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Partial regularity of solutions of nonlinear quasimonotone systems

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Abstract. We prove partial regularity of weak solutions u of the nonlinear *strictly quasimonotone* system $\operatorname{div} A(x, u, Du) + B(x, u, Du) = 0$ under natural polynomial growth, assuming that the coefficient function $A(x, u, P)$ is Hölder continuous in (x, u) and of class C^1 in P, and that $A(x, u, P) \cdot P \geq F(x, P)$ for some continuous function F which is strictly quasiconvex at zero.

Key words: partial regularity, weak solution, nonlinear system, quasilinear system, quasimonotonicity, ellipticity, natural growth.

1. Introduction

We are interested in the regularity of the vector-valued weak solutions $u \in W^{1,2}(\Omega, \mathbf{R}^N)$ of the nonlinear system

$$\operatorname{div} A(x, u, Du) + B(x, u, Du) = 0, \tag{1}$$

or, in components,

$$\sum_{\alpha=1}^{n} D_\alpha \left(A_i^\alpha (x, u, Du)\right) + B_i (x, u, Du) = 0 \quad \text{for} \quad i = 1, \ldots, N.$$

Here Ω is a bounded open subset of $\mathbf{R}^n$, $n \geq 2$, $N \geq 1$, and $Du(x) \in \mathbf{R}^{N \times n}$ denotes the gradient of u at a.e. point $x \in \Omega$. The coefficient functions A and B are defined on the set

$$\mathfrak{Z} = \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^{N \times n}$$

with values in $\mathbf{R}^{N \times n}$ and $\mathbf{R}^N$ respectively.

Definition 1 We say that $u \in W^{1,2}(\Omega, \mathbf{R}^N)$ is a *weak solution* of the system (1) if $A(x, u, Du)$ and $B(x, u, Du)$ are locally integrable and

$$\int_\Omega A(x, u, Du) \cdot D\varphi \, dx = \int_\Omega B(x, u, Du) \cdot \varphi \, dx \tag{2}$$

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for every $\varphi \in C_c^\infty(\Omega, \mathbf{R}^N)$. In components, (2) reads as

$$\int_\Omega \left(\sum_{i,\alpha} A_i^\alpha(x,u,Du)\, D_\alpha \varphi^i \right) dx = \int_\Omega \left(\sum_i B_i(x,u,Du)\, \varphi^i \right) dx.$$

The problem of regularity of a weak solution $u \in W^{1,2}(\Omega, \mathbf{R}^N)$ of the nonlinear superelliptic system (1) has been intensively investigated over the last 24 years. As we know (see Example II.3.2 of [5]), we can in general only expect *partial regularity* if $N > 1$, i.e. Hölder continuity of the gradient Du outside of a closed set of Lebesgue measure zero.

The standard hypotheses are that $A(x,u,P)$ be uniformly superelliptic, Hölder continuous in (x,u) and of class C^1 in P, and that $A(x,u,P)$, its partial derivative $A_P(x,u,P)$ and $B(x,u,P)$ have *natural quadratic growth* in the variable P, i.e.

$$\begin{aligned} &|A(x,u,P)| \le \Gamma(1+|P|), \quad |A_P(x,u,P)| \le \Gamma, \\ &|B(x,u,P)| \le a|P|^2 + b. \end{aligned} \tag{3}$$

Moreover, one assumes the smallness condition

$$2a \sup_\Omega |u| < \gamma,$$

the constant $\gamma > 0$ being determined by the superellipticity[1] of the system:

$$A_P(x,u,P)\cdot(\zeta,\zeta) = \sum_{i,j,\alpha,\beta} D_{P_\beta^j} A_i^\alpha(x,u,P)\, \zeta_\alpha^i \zeta_\beta^j \ge \gamma |\zeta|^2, \tag{4}$$

valid for all $(x,u,P) \in \mathfrak{Z}$ and $\zeta \in \mathbf{R}^{N\times n}$.

We start with a short account of the development of the partial regularity theory for nonlinear systems, always under the hypothesis of superellipticity (see also [5]). The general method of the proof is to compare the given solution u with a solution of a linear system with constant coefficients, for which standard elliptic estimates are available. For the direct approach, this comparison is carried out on an arbitrary ball under a Dirichlet boundary condition; for the indirect approach, it is shown that a sequence of blow-up functions $w_m \in W^{1,2}(B, \mathbf{R}^N)$, rescaled to the unit ball B, converges weakly

[1] *Superellipticity* is often referred to as *strong ellipticity* or simply *ellipticity*. We reserve the term *ellipticity*, which is also known as the *condition of Legendre-Hadamard*, for (9) or (24).

to such a solution.

Partial regularity of the solutions $u \in W^{1,2}\left(\Omega, \mathbf{R}^N\right)$ of the quasilinear system

$$\operatorname{div}\left(\mathcal{A}\left(x,u\right)\cdot Du\right)+B\left(x,u,Du\right)=0 \tag{5}$$

was shown, following the indirect approach, by Morrey [18] and by Giusti and Miranda [11]. A direct proof for this result was later given by Giaquinta and Giusti [7].

Partial regularity for the general nonlinear system (1) was obtained by Giaquinta and Modica [8] and by Ivert [17]. Their direct proofs are based on a reverse Hölder inequality with increasing supports for $Du - P_0$, for any constant $P_0 \in \mathbf{R}^{N\times n}$. This reverse Hölder inequality in turn is derived from Caccioppoli's second inequality by invoking the higher integrability theorem of Gehring, Giaquinta and Modica (see Theorem 3).

As the higher integrability theorem is rather involved, it is desirable to find a simpler partial regularity proof which avoids the use of a reverse Hölder inequality. Such direct proofs were supplied for the quasilinear system $\operatorname{div}\left(\mathcal{A}\left(u\right)\cdot Du\right)=0$ by Evans and Giarrusso [2], for the nonlinear system $\operatorname{div} A\left(Du\right)=0$ by Giarrusso [9], and for the general nonlinear system (1) by Duzaar and Grotowski [1]. An indirect partial regularity proof for (1), which does not even employ a Caccioppoli inequality, was proposed in [14]. Nevertheless, at the moment it seems that our main result, Theorem 1, is unattainable without resort to a reverse Hölder inequality.

Superellipticity (4) implies strict monotonicity

$$\left(A\left(x,u,P+Q\right)-A\left(x,u,P\right)\right)\cdot Q \geq \gamma\left|Q\right|^2. \tag{6}$$

Further, (3), (6) with $P=0$, and the Cauchy inequality yield control from below

$$A\left(x,u,P\right)\cdot P \geq F\left(x,P\right) \tag{7}$$

by the function $F\left(x,P\right)=\gamma\left|P\right|^2/2-$ const. In the present paper we replace the superellipticity hypothesis by weaker versions of (6) and (7). Moreover, as in [3], we dispense with the growth condition on A_P.

First, we assume strict *quasimonotonicity*, which is obtained by integrating (6) when Q is the gradient of a test function, and x, u, P are constants (see Hypothesis 3). Quasimonotonicity is weaker than superellipticity, reducing to superellipticity for $n=1$ or $N=1$. Also, quasiconvexity of a function $F\colon \mathfrak{Z} \to \mathbf{R}$ is weaker than quasimonotonicity of its partial derivative $F_P\colon \mathfrak{Z} \to \mathbf{R}^{N\times n}$. For a further discussion of this concept we refer to [12] (see also Remark 1 (b)). Zhang [19] proved the existence of a weak solution of the nonlinear quasimonotone system (1). Partial regularity of such solutions has to date only been achieved for the quasimonotone system

$$\operatorname{div} A\,(x, Du) + B\,(x, u, Du) = 0, \tag{8}$$

whose leading part does not depend on the variable u. This was shown indirectly by Fuchs [4], and directly by Frasca and Ivanov [3] and by Hamburger [12].

Secondly, we assume (7) with an arbitrary continuous function F that is strictly quasiconvex at zero (see Hypothesis 4). This estimate from below is the analog of the coercivity condition of Hong [16] in the calculus of variations. It will be put to use in the proof of Caccioppoli's first inequality and subsequent higher integrability of the gradient Du (see Theorem 4). While Hypothesis 4 is trivially satisfied for (8) by Remark 1 (a), it is to be considered as the missing link in the chain to partial regularity for the general quasimonotone system (1).

