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Calderón-Zygmund operators on weighted $H^p(R^n)$

Yasuo KOMORI

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Abstract. We consider the boundedness of Calderón-Zygmund operators from weighted $H^p(R^n)$ to weighted $h^p(R^n)$ (local Hardy space). We show Calderón's commutator is bounded from weighted H^p to weighted h^p.

Key words: Calderón-Zygmund operator, weighted Hardy space, local Hardy space.

1. Introduction

Consider the operator defined by

$$Tf(x) = \text{p.v.} \int_{R^n} K(x, y) f(y) dy,$$

where K is a Calderón-Zygmund kernel (see Section 2).

Quek and Yang [10] proved that if kernel $K(x, y)$ has some regularity then T is a bounded operator from $H_w^p(R^n)$ to $H_w^p(R^n)$ if $T^*1 = 0$.

In this paper we define the space $h_w^p(R^n)$ which is the local version of $H_w^p(R^n)$ and a weighted version of the local Hardy space $h^p(R^n)$ defined by Goldberg [4]. We show that if T^*1 belongs to Lipschitz class then T is a bounded operator from $H_w^p(R^n)$ to $h_w^p(R^n)$.

The author [8] proved the theorem when $w \equiv 1$ (see also [1], [2]).

2. Definitions and Notations

The following notation is used: For a set $E \subset R^n$ and a locally integrable function w, we denote the Lebesgue measure of E by $|E|$ and $w(E) = \int_E w(x) dx$. We denote the characteristic function of E by χ_E. We write a ball of radius r centered at x_0 by $B(x_0, r) = \{x; |x - x_0| < r\}$.

First we shall define two maximal functions and some Hardy spaces.

Let φ be a fixed Schwartz function in $\mathcal{S}(R^n)$ such that $\text{supp}(\varphi) \subset B(0, 1)$ and $\int \varphi(x) dx \neq 0$, then we define

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$$f^{++}(x) = \sup_{t>0} \left| \int f(y)\varphi_t(x-y)dy \right|,$$

$$f^{+}(x) = \sup_{1>t>0} \left| \int f(y)\varphi_t(x-y)dy \right|,$$

where $\varphi_t(x) = t^{-n}\varphi(x/t)$.

Definition 1 (Fefferman-Stein's Hardy space [3])

$$H^p(R^n) = \{f \in \mathcal{S}';\ \|f\|_{H^p} = \|f^{++}\|_{L^p} < \infty\}, \text{ where } 0 < p < \infty.$$

Definition 2 (local Hardy space [4])

$$h^p(R^n) = \{f \in \mathcal{S}';\ \|f\|_{h^p} = \|f^{+}\|_{L^p} < \infty\}, \text{ where } 0 < p < \infty.$$

Remark $\|f\|_{h^p} \le \|f\|_{H^p}$.

Definition 3 (Lipschitz space)

$$\mathrm{Lip}_\varepsilon(R^n) = \left\{ f;\ \|f\|_{\mathrm{Lip}_\varepsilon} = \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x-y|^\varepsilon} < \infty \right\} \text{ for } 0 < \varepsilon < 1.$$

Remark $(H^p)^* = \mathrm{Lip}_{n(1/p-1)}$ where $n/(n+1) < p < 1$ (For the duality, see [3] or [9], p. 54).

Before we define the weighted Hardy spaces we shall define Muckenhoupt A_q weight class (see [6], [12]).

Definition 4 Let $1 < q < \infty$. For a nonnegative locally integrable function w, we say $w \in A_q$ if

$$\left(\frac{1}{|B|} \int_B w(x)dx \right) \left(\frac{1}{|B|} \int_B w(x)^{-1/(q-1)}dx \right)^{q-1} \le C,$$

where C is a positive constant independent of a ball B.

We say $w \in A_1$ if

$$\frac{1}{|B|} \int_B w(x)dx \le C \operatorname*{essinf}_{x \in B} w(x).$$

We write $A_\infty = \bigcup_{q \ge 1} A_q$.

