{-\bar{\varepsilon} M s;\ A(M,R) \le s \le 1\} \tag{10.3}$$

and

$$J_B = \{\bar{\varepsilon} M s;\ B(M,R) \le s \le 1\}, \tag{10.4}$$

where

$$A(M,R) = \frac{1}{2}\left(M(4-R) - 2 - \sqrt{M^2(4-R)^2 - 4M(4-R)}\right) > 0$$

and

$$B(M,R) = \frac{1}{2}\left(MR + 2 - \sqrt{M^2R^2 + 4MR}\right) > 0.$$

Suppose that $M > 1$. Then, $A(M,0) = -\kappa_M(-1)/M < 1$. If $R > 0$, then $B(M,R) = K^{-1}(1/\{MR\}) < 1$. We then have the open arc $\Upsilon_B(R)$, $R > 0$, in the upper half of ∂D with one end-point 1, which, as well as its reflexion with respect to the real axis, is mapped onto the open interval $(MB(M,R), M)$ by the extension of $\kappa_{1,M,R}$ to $D \cup \partial D$, which we denote again by $\kappa_{1,M,R}$.

Our proof of Theorem 6 depends, in fact, on

Lemma 10.1 *Suppose tha $f \in S$ is bounded, $|f| < M$, in D. Then*

$$|Sw(f)(0)| \le 6\left(1 - \frac{1}{M^2}\right). \tag{10.5}$$

The equality holds in (10.5) if and only if $f = \kappa_{\varepsilon,M,R}$. See [Y2, p.114, Lemma 6.1] for the detailed proof of Lemma 10.1. In case $M = 1$, Lemma 10.1 is trivial because $f(z) \equiv z$.

General facts about the Schwarzian derivative will also be needed. We begin with the composed function $g \circ f$ in a domain of $\mathbf{C}$. If $f(z) \neq \infty$, then

$$Sw(g \circ f)(z) = Sw(g)(f(z))f'(z)^2 + Sw(f)(z).$$

Particularly if g is a Möbius transformation, $g(\zeta) = (a\zeta + b)/(c\zeta + d)$, $ad - bc \neq 0$, then $Sw(g \circ f)(z) = Sw(f)(z)$. Furthermore if $g = f^{-1}$, the inverse of f and $f(z) \neq \infty$, then

$$Sw(f^{-1})(f(z)) = -\frac{Sw(f)(z)}{f'(z)^2}.$$

Proof of Theorem 6. We may suppose that $f(z) \neq \infty$ because $Sw(f) = Sw(1/f)$. For g of (9.4) with $\delta = \delta^{\#}(z,f)$ again, we have

$$Sw(g)(0) = Sw(\phi)(f(z))\delta^2(1 + |f(z)|^2)^2$$

and

$$Sw(\phi)(f(z)) = -\frac{Sw(f)(z)}{f'(z)^2},$$

so that (10.5) for $g \in S$ with $|g| < M = \Phi_f(z)/\delta$ reads that

$$\frac{(1-|z|^2)^2|Sw(f)(z)|}{M^2} = |Sw(g)(0)| \le 6\left(1 - \frac{1}{M^2}\right).$$

Hence (10.2).

Suppose that the equality holds in (10.2) at z with $f^\#(z) \neq 0$. Then $g = \kappa_{\varepsilon,M,R}$, so that, this time again,

$$\frac{\delta w + f(z)}{1 - \overline{f(z)}\delta w} = f\left(\frac{A\kappa_{\varepsilon,M,R}(w) + z}{1 + \overline{z}A\kappa_{\varepsilon,M,R}(w)}\right), \quad w \in D, \tag{10.6}$$

where

$$A = \frac{\delta(1+|f(z)|^2)}{(1-|z|^2)f'(z)} \quad \text{with} \quad |A| = 1/M.$$

Suppose that $M > 1$. If $R = 0$, then for $t \in \big(-M, \kappa_M(-1)\big)$ we have $\zeta \in \Upsilon$ with $\kappa_M(\zeta) = \kappa_M(\overline{\zeta}) = t$. If $R > 0$, then for $t \in \big(MB(M,R), M\big)$ we have $\zeta \in \Upsilon_B(R)$ with $\kappa_{1,M,R}(\zeta) = \kappa_{1,M,R}(\overline{\zeta}) = t$.

