Hokkaido Mathematical Journal Vol. 32 (2003) p. 183–192

Another example of an invariant subspace of H^{∞} with index $\mathfrak{c}$

Norio NIWA

(Received January 7, 2002)

Abstract. A. Borichev gave an example of an invariant subspace $\mathcal{M}$ of H^{∞} with $\dim \mathcal{M}/z\mathcal{M} = \mathrm{card}[0,1] = \mathfrak{c}$, which is generated by an uncountable family of Blaschke products. In this paper, we construct singular inner functions which generate an invariant subspace $\mathcal{M}$ with $\dim \mathcal{M}/z\mathcal{M} = \mathrm{card}[0,1]$.

Key words: invariant subspace, index, singular inner function.

1. Introduction

Let $L_a^2(D)$ be the Bergman space of all analytic functions on the open unit disc D in the complex plane that satisfy the following condition:

$$\int_D |f(z)|^2 dA(z) < +\infty,$$

where dA is the normalized area measure in D. A closed subspace $\mathcal{M}$ of $L_a^2(D)$ is said to be (z-) invariant if $zf \in \mathcal{M}$ whenever $f \in \mathcal{M}$. Here, z is the coordinate function. The dimension of the quotient space $\mathcal{M}/z\mathcal{M}$ is called the index of $\mathcal{M}$.

In 1993, Hedenmalm [3] proved the existence of invariant subspaces of $L_a^2(D)$ with index n, $2 \le n < +\infty$, constructively. In the Hardy space $H^2(D)$, every invariant subspace, except $\{0\}$, has index 1. After Hedenmalm's work, many people have been interested in the structure of invariant subspaces of $L_a^2(D)$, see [4]. In 1996, by Hedenmalm, Richter and Seip [5], invariant subspaces of $L_a^2(D)$ with infinite index were constructed. So, in this paper, we study an invariant subspace of $H^{\infty}(D)$ with infinite index.

Let $H^{\infty} = H^{\infty}(D)$ be the Banach algebra of bounded analytic functions on D. Let $\mathfrak{M} = \mathfrak{M}(H^{\infty})$ be the maximal ideal space of H^{∞} endowed with the weak-$*$ topology. By natural identification, we may consider that $D \subset \mathfrak{M}$. It is known that $\mathfrak{M}$ is a compact Hausdorff space. We identify a function in H^{∞} with its Gelfand transform, so we view H^{∞} as a closed subalgebra of

---

2000 Mathematics Subject Classification : Primary 47A15; Secondary 46J15.

$C(\mathfrak{M})$, the space of complex valued continuous functions on $\mathfrak{M}$. A function $\varphi(z) \in H^\infty$ satisfying $|\varphi(e^{i\theta})| = 1$ almost everywhere on the unit circle ∂D is said to be inner. We know that every inner function $\varphi(z)$ has the form

$$\varphi(z) = e^{ic} b(z)\psi(z),$$

where c is a real constant, b is a Blaschke product, and ψ is a singular inner function. [2, 6] are nice references for the study of H^∞. A sup norm closed subspace $\mathcal{M}$ of H^∞ is called (z-) invariant if $z\mathcal{M} \subset \mathcal{M}$. The dimension of $\mathcal{M}/z\mathcal{M}$ is also called the index of $\mathcal{M}$.

In [1], Borichev gave an example of an invariant subspace of H^∞ with index $\mathfrak{c}$ (= card$[0,1]$), which is generated by Blaschke products. Our purpose of this paper is to construct an invariant subspace of H^∞ with index $\mathfrak{c}$ which is generated by singular inner functions. This construction is interesting in its own right in the study of singular inner functions.