Remark $A_{q_1} \subset A_{q_2}$ if $q_1 < q_2$.

Strömberg and Torchinsky [11] defined the weighted Hardy spaces as follows.

Definition 5 (H_w^p) Let $w \in A_\infty$.

$$H_w^p(R^n) = \{f \in \mathcal{S}';\ \|f\|_{H_w^p} = \|f^{++}\|_{L_w^p} < \infty\}, \quad \text{where } 0 < p < \infty.$$

We define weighted local Hardy spaces as follows.

Definition 6 (h_w^p) Let $w \in A_\infty$.

$$h_w^p(R^n) = \{f \in \mathcal{S}';\ \|f\|_{h_w^p} = \|f^{+}\|_{L_w^p} < \infty\}, \quad \text{where } 0 < p < \infty.$$

Next we shall define Calderón-Zygmund operator.

Definition 7 Let T be a bounded linear operator from $\mathcal{S}$ to $\mathcal{S}'$. T is called a standard operator if T satisfies the following conditions.

(i) T extends to a continuous operator on L^2.

(ii) There exists a function $K(x,y)$ defined on $\{(x,y) \in R^n \times R^n;\ x \neq y\}$ which satisfies $|K(x,y)| \leq \frac{C}{|x-y|^n}$.

(iii) $(Tf, g) = \int\int K(x,y)f(y)g(x)dydx$ for $f, g \in \mathcal{S}$ with disjoint supports.

Definition 8 A standard operator T is called a δ-Calderón-Zygmund operator if $K(x,y)$ satisfies

$$|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)| \leq C\frac{|y-z|^\delta}{|x-z|^{n+\delta}}$$

if $2|y-z| < |x-z|$, for some $0 < \delta \leq 1$.

Examples Let T be a classical singular integral operator defined by

$$Tf(x) = \text{p.v.} \int_{R^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy,$$

where Ω satisfies the following conditions.

(iv) $\Omega(rx) = \Omega(x)$ for $r > 0$, $x \neq 0$.

(v) $\int_{S^{n-1}} \Omega(x)d\sigma = 0$ where $d\sigma$ is the induced Euclidean measure on S^{n-1}.

(vi) $\Omega \in \text{Lip}_\delta$.

Then T is a δ-Calderón-Zygmund operator.

The Hilbert transform and the Riesz transforms are 1-Calderón-Zygmund operators ($\delta = 1$).

Remark If T is a δ-Calderón-Zygmund operator and $w \in A_q$, then T is bounded on L^q_w where $q > 1$ (see [5], [7], p. 52 and [10]).

3. Theorems

Quek and Yang [10] obtained next result.

Theorem *Let $1 \leq q < \frac{n+\delta}{n}$ and $\frac{nq}{n+\delta} < p \leq 1$. If $w \in A_q$ and T is a δ-Calderón-Zygmund operator such that $T^*1 = 0$ then T is a bounded operator from $H^p_w(R^n)$ to $H^p_w(R^n)$.*

Remark T^* is an adjoint operator of T. T and T^* are simultaneously δ-Calderón-Zygmund operators. For the definition of T^*1, see [12], p. 412.

We have the following:

Theorem 1 *Let $1 \leq q < \frac{n+\delta}{n}$, $q \leq \frac{n+\varepsilon}{n}$, $\frac{nq}{n+\delta} < p \leq 1$ and $\frac{nq}{n+\varepsilon} \leq p$. If $w \in A_q$ and T is a δ-Calderón-Zygmund operator such that $T^*1 \in \mathrm{Lip}_\varepsilon$ then T is a bounded operator from $H^p_w(R^n)$ to $h^p_w(R^n)$.*

Remark When $w \equiv 1$, that is $q = 1$, the conditions $\frac{n}{n+\delta} < p$ and $\frac{n}{n+\varepsilon} \leq p$ are the best possible (see [8], p. 70).

As a corollary of Theorem 1 we obtain the boundedness of Calderón's commutator.

