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$L^p(\mathbb{R}^n)$ boundedness for a class of g-functions and applications

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Abstract. A class of g-functions related to the commutators of convolution operators are considered, a sufficient condition implying the $L^p(\mathbb{R}^n)$ boundedness for these g-functions is obtained. As applications, some new results about the $L^p(\mathbb{R}^n)$ boundedness for the commutators of the Marcinkiewicz integrals and the maximal operators corresponding to the commutators of homogeneous singular integral operators are established.

Key words: commutator, convolution operator, g-function, singular integral, maximal operator.

1. Introduction

As well-known, commutators generated by some classical operators and BMO functions are of great interest in harmonic analysis and are useful in the study of some related topics (see [2] and [13]). Thus, it is meaningful to study the $L^p(\mathbb{R}^n)$ boundedness for these commutators. In their celebrated work [3], Coifman and Meyer observe that if T is standard Calderón-Zygmund singular integral operator, then for $b \in \mathrm{BMO}(\mathbb{R}^n)$, the $L^p(\mathbb{R}^n)$ boundedness for the commutator defined by

$$T_b f(x) = T\big(b(x) - b)f\big)(x)$$

can be obtained from the weighted $L^p(\mathbb{R}^n)$ estimate with A_p weights for the operator T, where A_p denotes the weight function class of Muckenhoupt (see [14, Chapter V] for definition and properties of A_p). Alvarez, Bagby, Kurtz and Pérez [1] developed the idea of Coifman and Meyer, and established a generalized boundedness result for the commutators of linear operators. Let E be a Banach space with norm $\|\cdot\|_E$, denote by $\mathcal{M}(E)$ the set of E-valued measurable functions on $\mathbb{R}^n$. For a weight function u on $\mathbb{R}^n$ and $1 \le p < \infty$, define the Banach space $L^p(E, u(x)dx)$ by

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$$L^p(E,u(x)dx) = \Big\{f\colon f\in\mathcal{M}(E),\ \|f\|_{L^p(E,u(x)dx)}=\Big(\int_{\mathbb{R}^n}\|f(x)\|_E^p u(x)dx\Big)^{1/p}<\infty\Big\}.$$

The result of Alvarez, Bagby, Kurtz and Pérez (see [1, Theorem 2.13]) states that if $1<p,\ q<\infty$ and the linear operator T is bounded from $L^p(\mathbb{R}^n,\ w(x)dx)$ to $L^p(E,\ w(x)dx)$ with bound independent of the weight w for any $w\in A_q$, then for $u\in A_q$, $b\in \mathrm{BMO}(\mathbb{R}^n)$ and positive integer k, the k-th order commutator of T defined by

$$T_{b,k}f(x)=T\big((b(x)-b)^k f\big)(x)$$

is bounded from $L^p(\mathbb{R}^n,\ u(x)dx)$ to $L^p(E,u(x)dx)$ with bound $C(n,k,p)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$. In [11], we considered the $L^p(\mathbb{R}^n)$ boundedness for the commutators of convolution operators and proved the following result.

Theorem HSW *Let $K(x)$ be a function on $\mathbb{R}^n\backslash\{0\}$ and $K_j(x)=K(x)\chi_{\{2^j<|x|\le 2^{j+1}\}}(x)$, where $\chi_{\{2^j<|x|\le 2^{j+1}\}}$ is the characteristic function of the set $\{2^j<|x|\le 2^{j+1}\}$. Suppose that there exist some constants $C>0$, $\alpha>k+1$ such that for each $j\in\mathbb{Z}$,*

$$\|K_j\|_1\le C,\quad \big|\widehat{K_j}(\xi)\big|\le C\min\big\{|2^j\xi|,\ \log^{-\alpha}(2+|2^j\xi|)\big\},$$
$$\|\nabla\widehat{K_j}\|_\infty\le C2^j,$$

where $\widehat{K_j}$ denotes the Fourier transform of K_j. Then for positive integer k, $b\in\mathrm{BMO}(\mathbb{R}^n)$ and $2\alpha/(2\alpha-(k+1))<p<2\alpha/(k+1)$, the commutator

$$T_{b,k}f(x)=\int_{\mathbb{R}^n}\big(b(x)-b(y)\big)^k K(x-y)f(y)dy$$

is bounded on $L^p(\mathbb{R}^n)$ with bound $C(n,k,p,\alpha)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.

This paper is a continuation of our previous works [10] and [11], we will consider the $L^p(\mathbb{R}^n)$ boundedness for a class of g-functions related to the commutators of convolution operators. Let $\{K_j\}_{j\in\mathbb{Z}}$ be a sequence of integrable functions on $\mathbb{R}^n$. Define the operator U_j by

$$U_jf(x)=\int_{\mathbb{R}^n}K_j(x-y)f(y)dy.$$

For $b\in\mathrm{BMO}(\mathbb{R}^n)$ and nonnegative integer k, the k-th order commutator of U_j is defined by

$$U_{j;\,b,\,k}f(x) = U_j\big((b(x) - b(y))^k f\big)(x),$$

(note that $U_{j;\,b,\,0} = U_j$). The operators we consider here are of the form

$$g_{b,\,k}(f)(x) = \Big(\sum_{j\in\mathbb{Z}} \big|U_{j;\,b,\,k}f(x)\big|^2\Big)^{1/2}. \tag{1}$$

We will see in Section 4 that the operator $g_{b,\,k}$ plays an important role in the study of some maximal operators associated with the commutators of convolution operators. Our main result can be stated as follows.

Theorem 1 *Let k be a nonnegative integer, $\{K_j\}_{j\in\mathbb{Z}}$ be a sequence of integrable functions on $\mathbb{R}^n$. Suppose that there are some constants $C > 0$, $0 < A \le 1/2$ and $\alpha > k + 1/2$ such that for each $j \in \mathbb{Z}$*

$$\|K_j\|_1 \le C, \quad \big|\widehat{K_j}(\xi)\big| \le C \min\big\{A|2^j\xi|,\, \log^{-\alpha}(2 + |2^j\xi|)\big\},$$
$$\|\nabla\widehat{K_j}\|_\infty \le C2^j. \tag{2}$$

Then for $b \in \mathrm{BMO}(\mathbb{R}^n)$ and any $0 < \nu < 1$ such that $\alpha\nu > k + 1/2$, the operator $g_{b,\,k}$ defined by (1) is bounded on $L^2(\mathbb{R}^n)$ with bound $C(n, k, \alpha, \nu)\log^{-\alpha\nu+k+1/2}\big(\frac{1}{A}\big)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.

Theorem 2 *Let k be a nonnegative integer, $\{K_j\}_{j\in\mathbb{Z}}$ be a sequence of integrable functions on $\mathbb{R}^n$ such that $\operatorname{supp} K_j \subset \{x\colon 2^j < |x| \le 2^{j+1}\}$. Suppose that the maximal operator*

$$\overline{M}f(x) = \sup_{j\in\mathbb{Z}} \big||K_j| * f(x)\big|$$

is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, and that there exist some constants $C > 0$, $\alpha > k + 1/2$ such that for each $j \in \mathbb{Z}$,

$$\|K_j\|_1 \le C, \quad \big|\widehat{K_j}(\xi)\big| \le C \min\big\{|2^j\xi|,\, \log^{-\alpha}(2 + |2^j\xi|)\big\},$$
$$\|\nabla\widehat{K_j}\|_\infty \le C2^j. \tag{3}$$

Then for $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $4\alpha/(4\alpha - (2k+1)) < p < 4\alpha/(2k+1)$, the operator $g_{b,\,k}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C(n, k, p, \alpha)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.

This paper is arranged as follows. In Section 2 and Section 3, we will give the proof of Theorem 1 and Theorem 2 respectively. We will see that

Theorem 1 for $A < 1/2$ is very useful in the proof of Theorem 2. Section 4 is devoted to some applications of Theorem 1 and Theorem 2. We will consider the $L^p(\mathbb{R}^n)$ boundedness for the maximal operator associated with the commutator of homogeneous singular integral operator and the commutator of the Marcinkiewicz integral.

Throughout this paper, C denotes the constants that are independent of the main parameters involved but whose values may differ from line to line. For any locally integrable function f, we will denote by Mf the standard Hardy-Littlewood maximal function of f, and $f^{\#}$ the Fefferman-Stein sharp function of f (see [14, Chapter IV]). For a power exponent p with $1 \le p < \infty$, we denote the dual exponent of p by p', that is, $p' = p/(p-1)$.

2. Proof of Theorem 1

We begin with some preliminary lemmas.