2. Preliminaries

A singular inner function is of the form

$$\psi_\mu(z) = \exp\left(-\int_{\partial D} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(e^{i\theta})\right), \quad z \in D,$$

where μ is a finite positive measure on ∂D and singular with respect to the Lebesgue measure on ∂D. We note that

$$|\psi_\mu(z)| = \exp\left(-\int_{\partial D} P_z(e^{i\theta}) d\mu(e^{i\theta})\right), \quad z \in D,$$

where $P_z(e^{i\theta}) = (1-|z|^2)/(|1-e^{-i\theta}z|^2)$ is the Poisson kernel. This implies that if μ and ν are singular measures and if $0 \le \nu \le \mu$, then

$$|\psi_\mu| \le |\psi_\nu| \quad \text{on } \mathfrak{M}. \tag{2.1}$$

We often use the following notations and facts. For a function $f \in H^\infty$, we put

$$\{|f|<1\} = \{x \in \mathfrak{M} \setminus D : |f(x)| < 1\}$$

and

$$Z(f) = \{x \in \mathfrak{M} \setminus D : f(x) = 0\}.$$

For a point $\lambda \in \partial D$, let $\mathfrak{M}_\lambda = \{x \in \mathfrak{M} : z(x) = \lambda\}$, where z is the identity function on D. It is known that $\mathfrak{M} \setminus D = \bigcup_{\lambda \in \partial D} \mathfrak{M}_\lambda$. We call $\mathfrak{M}_\lambda$ the fiber of $\mathfrak{M}$ over λ. We denote by $S(\mu)$ the closed support set of a singular measure μ on ∂D. It is well known ([6], p. 69) that

$$Z(\psi_\mu) \subset \{|\psi_\mu| < 1\} \subset \bigcup_{\lambda \in S(\mu)} \mathfrak{M}_\lambda, \tag{2.2}$$

and that

$$|\psi_\mu| = 1 \quad \text{on} \quad \bigcup_{\lambda \notin S(\mu)} \mathfrak{M}_\lambda. \tag{2.3}$$

For a positive constant c, it is easy to see that

$$Z(\psi_\mu) = Z(\psi_{c\mu}) \quad \text{and} \quad \{|\psi_\mu| < 1\} = \{|\psi_{c\mu}| < 1\}. \tag{2.4}$$

Let $\delta_{e^{i\theta}}$ denote the unit point measure at $e^{i\theta}$. In this paper, we deal with discrete singular measures. Let

$$\mu = \sum_{k=1}^{\infty} a_k \delta_{e^{i\theta_k}},$$

where $\sum_{k=1}^\infty a_k < \infty$, $a_k > 0$ for all k, and $e^{i\theta_k} \neq e^{i\theta_n}$ if $k \neq n$. Then

$$|\psi_\mu(z)| = \prod_{k=1}^{\infty} |\psi_{\delta_{e^{i\theta_k}}}(z)|^{a_k}, \quad z \in D.$$

Let l_+^∞ be the set of sequences of bounded positive numbers. For $p = (p_1, p_2, \ldots) \in l_+^\infty$, we define μ^p as $\sum_{k=1}^\infty p_k a_k \delta_{e^{i\theta_k}}$, and we put $\|p\|_\infty = \sup\{p_k : k \in \mathbb{N}\}$. Then $\mu^p \leq \|p\|_\infty \cdot \mu$. Thus by (2.1) and (2.4), we have

$$Z(\psi_{\mu^p}) \subset Z(\psi_\mu) \quad \text{and} \quad \{|\psi_{\mu^p}| < 1\} \subset \{|\psi_\mu| < 1\}. \tag{2.5}$$

Singular inner functions defined by μ^p, $p \in l_+^\infty$, were studied by K. Izuchi in [7].

We use the following theorem.

Theorem 2.1 ([7]) *Let μ and ν be positive singular measures on ∂D that are sums of infinitely many point measures, respectively. Then $\mu \perp \nu$ if and only if*

$$\bigcap_{p \in l_+^\infty} \{|\psi_{\mu^p}| < 1\} \cap \bigcap_{q \in l_+^\infty} \{|\psi_{\nu^q}| < 1\} = \emptyset.$$

By the above theorem, we obtain the following lemma, which is one of key lemmas for constructing the desired singular inner functions.