Our indirect proof of partial regularity employs the method which was introduced by Hamburger [13] in the context of minimizers of variational integrals in establishing convergence $w_m \to w$ in $W^{1,2}_{\mathrm{loc}}$ of a sequence of blow-up functions $w_m \in W^{1,2}\left(B, \mathbf{R}^N\right)$, which is known to converge only weakly. This technique has already been applied to nonlinear superelliptic systems in [14], and to polyconvex variational integrals in [15]. Since we are not assuming any growth condition on A_P, we need to define sets $E_{r,m} \subset B_r$, satisfying $\lim_{m\to\infty} |E_{r,m}| = 0$, where the functions w_m or Dw_m exceed a certain bound (cf. [6], [10]). A reverse Hölder inequality for $Du - P_0$, for any constant $P_0 \in \mathbf{R}^{N\times n}$, allows us to control the error integral of a rescaled power of $|Dw_m|$ over the set $E_{r,m}$. We show that the blow-up functions w_m are approximate solutions of suitable rescaled systems. This has two consequences. First, passing to the limit as $m \to \infty$ we infer that w solves a linear elliptic system with constant coefficients. Secondly, we derive the key estimate

$$\limsup_{m\to\infty} \int_{B_r} \zeta^2 G\,(Y_0) \cdot (Dw_m, Dw_m)\, dz \le \int_{B_r} \zeta^2 G\,(Y_0) \cdot (Dw, Dw)\, dz.$$

Here ζ is a cut-off function, the bilinear form $G\,(Y)$ depends continuously on Y, and the constant function Y_0 is the limit in L^2 of a suitable sequence of functions $\{Y_m\}$. We finally deduce from this estimate with the help of strict quasimonotonicity that $w_m \to w$ in $W^{1,2}_{\text{loc}}$. In this manner we achieve partial regularity of weak solutions of fully nonlinear quasimonotone systems in divergence form.

For the coefficient function $A : \mathfrak{Z} \to \mathbf{R}^{N\times n}$ we shall assume the following hypotheses, for an exponent $q \geq 2$.

Hypothesis 1 *We suppose that $A(x,u,P)$ is of class C^1 in P and of polynomial growth*

$$|A(x,u,P)| \leq \Gamma\big(1 + |P|^{q-1}\big),$$

and we assume that A_P is continuous.

Hypothesis 2 *We suppose that $\big(1+|P|^{q-1}\big)^{-1} A(x,u,P)$ is Hölder continuous in (x,u) uniformly with respect to P:*

$$|A(x,u,P) - A(y,v,P)| \leq \big(1+|P|^{q-1}\big)\omega\left(|u|, |x-y| + |u-v|\right)$$

for all $(x,u,P), (y,v,P) \in \mathfrak{Z}$. Here $\omega(s,t) = K(s)\min\big(t^{\delta},1\big)$ for $0<\delta<1$ and for a nondecreasing function $K(s)$; we note that $\omega(s,t)$, for fixed s, is concave, nondecreasing and bounded in t.

Hypothesis 3 *We suppose that A is uniformly strictly quasimonotone*

$$\int_{\mathbf{R}^n} A(x_0,u_0,P_0 + D\varphi)\cdot D\varphi\, dx \geq \int_{\mathbf{R}^n}\Big(\kappa|D\varphi|^2 + \gamma|D\varphi|^q\Big)\,dx$$

for some $\kappa,\gamma > 0$, and all $(x_0,u_0,P_0) \in \mathfrak{Z}$ and $\varphi \in C_c^{\infty}\big(\mathbf{R}^n,\mathbf{R}^N\big)$.

Hypothesis 4 *We suppose that*

$$A(x,u,P)\cdot P \geq F(x,P)$$

for all $(x,u,P) \in \mathfrak{Z}$, and for some function $F(x,P)$, satisfying $|F(x,0)| \leq c$, which is strictly quasiconvex at $P=0$, and for which $(1+|P|^q)^{-1}F(x,P)$ is continuous in x uniformly with respect to P:

$$\int_{\mathbf{R}^n}\left(F(x_0,D\varphi) - F(x_0,0)\right)dx \geq \tilde{\gamma}\int_{\mathbf{R}^n}|D\varphi|^q\,dx$$

for some $\tilde{\gamma} > 0$, and all $x_0 \in \overline{\Omega}$ and $\varphi \in C_c^{\infty}\big(\mathbf{R}^n,\mathbf{R}^N\big)$;

$$|F(x,P) - F(y,P)| \leq (1+|P|^q)\,\tilde{\omega}(|x-y|)$$

for all $x, y \in \overline{\Omega}$ *and* $P \in \mathbf{R}^{N \times n}$, *where* $\tilde{\omega}$ *is continuous and nondecreasing with* $\tilde{\omega}(0) = 0$.

Remark 1 (a) If $A(x, P)$ is independent of the variable u then Hypothesis 4, with $F(x, P) = A(x, P) \cdot P$, is already a consequence of Hypotheses 2 and 3.

(b) If $A(x, u, P) = \mathcal{A}(x, u) \cdot P$ is linear in the variable P, i.e. for the quasilinear system (5), Hypothesis 3 is equivalent to *uniform ellipticity*

$$\mathcal{A}(x, u) \cdot (\eta \otimes \xi, \eta \otimes \xi) \geq \gamma |\eta \otimes \xi|^2 \tag{9}$$

for all $(x, u) \in \overline{\Omega} \times \mathbf{R}^N$ and $\eta \in \mathbf{R}^N$, $\xi \in \mathbf{R}^n$ (see [12]). In general, ellipticity is weaker than quasimonotonicity.

(c) If $A(x, P) = \mathcal{A}(x) \cdot P$ is independent of the variable u and linear in the variable P, i.e. for the semilinear system

$$\operatorname{div}(\mathcal{A}(x) \cdot Du) + B(x, u, Du) = 0,$$

Hypotheses 1 to 4 are equivalent to Hölder continuity and uniform ellipticity (9) of the coefficient matrix $\mathcal{A}(x)$. This follows from (a) and (b).

Example 1 (a) If a coefficient function A verifies Hypotheses 1 to 4, with $n = N \leq q$, then so does

$$A_1(x, u, P) = A(x, u, P) + b(x) \operatorname{cof} P,$$

where b is of class $C^{0,\delta}$, and $\operatorname{cof} P$ denotes the matrix of co-factors[2] of P. Here we set $F_1(x, P) = F(x, P) + nb(x) \det P$ in Hypothesis 4. By the Piola identity $\operatorname{div} \operatorname{cof} Du = 0$, the added expression only contributes a lower order term to (1) if b is Lipschit continuous.

(b) A nonlinear coefficient function verifying Hypotheses 1 to 4, with $n = N = 2$, $q = 4$ and $F(x, P) = (1 + \gamma) |P|^4$, is given by

$$A(x, u, P) = \kappa P + (1 + \gamma) |P|^2 P + \eta(x, u) \det P \operatorname{cof} P,$$

where $\kappa, \gamma > 0$, and η of class $C^{0,\delta}$ satisfies $0 \leq \eta \leq 4$ (see [12], (1.13)).

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[2]It may be helpful to recall the formula

$$\operatorname{cof} Du \cdot D\varphi \, dx^1 \wedge \cdots \wedge dx^n = \sum_{i=1}^{n} du^1 \wedge \cdots \wedge du^{i-1} \wedge d\varphi^i \wedge du^{i+1} \wedge \cdots \wedge du^n.$$

We also assume the coefficient function $B : \mathfrak{Z} \to \mathbf{R}^N$ has natural polynomial growth. In order to save ourselves a separate treatment of the case $B = 0$, we agree that $2aM = 0$ if $a = 0$ and $M = \infty$. Of course, the condition (11) for $M = \infty$ is void.

Hypothesis 5 *We suppose that $B(x,u,P)$ is a Carathéodory function, i.e. measurable in x and continuous in (u,P). Moreover, we assume that, for constants $a, b \in [0,\infty[$ and $M \in [0,\infty]$,*

$$|B(x,u,P)| \le a\,|P|^q + b \quad \text{and} \quad 2aM < \min\{\gamma, \tilde{\gamma}\} \tag{10}$$

for all $(x,u,P) \in \mathfrak{Z}$ with $|u| \le M$.