Lemma 1 (see [10]) *Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that* $\operatorname{supp}\phi \subset \{\xi\colon 1/4 \le |\xi| \le 4\}$ *and*

$$\sum_{l\in\mathbb{Z}} \phi^2(2^{-l}\xi) = 1, \quad |\xi| \neq 0.$$

Denote by S_l the multiplier operator $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\widehat{f}(\xi)$. For $b \in \mathrm{BMO}(\mathbb{R}^n)$ and positive integer k, denote by $S_{l;\,b,\,k}$ the k-th order commutator of S_l as defined in (1). *Then the inequalities*

$$\Big\|\Big(\sum_{l\in\mathbb{Z}} \big|S_{l;\,b,\,k}f\big|^2\Big)^{1/2}\Big\|_p \le C(n,k,p)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p$$

and

$$\Big\|\sum_{l\in\mathbb{Z}} S_{l;\,b,\,k}f_l\Big\|_p \le C(n,k,p)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\Big\|\Big(\sum_{l\in\mathbb{Z}}|f_l|^2\Big)^{1/2}\Big\|_p$$

holds for all $1 < p < \infty$.

Lemma 2 (see [10]) *Let $m_\delta \in C^1(\mathbb{R}^n)$ $(0<\delta<\infty)$ be a multiplier such that* $\operatorname{supp} m_\delta \subset \{\xi\colon |\xi| \le \delta\}$ *and for some constants C, $0 < A \le 1/2$ and $\alpha > 1$,*

$$\|m_\delta\|_\infty \le C\min\big\{A\delta,\, \log^{-\alpha}(2+\delta)\big\}, \quad \big\|\nabla m_\delta\big\|_\infty \le C.$$

Let T_δ be the multiplier operator defined by $\widehat{T_\delta f}(\xi) = m_\delta(\xi)\widehat{f}(\xi)$. For positive intger k and $b \in \mathrm{BMO}(\mathbb{R}^n)$, denote by $T_{\delta;\,b,\,k}$ the k-th order commutator of T_δ. Then for any $0 < \varepsilon < 1$, there exists a positive constant $C = C(n, k, \varepsilon)$ such that

$$\|T_{\delta;\,b,\,k}f\|_2 \le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}(A\delta)^{1-\varepsilon}\log^k\Big(\frac{1}{A}\Big)\|f\|_2, \quad \textit{if } \delta < 10/\sqrt{A};$$

$$\|T_{\delta;\,b,\,k}f\|_2 \le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\log^{-\alpha(1-\varepsilon)+k}(2+\delta)\|f\|_2, \quad \textit{if } \delta > 1/\sqrt{A}.$$

Proof of Theorem 1. We shall carry out the argument by induction on the order k. If $k = 0$, the operator $g_{b,\,k}$ is exactly the operator g defined by

$$g(f)(x) = \Big(\sum_{j\in\mathbb{Z}}|U_jf(x)|^2\Big)^{1/2}.$$

We claim that g is bounded on $L^2(\mathbb{R}^n)$ with bound $C\log^{-\alpha+1/2}\big(\frac{1}{A}\big)$. In fact, by the Plancherel theorem, it suffices to show that for each $\xi \in \mathbb{R}^n\backslash\{0\}$,

$$\sum_{j\in\mathbb{Z}}|\widehat{K_j}(\xi)|^2 \le C\log^{-2\alpha+1}\Big(\frac{1}{A}\Big).$$

Let j_0 be the integer such that $2^{j_0} \le |\xi| < 2^{j_0+1}$. It follows that

$$\begin{aligned}\sum_{j\in\mathbb{Z}}|\widehat{K_j}(\xi)|^2 &= \sum_{j\le -j_0+\left[\log\left(\frac{1}{\sqrt{A}}\right)\right]}|\widehat{K_j}(\xi)|^2 + \sum_{j> -j_0+\left[\log\left(\frac{1}{\sqrt{A}}\right)\right]}|\widehat{K_j}(\xi)|^2\\ &\le CA^2\sum_{j\le\left[\log\left(\frac{1}{\sqrt{A}}\right)\right]}2^{2j} + \sum_{j>\left[\log\left(\frac{1}{\sqrt{A}}\right)\right]}j^{-2\alpha}\\ &\le CA + C\log^{-2\alpha+1}\Big(\frac{1}{A}\Big) \le C\log^{-2\alpha+1}\Big(\frac{1}{A}\Big),\end{aligned}$$

where $[a]$ denotes the integer part of the real number a. Now let k be a positive integer, we assume that the estimate

$$\|g_{b,\,m}(f)\|_2 \le C\log^{-\alpha\nu+m+1/2}\Big(\frac{1}{A}\Big)\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2$$

holds for all $0 \le m \le k-1$. Let S_l be the multiplier operator defined in Lemma 1. Define the operator S_l^2 by $S_l^2f(x) = S_l(S_lf)(x)$. Write

$$\|g_{b,\,k}f\|_2 = \Big\|\Big(\sum_{j\in\mathbb{Z}}\Big|\sum_{l\in\mathbb{Z}}S^2_{l-j}U_{j;\,b,\,k}f\Big|^2\Big)^{1/2}\Big\|_2, \quad f \in C_0^\infty(\mathbb{R}^n).$$

Again by the Plancherel theorem, we see that for $h \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\Big\|\sum_{l\in\mathbb{Z}} S_l^2 h\Big\|_2^2 &= \sum_{l\in\mathbb{Z}}\sum_{d\in\mathbb{Z}}\int_{\mathbb{R}^n}\widehat{S_l^2 h}(\xi)\overline{\widehat{S_d^2 h}(\xi)}d\xi \\ &= \sum_{l\in\mathbb{Z}}\sum_{d\in\mathbb{Z}}\int_{\mathbb{R}^n}|h(\xi)|^2\big(\widehat{\phi_l}(\xi)\big)^2\overline{\big(\widehat{\phi_d}(\xi)\big)^2}d\xi \\ &= \sum_{l\in\mathbb{Z}}\sum_{d\in\mathbb{Z}:|d-l|\le 4}\int_{\mathbb{R}^n}S_l^2h(x)\overline{S_d^2h(x)}dx \\ &\le C\sum_{l\in\mathbb{Z}}\sum_{d\in\mathbb{Z}:|d-l|\le 4}\int_{\mathbb{R}^n}\big(|S_l^2h(x)|^2+|S_d^2h(x)|^2\big)dx \\ &\le C\sum_{l\in\mathbb{Z}}\big\|S_l^2h\big\|_2^2.\end{aligned}$$

Therefore,

$$\|g_{b,k}(f)\|_2^2 \le C\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\|S_{l-j}^2U_{j;b,k}f\|_2^2.$$

With the aid of the formula

$$(b(x)-b(y))^k = \sum_{m=0}^{k} C_k^m(b(x)-b(z))^{k-m}(b(z)-b(y))^m, \qquad x,\ y,\ z\in\mathbb{R}^n,$$

the Fubini theorem and a straightforward computation gives that

$$S_{l-j}^2U_{j;b,k}f(x) = \big(S_{l-j}^2U_j\big)_{b,k}f(x) - \sum_{m=0}^{k-1}C_k^mS_{l-j;b,k-m}^2\big(U_{j;b,m}f\big)(x).$$

This in turn implies

$$\begin{aligned}\|g_{b,k}f\|_2^2 &\le C\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\big\|\big(S_{l-j}^2U_j\big)_{b,k}f\big\|_2^2 \\ &\quad + C\sum_{m=0}^{k-1}\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\big\|S_{l-j;b,k-m}^2\big(U_{j;b,m}f\big)\big\|_2^2.\end{aligned}$$

Lemma 1 together with our induction hypothesis says that for each m with

$0 \leq m \leq k-1$,

$$\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\|S^2_{l-j;\,b,\,k-m}(U_{j;\,b,\,m}f)\|_2^2$$
$$\leq C\|b\|^{2(k-m)}_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{j\in\mathbb{Z}}\|U_{j;\,b,\,m}f\|_2^2$$
$$\leq C\log^{-2\alpha\nu+2m+1}\Big(\frac{1}{A}\Big)\|b\|^{2k}_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2^2.$$

To estimate the $L^2(\mathbb{R}^n)$ bound for the operator $(S^2_{l-j}U_j)_{b,k}$, set $m_j(\xi) = \widehat{K_j}(\xi)$, $m_j^l(\xi) = m_j(\xi)\phi(2^{j-l}\xi)$, and define the operator U_j^l by

$$\widehat{U_j^l f}(\xi) = m_j^l(\xi)\widehat{f}(\xi).$$

Obviously, $\operatorname{supp} m_j^l(2^{-j}\xi) \subset \{|\xi| \leq 2^{l+2}\}$ and

$$\|m_j^l(2^{-j}\cdot)\|_\infty \leq C\min\{A2^l,\ \log^{-\alpha}(2+2^l)\},\quad \|\nabla m_j^l(2^{-j}\cdot)\|_\infty \leq C.$$