Lemma 2.2 *Let μ and ν be positive singular measures on ∂D that are sums of infinitely many point measures, respectively. If $\mu \perp \nu$, then there exist $p \in l^\infty_+$ and $q \in l^\infty_+$ such that $\|p\|_\infty \leq 1$, $\|q\|_\infty \leq 1$, and*

$$Z(\psi_{\mu^p}) \cap Z(\psi_{\nu^q}) = \emptyset.$$

Proof. By Theorem 2.1,

$$\bigcap_{p \in l^\infty_+} Z(\psi_{\mu^p}) \cap \bigcap_{q \in l^\infty_+} Z(\psi_{\nu^q}) = \emptyset. \tag{2.6}$$

For each $p \in l^\infty_+$, $Z(\psi_{\mu^p})$ is a closed subset of $\mathfrak{M} \setminus D$. Since $\mathfrak{M} \setminus D$ is compact, $\bigcap_{p \in l^\infty_+} Z(\psi_{\mu^p})$ is a compact subset of $\mathfrak{M} \setminus D$. By (2.6), we have

$$\bigcap_{p \in l^\infty_+} Z(\psi_{\mu^p}) \subset \bigcup_{q \in l^\infty_+} (Z(\psi_{\nu^q}))^c,$$

where $(Z(\psi_{\nu^q}))^c$ is the complement of $Z(\psi_{\nu^q})$ in $\mathfrak{M} \setminus D$. Then there exist $q^{(j)} = (q_1^{(j)}, q_2^{(j)}, \ldots) \in l^\infty_+$, $1 \leq j \leq m$, such that

$$\bigcap_{p \in l^\infty_+} Z(\psi_{\mu^p}) \subset \bigcup_{j=1}^{m} (Z(\psi_{\nu^{q^{(j)}}}))^c.$$

Therefore

$$\bigcap_{p \in l^\infty_+} Z(\psi_{\mu^p}) \cap \bigcap_{j=1}^{m} Z(\psi_{\nu^{q^{(j)}}}) = \emptyset.$$

In the same way, there exist $p^{(i)} \in l^\infty_+$, $1 \leq i \leq n$, such that

$$\bigcap_{i=1}^{n} Z(\psi_{\mu^{p^{(i)}}}) \cap \bigcap_{j=1}^{m} Z(\psi_{\nu^{q^{(j)}}}) = \emptyset.$$

We put $p_k = \min\{1, p_k^{(1)}, p_k^{(2)}, \ldots, p_k^{(n)}\}$ for each k. Then we have a new sequence $p = (p_1, p_2, \ldots) \in l^\infty_+$ with $\|p\|_\infty \leq 1$. It is clear that $\mu^p \leq \mu^{p^{(i)}}$ for all $i = 1, 2, \ldots, n$. Hence $Z(\psi_{\mu^p}) \subset Z(\psi_{\mu^{p^{(i)}}})$ for all i. A similar consideration gives us a singular measure ν^q. Therefore we obtain $Z(\psi_{\mu^p}) \cap Z(\psi_{\nu^q}) = \emptyset$. This completes the proof. $\square$

3. An invariant subspace with index $\mathfrak{c}$

In this section, we prove the following theorem.

Theorem 3.1 *There exists an invariant subspace of H^∞ with index $\mathfrak{c}$ which is generated by singular inner functions.*

To prove our theorem, we need the following fact, which was used in Borichev [1, p. 42] without proof. We denote by $\mathbb{N}$ the set of positive integers.

Lemma 3.2 *There exists a family $\{N_\alpha : \alpha \in [0,1]\}$ such that $N_\alpha \subset \mathbb{N}$ for each $\alpha \in [0,1]$, and such that for every finite family $\alpha_0, \alpha_1, \ldots, \alpha_n \in [0,1]$, $\alpha_0 \neq \alpha_i$, $1 \leq i \leq n$,*

$$\operatorname{card}\left(N_{\alpha_0} \setminus \bigcup_{i=1}^{n} N_{\alpha_i}\right) = \infty. \tag{3.1}$$

For the convenience of the reader, we include a proof.