The remaining part of the proof is established on following the same argument as in the proof of Theorem 4 with the obvious modification. □

Remark 5 Suppose that f is meromorphic with nonvanishing $f^\#$ in D and suppose further that

$$\sup_{z\in D} \frac{\Phi_f(z)}{\delta^\#(z,f)} \le \frac{2}{\sqrt{3}} = 1.154\ldots.$$

Then f is univalent in D. This follows on combining (10.1) and (10.2).

We note that, for an arbitrary $c > 0$, there exists $f \in \mathcal{M}$ such that

$$\sup_{z\in D} \frac{\Phi_f(z)}{\delta^\#(z,f)} > c.$$

Indeed, choose p, $0 < p < 1$, such that $(1+p)^2/p > c$. Then for $f = H_{p,1}$ of $\mathcal{M}$,

$$\frac{\Phi_f(0)}{\delta^\#(0,f)} = \frac{(1+p)^2}{p} > c.$$

11. Poincaré density

According to the Koebe uniformization theory a plane domain $\Omega \subset \mathbf{C}$ whose complement $\mathbf{C} \setminus \Omega$ contains at least two points admits a universal covering projection f from D onto Ω, in notation, $f \in \mathrm{Proj}(\Omega)$, which is holomorphic with nonvanishing derivative in D, in particular. Furthermore, f is normal. We set

$$P_\Omega(z) = \frac{1}{(1-|w|^2)|f'(w)|}, \quad z \in \Omega,$$

where $z = f(w)$; the choice of $f \in \mathrm{Proj}(\Omega)$ and w is immaterial as far as $z = f(w)$ is satisfied. We call $P_\Omega(z)$ the Poincaré density of Ω at $z \in \Omega$; the Poincaré metric in Ω in the differential form is $P_\Omega(z)|dz|$. For the specified case $\Omega = D$ one has $P_D(\zeta) = 1/(1-|\zeta|^2)$; the integral of $P_D(\zeta)|d\zeta|$ along $\gamma(z,w)$ defines the Poincaré distance $\sigma(z,w)$ in D.

Similarly, set for $z = f(w) \in \Omega$,

$$\rho_\Omega(z) = \rho(w,f), \quad \rho_{\Omega,a}(z) = \rho_a(w,f), \quad \rho^*_{\Omega,a}(z) = \rho^*_a(w,f),$$
$$\text{and} \quad \rho_{\Omega,au}(z) = \rho_{au}(w,f).$$

Then all are well defined in Ω and none of them depends on a particular choice of $f \in \mathrm{Proj}(\Omega)$ and w as far as $z = f(w)$ is satisfied. Since each $f \in \mathrm{Proj}(\Omega)$ is normal, it follows from Theorem 1 that

$$\rho_a(\Omega) \equiv \inf_{z\in\Omega} \rho_{\Omega,a}(z) = \rho_a(f) > 0,$$

or equivalently, by (**I**),

$$\rho^*_a(\Omega) \equiv \inf_{z\in\Omega} \rho^*_{\Omega,a}(z) = \rho^*_a(f) > 0.$$

Furthermore,

$$\delta^\#_\Omega(z) \equiv \inf_{\zeta\in\partial^\#\Omega} \left|\frac{\zeta - z}{1+\bar{z}\zeta}\right| = \delta^\#(w,f)$$

for $z = f(w)$, where $\partial^\#\Omega$ is the boundary of Ω in $\mathbf{C}^\#$.

Recall here that the spherical distance $\chi(z,w)$ is the line integral of $\chi(\zeta)|d\zeta|$ along curve(s), where $\chi(\zeta) = 1/(1+|\zeta|^2)$; see (2.1) and its following. We therefore call $\chi(\zeta)$ the spherical density at $\zeta \in \mathbf{C}^\#$, where $\chi(\infty) = 0$.

As relations among the functions Φ_f, χ, and P_Ω, we have at $z = f(w) \in \Omega$ for $f \in \mathrm{Proj}(\Omega)$ that $\Phi_f(w) = \chi(z)/P_\Omega(z)$ and consequently,

$$\Lambda_f(w) = \frac{1}{P_\Omega(z)}\left|\frac{\partial}{\partial z}\log\frac{\chi(z)}{P_\Omega(z)}\right|.$$

We can now apply preceding theorems to an $f \in \mathrm{Proj}(\Omega)$ and translate them in terms of P_Ω and others.