Let $\widetilde{U_j^l}$ be the operator defined by

$$\widehat{\widetilde{U_j^l} f}(\xi) = m_j^l(2^{-j}\xi)\widehat{f}(\xi).$$

The Fourier transform estimate for m_j^l via Lemma 2 states that for any $0<\nu<1$ and positive integer m,

$$\|\widetilde{U}^l_{j;\,b,\,m}f\|_2 \leq C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\log^m\Big(\frac{1}{A}\Big)(A2^l)^\nu\|f\|_2,$$
$$l \leq \Big[\log\Big(\frac{1}{\sqrt{A}}\Big)\Big]+1.$$

Note that if $b\in\mathrm{BMO}(\mathbb{R}^n)$, then for any $t>0$, $b_t(x) = b(tx)\in\mathrm{BMO}(\mathbb{R}^n)$ and $\|b_t\|_{\mathrm{BMO}(\mathbb{R}^n)} = \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. By dilation-invariance,

$$\|U^l_{j;\,b,\,m}f\|_2 \leq C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\log^m\Big(\frac{1}{A}\Big)(A2^l)^\nu\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2,$$
$$l \leq \Big[\log\Big(\frac{1}{\sqrt{A}}\Big)\Big]+1. \tag{4}$$

Since $|m_j^l(\xi)| \leq C\min\{A2^l,\,1\} \leq C(A2^l)^\nu$, the Plancherel theorem tells us that

$$\|U_j^l f\|_2 \leq C(A2^l)^\nu\|f\|_2. \tag{5}$$

Observe that for $f,\, h \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} h(x)\big(S_{l-j}^2 U_j\big)_{b,\,k} f(x)dx$$
$$= \sum_{m=0}^{k} C_k^m \int_{\mathbb{R}^n} h(x) U^l_{j;\,b,\,m}\big(S_{l-j;\,b,\,k-m} f\big)(x)dx.$$

It follows from the estimate (4) and (5) that

$$\sum_{j\in\mathbb{Z}} \big\|\big(S_{l-j}^2 U_j\big)_{b,\,k} f\big\|_2^2$$
$$\le C \sum_{m=0}^{k} \sum_{j\in\mathbb{Z}} \big\|U^l_{j;\,b,\,m}\big(S_{l-j;\,b,\,k-m} f\big)\big\|_2^2$$
$$\le C(A2^l)^{2\nu} \sum_{m=0}^{k} \log^{2m}\Big(\frac{1}{A}\Big) \|b\|^{2m}_{\mathrm{BMO}(\mathbb{R}^n)} \sum_{j\in\mathbb{Z}} \|S_{l-j;\,b,\,k-m} f\|_2^2$$
$$\le C(A2^l)^{2\nu} \log^{2k}\Big(\frac{1}{A}\Big) \|b\|^{2k}_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_2^2, \quad l \le \Big[\log\Big(\frac{1}{A}\Big)\Big] + 1.$$

On the other hand, by Lemma 2 and the same argument as above, we can obtain

$$\sum_{j\in\mathbb{Z}} \big\|\big(S_{l-j}^2 U_j\big)_{b,\,k} f\big\|_2^2 \le C \log^{2(-\alpha\nu+k)}(2+2^l) \|b\|^{2k}_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_2^2,$$
$$l > \Big[\log\Big(\frac{1}{\sqrt{A}}\Big)\Big] + 1.$$

Recall that $\alpha\nu > k + 1/2$. Therefore,

$$\sum_{l\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \big\|\big(S_{l-j}^2 U_j\big)_{b,\,k} f\big\|_2^2$$
$$\le \sum_{l>\left[\log\left(\frac{1}{\sqrt{A}}\right)\right]+1} \sum_{j\in\mathbb{Z}} \big\|\big(S_{l-j}^2 U_j\big)_{b,\,k} f\big\|_2^2$$
$$+ \sum_{l\le\left[\log\left(\frac{1}{\sqrt{A}}\right)\right]+1} \sum_{j\in\mathbb{Z}} \big\|\big(S_{l-j}^2 U_j\big)_{b,\,k} f\big\|_2^2$$
$$\le C\|b\|^{2k}_{\mathrm{BMO}(\mathbb{R}^n)} A^{2\nu} \log^{2k}\Big(\frac{1}{A}\Big) \|f\|_2^2 \sum_{l\le\left[\log\left(\frac{1}{\sqrt{A}}\right)\right]+1} 2^{2\nu l}$$

$$+ C\|b\|^{2k}_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2^2 \sum_{l > \left[\log\left(\frac{1}{\sqrt{A}}\right)\right]+1} \log^{2(-\alpha\nu+k)}(2+2^l)$$

$$\leq C\|b\|^{2k}_{\mathrm{BMO}(\mathbb{R}^n)} \log^{2(-\alpha\nu+k)+1}\left(\frac{1}{A}\right)\|f\|_2^2.$$

This finishes the proof of Theorem 1. □

3. Proof of Theorem 2

As in [16] and [11], let $\psi \in C_0^\infty(\mathbb{R}^n)$ be a radial and nonnegative function such that $\int_{\mathbb{R}^n}\psi(x)dx = 1$, $\operatorname{supp}\psi \subset \{x\colon |x| \leq 1/4\}$. For $j \in \mathbb{Z}$ and positive integer l, set

$$\psi_j(x) = 2^{-jn}\psi(2^{-j}x), \quad K_j^l(x) = K_j * \psi_{j-l}(x).$$

Denote by V_j^l the convolution operator whose kernel is K_j^l, and $V_{j,b,k}^l$ the k-th order commutator of V_j^l. Define the operator $g_{l,b,k}$ by

$$g_{l;b,k}(f)(x) = \Big(\sum_{j\in\mathbb{Z}} |V_{j;b,k}^l f(x)|^2\Big)^{1/2}.$$

We have

Lemma 3 *For any nonnegative integer m with $0 \leq m \leq k$, $g_{l;b,m}$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}$. Furthermore, for $0 < \nu < 1$ such that $\alpha\nu > k + 1/2$,*

$$\|g_{l;b,m}(f) - g_{b,m}(f)\|_2 \leq C\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)} l^{-\alpha\nu+m+1/2}\|f\|_2.$$

Proof. By the Minkowski inequality, we have

$$\|g_{l;b,m}(f) - g_{b,m}(f)\|_2 \leq \Big\|\Big(\sum_{j\in\mathbb{Z}} \big|V_{j;b,m}^l f(x) - U_{j;b,m}f(x)\big|^2\Big)^{1/2}\Big\|_2.$$

The Fourier transform estimate of K_j now states that

$$\begin{aligned}
|\widehat{K_j^l}(\xi) - \widehat{K_j}(\xi)| &= |\widehat{K_j}(\xi)||\widehat{\psi_{j-l}}(\xi) - 1| \\
&\leq C\min\{|2^j\xi|, \log^{-\alpha}(2+|2^j\xi|)\}\min\{|2^{j-l}\xi|, 1\} \\
&\leq C\min\{2^{-l}|2^j\xi|, \log^{-\alpha}(2+|2^j\xi|)\},
\end{aligned}$$

and

$$\|\nabla\widehat{K_j^l}-\nabla\widehat{K_j}\|_\infty \le \|\nabla\widehat{K_j}\|_\infty\|\widehat{\psi}-1\|_\infty + \|\widehat{K_j}\|_\infty\|\nabla\widehat{\psi_{j-l}}\|_\infty \le C2^j.$$

This together with Theorem 1 says that for each $0 \le m \le k$, $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $0 < \nu < 1$ such that $\alpha\nu > k + 1/2$,

$$\|g_{l;\,b,\,m}(f) - g_{b,\,m}(f)\|_2 \le Cl^{-\alpha\nu+m+1/2}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2.$$

On the other hand, Theorem 1 tells us that the operator $g_{b,m}$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$. Therefore, for each $0 \le m \le k$, $g_{l;\,b,\,m}$ is also bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$. □

Lemma 4 *For any nonnegative integer k and $1 < p < \infty$, $g_{l;\,b,\,k}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $Cl^{k+1/2}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$, and C depends only on n, p, k.*

Proof. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. By the same argument as in the kernel estimate used in the proof of Theorem 1 in [11], we can verify that there exists a positive constant A which is independent of l, such that for each nonnegative integer m, $R > 0$ and $|y| \le R/4$,

$$\begin{aligned}\sum_{d\ge1}|B(0,\,2^dR)|&\sum_{j\in\mathbb{Z}}\big\|K_j^l(\cdot-y)-K_j^l(\cdot)\big\|_{L(\log L)^{2m},\,B(0,2^{d+1}R)\setminus B(0,2^dR)}\\&+\sum_{d\ge1}d^{2m}\sum_{j\in\mathbb{Z}}\int_{B(0,\,2^{d+1}R)\setminus B(0,2^dR)}|K_j^l(x-y)-K_j^l(x)|dx\\&\le Al^{2m+1},\end{aligned}\tag{6}$$

where

$$\begin{aligned}&\big\|K_j^l(\cdot-y)-K_j^l(\cdot)\big\|_{L(\log L)^{2m},\,B(0,2^{d+1}R)\setminus B(0,2^dR)}\\&=\inf\Big\{\lambda>0\colon \int_{B(0,2^{d+1}R)\setminus B(0,2^dR)}\Psi_m\Big(\frac{|K_j^l(x-y)-K_j^l(x)|}{\lambda}\Big)dx\le1\Big\},\end{aligned}$$

and $\Psi_m(t) = t\log^m(2+t)$ for $t > 0$ (see also [12, page 168]).