Our first result concerns the fully nonlinear system (1).

Theorem 1 *Let A and B satisfy Hypotheses 1 to 5, with exponent $q \ge 2$. Let $u \in W^{1,q}\left(\Omega, \mathbf{R}^N\right)$, with*

$$\sup_{\Omega} |u| \le M, \tag{11}$$

be a weak solution of the system

$$\operatorname{div} A(x,u,Du) + B(x,u,Du) = 0.$$

Then there exists an open set $\Omega_0 \subset \Omega$, whose complement has Lebesgue measure zero, such that the gradient Du is locally Hölder continuous in Ω_0, with the exponent $0 < \delta < 1$ of Hypothesis 2:

$$u \in C^{1,\delta}\left(\Omega_0, \mathbf{R}^N\right) \quad \text{and} \quad \mathcal{L}^n(\Omega \setminus \Omega_0) = 0.$$

Moreover, the regular set is characterized by

$$\Omega_0 = \left\{ x_0 \in \Omega : \sup_{r>0} \left(|u_{x_0,r}| + |Du_{x_0,r}|\right) < \infty \right. \\ \left. \text{and} \quad \liminf_{r \searrow 0} \fint_{B_r(x_0)} |Du - Du_{x_0,r}|^q \, dx = 0 \right\}.$$

We now turn to the quasilinear case with $A(x,u,P) = \mathcal{A}(x,u) \cdot P$, for which we restate Hypotheses 1 to 4 with exponent $q = 2$.

Hypothesis 1* *We suppose that $\mathcal{A}$ is continuous and bounded:*

$$|\mathcal{A}(x,u)| \le \Gamma.$$

Hypothesis 2* *We suppose that $\mathcal{A}$ is Hölder continuous with exponent $0 < \delta < 1$.*

Hypothesis 3* *We suppose that $\mathcal{A}$ is uniformly elliptic:*

$$\mathcal{A}(x,u) \cdot (\eta \otimes \xi, \eta \otimes \xi) \geq \gamma |\eta \otimes \xi|^2$$

for some $\gamma > 0$, and all $(x,u) \in \overline{\Omega} \times \mathbf{R}^N$ and $\eta \in \mathbf{R}^N$, $\xi \in \mathbf{R}^n$.

Hypothesis 4* *We suppose that*

$$\mathcal{A}(x,u) \cdot (P,P) \geq F(x,P)$$

for all $(x,u,P) \in \mathfrak{Z}$, and for some function $F(x,P)$, satisfying $|F(x,0)| \leq c$, which is strictly quasiconvex with constant $\tilde{\gamma} > 0$ at $P = 0$, and for which $\left(1 + |P|^2\right)^{-1} F(x,P)$ is continuous in x uniformly with respect to P.

Remark 2 Hypothesis 4* reduces for $F(x,P) = \gamma |P|^2$ to uniform superellipticity of $\mathcal{A}$.

For the quasilinear system (5) we then have

Theorem 2 *Let $\mathcal{A}$ and B satisfy Hypotheses 1*, 3*, 4* and 5. Let $u \in W^{1,2}\left(\Omega, \mathbf{R}^N\right)$, with*

$$\sup_{\Omega} |u| \leq M,$$

be a weak solution of the quasilinear system

$$\operatorname{div}(\mathcal{A}(x,u) \cdot Du) + B(x,u,Du) = 0.$$

Then there exists an open set $\Omega_0 \subset \Omega$, whose complement has Hausdorff dimension less than $n-2$, such that the solution u is locally Hölder continuous in Ω_0, with any exponent $0 < \alpha < 1$:

$$u \in \bigcap_{0<\alpha<1} C^{0,\alpha}\left(\Omega_0, \mathbf{R}^N\right) \quad \text{and} \quad \dim_{\mathcal{H}}(\Omega \setminus \Omega_0) < n-2.$$

Moreover, the regular set is characterized by

$$\Omega_0 = \left\{ x_0 \in \Omega : \sup_{r>0} |u_{x_0,r}| < \infty \right.$$
$$\left. \text{and} \quad \liminf_{r \searrow 0} \fint_{B_r(x_0)} |u - u_{x_0,r}|^2 \, dx = 0 \right\}.$$

If $\mathcal{A}$ also satisfies Hypothesis 2^* with exponent $0 < \delta < 1$ then the gradient Du is locally Hölder continuous in Ω_0, with the same exponent δ:

$$u \in C^{1,\delta}\left(\Omega_0, \mathbf{R}^N\right).$$

Proof. The proofs of Theorems VI.1.3 and VI.1.5 of [5] hold verbatim with ellipticity in place of superellipticity, once the reverse Hölder inequality

$$\left\{ \fint_{B_{R/2}(x_0)} |Du|^{2(1+\epsilon)}\, dx \right\}^{1/(1+\epsilon)} \leq c \fint_{B_R(x_0)} \left(1 + |Du|^2\right) dx,$$

for some $\epsilon > 0$ and every ball $B_R(x_0) \subset\subset \Omega$, is known. Here we derive this inequality from Hypotheses 1^*, 4^* and 5 (see Theorem 4 for $q = 2$). □

2. A decay estimate for the excess

In what follows, all constants c may depend on the data and on the number L from the proof of Proposition 1. The Landau symbol $o(1)$ stands for any quantity for which $\lim_{m\to\infty} o(1) = 0$; this may in Section 4 also depend on the numbers $0 < s < r < 1$ and $\beta > 0$. We write $B_r(x_0) = \{x \in \mathbf{R}^n : |x - x_0| < r\}$, $B_r = B_r(0)$, and $B = B_1$ for the unit ball (we also used the symbol B in (1)). We denote the mean of a function f on the ball $B_r(x_0)$ by

$$f_{x_0,r} = \fint_{B_r(x_0)} f\, dx = \frac{1}{\mathcal{L}^n\left(B_r(x_0)\right)} \int_{B_r(x_0)} f\, dx.$$

In this section we assume Hypotheses 1 to 5 with $q \geq 2$. We let $u \in W^{1,q}\left(\Omega, \mathbf{R}^N\right)$, subject to (11), be a weak solution of the system (1). For the positive exponent $\alpha < \delta/2$ appearing in Theorem 5, we define the excess of Du on the ball $B_r(x_0) \subset\subset \Omega$:

$$U(x_0, r) = r^{2\alpha} + \fint_{B_r(x_0)} \left(|Du - Du_{x_0,r}|^2 + |Du - Du_{x_0,r}|^q\right) dx.$$

The conclusions of Theorem 1, as yet with exponent α instead of δ, follow in a routine way from the next proposition (see [5], pp. 197–199, [6], Section 3, [10], pp. 349–352, [12], Section 5).

Showing that $u \in C^{1,\delta}\left(\Omega_0, \mathbf{R}^N\right)$ with the optimal exponent δ requires a second step. For any open set $\Sigma \subset\subset \Omega_0$, we consider $u \in W^{1,q}\left(\Sigma, \mathbf{R}^N\right)$

as a solution of the system

$$\operatorname{div} \tilde{A}(x, Du) + B(x, u, Du) = 0$$

with the composite coefficient function $\tilde{A}(x, P) = A(x, u(x), P)$. We note that $\tilde{A} : \overline{\Sigma} \times \mathbf{R}^{N\times n} \to \mathbf{R}^{N\times n}$ satisfies Hypotheses 1 to 4 with the same exponent δ, as soon as we know that $u \in C^{0,1}(\Omega_0, \mathbf{R}^N)$. Thus, by invoking Theorem 1.2 of [12], we conclude that $u \in C^{1,\delta}(\Sigma, \mathbf{R}^N)$.[3]

Proposition 1 *Let $L > 0$ and $\tau \in]0,1[$ be given. Then there exist positive constants $c_1(L)$ and $\epsilon(L, \tau)$ such that if*

$$B_r(x_0) \subset\subset \Omega, \ |u_{x_0,r}| \leq L, \ |Du_{x_0,r}| \leq L \ \text{ and } \ U(x_0, r) \leq \epsilon$$

then

$$U(x_0, \tau r) \leq c_1 \tau^{2\alpha} U(x_0, r).$$

Proof. We will determine the constant c_1 later on. If the proposition were not true then there would exist a sequence of balls $B_{r_m}(x_m) \subset\subset \Omega$ such that, setting

$$u_m = u_{x_m,r_m}, \quad P_m = Du_{x_m,r_m}, \quad \lambda_m^2 = U(x_m, r_m), \tag{12}$$

we have

$$|u_m| \leq L, \quad |P_m| \leq L, \quad \lambda_m \searrow 0, \tag{13}$$

but

$$U(x_m, \tau r_m) > c_1 \tau^{2\alpha} \lambda_m^2. \tag{14}$$

We define the rescaled functions

$$w_m(z) = \frac{u(x_m + r_m z) - u_m - r_m P_m \cdot z}{r_m \lambda_m}$$

for $z \in B$. We notice that

$$Dw_m(z) = \frac{Du(x_m + r_m z) - P_m}{\lambda_m}, \tag{15}$$

$$(w_m)_{0,1} = 0, \quad (Dw_m)_{0,1} = 0. \tag{16}$$

[3]Theorem 1.2 of [12] has only been proved under a growth condition on $\tilde{A}_P$, which may however be eliminated following [6].