Definition 9 Calderón's commutator is defined by

$$T_b f(x) = \text{p.v.} \int_{R^1} \frac{b(x) - b(y)}{(x-y)^2} f(y)dy.$$

Theorem 2 *Let $w \in A_1$. If $b' \in L^\infty \cap \mathrm{Lip}_\varepsilon$, then T_b is a bounded operator from $H^p_w(R^1)$ to $h^p_w(R^1)$ where $\frac{1}{1+\varepsilon} \leq p \leq 1$.*

Proof. If $b' \in L^\infty$ then T_b is bounded on L^2 (see [12], p. 408) and a 1-Calderón-Zygmund operator ($\delta = 1$). We can write $T^*_b 1(x) = -H(b')(x)$ where H is the Hilbert transform. Since H is bounded on Lip_ε (see [12], p. 214), we have $T^*_b 1(x) \in \mathrm{Lip}_\varepsilon$. By Theorem 1 we obtain the desired result. $\square$

4. Lemmas

4.1. Weight

First we shall show two elementary lemmas about weight functions without proof (see [6] or [12], p. 226).

Lemma 1 *If $w \in A_q$ then w satisfies the following*:

$$\frac{w(B(x_0,r))}{w(B(x_0,s))} \le C\left(\frac{|B(x_0,r)|}{|B(x_0,s)|}\right)^q \quad \textit{for all} \quad r > s \quad \textit{and} \quad x_0 \in R^n.$$

where C is a positive constant independent of r, s and x_0. Especially

$$w(B(x_0, 2^j r)) \le C\ 2^{nqj}\ w(B(x_0,r)).$$

Lemma 2 *Let f be a nonnegative locally integrable function. If $w \in A_q$ then*

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} f(x)dx \le \left(\frac{C}{w(B(x_0,r))}\int_{B(x_0,r)} f(x)^q w(x)dx\right)^{1/q}.$$

4.2. Atom

Next we shall define atom on H_w^p and show the atomic decomposition of H_w^p.

Definition 10 Let $1 \le q \le \infty$. A function $a(x)$ is a (H_w^p, q)-atom centered at x_0 if there exists a ball $B(x_0,r)$ such that the following conditions are satisfied

$$\operatorname{supp}(a) \subset B(x_0,r), \tag{1}$$

$$\|a\|_{L_w^q} \le w(B(x_0,r))^{1/q-1/p}, \tag{2}$$

$$\int a(x)dx = 0. \tag{3}$$

The following Lemma 3 is trivial.

Lemma 3 *If a function $a(x)$ is a (H_w^p, ∞)-atom supported in $B(x_0,r)$, then $\|a\|_{H^{p_1}} \le C_{n,p_1}|B(x_0,r)|^{1/p_1} w(B(x_0,r))^{-1/p}$ where $\frac{n}{n+1} < p_1 \le 1$ and C_{n,p_1} is a constant depending only on n and p_1.*

Lemma 4 ([5], [11], p. 111) *Let $1 \le q < \frac{n+1}{n}$, $\frac{nq}{n+1} < p \le 1$ and $p < q$. If $w \in A_q$ and a function $a(x)$ is a (H_w^p, q)-atom, then $\|a\|_{H_w^p} \le C_{n,p,q,w}$ where $C_{n,p,q,w}$ is a constant depending only on n, p, q and w.*

Proof. We assume $\mathrm{supp}(a) \subset B(x_0, r)$. By using L^q_w-boundedness of the Hardy-Littlewood maximal function and Kolmogorov's inequality (see [12], p. 104), we obtain

$$\int_{B(x_0,2r)} a^{++}(x)^p w(x)dx$$
$$\leq C_{n,p,q,w} w(B(x_0, 2r))^{1-q/p} \|a\|^p_{L^q_w} \leq C_{n,p,q,w} \quad \text{if} \quad w \in A_q.$$