For example, it follows from Theorem 2 that

$$\frac{\rho_\Omega(z)}{P_\Omega(z)}\left|\frac{\partial}{\partial z}\log\frac{\chi(z)}{P_\Omega(z)}\right| \leq \frac{\rho_{\Omega,au}(z)}{\rho_\Omega(z)} + \frac{\rho_\Omega(z)}{\rho_{\Omega,au}(z)}, \quad z \in \Omega. \tag{11.1}$$

The equality holds at $z \in \Omega$ if and only if $\Omega = f(D) \subset \mathbf{C}$, where

$$f(\zeta) = \frac{\lambda H_{p,\varepsilon}\left(\dfrac{\zeta - w}{1 - \overline{w}\zeta}\right) + \mu}{1 - \overline{\mu}\lambda H_{p,\varepsilon}\left(\dfrac{\zeta - w}{1 - \overline{w}\zeta}\right)}$$

with four parameters $0 < p \leq 1$, $\varepsilon \in \mathbf{C}$, $|\varepsilon| = 1$, $\lambda \in \mathbf{C} \setminus \{0\}$, and $\mu \in \mathbf{C}^\#$. In this case Ω is $\mathbf{C}^\#$ (or Σ) minus a circular arc containing ∞.

It follows from Theorem 4 that

$$\delta_\Omega^\#(z) \leq \frac{2\chi(z)}{2P_\Omega(z) + \big|(\partial/\partial z)\log\big(\chi(z)/P_\Omega(z)\big)\big|}, \quad z \in \Omega. \tag{11.2}$$

The equality holds at $z \in \Omega$ if and only if Ω is a spherical cap $\mathrm{Cap}(z, R) \subset \mathbf{C}$ with $0 < R < +\infty$. In this case $\delta_\Omega^\#(z) = \chi(z)/P_\Omega(z) = R$ and $(\partial/\partial z)\big(\chi(z)/P_\Omega(z)\big) = 0$.

A hyperbolic domain $\Omega \subset \mathbf{C}$ is said to be of finite type [Y2] if $\rho(\Omega) = \inf_{z\in\Omega} \rho_\Omega(z) = (= \rho(f))$ is strictly positive. This notion was essentially introduced in [Y1]. We prove here that Ω is of finite type if and only if

$$\inf_{z\in\Omega} \frac{\delta_\Omega^\#(z)P_\Omega(z)}{\chi(z)} > 0. \tag{11.3}$$

The infimum is not greater than 1 because (C) holds for $f \in \mathrm{Proj}(\Omega)$.

Suppose that $\rho(\Omega) > 0$. Then

$$\rho_{au}(\Omega) \equiv \inf_{z\in\Omega} \rho_{\Omega,au}(z) > 0.$$

Hence it follows from (7.7) that

$$\frac{\chi(z)}{\delta_\Omega^\#(z)P_\Omega(z)} \leq 2\left(\frac{1}{\rho(\Omega)} + \frac{1}{\rho_{au}(\Omega)}\right) < +\infty,$$

so that we have (11.3). Conversely suppose (11.3). Then it follows from

(8.3) that there exists c, $0 < c \leq 1$, with

$$c \leq \frac{4\rho_{au}(z,f)}{\left(1+\rho_{au}(z,f)\right)^2} \leq \frac{4\rho(z,f)}{\left(1+\rho(z,f)\right)^2}$$

at each $z \in \Omega$. Hence $\rho(\Omega) \geq \left(2 - c - 2\sqrt{1-c}\right)/c > 0$.

Since

$$\frac{1}{2}(1-|w|^2)^2|Sw(f)(w)| = \frac{1}{P_\Omega(z)}\left|\frac{\partial^2}{\partial z^2}\frac{1}{P_\Omega(z)}\right|$$

at $z = f(w) \in \Omega$ for $f \in \mathrm{Proj}(\Omega)$, it follows from (10.2) that

$$\frac{1}{P_\Omega(z)}\left|\frac{\partial^2}{\partial z^2}\frac{1}{P_\Omega(z)}\right| \leq 3\left\{\left(\frac{\chi(z)}{\delta^{\#}_\Omega(z)P_\Omega(z)}\right)^2 - 1\right\}. \tag{11.4}$$

The equality holds in (11.4) if and only if Ω is a spherical cap $\mathrm{Cap}(z,R) \subset \mathbf{C}$, $0 < R < +\infty$; both sides of (11.4) are zero in this case.