Then (12) and (14) become

$$\lambda_m^{-2} r_m^{2\alpha} + \fint_B |Dw_m|^2 \, dz + \lambda_m^{q-2} \fint_B |Dw_m|^q \, dz = 1, \tag{17}$$

$$\begin{aligned} c_1 \tau^{2\alpha} < \lambda_m^{-2} r_m^{2\alpha} \tau^{2\alpha} &+ \fint_{B_\tau} \left| Dw_m - (Dw_m)_{0,\tau} \right|^2 dz \\ &+ \lambda_m^{q-2} \fint_{B_\tau} \left| Dw_m - (Dw_m)_{0,\tau} \right|^q dz. \end{aligned} \tag{18}$$

From (16), (17) and the Poincaré inequality we immediately have

$$\lambda_m^{-1} r_m^{\alpha} \le 1, \quad \|w_m\|_{W^{1,2}(B)} \le c, \quad \lambda_m^{(q-2)/q} \|w_m\|_{W^{1,q}(B)} \le c. \tag{19}$$

Since $\alpha < \delta/2$, we infer from (13) and (19) that

$$r_m \searrow 0 \quad \text{and} \quad \lambda_m^{-1} r_m^{\delta/2} \searrow 0. \tag{20}$$

By assumption (11) on u, we also note that

$$\sup_B |r_m \lambda_m w_m| \le 2M + o(1). \tag{21}$$

We denote the rescaled quantity $(x, u, sDu + (1-s) P_m) \in \mathfrak{Z}$:

$$\begin{aligned} &Z_m(s,z) \\ &\quad = (x_m + r_m z,\ u_m + r_m P_m \cdot z + r_m \lambda_m w_m,\ P_m + s \lambda_m D w_m). \end{aligned}$$

It follows from (13) and (19) that, on passing to a subsequence and relabelling, we have

$$\begin{aligned} &Dw_m \rightharpoonup Dw && \text{weakly in } L^2\left(B, \mathbf{R}^{N\times n}\right), \\ &w_m \to w && \text{in } L^2\left(B, \mathbf{R}^{N}\right), \\ &\lambda_m D w_m \to 0 && \text{in } L^2\left(B, \mathbf{R}^{N\times n}\right); \\ \\ &\lambda_m^{(q-2)/q} D w_m \rightharpoonup 0 && \text{weakly in } L^q\left(B, \mathbf{R}^{N\times n}\right), \\ &\lambda_m^{(q-2)/q} w_m \to 0 && \text{in } L^q\left(B, \mathbf{R}^{N}\right) \quad (\text{for } q > 2); \\ \\ &(x_m, u_m, P_m) \to Z_0 = (x_0, u_0, P_0) && \text{in } \mathfrak{Z}, \\ &Z_m \to Z_0 && \text{in } L^2([0,1] \times B, \mathfrak{Z}). \end{aligned} \tag{22}$$

Now suppose that we can show that $w \in W^{1,2}(B, \mathbf{R}^N)$ is a weak solution of the following linear system with constant coefficients:

$$\operatorname{div}(A_P(Z_0) \cdot Dw) = 0. \tag{23}$$

We infer from Hypothesis 1 and (13) that

$$|A_P(Z_0)| \leq c,$$

and from Hypothesis 3 that (23) is uniformly elliptic (see [12]):

$$A_P(Z_0) \cdot (\eta \otimes \xi, \eta \otimes \xi) \geq \gamma |\eta \otimes \xi|^2 \quad \text{for all} \quad \eta \in \mathbf{R}^N, \ \xi \in \mathbf{R}^n. \tag{24}$$

Hence, from the relevant regularity theory (see [5], Theorem III.2.1, Remarks III.2.2, III.2.3) we conclude that w is smooth and

$$\fint_{B_\tau} |Dw - Dw_{0,\tau}|^2 \, dz \leq c_2 \tau^2 \fint_B |Dw - Dw_{0,1}|^2 \, dz, \tag{25}$$

where by (16), (17) and (22)

$$Dw_{0,1} = 0 \quad \text{and} \quad \fint_B |Dw|^2 \, dz \leq \liminf_{m \to \infty} \fint_B |Dw_m|^2 \, dz \leq 1. \tag{26}$$

On the other hand, if we also know that

$$Dw_m \to Dw \qquad \text{in} \ \ L^2_{\text{loc}}\left(B, \mathbf{R}^{N \times n}\right), \tag{27}$$

$$\lambda_m^{(q-2)/q} Dw_m \to 0 \quad \text{in} \ \ L^q_{\text{loc}}\left(B, \mathbf{R}^{N \times n}\right) \ \ (\text{for } q > 2) \tag{28}$$

then it would follow from (18) and (19) that

$$c_1 \tau^{2\alpha} \leq \tau^{2\alpha} + \fint_{B_\tau} |Dw - Dw_{0,\tau}|^2 \, dz.$$

If we now choose $c_1 > 1 + c_2$, we obtain a contradiction to (25) and (26). This proves the proposition. □

The remainder of this work is devoted to showing (23), and (27), (28), which are the assertions of Lemmas 4 and 5 respectively.

We introduce some further notation. We define the set

$$\mathfrak{Y} = \overline{\Omega} \times \mathbf{R}^N \times \mathbf{R}^N \times \mathbf{R}^{N \times n} \times \mathbf{R}^{N \times n}.$$

Next, we define a bilinear form $G(Y)$ on $\mathbf{R}^{N\times n}$, for $Y=(x,u,v,P,Q)\in\mathfrak{Y}$, by

$$G(Y)=\int_0^1 A_P(x,u,P+sQ)\,ds-2a\,|u-v|\,|Q|^{q-2}\,I,$$

where $I=\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbf{R}^{N\times n}$. By Hypothesis 1, the bilinear form $G(Y)$ depends continuously on $Y\in\mathfrak{Y}$. We observe that

$$\begin{aligned}G(x,u,u,P,Q)\cdot Q&=\int_0^1 A_P(x,u,P+sQ)\cdot Q\,ds\\&=A(x,u,P+Q)-A(x,u,P).\end{aligned}\tag{29}$$

Therefore, by Hypothesis 1 and Young's inequality, we infer that

$$\begin{aligned}&\left|\int_0^1 A_P(x,u,P+sQ)\cdot(Q,R)\,ds\right|\\&\quad=|(A(x,u,P+Q)-A(x,u,P))\cdot R|\\&\quad\le c\left(1+|P|^q+|Q|^q+|R|^q\right).\end{aligned}\tag{30}$$

We end this section by showing that w_m is, to order zero as $m\to\infty$, a weak solution of a rescaled system of inequalities.