If $x \notin B(x_0, 2r)$ we have

$$a^{++}(x) \leq C \frac{r^{n+1} w(B(x_0, r))^{-1/p}}{|x - x_0|^{n+1}}.$$

By Lemma 1, we obtain

$$\int_{|x-x_0|\geq 2r} a^{++}(x)^p w(x)dx$$
$$= \sum_{j=1}^{\infty} \int_{2^j r \leq |x-x_0| < 2^{j+1} r} a^{++}(x)^p w(x)dx$$
$$\leq C_n w(B(x_0, 2r))^{-1} \sum_{j=1}^{\infty} 2^{-(n+1)pj} w(B(x_0, 2^{j+1} r))$$
$$\leq C_{n,w} \sum_{j=1}^{\infty} 2^{(-(n+1)p+nq)j} \leq C_{n,p,q,w},$$

where $p > nq/(n + 1)$. □

Proposition (The atomic decomposition of H^p_w, [5], [11]) *Let $1 \leq q < \frac{n+1}{n}$ and $\frac{nq}{n+1} < p \leq 1$. If $w \in A_q$ and $f \in H^p_w(R^n)$ then f can be written as $f = \sum_{j=1}^{\infty} \lambda_j a_j$ where a_j is (H^p_w, ∞)-atom and $\sum_{j=1}^{\infty} |\lambda_j|^p \sim \|f\|^p_{H^p_w}$.*

4.3. Molecule

We shall define atom and molecule on $h^p_w(R^n)$ and prove some properties.

Definition 11 Let $1 \leq q \leq \infty$. A function $a(x)$ is a (h^p_w, q)-atom centered at x_0 if there exists a ball $B(x_0, r)$ of radius $r \geq 1$ such that the conditions (1) and (2) are satisfied.

The following Lemma 5 is essentially proved in [4] when $w \equiv 1$.

Lemma 5 *Let $\frac{n}{n+1} < p \leq 1 \leq q$ and $p < q$. If $w \in A_q$ and a function $a(x)$ is a (h^p_w, q)-atom, then $\|a\|_{h^p_w} \leq C_{n,p,q,w}$.*

Proof. We assume $\mathrm{supp}(a) \subset B(x_0, r)$, then $a^+(x) = 0$ if $x \notin B(x_0, 2r)$. So we can prove the lemma by the same argument with the proof of Lemma 4. □

Lemma 6 *Let $1 \leq q < \frac{n+1}{n}$, $\frac{nq}{n+1} < p \leq 1$, $p < q$ and $w \in A_q$. Let $a(x)$ be a function such that there exists a ball $B(x_0, r)$, $0 < r < 2$, which satisfies the conditions* (1), (2) *and*

$$\left| \int a(x)dx \right| \leq r^{n(q-1)/p} \left(\frac{|B(x_0, r)|}{w(B(x_0, r))} \right)^{1/p}. \tag{3'}$$

Then $\|a\|_{h^p_w} \leq C_{n,p,q,w}$.

Proof. We write

$$a(x) = (a(x) - a_B)\chi_B(x) + a_B\chi_B(x) = a_1(x) + a_2(x),$$

where $B = B(x_0, r)$ and $a_B = \frac{1}{|B|}\int_B a(y)dy$.

By using Lemma 2, we have

$$\begin{aligned} &\int |a_1(x)|^q w(x)dx \\ &\leq C_q \Bigg(\int_{B(x_0,r)} |a(x)|^q w(x)dx \\ &\qquad + \Big(\frac{1}{|B(x_0, r)|} \int_{B(x_0,r)} |a(x)|dx \Big)^q w(B(x_0, r)) \Bigg) \\ &\leq C_{n,q,w} \int_{B(x_0,r)} |a(x)|^q w(x)dx \\ &\leq C_{n,q,w}\ w(B(x_0, r))^{1-q/p}. \end{aligned}$$

So a_1 is a constant multiple of (H^p_w, q)-atom, and we have $\|a_1\|_{H^p_w} \leq C_{n,p,q,w}$ by Lemma 4.