Finally, the quantity $\nu(\Omega) = \nu(f)$ for $f \in \mathrm{Proj}(\Omega)$ is called the normal constant of Ω in [Y3, p.302]; this is independent of a particular choice of f. It is known, for example, that $\nu\left(\mathbf{C}\setminus\{0,1\}\right) \leq 4.487\ldots$. An immediate consequence of (**I**) is then that

$$\tanh\frac{\pi}{4\nu(\Omega)} \leq \rho^*_a(\Omega) \leq \rho_a(\Omega) \leq 2\rho^*_a(\Omega) \leq \frac{2}{\nu(\Omega)}.$$

12. Univalent meromorphic function

As estimates of $\Phi_f(z)$ of the type somewhat different from previous ones we propose here

Theorem 7 *Let f be meromorphic and univalent in D. Suppose that $f(w) = f(z)^*$. Then*

$$\left(\frac{1-\tau(z,w)^2}{\tau(z,w)}\right)^2 \leq \Phi_f(z)\Phi_f(w) \leq \frac{1}{\tau(z,w)^2}. \tag{12.1}$$

Note that $\tau(z,w) = \rho_a(z,f) = \rho_a(w,f) < 1$ in this case.

To describe an equality condition for the right-hand side of (12.1) we set

$$G_p(z) = \frac{pz(1-pz)}{p-z}, \quad 0 < p < 1.$$

Then the rational function G_p of z maps D univalently onto $\mathbf{C}^{\#}$ minus the subarc on the circle $\{|z| = p\}$ bisected by $-p$ and having the end-points

$$p\left(2p^2 - 1 \pm 2p\sqrt{1-p^2}\, i\right).$$

To be more precise, $G_p = f_3 \circ f_2 \circ f_1$, where

$$f_1(\zeta) = \frac{\zeta - b}{1 - b\zeta} \quad \text{with } b = \frac{1 - \sqrt{1-p^2}}{p};$$

$$f_2(\zeta) = \frac{p}{2\sqrt{1-p^2}}\left(\zeta - \frac{1}{\zeta}\right); \quad f_3(\zeta) = \frac{p(\zeta - 1)}{\zeta + 1}.$$

We set $G_{p,\varepsilon}(z) = \overline{\varepsilon} G_p(\varepsilon z)$ for $\varepsilon \in \mathbf{C}$ with $|\varepsilon| = 1$. Specifically, $G_{p,\varepsilon} \in S(p)$.

The equality in the right of (12.1) holds if and only if

$$f(\zeta) = \frac{\lambda G_{p,\varepsilon}\left(\dfrac{\zeta - z}{1 - \overline{z}\zeta}\right) + \mu}{1 - \overline{\mu}\lambda G_{p,\varepsilon}\left(\dfrac{\zeta - z}{1 - \overline{z}\zeta}\right)}, \tag{12.2}$$

where $\lambda \in \mathbf{C} \setminus \{0\}$, $\mu \in \mathbf{C}^{\#}$, $\varepsilon \in \mathbf{C}$, $|\varepsilon| = 1$, and p, $0 < p < 1$, all are constants. In particular, $f(w) = f(z)^*$ for $w = (\overline{\varepsilon}p + z)/(1 + \overline{z\varepsilon}p)$. For f of (12.2), the equality $\Phi_f(z) = |\lambda|$ is valid, so that $\Phi_f(w) = (p^2|\lambda|)^{-1}$.

The equality in the left of (12.1) holds if and only if

$$f(\zeta) = \frac{\lambda H_{p,\varepsilon}\left(\dfrac{\zeta - z}{1 - \overline{z}\zeta}\right) + \mu}{1 - \overline{\mu}\lambda H_{p,\varepsilon}\left(\dfrac{\zeta - z}{1 - \overline{z}\zeta}\right)} \tag{12.3}$$

where $\lambda \in \mathbf{C} \setminus \{0\}$, $\mu \in \mathbf{C}^{\#}$, $\varepsilon \in \mathbf{C}$, $|\varepsilon| = 1$, and p, $0 < p < 1$, all are constants. In particular, $f(w) = f(z)^*$ for $w = (\overline{\varepsilon}p + z)/(1 + \overline{z\varepsilon}p)$. For f of (12.3), the equality $\Phi_f(z) = |\lambda|$ is valid, so that $\Phi_f(w) = (1-p^2)^2/(p^2|\lambda|)$.