Lemma 1 *Suppose that A and B satisfy Hypotheses* 1, 2, 3 *and* 5. *For $\varphi\in W_0^{1,q}\left(B,\mathbf{R}^N\right)$, with $\sup_B|\varphi|<\infty$ if $a\neq 0$, we then have*

$$\begin{aligned}&\int_B\int_0^1 A_P(Z_m(s,z))\cdot(Dw_m,D\varphi)\,ds\,dz\\&\qquad\le 2a\int_B\lambda_m^{q-2}\,|Dw_m|^q\,r_m\lambda_m\,|\varphi|\;dz+c\lambda_m^{-1}r_m^{\delta/2}\,\|\varphi\|_{W^{1,2}}.\end{aligned}\tag{31}$$

Proof. Rescaling the system (1) we find

$$\int_B A(Z_m(1,z))\cdot D\varphi\,dz=r_m\int_B B(Z_m(1,z))\cdot\varphi\,dz$$

for every $\varphi\in C_c^\infty\left(B,\mathbf{R}^N\right)$. So it follows from (29) that

$$\begin{aligned}&\int_B\int_0^1 A_P(Z_m(s,z))\cdot(Dw_m,D\varphi)\,ds\,dz\\&\quad=\lambda_m^{-1}\int_B\big(A(Z_m(1,z))-A(Z_m(0,z))\big)\cdot D\varphi\,dz\end{aligned}$$

$$
\begin{aligned}
&= \lambda_m^{-1} r_m \int_B B\left(Z_m(1,z)\right) \cdot \varphi \, dz \\
&\quad - \lambda_m^{-1} \int_B \big(A\left(x_m + r_m z,\ u_m + r_m P_m \cdot z + r_m \lambda_m w_m,\ P_m\right) \\
&\qquad\qquad - A\left(x_m, u_m, P_m\right)\big) \cdot D\varphi \, dz = (\mathrm{I}) + (\mathrm{II}) .
\end{aligned}
$$

By virtue of Hypothesis 2, we estimate the term (II) as follows (using (13), (19), Hölder's and Jensen's inequalities in combination with the boundedness and concavity of $\omega(L,\cdot)$, and the inequality $\omega(L,t) \le ct^{\delta}$):

$$
\begin{aligned}
(\mathrm{II}) &\le c\big(1+|P_m|^{q-1}\big)\lambda_m^{-1} \fint_B \omega\left(|u_m|,\ r_m + r_m |P_m| + r_m \lambda_m |w_m|\right) |D\varphi| \, dz \\
&\le c\lambda_m^{-1} \left(\fint_B \omega\left(L,\ r_m + r_m L + r_m \lambda_m |w_m|\right) dz \right)^{1/2} \|\varphi\|_{W^{1,2}} \\
&\le c\lambda_m^{-1} \omega \left(L,\ r_m + r_m L + r_m \lambda_m \fint_B |w_m| \, dz \right)^{1/2} \|\varphi\|_{W^{1,2}} \\
&\le c\lambda_m^{-1} r_m^{\delta/2} \|\varphi\|_{W^{1,2}} .
\end{aligned}
$$

We next apply the estimate (10) to the term (I) and we use the inequality $(x+y)^q \le 2x^q + cy^q$. This gives

$$
\begin{aligned}
(\mathrm{I}) &\le \lambda_m^{-1} r_m \int_B \left(a\,|P_m + \lambda_m D w_m|^q + b\right) |\varphi| \, dz \\
&\le 2a \int_B \lambda_m^{q-2} |Dw_m|^q \, r_m \lambda_m |\varphi| \, dz + c\lambda_m^{-1} r_m \|\varphi\|_{L^2} .
\end{aligned}
$$

□

3. Caccioppoli and reverse Hölder inequalities

We recall a simple algebraic lemma (see [5], Lemma V.3.1, [10], Lemma 6.1) and the higher integrability theorem of Gehring, Giaquinta and Modica (see [5], Proposition V.1.1, [10], Theorem 6.6).

Lemma 2 *Let $f(t)$ be a bounded nonnegative function defined for $R/2 \le t \le R$. Suppose that*

$$
f(t) \le \theta f(s) + A(s-t)^{-2} + B(s-t)^{-q} + C
$$

for $R/2 \le t < s \le R$, where θ, A, B, C are nonnegative constants with $\theta < 1$. Then

$$f\left(\frac{R}{2}\right) \leq c(\theta, q)\left(AR^{-2} + BR^{-q} + C\right).$$

Theorem 3 *Let Ω be a bounded open subset of $\mathbf{R}^n$, and let $f \in L^1_{\mathrm{loc}}(\Omega)$ and $g \in L^t_{\mathrm{loc}}(\Omega)$ be nonnegative functions with $0 < s < 1 < t < \infty$. Suppose that*

$$\fint_{B_{R/2}(x_0)} f\,dx \leq b\left\{\fint_{B_R(x_0)} f^s\,dx\right\}^{1/s} + \fint_{B_R(x_0)} g\,dx \tag{32}$$

for every ball $B_R(x_0) \subset\subset \Omega$ with $R \leq R_0$. Then $f \in L^{1+\epsilon}_{\mathrm{loc}}(\Omega)$ for any $0 \leq \epsilon < \epsilon_0$, and

$$\left\{\fint_{B_{\mu R}(x_0)} f^{1+\epsilon} dx\right\}^{1/(1+\epsilon)} \leq c \fint_{B_R(x_0)} f\,dx + c\left\{\fint_{B_R(x_0)} g^{1+\epsilon} dx\right\}^{1/(1+\epsilon)}$$

for every ball $B_R(x_0) \subset\subset \Omega$ with $R \leq R_0$, and $0 < \mu < 1$, where $\epsilon_0 = \epsilon_0(n, s, t, b)$ and $c = c(n, s, t, b, \mu, \epsilon)$ are positive constants.

The next lemma is similar to [6], Lemma 2.1, [10], Lemma 9.1.

Lemma 3 *There exists a constant c such that the estimate*

$$|A(x_0, u_0, P_0 + P) - A(x_0, u_0, P_0)| \leq c\left(|P| + |P|^{q-1}\right)$$

holds for all $P \in \mathbf{R}^{N\times n}$, and $(x_0, u_0, P_0) \in \mathfrak{Z}$ with $|u_0| \leq L$ and $|P_0| \leq L$.

Proof. We let $K = \sup_{\mathfrak{Z}_L} |A_P|$ for the compact set

$$\mathfrak{Z}_L = \{(x, u, P) \in \mathfrak{Z} : |u|, |P| \leq L + 1\}.$$

For $|P| \leq 1$, we then have

$$\begin{aligned}
&|A(x_0, u_0, P_0 + P) - A(x_0, u_0, P_0)| \\
&\quad = \left|\int_0^1 A_P(x_0, u_0, P_0 + sP) \cdot P\,ds\right| \leq K|P|,
\end{aligned}$$

while for $|P| \geq 1$,

$$|A(x_0, u_0, P_0 + P) - A(x_0, u_0, P_0)| \leq c\left(1 + |P|^{q-1}\right) \leq c|P|^{q-1}.$$

□

We first prove a reverse Hölder inequality for Du (cf. [6], Proposition 2.2 and (2.18), [10], Theorem 6.7 and Proposition 9.1).

Theorem 4 *Let $u \in W^{1,q}(\Omega, \mathbf{R}^N)$ be a weak solution of the system (1), whose coefficient functions A and B satisfy Hypotheses 1, 4 and 5.*

Then $Du \in L^{q(1+\epsilon')}_{\mathrm{loc}}(\Omega, \mathbf{R}^{N\times n})$ for any $0 \le \epsilon' < \epsilon_1$, and

$$\left\{ \fint_{B_{\mu R}(x_0)} (1 + |Du|^q)^{1+\epsilon'}\, dx \right\}^{1/(1+\epsilon')} \le c \fint_{B_R(x_0)} (1 + |Du|^q)\, dx$$

for every ball $B_R(x_0) \subset\subset \Omega$ with $R \le R_0$, and $0 < \mu < 1$, where ϵ_1, R_0 and $c(\mu, \epsilon')$ are positive constants.