$\mathrm{supp}(a_2) \subset B(x_0, 2)$ and

$$\|a_2\|_{L^q_w} \leq |a_B| w(B)^{1/q} \leq C_n \frac{|B(x_0, r)|^{q(1/p-1/q)}}{w(B(x_0, r))^{1/p-1/q}}.$$

By Lemma 1,

$$\left(\frac{|B(x_0,r)|^q}{w(B(x_0,r))}\right)^{1/p-1/q} \le C_{n,p,q,w}\left(\frac{|B(x_0,2)|^q}{w(B(x_0,2))}\right)^{1/p-1/q}$$
$$\le C_{n,p,q,w}\ w(B(x_0,2))^{1/q-1/p}.$$

Therefore a_2 is a constant multiple of (h^p_w,q)-atom. By Lemma 5 we have $\|a_2\|_{h^p_w} \le C_{n,p,q,w}$. □

Definition 12 Let $\delta > 0$ and $w \in A_q$. A function $M(x)$ is a large (h^p_w,q,δ)-molecule centered at x_0 if there exists a ball $B(x_0,r), r \ge 1$, such that the conditions (M_1) and (M_2) are satisfied:

$$(\mathrm{M}_1)\quad \left(\int_{|x-x_0|<2r}|M(x)|^q w(x)dx\right)^{1/q} \le w(B(x_0,r))^{(1/q-1/p)},$$

$$(\mathrm{M}_2)\quad |M(x)| \le \frac{r^{n+\delta}w(B(x_0,r))^{-1/p}}{|x-x_0|^{n+\delta}} \quad \text{where } |x-x_0| \ge 2r.$$

A function $M(x)$ is a small (h^p_w,q,δ)-molecule centered at x_0 if there exists a ball $B(x_0,r), 0<r<1$, such that the conditions (M_1), (M_2) are satisfied and the following condition (M_3) is satisfied:

$$(\mathrm{M}_3)\quad \left|\int M(x)dx\right| \le r^{n(q-1)/p}\left(\frac{|B(x_0,r)|}{w(B(x_0,r))}\right)^{1/p}.$$

Remark For the definition of H^p-molecule, see [9], p. 83.

Lemma 7 *Let $1 \le q < \frac{n+\delta}{n}$, $\frac{nq}{n+\delta} < p \le 1$ and $p<q$. If $w \in A_q$ and a function $M(x)$ is a large or small (h^p_w,q,δ)-molecule centered at $B(x_0,r)$, then $\|M\|_{h^p_w} \le C_{n,p,q,\delta,w}$.*

Proof. Let $E_0 = \{x; |x-x_0| < 2r\}$ and $E_j = \{x; 2^j r \le |x-x_0| < 2^{j+1}r\}$, $j=1,2,3,\ldots$, and let $\chi_j(x) = \chi_{E_j}(x)$, $\tilde{\chi}_j(x) = \frac{1}{|E_j|}\chi_{E_j}(x)$, $m_j = \frac{1}{|E_j|}\int_{E_j} M(y)dy$, $\tilde{m}_j = \int_{E_j} M(y)dy$ and $M_j(x) = (M(x)-m_j)\chi_j(x)$.

We write

$$M(x) = \sum_{j=0}^{\infty} M_j(x) + \sum_{j=0}^{\infty} m_j\chi_j(x) = \sum_{j=0}^{\infty} M_j(x) + \sum_{j=0}^{\infty} \tilde{m}_j\tilde{\chi}_j(x).$$

Let $N_j = \sum_{k=j}^{\infty} \tilde{m}_k$ and we write

$$\begin{aligned} M(x) &= \sum_{j=0}^{\infty} M_j(x) + \sum_{j=1}^{\infty} N_j(\tilde{\chi}_j(x) - \tilde{\chi}_{j-1}(x)) + N_0\tilde{\chi}_0(x) \\ &= I + II + III. \end{aligned}$$

We shall show $\|I\|_{H^p_w} \le C_{n,p,q,\delta,w}$, $\|II\|_{H^p_w} \le C_{n,p,q,\delta,w}$ and $\|III\|_{h^p_w} \le C_{n,p,q,\delta,w}$.