We recall here the result of Komatu on $S(p)$ [K, p.278, (4.4)] on which Theorem 7 depends; see [G, II, p.263].

Lemma 12.1 *Suppose that $f(z_o) = \infty$ for $f \in S(p)$ with $0 < p < 1$, so that $|z_o| = p$. Then*

$$p^2(1-p^2) \le |\operatorname{Res}(f, z_o)| \le \frac{p^2}{1-p^2}, \tag{12.4}$$

where

$$\operatorname{Res}(f, z_o) = \lim_{z \to z_o} (z - z_o) f(z).$$

The equality in the left (right, respectively) of (12.4) holds if and only if $f = G_{p,\varepsilon}$ ($f = H_{p,\varepsilon}$, respectively); see [K, p.279, (4.8)].

Before passing to the proof of Theorem 7 let us note that

$$1/\operatorname{Res}(f, z_o) = (1/f)'(z_o) \tag{12.5}$$

for $f \in S(p)$ with $f(z_o) = \infty$, $|z_o| = p < 1$. We then have $\Phi_f(z_o) = (1 - p^2)/|\operatorname{Res}(f, z_o)|$.

Proof of Theorem 7. We may suppose, without loss of generality, that $0 \neq f(z) \neq \infty$. Indeed, otherwise, consider $(f - b)/(1 + \bar{b}f)$ for a constant $b \in \mathbf{C} \setminus \{0\}$. Set $T(\zeta) = (\zeta + z)/(1 + \bar{z}\zeta)$, $\zeta \in D$, and choose $\eta \in D$ such that $w = T(\eta)$. Then $|\eta| = \tau(z, w)$. Set

$$g(\zeta) = \frac{1 + |f(z)|^2}{(1 - |z|^2) f'(z)} \cdot \frac{f \circ T(\zeta) - f(z)}{1 + \overline{f(z)} f \circ T(\zeta)}, \quad \zeta \in D.$$

Then $g \in S(p)$ with $g(\eta) = \infty$ and $p = |\eta|$. It follows therefore from (12.5) for g and $z_o = \eta$ that

$$\begin{aligned}\frac{1}{|\operatorname{Res}(g, \eta)|} &= \left| \left(\frac{1}{g}\right)'(\eta) \right| = \Phi_f(z) f^{\#}(w) |T'(\eta)| \\ &= \frac{\Phi_f(z)\Phi_f(w)}{1 - |\eta|^2} = \frac{\Phi_f(z)\Phi_f(w)}{1 - p^2}.\end{aligned}$$

Here we make use of

$$\begin{aligned}|f(w) - f(z)|^2 &= \frac{(1 + |f(z)|^2)}{|f(z)|} \frac{(1 + |f(w)|^2)}{|f(w)|} \\ &= (1 + |f(z)|^2)(1 + |f(w)|^2)\end{aligned}$$

and

$$(1 - |\eta|^2)|T'(\eta)| = 1 - |T(\eta)|^2 = 1 - |w|^2.$$

One now takes advantage of (12.4) for g, $z_o = \eta$, and $p = \tau(z, w)$ to have (12.1). It is now an exercise to have the described equality condition. □

Remark 6 A consequence of the left of (12.1) is that

$$\frac{\sqrt{\nu(f)^2+4}-\nu(f)}{2} \le \rho_a(z,f)$$

if $\rho_a(z,f) < 1$; this is obviously true in case $\rho_a(z,f) = 1$. It then follows that

$$\frac{\sqrt{\nu(f)^2+4}-\nu(f)}{2} \le \rho_a(f)$$

if f is meromorphic and univalent in D.

We can translate these results in terms of $\nu(\Omega)$ and $\rho_a(\Omega)$ for a simply connected domain $\Omega \neq \mathbf{C}$.

13. Bloch function

What are reasonable holomorphic versions of (**I**) and (**II**) in Section 3? The situation appears to be very much different. For f holomorphic in D we set

$$\beta(f) = \sup_{z\in D}(1-|z|^2)|f'(z)|$$

and call f Bloch if $\beta(f) < +\infty$.