Proof. We fix some ball $B_R(x_0) \subset\subset \Omega$ with $R \le R_0$, and we set $u_0 = u_{x_0,R}$. For $R/2 \le t < s \le R$, we let $\zeta \in C_c^\infty(B_s(x_0))$ be a cut-off function with $0 \le \zeta \le 1$, $\zeta = 1$ on $B_t(x_0)$ and $|D\zeta| \le c(s-t)^{-1}$. We set $\varphi = \zeta(u - u_0)$, which by (11) satisfies $|\varphi| \le 2M$. By Hypothesis 4 and (2), we then have

$$\begin{aligned}
\tilde\gamma \int_{B_s} |D\varphi|^q\, dx &\le \int_{B_s} (F(x_0, D\varphi) - F(x_0, 0))\, dx \\
&\le \int_{B_s} F(x, D\varphi)\, dx + \tilde\omega(R_0) \int_{B_s} |D\varphi|^q\, dx + cR^n \\
&\le \int_{B_s} (A(x, u, D\varphi) - A(x, u, Du)) \cdot D\varphi\, dx \\
&\quad + \int_{B_s} B(x, u, Du) \cdot \varphi\, dx + \tilde\omega(R_0) \int_{B_s} |D\varphi|^q\, dx + cR^n.
\end{aligned}$$

Thus, by Hypotheses 1 and 5, and Young's inequality, we obtain

$$\begin{aligned}
\int_{B_t} |Du|^q\, dx &\le c_1 \int_{B_s \setminus B_t} |Du|^q\, dx + \frac{2aM + 2\tilde\omega(R_0)}{\tilde\gamma} \int_{B_s} |Du|^q\, dx \\
&\quad + c(s-t)^{-q} \int_{B_R} |u - u_0|^q\, dx + cR^n.
\end{aligned}$$

We now "fill the hole", that is, we add c_1 times the left hand side to both sides and we divide the resulting inequality by $1 + c_1$. This yields

$$\int_{B_t} |Du|^q\, dx \le \theta \int_{B_s} |Du|^q\, dx + c(s-t)^{-q} \int_{B_R} |u - u_0|^q\, dx + cR^n,$$

where by (10)

$$\theta = \frac{\frac{2aM + 2\tilde\omega(R_0)}{\tilde\gamma} + c_1}{1 + c_1} < 1$$

for sufficiently small R_0. By an application of Lemma 2, we arrive at *Caccioppoli's first inequality*

$$\int_{B_{R/2}} |Du|^q \, dx \leq cR^{-q} \int_{B_R} |u - u_0|^q \, dx + cR^n.$$

By the Poincaré-Sobolev inequality, we deduce the estimate (32) with $f = 1 + |Du|^q$, $g = 0$ and $s = q_*/q = n/(n+q) < 1$. The result now follows by Theorem 3. $\square$

We are ready for a reverse Hölder inequality for $Du - P_0$ plus an error term (cf. [12], Corollary 4.1, [6], Theorems 2.2 and 2.5)[4]. It provides a uniform bound in $L^{2(1+\epsilon)}_{\mathrm{loc}}$ for the gradients of the blow-up functions w_m (see Corollary 1).

Theorem 5 *Let $u \in W^{1,q}(\Omega, \mathbf{R}^N)$ be a weak solution of the system* (1), *whose coefficient functions A and B satisfy Hypotheses* 1 *to* 5.

Then there exist positive constants ϵ, $\alpha < \delta/2$, R_0 and $c(\mu)$ such that

$$\begin{aligned} &\left\{ ⨍_{B_{\mu R}(x_0)} \left(|Du - P_0|^2 + |Du - P_0|^q \right)^{1+\epsilon} dx \right\}^{1/(1+\epsilon)} \\ &\quad \leq c ⨍_{B_R(x_0)} \left(|Du - P_0|^2 + |Du - P_0|^q \right) dx \\ &\qquad + cR^{2\alpha} \left\{ ⨍_{B_R(x_0)} (1 + |Du|^q) \, dx \right\}^{1+2\alpha/q} \end{aligned}$$

holds for every ball $B_R(x_0) \subset\subset \Omega$, $0 < \mu < 1$ and $P_0 \in \mathbf{R}^{N \times n}$, with $R \leq R_0$, $|u_{x_0,R}| \leq L$ and $|P_0| \leq L$.

Proof. We fix $B_R(x_0) \subset\subset \Omega$ and $P_0 \in \mathbf{R}^{N \times n}$, subject to the conditions $R \leq R_0$, $|u_0| \leq L$ and $|P_0| \leq L$, where $u_0 = u_{x_0,R}$.

We next fix some ball $B_r(y_0) \subset\subset B_R(x_0)$. For $r/2 \leq t < s \leq r$, we let $\zeta \in C_c^\infty(B_s(y_0))$ be a cut-off function with $0 \leq \zeta \leq 1$, $\zeta = 1$ on $B_t(y_0)$ and

[4] The efforts of both Giusti and Hamburger to simplify the argument in [6] are fruitless: the estimate in line 14 on p. 337 of the excellent book [10] is incorrect; in the proof of [13], Theorem 4.3, we need to define $P(x)$ by (33) in line 4 on p. 274, to add an extra term $|u - P|$ to the second argument of ω in line 18 on p. 275, and thereafter to complete the proof as in [6]. This extra term does not occur in our proof of Theorem 5, so here we can follow [13].

$|D\zeta| \leq c\,(s-t)^{-1}$. We set

$$\begin{aligned} &P(x) = u_{y_0,r} + P_0 \cdot (x - y_0)\,, \\ &\varphi = \zeta\,(u - P)\,, \quad \psi = (1-\zeta)\,(u-P)\,, \end{aligned} \tag{33}$$

for which

$$\varphi + \psi = u - P, \quad D\varphi + D\psi = Du - P_0,$$

and

$$a\,|\varphi| \leq 2aM + cr. \tag{34}$$

Hypotheses 2, 3 and 5, (2) and Lemma 3 assert that

$$\begin{aligned} &\int_{B_s} \left(\kappa\,|D\varphi|^2 + \gamma\,|D\varphi|^q\right) dx \\ &\quad \leq \int_{B_s} \left(A\,(x_0,u_0,P_0+D\varphi) - A\,(x_0,u_0,P_0+D\varphi+D\psi)\right)\cdot D\varphi\,dx \\ &\qquad + \int_{B_s} \left(A\,(x_0,u_0,Du) - A\,(x,u,Du)\right)\cdot D\varphi\,dx \\ &\qquad + \int_{B_s} B\,(x,u,Du)\cdot\varphi\,dx \\ &\quad \leq c\int_{B_s\setminus B_t} \left(|D\varphi| + |D\varphi|^{q-1} + |D\psi| + |D\psi|^{q-1}\right)|D\varphi|\;dx \\ &\qquad + \int_{B_s} \omega\left(|u_0|\,, |x-x_0| + |u-u_0|\right)\left(1+|Du|^{q-1}\right)|D\varphi\,|\;dx \\ &\qquad + \int_{B_s} \left(a\,|Du|^q + b\right)|\varphi|\;dx. \end{aligned}$$

We estimate the last term, using the inequality $(x+y)^q \leq (1+\epsilon)\,x^q + c\,(\epsilon)\,y^q$, (34), the Poincaré inequality for φ on $B_s\,(y_0)$ and the Young inequality, as

$$\begin{aligned} \int_{B_s} \left(a\,|Du|^q + b\right)|\varphi|\;dx &\leq \left((1+\epsilon)\,2aM + c\,(\epsilon)\,R_0\right)\int_{B_s} |D\varphi|^q\;dx \\ &\quad + c\,(\epsilon)\int_{B_s} |D\psi|^q\;dx + c\,(\epsilon)\,r^{n+1}. \end{aligned}$$

By choosing ϵ and R_0 sufficiently small so that $(1+\epsilon)\,2aM + c\,(\epsilon)\,R_0 < \gamma$, we conclude that

$$\int_{B_t} f\,dx \le c_1 \int_{B_s \setminus B_t} f\,dx + c\,(s-t)^{-2} \int_{B_r} |u-P|^2\,dx$$
$$+\,c\,(s-t)^{-q} \int_{B_r} |u-P|^q\,dx + c \int_{B_r} g\,dx$$

for the functions

$$f = |Du - P_0|^2 + |Du - P_0|^q,$$
$$g = \omega\left(|u_0|, |x - x_0| + |u - u_0|\right)\left(1 + |Du|^q\right) + R.$$

We note that the definitions of f and g do not involve y_0 or r. "Filling the hole" and applying Lemma 2, with $\theta = c_1/(1+c_1) < 1$, results in *Caccioppoli's second inequality*

$$\int_{B_{r/2}} f\,dx \le cr^{-2} \int_{B_r} |u-P|^2\,dx + cr^{-q} \int_{B_r} |u-P|^q\,dx + c \int_{B_r} g\,dx.$$

By means of the Poincaré-Sobolev and Hölder inequalities, we deduce, for $s = 2_*/2 = n/(n+2) < 1$, that

$$\fint_{B_{r/2}(y_0)} f\,dx \le c\left\{\fint_{B_r(y_0)} f^s\,dx\right\}^{1/s} + c\fint_{B_r(y_0)} g\,dx$$

for all $B_r(y_0) \subset\subset B_R(x_0)$. Invoking Theorem 3 we finally arrive at

$$\left\{\fint_{B_{\mu R}(x_0)} f^{1+\epsilon}\,dx\right\}^{1/(1+\epsilon)}$$
$$\le c\fint_{B_{\nu R}(x_0)} f\,dx + c\left\{\fint_{B_{\nu R}(x_0)} g^{1+\epsilon}dx\right\}^{1/(1+\epsilon)} \tag{35}$$

for some exponent $0 < \epsilon < \epsilon'$ and $0 < \mu < \nu < 1$. Here the constant c also depends on μ/ν and ϵ, and ϵ' is the exponent from Theorem 4.

