Hokkaido Mathematical Journal Vol. 32 (2003) p. 127–136

Subclasses of certain analytic functions

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(Received August 31, 2001; Revised February 12, 2002)

Abstract. Let $\mathcal{A}$ be the class of functins $f(z)$ which are analytic in the open unit disc $\mathbb{E}$ with $f(0)=0$ and $f'(0)=1$. Two subclasses of $\mathcal{A}$ with some inequalities are defined. The object of the present paper is to consider some properties for functions $f(z)$ belonging to these classes.

Key words: analytic function, univalent function, starlike function, subordination.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_{n}z^{n}$$

which are analytic in the open unit disc $\mathbb{E}=\{z\in\mathbb{C}:|z|<1\}$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{E}$. A function $f(z)\in\mathcal{A}$ is called starlike in $|z|<r$ $(0<r\leqq 1)$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \qquad (|z|<r).$$

For a function $f(z)\in\mathcal{A}$, we say that $f(z)$ is in the class $\mathcal{H}(\lambda,\mu)$ if and only if it satisfies the conditions $\frac{f(z)}{z}\neq 0$ $(z\in\mathbb{E})$ and

$$\left|\frac{z^{2}f'(z)}{f(z)^{2}}-\lambda z^{2}\left(\frac{z}{f(z)}\right)''-1\right|<\mu \qquad (z\in\mathbb{E}), \tag{1}$$

where λ is a complex number with $\operatorname{Re}(\lambda)\geqq 0$ and μ is a positive real number. Also we define the class $\mathcal{H}_{0}(\lambda,\mu)$ by

$$\mathcal{H}_{0}(\lambda,\mu)=\{f(z)\in\mathcal{H}(\lambda,\mu):f''(0)=0\}.$$

Nunokawa, Obradović and Owa [2] proved that if $f(z)\in\mathcal{A}$ with

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2000 Mathematics Subject Classification : 30C45.

$\frac{f(z)}{z} \neq 0$ $(z \in \mathbb{E})$ and $\left|\left(\frac{z}{f(z)}\right)''\right| \leqq 1$ in $\mathbb{E}$, then $f(z) \in \mathcal{S}$. Ozaki and Nunokawa [4] showed that $\mathcal{H}(0,1) \subset \mathcal{S}$ and Obradović, Pascu and Radomir [3] considered the classes $\mathcal{H}(0,1)$ and $\mathcal{H}_0(0,1)$.

In the present paper, we investigate certain properties for the classes $\mathcal{H}(\lambda,\mu)$ and $\mathcal{H}_0(\lambda,\mu)$. Our results generalize or improve the results obtained in [2], [3] and [4]. Also some other new results are given in this paper.

2. Properties of the class $\mathcal{H}(\lambda,\mu)$

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{E}$. Then we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{E}$, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in $\mathbb{E}$ such that $|w(z)| \leqq |z|$ and $f(z) = g(w(z))$ for $z \in \mathbb{E}$. If $g(z)$ is univalent in $\mathbb{E}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$. To derive our results, we need the following lemma due to Miller and Mocanu ([1], p. 170).

Lemma *Let $h(z)$ be analytic and convex univalent in $\mathbb{E}$, $h(0) = 1$, and let*

$$p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \cdots \qquad (n \in \mathbb{N} = \{1,2,3,\cdots\})$$

be analytic in $\mathbb{E}$. If

$$p(z) + \frac{nzp'(z)}{c} \prec h(z),$$

where $c \neq 0$ and $\mathrm{Re}(c) \geqq 0$, then

$$p(z) \prec \left(\frac{c}{n}\right) z^{-\frac{c}{n}} \int_0^z t^{\frac{c}{n}-1} h(t)dt.$$

For $\mathrm{Re}(\lambda) \geqq 0$ and $\mu > 0$, it is easy to verify that the function

$$f(z) = \frac{z}{\left(1 + \sqrt{\frac{\mu}{|1+2\lambda|}}\, z\right)^2} \tag{2}$$

belongs to $\mathcal{H}(\lambda,\mu)$ if and only if $\mu \leqq |1+2\lambda|$. Applying the lemma, we derive

Theorem 1 *Let $\mathrm{Re}(\lambda) \geqq 0$ and $0 < \mu \leqq |1+2\lambda|$. Then $\mathcal{H}(\lambda,\mu) \subset \mathcal{S}$.*