Proof. Take a countable dense subset $\{a_k : k \in \mathbb{N}\}$ in the open square $(0,1) \times (0,1)$. For $\alpha \in [0,1]$, let $A_\alpha = \{(x,y) : |x - \alpha| < y\}$ be the angular domain at vertex α, and put $N_\alpha = \{k : a_k \in A_\alpha\}$, which gives the desired family. □

Proof of Theorem 3.1. Let $\{e^{i\theta_k} : k \in \mathbb{N}\}$ be a dense subset of distinct points in ∂D. For each k, let

$$\{\lambda_{k,j}\}_j \text{ be a sequence of distinct points in } \partial D \tag{3.2}$$

such that

$$\lim_{j\to\infty} \lambda_{k,j} = e^{i\theta_k}. \tag{3.3}$$

Furthermore, we may assume that

$$\{\lambda_{k,j} : j \in \mathbb{N}\} \cap \{\lambda_{l,j} : j \in \mathbb{N}\} = \emptyset \quad \text{if } k \neq l, \tag{3.4}$$

and

$$\{\lambda_{k,j} : j, k \in \mathbb{N}\} \cap \{e^{i\theta_k} : k \in \mathbb{N}\} = \emptyset. \tag{3.5}$$

First, we set up a singular measure

$$\nu = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} c_{k,j}\delta_{\lambda_{k,j}},$$

where $\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} c_{k,j} < \infty$ and $c_{k,j} > 0$ for every k, j. Put

$$\nu_k = \sum_{j=1}^{\infty} c_{k,j}\delta_{\lambda_{k,j}}.$$

By induction, we show the existence of singular measures μ_k, $k = 1, 2, \ldots$, satisfying the following conditions:

$$\mu_k = \sum_{j=1}^{\infty} a_{k,j}\delta_{\lambda_{k,j}}, \tag{3.6}$$

$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{k,j} < \infty, \quad a_{k,j} > 0 \quad \text{for all} \quad k,\ j,$$

and

$$Z(\psi_{\mu_k}) \cap Z(\psi_{\sum_{i=k+1}^{\infty}\mu_i}) = \emptyset \quad \text{for every} \quad k = 1, 2, \ldots . \tag{3.7}$$

Apply Lemma 2.2 for singular measures ν_1 and $\sum_{k=2}^{\infty}\nu_k$, then there exist $p \in l_+^{\infty}$ and $q \in l_+^{\infty}$ such that $\|p\|_{\infty} \le 1$, $\|q\|_{\infty} \le 1$, and

$$Z(\psi_{\nu_1^p}) \cap Z\big(\psi_{(\sum_{k=2}^{\infty}\nu_k)^q}\big) = \emptyset.$$

Put $\mu_1 = \nu_1^p$. Then $0 \le \mu_1 \le \nu_1$, $0 \le \big(\sum_{k=2}^{\infty}\nu_k\big)^q \le \sum_{k=2}^{\infty}\nu_k$, and

$$Z(\psi_{\mu_1}) \cap Z\big(\psi_{(\sum_{k=2}^{\infty}\nu_k)^q}\big) = \emptyset. \tag{3.8}$$