Let $\rho_\omega(z,f)$ be the maximum of r, $0 < r \le 1$, such that $e^{f(w)}+e^{f(z)} \neq 0$ for all $w \in \Delta(z,r)$, whereas let $\rho_\omega^*(z,f)$ be the maximum of r, $0 < r \le 1$, such that

$$e^{f(\zeta)} + e^{f(\eta)} \neq 0 \tag{13.1}$$

for all ζ, $\eta \in \Delta(z,r)$.

Since

$$0 < \rho_\omega^*(z,f) \le \rho_\omega(z,f) \le 1, \quad z \in D,$$

it follows that

$$0 \le \rho_\omega^*(f) \le \rho_\omega(f) \le 1,$$

where

$$\rho_\omega(f) = \inf_{z\in D}\rho_\omega(z,f) \quad \text{and} \quad \rho_\omega^*(f) = \inf_{z\in D}\rho_\omega^*(z,f).$$

Theorem 8 *A holomorphic function f defined in D is Bloch if and only if $\rho_\omega(f) > 0$.*

For the proof it suffices to observe that

$$\textbf{(III)} \qquad \tanh\frac{\pi}{2\beta(f)} \le \rho^*_\omega(f) \le \rho_\omega(f) \le 2\rho^*_\omega(f) \le \frac{4}{\beta(f)}$$

for f holomorphic in D.

THE FIRST INEQUALITY IN **(III)**. We may suppose that $0 < \beta(f) < +\infty$. Then for all ζ, $\eta \in D$, one has

$$|\operatorname{Im} f(\zeta) - \operatorname{Im} f(\eta)| \le |f(\zeta) - f(\eta)| \le \beta(f)\sigma(\zeta,\eta).$$

Hence, for all ζ, $\eta \in \Delta(z,r)$ with $r = \tanh\big(\pi/(2\beta(f))\big)$, one further has

$$|\operatorname{Im} f(\zeta) - \operatorname{Im} f(\eta)| \le \beta(f)\big(\sigma(\zeta,z) + \sigma(z,\eta)\big) < \pi.$$

Hence (13.1) is true, and consequently,

$$\tanh\frac{\pi}{2\beta(f)} \le \rho^*_\omega(z,f),$$

whence the first follows.

THE THIRD INEQUALITY IN **(III)**. We may suppose that $\rho_\omega(f) > 0$. For $z \in D$ and for all ζ, $\eta \in \Delta(z, \rho_\omega(f)/2)$, we have $\zeta \in \Delta(\eta, \rho_\omega(\eta, f))$, so that (13.1) is valid. Hence

$$\rho_\omega(f)/2 \le \rho^*_\omega(z,f),$$

from which the third follows.

THE FOURTH INEQUALITY IN **(III)**. We shall make use of

$$\textbf{(IV)} \qquad \rho^*_\omega(z,f)(1-|z|^2)|f'(z)| \le 2, \quad z \in D$$

for f holomorphic in D.

For the proof of **(IV)** we may suppose that $a = \rho^*_\omega(z,f) > 0$. Then for each fixed $z \in D$, the function

$$g(w) = \exp\left(f\left(\frac{aw+z}{1+\bar{z}aw}\right) - f(z)\right)$$

of $w \in D$ satisfies $g(0) = 1$ and $g(\zeta) + g(\eta) \ne 0$ for all ζ, η in D. In other words, g is a Gel'fer function [G, II, p.73]. Hence the holomorphic function $h = (g-1)/(g+1)$ is a Bieberbach-Eilenberg function in the sense that $h(0) = 0$ and $h(\zeta)h(\eta) \ne 1$ for all ζ, η in D; see [G, II, p.61]. It then follows from the Eilenberg theorem [G, II, p.63, Theorem 31] that $|h'(0)| \leq 1$ and, moreover, $|h'(0)| = 1$ if and only if $h(w) \equiv \varepsilon w$, $|\varepsilon| = 1$. Consequently,

$$(1 - |z|^2)|f'(z)| = \frac{|g'(0)|}{a} = \frac{2|h'(0)|}{a} \leq \frac{2}{a},$$

whence (**IV**). The equality holds in (**IV**) at z if and only if

$$f(w) = \log \frac{1 + \varepsilon \cdot \dfrac{w - z}{1 - \bar{z}w}}{1 - \varepsilon \cdot \dfrac{w - z}{1 - \bar{z}w}} + \mu, \quad \varepsilon \in \mathbf{C}, \quad |\varepsilon| = 1, \quad \text{and} \quad \mu \in \mathbf{C}. \tag{13.2}$$