We first consider the $L^p(\mathbb{R}^n)$ bound of $g_{l;\,b,\,k}$ for $1 < p < 2$. By the Marcinkiewicz interpolation theorem, it is enough to show that for each $1 < p < 2$, $g_{l;\,b,\,k}$ is a bounded mapping from $L^p(\mathbb{R}^n)$ to weak $L^p(\mathbb{R}^n)$ with bound $Cl^{k+1/2}$, that is, for each $1 < p < 2$ and $\lambda > 0$,

$$|\{x \in \mathbb{R}^n : g_{l;\,b,\,k}(f)(x) > \lambda\}| \leq C\big(l^{k+1/2}\lambda^{-1}\big)^p \|f\|_p^p.$$

For $f \in L^p(\mathbb{R}^n)$ and fixed $\lambda > 0$, applying the Calderón-Zygmund decomposition to $|f|^p$ at level λ^p, we can decompose f as $f = f^{\mathrm{I}} + f^{\mathrm{II}} = f^{\mathrm{I}} + \sum_d f_d^{\mathrm{II}}$, where $\|f^{\mathrm{I}}\|_2^2 \leq C\lambda^{2-p}\|f\|_p^p$, each f_d^{II} is supported on some cube Q_d, the cubes Q_d have disjoint interiors, $\int f_d^{\mathrm{II}}(x)dx = 0$, $\|f_d^{\mathrm{II}}\|_p^p \leq C\lambda^p|Q_d|$ and $\sum_d |Q_d| \leq C\lambda^{-p}\|f\|_p^p$. Let x_d and r_d be the center and the side length of Q_d. Set $B_d = B(x_d, 4nr_d)$ and $E = \cup_d B_d$, then $|E| \leq C\lambda^{-p}\|f\|_p^p$. By the $L^2(\mathbb{R}^n)$ boundedness of $g_{l;\,b,\,k}$, it is easy to see that

$$|\{x \in \mathbb{R}^n : g_{l;\,b,\,k}(f^{\mathrm{I}})(x) > \lambda\}| \leq C\lambda^{-p}\|f\|_p^p.$$

Thus, it suffices to prove that

$$|\{x \in \mathbb{R}^n \backslash E : g_{l;\,b,\,k}(f^{\mathrm{II}})(x) > \lambda\}| \leq C(l^{k+1/2}\lambda^{-1})^p\|f\|_p^p, \quad 1 < p < 2. \tag{7}$$

We shall carry out the argument by induction on the order k. If $k = 0$, then $g_{l;\,b,\,k}$ is the operator

$$g_l(f)(x) = \Big(\sum_{j\in\mathbb{Z}} |V_j^l f(x)|^2\Big)^{1/2}.$$

Note that

$$\big\{x \in \mathbb{R}^n \backslash E : g_l(f^{\mathrm{II}})(x) > \lambda\big\} \subset \Big\{x \in \mathbb{R}^n, \sup_{j\in\mathbb{Z}} |K_j^l * (f^{\mathrm{II}})(x)| > l^{-1/2}\lambda\Big\}$$
$$\cup \Big\{x \in \mathbb{R}^n \backslash E : \sum_{j\in\mathbb{Z}} |V_j^l(f^{\mathrm{II}})(x)| > l^{1/2}\lambda\Big\},$$

and

$$\sup_{j\in\mathbb{Z}} |K_j^l * (f^{\mathrm{II}})(x)| \leq C \sup_{j\in\mathbb{Z}} \big(|K_j| * (Mf^{\mathrm{II}})\big)(x).$$

It follows that

$$\Big|\Big\{x \in \mathbb{R}^n : \sup_{j\in\mathbb{Z}} |K_j^l * (f^{\mathrm{II}})(x)| > l^{-1/2}\lambda\Big\}\Big| \leq C\big(l^{1/2}\lambda^{-1}\big)^p\|f\|_p^p.$$

The inequality (6) (with $m=0$) now gives us that

$$\begin{aligned}
&\Big|\Big\{x\in\mathbb{R}^n\backslash E\colon \sum_{j\in\mathbb{Z}}|K_j^l*(f^{\mathrm{II}})(x)|>l^{1/2}\lambda\Big\}\Big|\\
&\le l^{-1/2}\lambda^{-1}\sum_d\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n\backslash B_d}|K_j^l*(f_d^{\mathrm{II}})(x)|dx\\
&\le l^{-1/2}\lambda^{-1}\sum_d\sum_{j\in\mathbb{Z}}\int_{Q_d}\int_{\mathbb{R}^n\backslash B_d}|K_j^l(x-x_d)-K_j^l(x-y)|dx|f_d^{\mathrm{II}}(y)|dy\\
&\le Cl^{1/2}\sum_d|Q_d|\le Cl^{1/2}\lambda^{-p}\|f\|_p^p,
\end{aligned}$$

and the estimate (7) holds for the case of $k=0$.

Now let k be a positive integer. We assume that the inequality (7) holds for all $1<p<2$ and integer m with $0\le m\le k-1$. Denote by $m_{B_d}(b)$ the mean value of b on the ball B_d. Write

$$\begin{aligned}
\big(b(x)-b(y)\big)^k&=\big(b(x)-m_{B_d}(b)\big)^k\\
&\quad-\sum_{m=0}^{k-1}C_k^m\big(b(x)-b(y)\big)^m\big(b(y)-m_{B_d}(b)\big)^{k-m},\\
&\qquad\qquad x,\,y\in\mathbb{R}^n.
\end{aligned}$$

It follows that for $j,\,l\in\mathbb{Z}$,

$$\begin{aligned}
V_{j;\,b,\,k}^lf^{\mathrm{II}}(x)&=\sum_d\big(b(x)-m_{B_d}(b)\big)^kV_j^lf_d^{\mathrm{II}}(x)\\
&\quad-\sum_{m=0}^{k-1}C_k^mV_{j;\,b,\,m}^l\Big(\sum_d\big(b(\cdot)-m_{B_d}(b)\big)^{k-m}f_d^{\mathrm{II}}\Big)(x).
\end{aligned}$$

Therefore, by the Schwarz inequality,

$$\begin{aligned}
&g_{l;\,b,\,k}\big(f^{\mathrm{II}}\big)(x)\\
&\le\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_d|b(x)-m_{B_d}(b)|^k|V_j^lf_d^{\mathrm{II}}(x)|\Big)^2\Big)^{1/2}\\
&\quad+C\sum_{m=0}^{k-1}\Big(\sum_{j\in\mathbb{Z}}\Big|V_{j;\,b,\,m}^l\Big(\sum_d\big(b(\cdot)-m_{B_d}(b)\big)^{k-m}f_d^{\mathrm{II}}\Big)(x)\Big|^2\Big)^{1/2}
\end{aligned}$$

$$\leq \Big(\sum_{j\in\mathbb{Z}}\sum_{d}|b(x)-m_{B_d}(b)|^{2k}|V_j^l f_d^{\mathrm{II}}(x)|\sum_{d}|V_j^l f_d^{\mathrm{II}}(x)|\Big)^{1/2}$$

$$+C\sum_{m=0}^{k-1} g_{l;\,b,\,m}\Big(\sum_{d}\big(b(\cdot)-m_{B_d}(b)\big)^{k-m} f_d^{\mathrm{II}}\Big)(x)$$

$$\leq C\Big(\sup_{j\in\mathbb{Z}}\sum_{d}|V_j^l f_d^{\mathrm{II}}(x)|\Big)^{1/2}\Big(\sum_{j\in\mathbb{Z}}\sum_{d}|b(x)-m_{B_d}(b)|^{2k}|V_j^l f_d^{\mathrm{II}}(x)|\Big)^{1/2}$$

$$+C\sum_{m=0}^{k-1} g_{l;\,b,\,m}\Big(\sum_{d}\big(b(\cdot)-m_{B_d}(b)\big)^{k-m} f_d^{\mathrm{II}}\Big)(x).$$