We write $\big(\sum_{k=2}^{\infty}\nu_k\big)^q$ as $\sum_{k=2}^{\infty}\sum_{j=1}^{\infty} d_{k,j}\delta_{\lambda_{k,j}}$. For each $k \ge 2$, put $\nu_k' = \sum_{j=1}^{\infty} d_{k,j}\delta_{\lambda_{k,j}}$. And, apply Lemma 2.2 for measures ν_2' and $\sum_{k=3}^{\infty}\nu_k'$, then

$$Z(\psi_{(\nu_2')^r}) \cap Z\big(\psi_{(\sum_{k=3}^{\infty}\nu_k')^s}\big) = \emptyset$$

holds for some $r \in l_+^{\infty}$ and $s \in l_+^{\infty}$ with $\|r\|_{\infty} \le 1$, $\|s\|_{\infty} \le 1$. Put $\mu_2 = (\nu_2')^r$. Then we have

$$Z(\psi_{\mu_2}) \cap Z\big(\psi_{(\sum_{k=3}^{\infty}\nu_k')^s}\big) = \emptyset,$$

$0 \le \mu_2 \le \nu_2$, and $0 \le \big(\sum_{k=3}^{\infty}\nu_k'\big)^s \le \sum_{k=3}^{\infty}\nu_k$.

Repeating the same argument, we obtain the desired singular measures μ_k, $k = 1, 2, \ldots$.

By (2.4) and (3.7), multiplying the measure $\sum_{i=k+1}^{\infty} \mu_i$ by a suitable positive constant less than 1 if necessary, we may assume that

$$\left|\psi_{\sum_{i=k+1}^{\infty} \mu_i}\right| \geq \frac{1}{2} \quad \text{on} \quad Z(\psi_{\mu_k}) \tag{3.9}$$

for each k.

Let $\{N_\alpha : \alpha \in [0,1]\}$ be the family given in Lemma 3.2. For each $\alpha \in [0,1]$, put

$$\sigma_\alpha = \sum_{k=1}^{\infty} \sum_{j \in \mathbb{N} \setminus N_\alpha} a_{k,j} \delta_{\lambda_{k,j}} \tag{3.10}$$

and

$$\sigma_{k,\alpha} = \sum_{j \in \mathbb{N} \setminus N_\alpha} a_{k,j} \delta_{\lambda_{k,j}}. \tag{3.11}$$

Then $\sigma_\alpha = \sum_{k=1}^{\infty} \sigma_{k,\alpha}$, and by (3.6)

$$\sigma_{k,\alpha} \leq \sigma_\alpha \quad \text{and} \quad \sigma_{k,\alpha} \leq \mu_k \quad \text{for every} \quad k. \tag{3.12}$$

Let $\mathcal{M}$ be the invariant subspace of H^∞ generated by singular inner functions $\psi_{\sigma_\alpha}, \alpha \in [0,1]$. Let $\alpha_0 \in [0,1]$. We claim that the vector $\psi_{\sigma_{\alpha_0}} + z\mathcal{M}$ does not belong to the closure of the linear span of $\{\psi_{\sigma_\alpha} + z\mathcal{M} : \alpha \in [0,1], \alpha \neq \alpha_0\}$ in $\mathcal{M}/z\mathcal{M}$. This implies that $\dim \mathcal{M}/z\mathcal{M} = \operatorname{card}[0,1]$, that is, the index of $\mathcal{M}$ equals $\mathfrak{c}$.

To prove the above, suppose not. Then we have

$$\left\| \psi_{\sigma_{\alpha_0}} + p_0 \psi_{\sigma_{\alpha_0}} + \sum_{i=1}^{n} p_i \psi_{\sigma_{\alpha_i}} \right\|_{H^\infty} < \frac{1}{2} \tag{3.13}$$

for some finite set of elements $\alpha_i \in [0,1]$, $\alpha_i \neq \alpha_0$, $1 \leq i \leq n$, and polynomials $p_0, p_1, \ldots, p_n$. Here $p_0(0) = 0$.