We next set

$$\beta = \frac{\epsilon' - \epsilon}{(1+\epsilon')(1+\epsilon)} \quad \text{and} \quad 2\alpha = \delta\beta.$$

Then, using the Hölder, Jensen and Poincaré inequalities, the boundedness and concavity of $\omega(L,\cdot)$, Theorem 4 and the estimate $\omega(L,t) \le ct^\delta$ we control the last term of (35) by

$$\begin{aligned}
&\left\{ \fint_{B_{\nu R}} g^{1+\epsilon} dx \right\}^{1/(1+\epsilon)} \\
&\leq c \left\{ \fint_{B_{\nu R}} \omega \left(|u_0|, |x - x_0| + |u - u_0| \right) dx \right\}^{\beta} \\
&\qquad\qquad \times \left\{ \fint_{B_{\nu R}} \left(1 + |Du|^q\right)^{1+\epsilon'} dx \right\}^{1/(1+\epsilon')} + cR \\
&\leq c\omega \left(L,\ R + \fint_{B_R} |u - u_0|\ dx \right)^{\beta} \fint_{B_R} \left(1 + |Du|^q\right) dx + cR \\
&\leq c\omega \left(L,\ cR \left\{ \fint_{B_R} \left(1 + |Du|^q\right) dx \right\}^{1/q} \right)^{\beta} \fint_{B_R} \left(1 + |Du|^q\right) dx + cR \\
&\leq cR^{2\alpha} \left\{ \fint_{B_R} \left(1 + |Du|^q\right) dx \right\}^{1+2\alpha/q} .
\end{aligned}$$

□

Corollary 1 *In terms of* $w_m \in W^{1,q}\left(B, \mathbf{R}^N\right)$*, we have, for* $0 < r < 1$*,*

$$\left\{ \fint_{B_r} \left(|Dw_m|^2 + \lambda_m^{q-2} |Dw_m|^q \right)^{1+\epsilon} dz \right\}^{1/(1+\epsilon)} \leq c\,(r)\,.$$

Proof. Substituting (15), $x_0 = x_m$, $R = r_m$, $\mu = r$ and $P_0 = P_m$ in Theorem 4, and using (13) and (19) yields

$$\begin{aligned}
&\left\{ \fint_{B_r} \left(|Dw_m|^2 + \lambda_m^{q-2} |Dw_m|^q \right)^{1+\epsilon} dz \right\}^{1/(1+\epsilon)} \\
&\quad \leq c\,(r) \fint_B \left(|Dw_m|^2 + \lambda_m^{q-2} |Dw_m|^q \right) dz \\
&\qquad + c\,(r)\, \lambda_m^{-2} r_m^{2\alpha} \left\{ \fint_B \left(1 + |P_m + \lambda_m Dw_m|^q\right) dz \right\}^{1+2\alpha/q} \\
&\quad \leq c\,(r)\,.
\end{aligned}$$

□

4. Convergence of the blow-up functions

For $0 < r < 1$ and $\beta > 0$, we define the sets

$$E_{r,m} = \{z \in B_r : \lambda_m (|w_m| + |Dw_m|) \geq \beta\} .$$

By (13) and (19), we infer that

$$|E_{r,m}| \leq \beta^{-2} \lambda_m^2 \int_{E_{r,m}} (|w_m| + |Dw_m|)^2 \, dz \leq c\beta^{-2} \lambda_m^2 = o(1) . \quad (36)$$

Moreover, by the Hölder inequality, Corollary 1 and (36), we deduce

$$\int_{E_{r,m}} \lambda_m^{q-2} |Dw_m|^q \, dz$$
$$\leq \left\{ \int_{B_r} \left(\lambda_m^{q-2} |Dw_m|^q \right)^{1+\epsilon} dz \right\}^{1/(1+\epsilon)} |E_{r,m}|^{\epsilon/(1+\epsilon)} = o(1) .$$

In summary,

$$\int_{E_{r,m}} \left(\lambda_m^{-2} + \lambda_m^{q-2} |Dw_m|^q \right) dz \leq c\beta^{-2} + o(1) . \quad (37)$$

By choosing for φ a test function in Lemma 1 and taking the limit as $m \to \infty$, we derive

Lemma 4 *The function $w \in W^{1,2}(B, \mathbf{R}^N)$ is a weak solution of the linear elliptic system with constant coefficients*

$$\operatorname{div}(A_P(Z_0) \cdot Dw) = 0.$$

In particular, we conclude that w is smooth.

Proof. We fix $\varphi \in C_c^\infty(B_r, \mathbf{R}^N)$ with $0 < r < 1$, and $\beta > 0$. We define the compact set

$$\mathfrak{Z}_\beta = \{(x, u, P) \in \mathfrak{Z} : |u| , |P| \leq 2L + \beta\}$$

and note that $Z_m(s, z) \in \mathfrak{Z}_\beta$ for every $s \in [0, 1]$ and a.e. $z \in B_r \setminus E_{r,m}$. Therefore

$$\sup_{[0,1]\times(B_r\setminus E_{r,m})} |A_P(Z_m)| \leq \sup_{\mathfrak{Z}_\beta} |A_P| = c(\beta) . \quad (38)$$

We claim by Lebesgue's dominated convergence that

$$\left(1 - \chi_{E_{r,m}}\right) A_P(Z_m) \to A_P(Z_0) \text{ in } L^2([0, 1] \times B_r) , \quad (39)$$

where $\chi_{E_{r,m}}$ is the characteristic function of the set $E_{r,m}$. Indeed, the left hand side of (39) is bounded and converges pointwise a.e. on $[0, 1] \times B_r$, by

(36), which asserts that $\chi_{E_{r,m}} \to 0$ in $L^1(B_r)$, and by (22), (38) and the continuity of A_P.

As a consequence of (22) and (39), we have

$$\lim_{m\to\infty} \int_{B_r \setminus E_{r,m}} \int_0^1 A_P(Z_m) \cdot (Dw_m, D\varphi)\, ds\, dz$$
$$= \int_{B_r} A_P(Z_0) \cdot (Dw, D\varphi)\, dz. \tag{40}$$

It next follows from (13) and (30) that

$$\left| \int_{E_{r,m}} \int_0^1 A_P(Z_m) \cdot (Dw_m, D\varphi)\, ds\, dz \right|$$
$$\le c \int_{E_{r,m}} \left(\lambda_m^{-2} + \lambda_m^{q-2} |Dw_m|^q + \lambda_m^{q-2} |D\varphi|^q \right) dz. \tag{41}$$

For $\varphi \in C_c^\infty(B_r, \mathbf{R}^N)$, the right hand side of (31) is easily seen to approach zero as $m \to \infty$. Therefore, using (13), (37) and (41) we obtain

$$\int_{B_r \setminus E_{r,m}} \int_0^1 A_P(Z_m) \cdot (Dw_m, D\varphi)\, ds\, dz$$
$$= o(1) - \int_{E_{r,m}} \int_0^1 A_P(Z_m) \cdot (Dw_m, D\varphi)\, ds\, dz \le c\beta^{-2} + o(1).$$

We conclude by (40) and since $\beta > 0$ was arbitrary that

$$\int_{B_r} A_P(Z_0) \cdot (Dw, D\varphi)\, dz \le 0,$$

and the result follows by replacing φ by $-\varphi$. □

Lemma 5 *We have the limits*

$$Dw_m \to Dw \qquad \text{in } L^2_{\text{loc}}\left(B, \mathbf{R}^{N\times n}\right), \tag{42}$$
$$\lambda_m^{(q-2)/q} Dw_m \to 0 \quad \text{in } L^q_{\text{loc}}\left(B, \mathbf{R}^{N\times n}\right) \quad (\text{for } q > 2). \tag{43}$$

In the proof we shall make use of the fact that, by Lemma 4, the function w and its gradient Dw are locally bounded on B.