Proof. Let

$$p(z) = \frac{z^2 f'(z)}{f(z)^2} = 1 + p_2 z^2 + \cdots \tag{3}$$

for $f(z) \in \mathcal{H}(\lambda,\mu)$. Then

$$zp'(z) = -z^2\left(\frac{z}{f(z)}\right)''$$

and it follows from the condition (1) that

$$p(z) + \lambda zp'(z) = \frac{z^2 f'(z)}{f(z)^2} - \lambda z^2\left(\frac{z}{f(z)}\right)'' \prec 1 + \mu z.$$

For $\lambda \neq 0$, $\mathrm{Re}(\lambda) \geqq 0$ and $\mu > 0$, an application of the lemma yields

$$p(z) \prec 1 + \frac{\mu}{1+2\lambda} z. \tag{4}$$

From (3), (4) and the Schwarz lemma, we have

$$\left|\frac{z^2 f'(z)}{f(z)^2} - 1\right| \leqq \frac{\mu}{|1+2\lambda|}|z|^2 \qquad (z \in \mathbb{E}) \tag{5}$$

for $\mathrm{Re}(\lambda) \geqq 0$ and $\mu > 0$. Hence

$$\left|\frac{z^2 f'(z)}{f(z)^2} - 1\right| < \frac{\mu}{|1+2\lambda|} \leqq 1 \qquad (z \in \mathbb{E}) \tag{6}$$

for $\mathrm{Re}(\lambda) \geqq 0$ and $0 < \mu \leqq |1+2\lambda|$. Now, using Theorem 2 in [4], from (6) we conclude that $f(z) \in \mathcal{S}$. □

Remark 1 If we let $\mu = |1+2\lambda|$ and $\lambda \to \infty$, then the condition (1) can be written as

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leqq 2 \qquad (z \in \mathbb{E}) \tag{7}$$

by the Schwarz lemma. Thus we obtain the following corollary which is an improvement of the main theorem by Nunokawa, Obradović and Owa in [2].

Corollary 1 *Let $f(z) \in \mathcal{A}$ with $\frac{f(z)}{z} \neq 0$ for $z \in \mathbb{E}$, and let $f(z)$ satisfy the inequality* (7)*. Then $f(z) \in \mathcal{S}$,*

Remark 2 Recently, Yang and Liu [5] showed Corollary 1 by using the different method.

Next we have

Corollary 2 *Let* $\mathrm{Re}(\lambda) \geqq 0$, $0 < \mu \leqq |1+2\lambda|$ *and*

$$f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n} \in \mathcal{A}. \tag{8}$$

If

$$\sum_{n=2}^{\infty}(n-1)|1+n\lambda||b_n| \leqq \mu, \tag{9}$$

then $f(z) \in \mathcal{S}$.

Proof. From (8) and (9), we have

$$\begin{aligned}\left|\frac{z^2 f'(z)}{f(z)^2} - \lambda z^2 \left(\frac{z}{f(z)}\right)'' - 1\right| &= \left|-\sum_{n=2}^{\infty}(n-1)(1+n\lambda)b_n z^n\right| \\ &\leqq \sum_{n=2}^{\infty}(n-1)|1+n\lambda||b_n| \leqq \mu\end{aligned}$$

for $z \in \mathbb{E}$. Therefore, we see that $f(z) \in \mathcal{H}(\lambda,\mu) \subset \mathcal{S}$ by using Theorem 1. □

Theorem 2 *Let* $0 \leqq \lambda_1 < \lambda_2$ *and* $\mu > 0$. *Then* $\mathcal{H}(\lambda_2,\mu) \subset \mathcal{H}(\lambda_1,\mu)$.

Proof. Let $f(z) \in \mathcal{H}(\lambda_2,\mu)$. Then, it follows that

$$\frac{z^2 f'(z)}{f(z)^2} - \lambda_2 z^2 \left(\frac{z}{f(z)}\right)'' \prec 1+\mu z.$$

From (4) in the proof of Theorem 1, we also see that

$$\frac{z^2 f'(z)}{f(z)^2} \prec 1 + \frac{\mu}{1+2\lambda_2} \prec 1+\mu z.$$

Hence, we have

$$\begin{aligned}&\frac{z^2 f'(z)}{f(z)^2} - \lambda_1 z^2 \left(\frac{z}{f(z)}\right)'' \\ &\quad = \frac{\lambda_1}{\lambda_2}\left\{\frac{z^2 f'(z)}{f(z)^2} - \lambda_2 z^2 \left(\frac{z}{f(z)}\right)''\right\} + \left(1-\frac{\lambda_1}{\lambda_2}\right)\frac{z^2 f'(z)}{f(z)^2} \\ &\quad \prec 1+\mu z\end{aligned}$$

for $0 \leqq \lambda_1 < \lambda_2$, which implies that $f(z) \in \mathcal{H}(\lambda_1,\mu)$. □