For each k, j, there exists a point $x_{k,j} \in \mathfrak{M}_{\lambda_{k,j}}$ such that

$$\psi_{\delta_{\lambda_{k,j}}}(x_{k,j}) = 0. \tag{3.14}$$

Then by (2.1) and (3.6), $|\psi_{\mu_k}(x_{k,j})| \leq |\psi_{\delta_{\lambda_{k,j}}}(x_{k,j})|^{a_{k,j}} = 0$ for each k, j. Hence

$$x_{k,j} \in Z(\psi_{\mu_k}) \quad \text{for every} \quad k,\ j. \tag{3.15}$$

Take $t \in (N_{\alpha_0} \setminus \bigcup_{i=1}^{n} N_{\alpha_i})$ arbitrary. Then by using (2.1), for $1 \leq i \leq n$

$$\begin{aligned}|\psi_{\sigma_{\alpha_i}}(x_{k,t})| &\leq |\psi_{\sigma_{k,\alpha_i}}(x_{k,t})| \qquad \text{by (3.12)}\\ &\leq |\psi_{\delta_{\lambda_{k,t}}}(x_{k,t})|^{a_{k,t}} \qquad \text{by (3.11)}\\ &= 0 \qquad \text{by (3.14).}\end{aligned}$$

Thus we get

$$\psi_{\sigma_{\alpha_i}}(x_{k,t}) = 0 \tag{3.16}$$

for every $k \in \mathbb{N}$, $1 \leq i \leq n$, and $t \in \left(N_{\alpha_0} \setminus \bigcup_{i=1}^{n} N_{\alpha_i}\right)$.

Let y_k be one of cluster points of $\{x_{k,t} : t \in N_{\alpha_0} \setminus \bigcup_{i=1}^{n} N_{\alpha_i}\}$. Since $\psi_{\sigma_{\alpha_i}}$ is continuous on $\mathfrak{M}$, by (3.16)

$$\psi_{\sigma_{\alpha_i}}(y_k) = 0 \tag{3.17}$$

for $i = 1, 2, \ldots, n$, and for $k = 1, 2, \ldots$.

Let π denote the fiber projection from $\mathfrak{M} \setminus D$ onto ∂D. By (2.2) and (3.14),

$$\pi(x_{k,j}) = \lambda_{k,j} \quad \text{and} \quad x_{k,j} \in \mathfrak{M}_{\lambda_{k,j}}. \tag{3.18}$$

By (3.1), there is a sequence $\{t_m\}_m \subset \left(N_{\alpha_0} \setminus \bigcup_{i=1}^{n} N_{\alpha_i}\right)$ such that $t_m \to \infty$ as $m \to \infty$. Then by (3.3),

$$\lim_{m \to \infty} \pi(x_{k,t_m}) = \lim_{m \to \infty} \lambda_{k,t_m} = e^{\mathrm{i}\theta_k}.$$

Since π is continuous on $\mathfrak{M} \setminus D$ ([6], p. 160), it follows that

$$\pi(y_k) = e^{\mathrm{i}\theta_k} \quad \text{and} \quad y_k \in \mathfrak{M}_{e^{\mathrm{i}\theta_k}}. \tag{3.19}$$

For every $k \in \mathbb{N}$ and $t \in (N_{\alpha_0} \setminus \bigcup_{i=1}^{n} N_{\alpha_i})$, we have

$$\begin{aligned}&|\psi_{\sigma_{\alpha_0}}(x_{k,t})|\\ &= \left(\prod_{i=1}^{k} |\psi_{\sigma_{i,\alpha_0}}(x_{k,t})|\right) \cdot \left|\psi_{\sum_{i=k+1}^{\infty} \sigma_{i,\alpha_0}}(x_{k,t})\right| \quad \text{by (3.10) and (3.11)}\\ &\geq \left(\prod_{i=1}^{k} |\psi_{\sigma_{i,\alpha_0}}(x_{k,t})|\right) \cdot \left|\psi_{\sum_{i=k+1}^{\infty} \mu_i}(x_{k,t})\right| \qquad \text{by (3.12)}\end{aligned}$$

$$\begin{aligned}&\geq \frac{1}{2}\prod_{i=1}^{k}|\psi_{\sigma_{i,\alpha_0}}(x_{k,t})| \qquad \text{by (3.9) and (3.15)}\\ &= \frac{1}{2}.\end{aligned}$$