First we estimate I.

It is clear that $\mathrm{supp}(M_j) \subset B(x_0, 2^{j+1}r)$, $\int M_j(x)dx = 0$.

By using the condition (M_1), the estimate of M_0 is the same as was given in the proof of Lemma 6 (the estimate of a_1) and we have

$$\int |M_0(x)|^q w(x)dx \le C_{n,q,w}\ w(B(x_0, 2r))^{1-q/p}.$$

Therefore we have $\|M_0\|_{H^p_w} \le C_{n,p,q,w}$ by Lemma 4.

Using the condition (M_2) and Lemma 1, we have for $j \ge 1$,

$$\begin{aligned} |M_j(x)| &\le 2^{(-n-\delta)j} w(B(x_0, r))^{-1/p} \\ &\le 2^{(-n-\delta)j} \left(\frac{w(B(x_0, 2^{j+1}r))}{w(B(x_0, r))} \right)^{1/p} w(B(x_0, 2^{j+1}r))^{-1/p} \\ &\le C_{n,w}\ 2^{(-n-\delta+nq/p)j}\ w(B(x_0, 2^{j+1}r))^{-1/p}. \end{aligned}$$

By Lemma 4, we have $\|M_j\|_{H^p_w} \le C_{n,p,q,w}\ 2^{(-n-\delta+nq/p)j}$.

Since $p > nq/(n+\delta)$, we obtain $\sum_{j=0}^{\infty} \|M_j\|^p_{H^p_w} \le C_{n,p,q,\delta,w}$ and $\|I\|_{H^p_w} \le C_{n,p,q,\delta,w}$.

Next we estimate II.

Let $A_j(x) = N_j(\tilde{\chi}_j(x) - \tilde{\chi}_{j-1}(x))$.

It is clear that $\mathrm{supp}(A_j) \subset B(x_0, 2^{j+1}r)$, $\int A_j(x)dx = 0$. By the same estimate with I we have

$$\begin{aligned} \|A_j\|_{L^\infty} &\le C_n (2^j r)^{-n} \int_{2^{j-1}r \le |x-x_0| < 2^{j+1}r} |M(x)|dx \\ &\le C_{n,w}\ 2^{(-n-\delta+nq/p)j}\ w(B(x_0, 2^{j+1}r))^{-1/p}. \end{aligned}$$

So we obtain $\sum_{j=1}^{\infty} \|A_j\|^p_{H^p_w} \le C_{n,p,q,\delta,w}$ and $\|II\|_{H^p} \le C_{n,p,q,\delta,w}$.

Finally we estimate III.

It is clear that $\operatorname{supp}(N_0\tilde{\chi}_0) \subset B(x_0, 2r)$.

By the same estimate with I (see also the proof of Lemma 4), we have

$$\begin{aligned}
\|N_0\tilde{\chi}_0\|_{L^\infty} &\leq \frac{1}{|B(x_0,2r)|}\int |M(x)|dx \\
&\leq \frac{1}{|B(x_0,2r)|}\left(\int_{|x-x_0|<2r}|M(x)|dx + \int_{|x-x_0|\geq 2r}|M(x)|dx\right) \\
&\leq C_{n,p,q,\delta,w}\ w(B(x_0,2r))^{-1/p}. \qquad (4)
\end{aligned}$$

If $r \geq 1$, by (4) and Lemma 5 we have $\|N_0\tilde{\chi}_0\|_{h^p_w} \leq C_{n,p,q,\delta,w}$.

If $r < 1$, using the condition (M_3), we have

$$\begin{aligned}
\left|\int N_0\tilde{\chi}_0(x)dx\right| = \left|\int M(x)dx\right| &\leq r^{n(q-1)/p}\left(\frac{|B(x_0,r)|}{w(B(x_0,r))}\right)^{1/p} \\
&\leq C_{n,w}\ (2r)^{n(q-1)/p}\left(\frac{|B(x_0,2r)|}{w(B(x_0,2r))}\right)^{1/p}. \qquad (5)
\end{aligned}$$

By (4), (5) and Lemma 6 we have $\|N_0\tilde{\chi}_0\|_{h^p_w} \leq C_{n,p,q,\delta,w}$.