Indeed, suppose that the equality holds in (**IV**). Then $h(w) \equiv \varepsilon w$, $\varepsilon \in \mathbf{C}$, $|\varepsilon| = 1$, so that

$$\frac{1 + \varepsilon w}{1 - \varepsilon w} (= g(w)) = \exp\left(f\left(\frac{aw + z}{1 + \bar{z}aw}\right) - f(z)\right)$$

for all $w \in D$. If $a < 1$, then we have a contradiction on letting $w \to \bar{\varepsilon}$. Hence $a = 1$, whence we have (13.2) with $\mu = f(z)$. The converse is obvious.

The fourth inequality in (**III**) now immediately follows from (**IV**).

Remark 7 The domain constants $\rho_\omega(\Omega) = \rho_\omega(f)$ and $\rho^*_\omega(\Omega) = \rho^*_\omega(f)$ both are well defined in Ω with the aid of $f \in \mathrm{Proj}(\Omega)$ in the sense that the choice of f is immaterial. Set

$$\beta(\Omega) = \frac{1}{\inf_{z \in \Omega} P_\Omega(z)}.$$

Then $\beta(\Omega) = \beta(f)$, $f \in \mathrm{Proj}(\Omega)$, and $0 < \beta(\Omega) \leq +\infty$. As the case $\Omega = \{z; \mathrm{Re}\, z > 0\}$ shows, it is possible that $\beta(\Omega) = +\infty$. For general hyperbolic domain Ω one then has

$$\tanh \frac{\pi}{2\beta(\Omega)} \leq \rho^*_\omega(\Omega) \leq \rho_\omega(\Omega) \leq 2\rho^*_\omega(\Omega) \leq \frac{4}{\beta(\Omega)}.$$

14. Absolute constants

We proved in Remark 2 in Section 5 that

$$\sup_{f \in \mathcal{M}} \nu(f) = +\infty.$$

For $S \subset \mathcal{M}$, in contrast, we are here able to prove that

$$\nu(S) \equiv \sup_{f \in S} \nu(f) \leq \Xi(\alpha) = 3.4569\ldots. \tag{14.1}$$

In the above

$$\Xi(x) = \frac{1}{(1-x)^4 + x^2}, \quad 0 \leq x \leq 1, \tag{14.2}$$

with

$$\alpha = \sqrt[3]{\frac{1}{4}\left(\sqrt{\frac{29}{27}} - 1\right)} - \sqrt[3]{\frac{1}{4}\left(\sqrt{\frac{29}{27}} + 1\right)} + 1 = 0.4102\ldots,$$

so that $\Xi(\alpha) = \max_{0 \leq x \leq 1} \Xi(x)$. The proof of (14.1) depends on two forthcoming inequalities. For the first we let $f \in S$ and set $g(w) = 1/f(z)$, $w = 1/z$. Then g is meromorphic and univalent in $\mathrm{Cap}(\infty, 1) = \{w;\ 1 < |w| \leq +\infty\}$ such that $\lim_{|w| \to +\infty} g(w)/w = 1$. It then follows from Loewner's inequality [D, p.127, Corollary 6] that

$$|g'(w)| \leq \frac{|w|^2}{|w|^2 - 1}, \quad 1 < |w| < +\infty,$$

which is reduced to the first inequality

$$(1 - |z|^2)\left|\frac{f'(z)}{f(z)^2}\right| |z|^2 \leq 1, \quad z \in D; \tag{14.3}$$

the left-hand side is 1 at $z = 0$. On the other hand, the familiar inequality

$$|f(z)| \leq K(|z|) = \frac{|z|}{(1 - |z|)^2} \tag{14.4}$$

for $z \in D$ holds [G, I, p.68, Theorem 8]. Hence the second is that

$$\frac{1}{|z|^2} \frac{|f(z)|^2}{1 + |f(z)|^2} \leq \Xi(|z|);$$

the left-hand side is again 1 at $z = 0$. It then follows from (14.3) and

$$\Phi_f(z) = (1 - |z|^2)\left|\frac{f'(z)}{f(z)^2}\right| |z|^2 \cdot \frac{1}{|z|^2} \frac{|f(z)|^2}{1 + |f(z)|^2}$$

that

$$\Phi_f(z) \leq \Xi(|z|), \qquad z \in D.$$

Since

$$\max_{0 \leq x \leq 1} \Xi(x) = \Xi(\alpha),$$

where $2(\alpha - 1)^3 + \alpha = 0$, we finally have (14.1).