Choose p_0, $1<p_0<p$, and set $r=p/p_0$. For each $0\leq m\leq k-1$, our induction hypothesis states that

$$\Big|\Big\{x\in\mathbb{R}^n\backslash E\colon g_{l;\,b,\,m}\Big(\sum_{d}\big(b(\cdot)-m_{B_d}(b)\big)^{k-m} f_d^{\mathrm{II}}\Big)(x)>\lambda\Big\}\Big|$$

$$\leq \lambda^{-p_0}\Big\|g_{l;\,b,\,m}\Big(\sum_{d}\big(b(\cdot)-m_{B_d}(b)\big)^{k-m} f_d^{\mathrm{II}}\Big)\Big\|_{p_0}^{p_0}$$

$$\leq C(l^{m+1/2}\lambda^{-1})^{p_0}\sum_{d}\|(b(\cdot)-m_{B_d}(b))^{k-m} f_d^{\mathrm{II}}\|_{p_0}^{p_0}$$

$$\leq C(l^{m+1/2}\lambda^{-1})^{p_0}\sum_{d}\|f_d^{\mathrm{II}}\|_p^{p_0}\Big(\int_{B_d}|b(y)-m_{B_d}(b)|^{(k-m)p_0 r'}dy\Big)^{1/r'}$$

$$\leq C(l^{m+1/2}\lambda^{-1})^{p}\|f\|_p^p.$$

Obviously,

$$\Big|\Big\{x\in\mathbb{R}^n\backslash E\colon \Big(\sup_{j\in\mathbb{Z}}\sum_{d}|V_j^l f_d^{\mathrm{II}}(x)|\Big)^{1/2}$$

$$\Big(\sum_{j\in\mathbb{Z}}\sum_{d}|b(x)-m_{B_d}(b)|^{2k}|V_j^l f_d^{\mathrm{II}}(x)|\Big)^{1/2}>\lambda\Big\}\Big|$$

$$\subset \Big|\Big\{x\in\mathbb{R}^n\backslash E\colon \sum_{j\in\mathbb{Z}}\sum_{d}|b(x)-m_{B_d}(b)|^{2k}V_j^l f_d^{\mathrm{II}}(x)|>l^{k+1/2}\lambda\Big\}\Big|$$

$$\cup\Big|\Big\{x\in\mathbb{R}^n\colon \sup_{j\in\mathbb{Z}}\sum_{d}|V_j^l f_d^{\mathrm{II}}(x)|>l^{-k-1/2}\lambda\Big\}\Big|.$$

Recall that the cubes Q_d have disjoint interiors, straightforward computation yields

$$\Big|\Big\{x\in\mathbb{R}^n\colon \sup_{j\in\mathbb{Z}}\sum_d |V_j^l f_d^{\rm II}(x)|>l^{-k-1/2}\lambda\Big\}\Big|$$
$$\le (l^{k+1/2}\lambda^{-1})^p\Big\|\sup_{j\in\mathbb{Z}}\Big(|K_j^l|*\sum_d|f_d^{\rm II}|\Big)\Big\|_p^p$$
$$\le C(l^{k+1/2}\lambda^{-1})^p\Big\|\sum_d|f_d^{\rm II}|\Big\|_p^p\le C(l^{k+1/2}\lambda^{-1})^p\|f\|_p^p.$$

Thus the proof of the inequality (7) for k can be reduced to proving that

$$\Big|\Big\{x\in\mathbb{R}^n\backslash E\colon \sum_{j\in\mathbb{Z}}\sum_d|b(x)-m_{B_d}(b)|^{2k}|V_j^l f_d^{\rm II}(x)|>l^{k+1/2}\lambda\Big\}\Big|$$
$$\le C(l^{k+1/2}\lambda^{-1})^p\|f\|_p^p. \tag{8}$$

By the kernel estimate (6), the same argument as in the proof of Lemma 2 in [11] leads to that for any $y\in Q_d$,

$$\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n\backslash B_d}|b(x)-m_{B_d}(b)|^{2k}|K_j^l(x-y)-K_j^l(x-x_d)|dx\le Cl^{2k+1}. \tag{9}$$

Therefore,

$$\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n\backslash B_d}|b(x)-m_{B_d}(b)|^{2k}|V_j^l f_d^{\rm II}(x)|dx$$
$$\le\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n\backslash B_d}|m_{B_d}(b)-b(x)|^{2k}\int_{Q_d}|K_j^l(x-y)-K_j^l(x-x_d)||f_d^{\rm II}(y)|dy\,dx$$
$$\le Cl^{2k+1}\int_{Q_d}|f_d^{\rm II}(y)|dy\le Cl^{2k+1}\lambda|Q_d|,$$

which in turn shows that

$$\Big|\Big\{x\in\mathbb{R}^n\backslash E\colon \sum_{j\in\mathbb{Z}}\sum_d|b(x)-m_{B_d}(b)|^{2k}|V_j^l f_d^{\rm II}(x)|>l^{k+1/2}\lambda\Big\}\Big|$$
$$\le l^{-k-1/2}\lambda^{-1}\sum_d\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n\backslash B_d}|b(x)-m_{B_d}(b)|^{2k}|V_j^l f_d^{\rm II}(x)|dx$$
$$\le Cl^{k+1/2}\sum_d|Q_d|\le Cl^{k+1/2}\lambda^{-p}\|f\|_p^p.$$

This establishes the inequality (8).

Since that the operator $g_{l;\,b,\,k}$ is not linear, we don't automatically get the $L^p(\mathbb{R}^n)$ boundedness of $g_{l;\,b,\,k}$ for $2 < p < \infty$ by duality argument. We claim that

$$\|(g_l(f))^{\#}\|_\infty \le Cl^{1/2}\|f\|_\infty, \quad f \in L_0^\infty(\mathbb{R}^n) \tag{10}$$

and for any $1 < s < \infty$ and positive integer k,

$$\begin{aligned}\big(g_{l;\,b,\,k}(f)\big)^{\#}(x) \le C_s \sum_{m=0}^{k-1} M_s\big(g_{l;\,b,\,m}(f)\big)(x)& \\ + Cl^{k+1/2}\|f\|_\infty, \quad f \in L_0^\infty(\mathbb{R}^n),&\end{aligned} \tag{11}$$

where C_s is a positive constant depending only on s, $M_s h(x) = \big(M(|h|^s)(x)\big)^{1/s}$. We only prove the inequality (11), the inequality (10) can be proved in the same way. For each $x \in \mathbb{R}^n$, let $B = B(x_0,\, R)$ be a ball containing x, with center x_0 and the radius R. Denote by $M_B(b)$ the mean value of b on B. Write

$$\begin{aligned}V^l_{j;\,b,\,k}h(y) &= V^l_j\big((m_B(b) - b)^k h\big)(y) \\ &\quad - \sum_{m=0}^{k-1} C_k^m \big(m_B(b) - (b(y)\big)^{k-m} V^l_{j;\,b,\,m}h(y).\end{aligned}$$

For $f \in L_0^\infty(\mathbb{R}^n)$, decompose f as

$$f(y) = f(y)\chi_{B(x_0,\,4R)}(y) + f(y)\chi_{\mathbb{R}^n\setminus B(x_0,\,4R)}(y) = f_1(y) + f_2(y).$$

Note that $(m_B(b) - b)^k f_2 \in L^2(\mathbb{R}^n)$. By the $L^2(\mathbb{R}^n)$ boundedness of g_l, we can take $y_0 \in B(x_0,\, R)$ such that $|g_l\big((m_B(b) - b)^k f_2\big)(y_0)| < \infty$. Write

$$\begin{aligned}&\frac{1}{|B|}\int_B |g_{l;\,b,\,k}(f)(y) - g_l\big((m_B(b) - b)^k f_2\big)(y_0)|dy \\ &\quad\le \frac{1}{|B|}\int_B \Big(\sum_{j\in\mathbb{Z}} |V^l_{j;\,b,\,k}f(y) - V^l_j\big((m_B(b) - b)^k f_2(y_0)|^2\Big)^{1/2} dy \\ &\quad\le C\sum_{m=0}^{k-1} \frac{1}{|B|}\int_B |m_B(b) - b(y)|^{k-m} g_{l;\,b,\,m}(f)(y)dy \\ &\qquad + \frac{1}{|B|}\int_B g_l\big((m_B(b) - b)^k f_1\big)(y)dy\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|B|}\int_B\Big(\sum_{j\in\mathbb{Z}}\big|V_j^l\big((m_B(b)-b)^k f_2\big)(y)\\
&\qquad\qquad - V_j^l\big((m_B(b)-b)^k f_2\big)(y_0)\big|^2\Big)^{1/2}dy\\
&= \mathrm{I}+\mathrm{II}+\mathrm{III}
\end{aligned}
$$