So we obtain $\|III\|_{h^p} \leq C_{n,p,q,\delta,w}$. □

5. Proof of Theorem 1

Applying the interpolation theorem between L^2_w and H^p_w or h^p_w, we may assume $p < 1$, so we may assume $p < q$. By the atomic decomposition of H^p_w, it suffices to show that there exists $C_{n,p,q,\varepsilon,\delta,w,T} > 0$ such that $\|Ta\|_{h^p_w} \leq C_{n,p,q,\varepsilon,\delta,w,T}$, for every (H^p_w,∞)-atom a, where $C_{n,p,q,\varepsilon,\delta,w,T}$ is a positive constant depending only on n, p, q, ε, δ, w and $\|T\|_{\mathrm{Lip}_\varepsilon}$.

We assume (H^p_w,∞)-atom a is supported in $B(x_0,r)$. We shall show that if $r \geq 1$ then $Ta(x)$ is a constant multiple of a large (h^p_w,q,δ)-molecule, and $r < 1$ then $Ta(x)$ is a constant multiple of a small (h^p_w,q,δ)-molecule.

We have to check that if $r \geq 1$ then Ta satisfies (M_1) and (M_2), and if $r < 1$ then Ta satisfies three conditions of Definition 12.

Since T is bounded on L^{2q}_w ([7], p. 52), we have

$$\begin{aligned}
&\left(\int_{|x-x_0|\leq 2r}|Ta(x)|^q w(x)dx\right)^{1/q} \\
&\leq \left(\int_{|x-x_0|\leq 2r}|Ta(x)|^{2q} w(x)dx\right)^{1/2q} w(B(x_0,2r))^{1/2q}
\end{aligned}$$

$$\begin{aligned}&\le C_{n,q,w}\|a\|_{L^{2q}_w}\ w(B(x_0,2r))^{1/2q}\\ &\le C_{n,q,w}\ w(B(x_0,r))^{1/q-1/p}.\end{aligned} \tag{6}$$

If $|x-x_0|\ge 2r$, we have

$$\begin{aligned}|Ta(x)| &= \left|\int (K(x,y)-K(x-x_0))a(y)dy\right|\\ &\le C_n\frac{r^{n+\delta}w(B(x_0,r))^{-1/p}}{|x-x_0|^{n+\delta}}.\end{aligned} \tag{7}$$

If $r\ge 1$, by (6), (7) and Lemma 7, we have $\|Ta\|_{h^p_w}\le C_{n,p,q,\delta,w}$.

If $r<1$, by the duality of $H^{n/(n+\varepsilon)}$ and Lip_ε and Lemma 3, we have

$$\begin{aligned}\left|\int Ta(x)dx\right| &= |(Ta,1)| = |(a,T^*1)| \le C_n\|a\|_{H^{n/(n+\varepsilon)}}\|T^*1\|_{\mathrm{Lip}_\varepsilon}\\ &\le C_n\|T^*1\|_{\mathrm{Lip}_\varepsilon}|B(x_0,r)|^{(n+\varepsilon)/n}w(B(x_0,r))^{-1/p}\\ &\le C_n\|T^*1\|_{\mathrm{Lip}_\varepsilon}\left(\frac{|B(x_0,r)|}{w(B(x_0,r))}\right)^{1/p}\cdot r^{n(q-1)/p},\end{aligned} \tag{8}$$

because $p\ge nq/(n+\varepsilon)$.

By (6), (7), (8) and Lemma 7, we obtain $\|Ta\|_{h^p_w}\le C_{n,p,q,\varepsilon,\delta,w,T}$.

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School of High Technology and
Human Welfare
Tokai University
317 Nishino Numazu
Shizuoka 410-0395, Japan
E-mail: komori@wing.ncc.u-tokai.ac.jp