As for the lower estimate of $\nu(S)$ we have

$$\nu(S) \geq \nu(K) \geq \max_{0 \leq x \leq 1} \Phi_K(x) = \max_{0 \leq x \leq 1} \frac{(1 - x^2)^2}{(1 - x)^4 + x^2} = 2.561\ldots,$$

the maximum actually being attained at $\beta = 0.340\ldots$ for which

$$2\beta^4 - 13\beta^3 + 12\beta^2 - 9\beta + 2 = 0.$$

It is of interest to determine the exact value of $\nu(S)$.

For $S(p)$, $0 < p < 1$, the situation is not very much different. Namely,

$$\nu(S(p)) \equiv \sup_{f \in S(p)} \nu(f) \leq C(p), \tag{14.5}$$

where

$$C(p) = \max_{0 \leq x \leq 1} \Xi_p(x),$$

$$\Xi_p(x) = \frac{p^2}{(1 - px)^2 (p - x)^2 + p^2 x^2}, \quad 0 \leq x \leq 1.$$

For the proof of (14.5) we have only to remember a counter part of (14.4) for $S(p)$, namely,

$$|f(z)| \leq |H_{p,-1}(|z|)| = \frac{p|z|}{|p - |z||(1 - p|z|)}$$

for $f \in S(p)$ and $z \in D$; see [KS, Theorem 3] and [G, II, p.248, Theorem 40]. The definition of $S(p)$ in [KS] is slightly different. If $f(z_o) = \infty$ for f of our $S(p)$, $z_o = \varepsilon p$, then $g(z) = \bar{\varepsilon} f(\varepsilon z)$ is in $S(p)$ defined in [KS]. We have $C(p) = \Xi_p\big(\alpha(p)\big)$, where the constant $\alpha = \alpha(p)$, $0 < \alpha(p) < 1$, satisfies the equation

$$2p^2\alpha^3 - 3p(1 + p^2)\alpha^2 + (p^4 + 5p^2 + 1)\alpha - p(1 + p^2) = 0.$$

Note that $1 < 1/p^2 \leq C(p)$.

As for the lower estimate of $\nu(S(p))$ we have, in a similar manner, that

$$\nu(S(p)) \geq \nu(H_{p,-1}) \geq \max_{0\leq x\leq 1} \Phi_{H_{p,-1}}(x)$$
$$= \max_{0\leq x\leq 1} \frac{p^2(1-x^2)^2}{(p-x)^2(1-px)^2+p^2x^2}.$$

Remark 8 Let $g(w) = 1/f(z)$, $w = 1/z$, again for $f \in S(p)$, $0 < p \leq 1$. It follows from [D, p.127, Corollary 6] that

$$\frac{|w|^2-1}{|w|^2} \leq |g'(w)|, \quad 1 < |w| < +\infty,$$

so that

$$(1-|z|^2)^2 \leq (1-|z|^2)\left|\frac{f'(z)}{f(z)^2}\right||z|^2, \quad z \in D.$$

On the other hand, it is known for $f \in S(p)$ that

$$\frac{p|z|}{(1+p|z|)(p+|z|)} = -H_{p,-1}(-|z|) \leq |f(z)|, \quad z \in D;$$

see [G, I, p.68, Theorem 8] for $p = 1$, whereas, see [F], [KS], and [G, II, p.248, (90)] for $0 < p < 1$. It is indeed an exercise to have the left-hand side of

$$\frac{p^2(1-|z|^2)^2}{(1+p|z|)^2(p+|z|)^2+p^2|z|^2}$$
$$\leq \Phi_f(z) \leq \frac{p^2}{(1-p|z|)^2(p-|z|)^2+p^2|z|^2}, \quad z \in D,$$

for $f \in S(p)$, $0 < p \leq 1$. In particular, let z_o be the pole of $f \in S(p)$, $0 < p < 1$, and set $z = z_o$ in the above. One then again has the left-hand side of (12.4).

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