By the Hölder inequality,

$$
\begin{aligned}
\mathrm{I} &\le C\sum_{m=0}^{k-1}\Big(\frac{1}{|B|}\int_B |m_B(b)-b(y)|^{(k-m)s'}dy\Big)^{1/s'}\\
&\qquad\qquad \Big(\frac{1}{|B|}\int_B \big(g_{l;\,b,\,m}(f)(y)\big)^s dy\Big)^{1/s}\\
&\le C\sum_{m=0}^{k-1} M_s\big(g_{l;\,b,m}(f)\big)(x).
\end{aligned}
$$

The $L^2(\mathbb{R}^n)$ boundedness of g_l now says that

$$
\mathrm{II}\le C\Big(\frac{1}{|B|}\int_{B(x_0,\,4R)} |m_B(b)-b(y)|^{2k}|f(y)|^2dy\Big)^{1/2}\le C\|f\|_\infty.
$$

Similar to the inequality (9), we have that for each $y\in B(x_0,\,R)$,

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n\setminus B(x_0,\,4R)} |m_B(b)-b(z)|^{2k}|K_j^l(y-z)-K_j^l(y_0-z)|dz\\
&\qquad \le Cl^{2k+1}.
\end{aligned}
$$

Thus, for $y\in B$,

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}}\big|V_j^l\big((m_b(b)-b)^k f_2\big)(y)-V_j^l\big((m_b(b)-b)^k f_2\big)(y_0)\big|^2\\
&\quad \le 2\sup_{j\in\mathbb{Z}}\|K_j^l * f_2\|_\infty \int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}|K_j^l(y-z)-K_j^l(y_0-z)|\\
&\qquad\qquad\qquad \times |m_B(b)-b(z)|^{2k}|f_2(z)|dz\\
&\quad \le Cl^{2k+1}\|f\|_\infty^2,
\end{aligned}
$$

which shows that

$$
\mathrm{III}\le Cl^{k+1/2}\|f\|_\infty,
$$

and gives the inequality (11).

We can now conclude the proof of Lemma 4 by proving that $g_{l;b,k}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $Cl^{k+1/2}$ when $2 < p < \infty$. Again we shall use the induction argument on the order k. If $k = 0$, the estimate (10) and the $L^2(\mathbb{R}^n)$ boundedness of g_l via the well-known interpolation theorem of Fefferman-Stein (see [14, Chapter IV]) give us the desired result directly. Now let k be a positive integer, and the estimate

$$\|g_{l;b,m}(f)\|_p \le Cl^{m+1/2}\|f\|_p, \quad 2 < p < \infty$$

hold for all $0 \le m \le k-1$. Our goal is to show that

$$\|g_{l;b,k}(f)\|_p \le Cl^{k+1/2}\|f\|_p, \quad 2 < p < \infty.$$

By the Marcinkiewicz interpolation theorem, the relationship of the Hardy-Littlewood maximal operator and sharp function ([14, Chapter V]), and a standard density argument, it is enough to show that for each $2 < p < \infty$ and $\lambda > 0$,

$$\begin{aligned}\big|\{x \in \mathbb{R}^n \colon \big(g_{l;b,k}(f)\big)^{\#}(x) > \lambda\}\big| \\ \le C(l^{k+1/2}\lambda^{-1})^p\|f\|_p^p, \quad f \in C_0^\infty(\mathbb{R}^n).\end{aligned} \tag{12}$$

For each given $f \in C_0^\infty(\mathbb{R}^n)$ and $\lambda > 0$, perform the Whitney decomposition of the set $\{x \in \mathbb{R}^n \colon M(|f|^p)(x) > (l^{-(k+1/2)}\lambda)^p\}$ into a union of non-overlapping cubes $\{Q_d\}$. We can write $f(x) = f^{\mathrm{I}}(x) + f^{\mathrm{II}}(x)$, where $\|f^{\mathrm{I}}\|_\infty \le Cl^{-(k+1/2)}\lambda$, $\|f^{\mathrm{I}}\|_p \le C\|f\|_p$; $f^{\mathrm{II}}(x) = \sum_d f_d^{\mathrm{II}}(x)$, with f_d^{II} supported on Q_d, and $\int |f_d^{\mathrm{II}}(x)|^p dx \le C(l^{-(k+1/2)}\lambda)^p|Q_d|$. Since $f \in C_0^\infty(\mathbb{R}^n)$, there exists a compact set E such that $\{x \colon M(|f|^p)(x) > (l^{-(k+1/2)}\lambda)^p\} \subset E$, and thus f^{I} has compact support. Choose $1 < s < \infty$ such that $2 < p/s < \infty$. By the inequality (11) and our induction hypothesis, it follows that

$$\begin{aligned}&\big|\{x \in \mathbb{R}^n \colon (g_{l;b,k}(f^{\mathrm{I}}))^{\#}(x) > (C+1)\lambda\}\big| \\ &\le \Big|\Big\{x \colon C_s \sum_{m=0}^{k-1} M_s(g_{b,m}f^{\mathrm{I}})(x) > \lambda\Big\}\Big| \\ &\le C\lambda^{-p}\sum_{m=0}^{k-1}\|M_s(g_{b,m}f^{\mathrm{I}})\|_p^p \le C\big(l^{k+1/2}\lambda^{-1}\big)^p\|f\|_p^p.\end{aligned}$$

The $L^2(\mathbb{R}^n)$ boundedness of $g_{l;b,k}$ states that

$$\begin{aligned}&\left|\{x \in \mathbb{R}^n, (g_{l;b,k}(f^{\mathrm{II}}))^{\#}(x) > \lambda\}\right|\\&\quad\le \lambda^{-2}\|(g_{l;b,k}f^{\mathrm{II}})^{\#}\|_2^2 \le C\lambda^{-2}\|f^{\mathrm{II}}\|_2^2\\&\quad\le Cl^{(k+1/2)p}\left|\{x\colon M(|f|^p)(x) > (l^{k+1/2}\lambda)^p\}\right|\\&\quad\le C\lambda^{-p}\|f\|_p^p.\end{aligned}$$

Combining the estimates above leads to the inequality (12). □

Proof of Theorem 2. We only consider the case of $2 < p < 4\alpha/(2k+1)$. For $4\alpha/(4\alpha - 2k - 1) < p < 2$, the proof is very similar. Let $g_{l;b,k}$ be the same as above. By Lemma 3,

$$\|g_{2^l;b,k}(f) - g_{2^{l+1};b,k}(f)\|_2 \le 2^{(-\alpha\nu+k+1/2)l}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|f\|_2. \tag{13}$$

Therefore, the series

$$g_{b,k} = g_{1;b,k} + \sum_{l=0}^{\infty}\left(g_{2^{l+1};b,k} - g_{2^l;b,k}\right) \tag{14}$$

converges strongly in the $L^2(\mathbb{R}^n)$-operator norm. On the other hand, invoking Lemma 4, we can obtain

$$\|g_{l;b,k}(f)\|_p \le Cl^{k+1/2}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|f\|_p, \quad 1 < p < \infty,$$

and so

$$\begin{aligned}&\|g_{2^l;b,k}(f) - g_{2^{l+1};b,k}(f)\|_p\\&\quad\le C2^{(k+1/2)l}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k\|f\|_p, \quad 1 < p < \infty.\end{aligned} \tag{15}$$

Interpolation the inequalities (13) and (15) gives us that for any $\theta > 0$,

$$\begin{aligned}&\|g_{2^{l+1};b,k}(f) - g_{2^l;b,k}(f)\|_p\\&\quad\le C_\theta\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k 2^{(-2\alpha\nu/p+k+1/2+\theta)l}\|f\|_p, \quad 2 < p < \infty.\end{aligned}$$

For each given p with $2 < p < 4\alpha/(2k+1)$, we can choose $\theta > 0$ and $0 < \nu < 1$ such that $2\alpha\nu/p > k + 1/2 + \theta$. Thus, the series (14) converges in the $L^p(\mathbb{R}^n)$-operator norm, and the operator $g_{b,k}$ is bounded on $L^p(\mathbb{R}^n)$. This completes the proof of Theorem 2. □

4. Some applications

In this section, we will give some applications of the theorems we have established. We begin with the maximal operator defined by

$$M_{\Omega;\,b,\,k}f(x)=\sup_{r>0}r^{-n}\int_{|x-y|<r}|b(x)-b(y)|^k|\Omega(x-y)f(y)|dy,$$

where Ω is homogeneous of degree zero, integrable on the unit sphere S^{n-1}, $b\in \mathrm{BMO}(\mathbb{R}^n)$ and k is a positive integer. We can prove that

Corollary 1 *Let Ω be homogeneous of degree zero, integrable on S^{n-1}, $b\in \mathrm{BMO}(\mathbb{R}^n)$. If k is a even number and for some $\alpha>k+1/2$,*

$$\sup_{\zeta\in S^{n-1}}\int_{S^{n-1}}|\Omega(\theta)|\log\left(\frac{1}{|\theta\cdot\zeta|}\right)^{\alpha}d\theta<\infty, \tag{16}$$

then $M_{\Omega;\,b,\,k}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$ for $4\alpha/(4\alpha-2k-1)<p<4\alpha/(2k+1)$. If k is a odd number and Ω satisfies (16) for some $\alpha>k+3/2$, then $M_{\Omega;\,b,\,k}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$ for $4\alpha/(4\alpha-2k-3)<p<4\alpha/(2k+3)$.