We fix $0 < s < r < 1$ and $\beta > 0$. We let $\zeta \in C_c^\infty(B_r)$ be a cut-off function with $0 \le \zeta \le 1$, $\zeta = 1$ on B_s and $|D\zeta| \le c(r-s)^{-1}$, and we define the functions

$$\begin{aligned} Y_m &= \left(x_m + r_m z,\ u_m + r_m P_m \cdot z + r_m \lambda_m w_m,\ u_m,\ P_m,\ \lambda_m D w_m\right), \\ \tilde{Y}_m &= \left(x_m,\ u_m,\ u_m,\ P_m,\ \lambda_m \zeta \left(D w_m - D w\right) + \lambda_m \left(w_m - w\right) \otimes D\zeta\right). \end{aligned}$$

By virtue of (13), (20) and (22), we notice the limits

$$Y_m, \tilde{Y}_m \to Y_0 = \left(x_0, u_0, u_0, P_0, 0\right) \quad \text{in} \quad L^2\left(B, \mathfrak{Y}\right).$$

We define the compact set

$$\begin{aligned} \mathfrak{Y}_\beta = \Big\{ (x, u, v, P, Q) \in \mathfrak{Y} : |u|, |v|, |P|, |Q| \le 2L + \beta \\ + \|Dw\|_{L^\infty(B_r)} + c\,(r - s)^{-1} \left(\beta + \|w\|_{L^\infty(B_r)}\right) \Big\} \end{aligned}$$

and note that $Y_m(z), \tilde{Y}_m(z) \in \mathfrak{Y}_\beta$ for a.e. $z \in B_r \setminus E_{r,m}$. Therefore

$$\sup_{B_r \setminus E_{r,m}} \left\{ \left|G\left(Y_m\right)\right|, \left|G\big(\tilde{Y}_m\big)\right| \right\} \le \sup_{\mathfrak{Y}_\beta} |G| = c\,(\beta, r, s). \tag{44}$$

By the same argument as that for (39), we show that

$$\begin{aligned} \left(1 - \chi_{E_{r,m}}\right) G\left(Y_m\right) &\to G\left(Y_0\right) \quad \text{in} \quad L^p\left(B_r\right), \\ \left(1 - \chi_{E_{r,m}}\right) G\big(\tilde{Y}_m\big) &\to G\left(Y_0\right) \quad \text{in} \quad L^p\left(B_r\right), \end{aligned}$$

for $1 \le p < \infty$. We then infer by Hölder's inequality and Corollary 1 that

$$\begin{aligned} &\int_{B_r \setminus E_{r,m}} \zeta^2 G\left(Y_m\right) \cdot \left(D w_m, D w_m\right) dz - \int_{B_r} \zeta^2 G\left(Y_0\right) \cdot \left(D w_m, D w_m\right) dz \\ &\quad \le \left\{ \int_{B_r} \left| \left(1 - \chi_{E_{r,m}}\right) G\left(Y_m\right) - G\left(Y_0\right) \right|^{(1+\epsilon)/\epsilon} dz \right\}^{\epsilon/(1+\epsilon)} \\ &\qquad\qquad \times \left\{ \int_{B_r} \left|D w_m\right|^{2(1+\epsilon)} dz \right\}^{1/(1+\epsilon)} \\ &\quad = o(1). \end{aligned} \tag{45}$$

Similarly, we obtain

$$\begin{aligned} &\int_{B_r \setminus E_{r,m}} \zeta^2 G\big(\tilde{Y}_m\big) \cdot \left(D w_m - D w, D w_m - D w\right) dz \\ &\quad \le \int_{B_r} \zeta^2 G\left(Y_0\right) \cdot \left(D w_m - D w, D w_m - D w\right) dz + o(1). \end{aligned} \tag{46}$$

We now insert $\varphi = \zeta^2 (w_m - w) \in W_0^{1,q}(B_r, \mathbf{R}^N)$ in (31), for which

$$D\varphi = \zeta^2 (Dw_m - Dw) + 2\zeta (w_m - w) \otimes D\zeta.$$

By (13), (19), (20), (21), (22), (37), (38) and (41), this easily yields

$$\begin{aligned}
&\int_{B_r \setminus E_{r,m}} \zeta^2 G(Y_m) \cdot (Dw_m, Dw_m)\, dz \\
&\quad = \int_{B_r \setminus E_{r,m}} \int_0^1 \zeta^2 A_P(Z_m) \cdot (Dw_m, Dw_m)\, ds\, dz \\
&\qquad - 2a \int_{B_r \setminus E_{r,m}} \zeta^2 |r_m P_m \cdot z + r_m \lambda_m w_m| \lambda_m^{q-2} |Dw_m|^q\, dz \\
&\quad \le \int_{B_r \setminus E_{r,m}} \int_0^1 \zeta^2 A_P(Z_m) \cdot (Dw_m, Dw)\, ds\, dz + c\beta^{-2} + o(1).
\end{aligned}$$

We also use (40) (with $\zeta^2 Dw$ in place of $D\varphi$, and noting that $A_P(Z_0) = G(Y_0)$) and (45), and we arrive at the key estimate

$$\begin{aligned}
&\limsup_{m\to\infty} \int_{B_r} \zeta^2 G(Y_0) \cdot (Dw_m, Dw_m)\, dz \\
&\qquad \le \int_{B_r} \zeta^2 G(Y_0) \cdot (Dw, Dw)\, dz + c\beta^{-2}.
\end{aligned} \tag{47}$$

According to Hypothesis 3, (13), (29), (30) and the definition of $\tilde{Y}_m$, we have

$$\begin{aligned}
&\int_{B_r} \left(\kappa |D\varphi|^2 + \gamma \lambda_m^{q-2} |D\varphi|^q\right) dz \\
&\le \lambda_m^{-1} \int_{B_r} \left(A(x_m, u_m, P_m + \lambda_m D\varphi) - A(x_m, u_m, P_m)\right) \cdot D\varphi\, dz \\
&\le c \int_{E_{r,m}} \left(\lambda_m^{-2} + \lambda_m^{q-2} |D\varphi|^q\right) dz + \int_{B_r \setminus E_{r,m}} G(\tilde{Y}_m) \cdot (D\varphi, D\varphi)\, dz,
\end{aligned}$$

where now $\varphi = \zeta (w_m - w)$. By (13), (22), (37), (44) and (46), this gives

$$\begin{aligned}
&\int_{B_s} \left(\kappa |Dw_m - Dw|^2 + \gamma \lambda_m^{q-2} |Dw_m - Dw|^q\right) dz \\
&\quad \le \int_{B_r} \zeta^2 G(Y_0) \cdot (Dw_m - Dw, Dw_m - Dw)\, dz + c\beta^{-2} + o(1).
\end{aligned}$$

Thus we infer, using (22) and (47), that

$$\limsup_{m\to\infty}\int_{B_s}\left(\kappa\,|Dw_m - Dw|^2 + \gamma\lambda_m^{q-2}\,|Dw_m - Dw|^q\right)dz$$
$$\le (1+1-1-1)\int_{B_r}\zeta^2 G\,(Y_0)\cdot(Dw, Dw)\,dz + c\beta^{-2}.$$

Bearing in mind that $\beta > 0$ was arbitrary we conclude that

$$\lim_{m\to\infty}\int_{B_s}|Dw_m - Dw|^2\,dz = 0,$$
$$\lim_{m\to\infty}\lambda_m^{q-2}\int_{B_s}|Dw_m - Dw|^q\,dz = 0.$$

The last equation implies

$$\lim_{m\to\infty}\lambda_m^{q-2}\int_{B_s}|Dw_m|^q\,dz = 0,$$

and we have shown that (42) and (43) hold. □

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