The proof of the last equality is the following. For each i, $1 \leq i \leq k$, we have

$$\begin{aligned}S(\sigma_{i,\alpha_0}) &= \overline{\{\lambda_{i,j} : j \in \mathbb{N} \setminus N_{\alpha_0}\}} \qquad \text{by (3.11)}\\ &= \{e^{i\theta_i}\} \cup \{\lambda_{i,j} : j \in \mathbb{N} \setminus N_{\alpha_0}\} \qquad \text{by (3.3)}\\ &\not\ni \lambda_{k,t} \qquad \text{by (3.2), (3.4) and (3.5).}\end{aligned}$$

Hence by (2.3) and (3.18), $|\psi_{\sigma_{i,\alpha_0}}(x_{k,t})| = 1$ for every i, $1 \leq i \leq k$.

Since y_k is one of cluster points of $\{x_{k,t} : t \in N_{\alpha_0} \setminus \bigcup_{i=1}^{n} N_{\alpha_i}\}$, by the above inequalities, we have

$$|\psi_{\sigma_{\alpha_0}}(y_k)| \geq \frac{1}{2} \quad \text{for every } \ k. \tag{3.20}$$

By (3.13) and (3.17), we have

$$|(1 + p_0(y_k))\psi_{\sigma_{\alpha_0}}(y_k)| < \frac{1}{2} \quad \text{for every } \ k.$$

Therefore by (3.20), we obtain

$$|1 + p_0(y_k)| < 1 \quad \text{for all } \ k \in \mathbb{N}.$$

By (3.19) and p_0 is a polynomial, we have $p_0(y_k) = p_0(e^{i\theta_k})$. Hence

$$|1 + p_0(e^{i\theta_k})| < 1 \quad \text{for all } \ k \in \mathbb{N}. \tag{3.21}$$

By the starting assumption, $\{e^{i\theta_k} : k \in \mathbb{N}\}$ is dense in ∂D, so that we obtain $\|1 + p_0\|_{H^\infty} \leq 1$. Since $1 + p_0(0) = 1$, by the maximality we have $1 + p_0 \equiv 1$. This implies that $p_0 \equiv 0$. By (3.21), we get $1 < 1$. This is the desired contradiction. □

Acknowledgements The author would like to thank Professor Mikihiro Hayashi for teaching me the proof of Lemma 3.2 and Professor Keiji Izuchi for his advice and suggestions.

References

[1] Borichev A., *Invariant subspaces of given index in Banach spaces of analytic functions*. J. Reine Angew. Math. **505** (1998), 23–44.

[2] Garnett J., *Bounded Analytic Functions*. Academic Press, New York, 1981.

[3] Hedenmalm H., *An invariant subspace of the Bergman space having the codimension two property*. J. Reine Angew. Math. **443** (1993), 1–9.

[4] Hedenmalm H., Korenblum B. and Zhu K., *Theory of Bergman spaces*. Springer-Verlag, New York, 2000.

[5] Hedenmalm H., Richter S. and Seip K., *Interpolating sequences and invariant subspaces of given index in the Bergman spaces*. J. Reine Angew. Math. **477** (1996), 13–30.

[6] Hoffman K., *Banach Spaces of Analytic Functions*. Prentice Hall, Englewood Cliffs, N. J., 1962.

[7] Izuchi K., *Singular inner functions of L^1- type II*. J. Math. Soc. Japan **53** (2001), 287–305.

[8] Richter S., *Invariant subspaces in Banach spaces of analytic functions*. Trans. Amer. Math. Soc. **304** (1987), 585–616.

Department of Mathematical Sciences
Graduate School of Science and Technology
Niigata University
8050 Ikarashi 2-no-chou, Niigata 950-2181
Japan
E-mail: niwa@m.sc.niigata-u.ac.jp