Proof. We first consider the case that k is even. Let $A=\|\Omega\|_1/|S^{n-1}|$ and $\widetilde{\Omega}(x)=|\Omega(x)|-A$. Note that $\widetilde{\Omega}$ has mean value zero on S^{n-1}. Set $K_j(x)=\widetilde{\Omega}(x)|x|^{-n}\chi_{\{2^j\le|x-y|<2^{j+1}\}}$. By the estimate of Grafakos and Stefanov [9], we know that if Ω satisfies (16) for some $\alpha>1$, then K_j satisfies the Fourier transform estimate (3) for the same α. Denote by U_j the convolution operator whose kernel is K_j. Write

$$\begin{aligned}M_{\Omega;\,b,\,k}f(x)&\le C\sup_{j\in\mathbb{Z}}\int_{2^j<|x-y|\le 2^{j+1}}\big(b(x)-b(y)\big)^k\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy\\&\le C\Big(\sum_{j\in\mathbb{Z}}|U_{j;\,b,\,k}(|f|)(x)|^2\Big)^{1/2}\\&\qquad+CA\sup_{r>0}r^{-n}\int_{|x-y|<r}(b(x)-b(y))^k|f(y)|dy.\end{aligned}$$

Note that the commutator

$$M_{b,\,k}f(x)=\sup_{r>0}r^{-n}\int_{|x-y|<r}|b(x)-b(y)|^k|f(y)|dy.$$

(see [8]) and the operator

$$M_{\Omega}f(x)=\sup_{r>0}r^{-n}\int_{|x-y|<r}|\Omega(x-y)||f(y)|dy$$

are all bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. Corollary 1 for even number k now follows from Theorem 2 directly.

Now let k be an odd positive integer. By the Hölder inequality,

$$M_{\Omega;\,b,\,k}f(x) \le \big(M_{\Omega;\,b,\,k+1}f(x)\big)^{k/(k+1)}\big(M_\Omega f(x)\big)^{1/(k+1)}.$$

Another application of the Hölder inequality gives the desired result for odd number k. □

We now turn our attention to the maximal operator associated with the commutator of homogeneous singular operator defined by

$$T^*_{b,\,k}f(x) = \sup_{\varepsilon>0}\Big|\int_{|x-y|>\varepsilon}\big(b(x)-b(y)\big)^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy\Big|.$$

We have

Corollary 2 *Let Ω be homogeneous of degree zero, integrable on S^{n-1} and have mean value zero, k be a nonnegative integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Suppose that Ω satisfies the inequality* (16) *for some $\alpha > k+3/2$. Then $T^*_{b,\,k}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$ for $4\alpha/(4\alpha-2k-3) < p < 4\alpha/(2k+3)$.*

Proof. Our argument will proceed by induction on k. If $k=0$, Corollary 2 can be obtained from Theorem 2 in [6]. Now we assume that Corollary 2 is true for all integer m with $0 \le m \le k-1$. Let $K_l(x) = \frac{\Omega(x)}{|x|^n}\chi_{\{2^l\le|x|<2^{l+1}\}}(x)$ and define the operator

$$T_{l;\,b,\,k}f(x) = \int_{2^l\le|x-y|<2^l}\big(b(x)-b(y)\big)^k\frac{\Omega(x-y)}{|x-y|^n}f(y)dy.$$

By Corollary 1, it suffices to consider the operator $\sup_j\big|\sum_{l=j}^{\infty}T_{l;\,b,\,k}f(x)\big|$. Take $\eta \in \mathcal{S}(\mathbb{R}^n)$ such that $\eta(x)\equiv 1$ when $|x|\le 1$ and set $\eta_j(x)=\eta(2^jx)$. Let $\Phi_j \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\Phi_j}(\xi)=\eta_j(\xi)$. Denote by W_j the convolution operator whose kernel is Φ_j and W_j^l the convolution operator whose kernel is $K_l - \Phi_j * K_l$. Write

$$\begin{aligned}\sum_{l=j}^{\infty}T_{j;\,b,\,k}f(x) &= \Phi_j * \Big(T_{b,\,k}f - \sum_{l=-\infty}^{j-1}T_{l;\,b,\,k}f\Big)(x)\\ &\quad + \Big(\sum_{l=j}^{\infty}T_{l;\,b,\,k}f(x) - \Phi_j * \Big(\sum_{l=j}^{\infty}T_{l;\,b,\,k}f\Big)(x)\Big)\\ &= \mathrm{I}_j(f)(x) + \mathrm{II}_j(f)(x).\end{aligned}$$

Observe that

$$\left|\Phi_j * \sum_{l=-\infty}^{j-1} K_l(x)\right| \le C2^{-jn}/(1+|2^{-j}x|)^{n+1}$$

(see [5]) and that

$$\Phi_j * \Big(\sum_{l=-\infty}^{j-1} T_{l;\,b,\,k}f\Big)(x) = \Big(\Phi_j * \sum_{l=-\infty}^{j-1} K_l\Big)_{b,\,k} f(x) \\ - \sum_{m=0}^{k-1} C_k^m W_{j;\,b,\,k-m}\Big(\sum_{l=-\infty}^{j-1} T_{l;\,b,\,m}f\Big)(x).$$

It follows that

$$\sup_{j\in\mathbb{Z}} |\mathrm{I}_j(f)(x)| \le \sum_{m=0}^{k-1} \big(M_{b,\,k-m}(T_{b,\,m}f)(x) + M_{b,\,k-m}(T^*_{b,\,m}f)(x)\big) \\ + CM_{b,\,k}f(x) + CM(T_{b,\,k}f)(x).$$

Theorem 2 in [11] tells us that $T_{b,\,k}$ is bounded on $L^p(\mathbb{R}^n)$ for $2\alpha/(2\alpha-k-1) < p < 2\alpha/(k+1)$. Thus, $\sup_{j\in\mathbb{Z}} |\mathrm{I}_j(f)(x)|$ is bounded by an operator that is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$ for $4\alpha/(4\alpha-2k-3) < p < 4\alpha/(2k+3)$. To estimate $\sup_{j\in\mathbb{Z}} |\mathrm{II}_j(f)(x)|$, write

$$\mathrm{II}_j f(x) = \sum_{l=j}^{\infty} T_{l;\,b,\,k}f(x) - \Big(\Phi_j * \sum_{l=j}^{\infty} T_l\Big)_{b,\,k} f(x) \\ + \sum_{m=0}^{k-1} C_k^m W_{j;\,b,\,k-m}\Big(\sum_{l=j}^{\infty} T_{l;\,b,\,m}f\Big)(x) \\ = \sum_{l=j}^{\infty} W^l_{j;\,b,\,k}f(x) + \sum_{m=0}^{k-1} C_k^m W_{j;\,b,\,k-m}\Big(\sum_{l=j}^{\infty} T_{l;\,b,\,m}f\Big)(x).$$

For each $0 \le m \le k-1$, it is easy to see that

$$\sup_{j\in\mathbb{Z}} \left|W_{j;\,b,\,k-m}\Big(\sum_{l=j}^{\infty} T_{l;\,b,\,m}f\Big)(x)\right| \le CM_{b,\,k-m}(T^*_{b,\,m}f)(x).$$

Thus, it suffices to estimate the $L^p(\mathbb{R}^n)$ norm of $\sup_{j\in\mathbb{Z}} \big|\sum_{l=j}^{\infty} W^l_{j;\,b,\,k}f(x)\big|$.

Write

$$\sup_{j\in\mathbb{Z}}\Big|\sum_{l=j}^{\infty}W^{l}_{j;\,b,\,k}f(x)\Big|\le\sum_{l=0}^{\infty}\sup_{j\in\mathbb{Z}}\big|W^{j}_{j-l;\,b,\,k}f(x)\big|$$
$$\le\sum_{l=0}^{\infty}\Big(\sum_{j\in\mathbb{Z}}\big|W^{j}_{j-l;\,b,\,k}f(x)\big|^2\Big)^{1/2}.$$

since $K_j-\Phi_{j-l}*K_j$ satisfies the Fourier transform (2) with $A=2^{-l}$, by Theorem 1, we know that

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\big|W^{j}_{j-l;\,b,\,k}f\big|^2\Big)^{1/2}\Big\|_2\le Cl^{-\alpha\nu+k+1/2}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2. \tag{17}$$

It follows from Theorem 2 that

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\big|W^{j}_{j-l;\,b,\,k}f\big|^2\Big)^{1/2}\Big\|_p\le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p,$$
$$4\alpha/(4\alpha-2k-1)<p<4\alpha/(2k+1). \tag{18}$$

Interpolation the inequalities (17) and (18) yields that for $2<p<4\alpha/(2k+1)$,

$$\Big\|\sup_{j\in\mathbb{Z}}\big|W^{j}_{j-l;\,b,\,k}f\big|\Big\|_p\le Cl^{-2\alpha\nu(1/p-(2k+1)/(4\alpha))+\theta_0}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p. \tag{19}$$

For each fixed $2<p<4\alpha/(2k+3)$, we can take $\theta_0>0$ and $0<\nu<1$, such that $2\alpha\nu(1/p-(2k+1)/(4\alpha))>1+\theta_0$. Summing over the inequality (19) for all integers $l\ge 0$ gives the desired result. For the case of $4\alpha/(4\alpha-2k-3)<p<2$, the proof is very similar and is omited. This finishes the proof of Corollary 2. □

At the end of this section, we consider the commutator of the Marcinkiewicz integral. For $t>0$, $b\in\mathrm{BMO}(\mathbb{R}^n)$ and nonnegative integer k, let

$$F_{t;\,b,\,k}f(x)=\int_{|x-y|\le t}\big(b(x)-b(y)\big)^k\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy,$$

and define the operator

$$\mu_{\Omega;\, b,\, k} f(x) = \Big(\int_0^\infty \big| F_{t;\, b,\, k} f(x) \big|^2 \frac{dt}{t^3} \Big)^{1/2}.$$

For $k = 0$, this is the higher-dimensional Marcinkiewicz integral introduced by Stein [15] and has been considered by many authors recently (see [4] and [7]). For $j \in \mathbb{Z}$ and $t \in [1,\, 2]$, let

$$U_{j,\, t;\, b,\, k} f(x) = \int_{2^{j-1}t < |x-y| \le 2^j t} \big(b(x) - b(y)\big)^k \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

The Minkowski inequality states that

$$\begin{aligned}
&\mu_{\Omega;\, b,\, k} f(x) \\
&\le \sum_{l=-\infty}^{0} \Big(\int_0^\infty \Big| \int_{2^{l-1}t < |x-y| \le 2^l t} \big(b(x) - b(y)\big)^k \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \Big|^2 \frac{dt}{t^3} \Big)^{1/2} \\
&\le \sum_{l=-\infty}^{0} 2^l \Big(\int_0^\infty \Big| \int_{t/2 < |x-y| \le t} \big(b(x) - b(y)\big)^k \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \Big|^2 \frac{dt}{t^3} \Big)^{1/2} \\
&= 2 \Big(\sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \Big| \int_{t/2 < |x-y| \le t} \big(b(x) - b(y)\big)^k \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \Big|^2 \frac{dt}{t^3} \Big)^{1/2} \\
&\le 4 \Big(\sum_{j \in \mathbb{Z}} \int_1^2 \big| U_{j,\, t;\, b,\, k} f(x) \big|^2 dt \Big)^{1/2}.
\end{aligned}$$

For each $t \in [1,\, 2]$ and $4\alpha/(4\alpha - 2k - 1) < p < 4\alpha/(2k+1)$, by Theorem 2 and rescaling,

$$\Big\| \Big(\sum_{j \in \mathbb{Z}} \big| U_{j,\, t;\, b,\, k} f \big|^2 \Big)^{1/2} \Big\|_p \le C \|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_p.$$

If $2 \le p < 4\alpha/(2k+1)$, it follows from the Minkowski inequality that

$$\begin{aligned}
\Big\| \Big(\int_1^2 \sum_{j \in \mathbb{Z}} \big| U_{j,\, t;\, b,\, k} f \big|^2 dt \Big)^{1/2} \Big\|_p^2 &\le \int_1^2 \Big\| \Big(\sum_{j \in \mathbb{Z}} \big| U_{j,\, t;\, b,\, k} f \big|^2 \Big)^{1/2} \Big\|_p^2 dt \\
&\le C \|b\|^{2k}_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_p^p.
\end{aligned}$$

Furthermore, repeating the proof of Theorem 2, with some suitable modifications, we can prove that for $4\alpha/(4\alpha-2k-1)<p<4\alpha/(2k+1)$,

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}\int_1^2 \big|U_{j,t;b,k}f(x)\big|^2dt\Big)^{1/2}\Big\|_p \le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p.$$

Therefore, we have

Corollary 3 *Let Ω be homogeneous of degree zero, integrable on S^{n-1} and have mean value zero, k be a nonnegative integer and $b\in\mathrm{BMO}(\mathbb{R}^n)$. Suppose that Ω satisfies the inequality* (16) *for some $\alpha>k+1/2$. Then $\mu_{\Omega;b,k}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$ for all $4\alpha/(4\alpha-(2k+1))<p<4\alpha/(2k+1)$.*

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References

[1] Alvarez J., Bagby R., Kurtz D. and Pérez C., *Weighted estimates for commutators of linear operators*. Studia Math. **104** (1993), 195–209.

[2] Coifman R., Lions P.L., Meyer Y. and Semmes S., *Compensated compactness and Hardy space*. J. Math. Pure Appl. **72** (1993), 247–286.

[3] Coifman R. and Meyer Y., *Au déla des opérateurs pseudo-différentiles*. Astérisque **57** (1978), 1–185.

[4] Ding Y., Fan D. and Pan Y., *Weighted boundedness for a class of Marcinkiewicz integrals*. Indiana Univ. Math. J. **48** (1999), 1037–1055.

[5] Duoandikoetxea J. and Rubio de Francia J.L., *Maximal and singular integrals via Fourier transform estimates*. Invent. Math. **84** (1986), 541–561.

[6] Fan D., Guo K. and Pan Y., *A note on rough singular integral operators*. Math. Inequal. Appl. **2** (1999), 73–81.

[7] Fan D. and Sato S., *Weak type* (1, 1) *estimates for Marcinkiewicz integrals with rough kernels*. Tohoku Math. J. **53** (2001), 265–284.

[8] Garcia-Cuerva J., Harboure E., Segovia C. and Torrea J.L., *Weighted norm inequalities for commutators of strongly singular integrals*. Indiana Univ. Math. J. **40** (1991), 1397–1420.

[9] Grafakos L. and Stefanov A., *L^p bounds for singular integrals and maximal singular integrals with rough kernels*. Indiana Univ. Math. J. **47** (1998), 455–469.

[10] Hu G., *$L^2(\mathbb{R}^n)$ boundedness for the commutators of convolution operators*. Nagoya Math. J. **163** (2001), 55–70.

[11] Hu G., Sun Q. and Wang X., *$L^p(\mathbb{R}^n)$ bounds for the commutators of convolution operators*. Colloquium Math. **93** (2002), 11–20.

[12] Pérez C., *Endpoint estimate for commutators of singular integral operators*. J. Funct. Anal. **128** (1995), 163–185.

[13] Segovia C. and Torrea J.L., *Vector-valued commutators and applications*. Indiana Univ. Math. J. **38** (1989), 959–971.

[14] Stein E.M., *Harmonic Analysis: Real-Variable Mathods, Orthogonolity and Oscillatory Integrals*. Princeton Univ. Press, Princeton, New Jersey, 1993.

[15] Stein E.M., *On the functions of Littlewood-Paley, Lusin and Marcinkiewicz*. Trans. Amer. Math. Soc. **88** (1958), 430–466.

[16] Watson D.K., *Weighted estimate for singular integrals via Fourier transform estimates*. Duke Math. J. **60** (1990), 389–399.

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