Further, we derive

Theorem 3 *Let* $\mathrm{Re}(\lambda) \geqq 0$, $\mu > 0$ *and*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

be in the class $\mathcal{H}(\lambda, \mu)$. *Then*

$$\left| \frac{z}{f(z)} - 1 + a_2 z \right| \leqq \frac{\mu}{|1+2\lambda|} |z|^2, \tag{10}$$

$$\left| \frac{z}{f(z)} - 1 \right| \leqq |z| \left(|a_2| + \frac{\mu}{|1+2\lambda|} |z| \right), \tag{11}$$

$$\begin{aligned} 1 - |z| \left(|a_2| + \frac{\mu}{|1+2\lambda|} |z| \right) &\leqq \mathrm{Re}\left(\frac{z}{f(z)} \right) \\ &\leqq 1 + |z| \left(|a_2| + \frac{\mu}{|1+2\lambda|} |z| \right), \end{aligned} \tag{12}$$

and

$$|f(z)| \geqq \frac{|z|}{1 + |a_2||z| + \left(\frac{\mu}{|1+2\lambda|} \right) |z|^2}. \tag{13}$$

Equalities in (10), (11), (12) *and* (13) *are attended for the function* $f(z)$ *given by*

$$f(z) = \frac{z}{1 \pm bz + \left(\frac{\mu}{|1+2\lambda|} \right) z^2} \in \mathcal{H}(\lambda, \mu)$$

with $0 < \mu \leqq |1+2\lambda|$ *and* $0 \leqq b \leqq 2\sqrt{\frac{\mu}{|1+2\lambda|}}$.

Proof. For the function $f(z) \in \mathcal{H}(\lambda, \mu)$, we find that

$$\int_0^z \left(\frac{f'(t)}{f(t)^2} - \frac{1}{t^2} \right) dt = \frac{1}{z} - \frac{1}{f(z)} - a_2. \tag{14}$$

Using (5) in the proof of Theorem 1, it follows from (14) that

$$\left| \frac{1}{f(z)} - \frac{1}{z} + a_2 \right| \leqq \int_0^{|z|} \left| \frac{f'(t)}{f(t)^2} - \frac{1}{t^2} \right| dt \leqq \frac{\mu}{|1+2\lambda|} |z| \qquad (z \in \mathbb{E}),$$

which gives (10). In view of (10), we easily have (11), (12) and (13). □

Remark 3 Taking $\lambda = 0$ and $\mu = 1$ in (11) and (12), we have the corresponding results by Obradović, Pascu and Radamir [3].

The inequlity (10) in Theorem 3 leads us the following theorem.

Theorem 4 *Let* $\mathrm{Re}(\lambda) \geqq 0$, $\mu > 0$ *and* $f(z) \in \mathcal{H}(\lambda, \mu)$. *Then*

$$|a_2^2 - a_3| \leqq \frac{\mu}{|1+2\lambda|}. \tag{15}$$

The result is sharp for $0 < \mu \leqq |1+2\lambda|$.

Proof. Since

$$\frac{z}{f(z)} - 1 + a_2 z = (a_2^2 - a_3)z^2 + \sum_{n=3}^{\infty} b_n z^n,$$

from (10) in Theorem 3, we deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} \left| \frac{re^{i\theta}}{f(re^{i\theta})} - 1 + a_2 re^{i\theta} \right|^2 d\theta$$
$$= |a_2^2 - a_3|^2 r^4 + \sum_{n=3}^{\infty} |b_n|^2 r^{2n} \leqq \left(\frac{\mu}{|1+2\lambda|} \right)^2 r^4 \quad (0 < r < 1),$$

which leads to (15). Further, it is easy to check that the estimate (15) is best possible for the function $f(z)$ given by (2) with $0 < \mu \leqq |1+2\lambda|$. □

3. Properties of the class $\mathcal{H}_0(\lambda, \mu)$

For functions $f(z)$ belonging to the class $\mathcal{H}_0(\lambda, \mu)$, we may have

Theorem 5 *Let* $\mathrm{Re}(\lambda) \geqq 0$ *and* $f(z) \in \mathcal{H}_0(\lambda, \mu)$.

(a) *If* $\frac{|1+2\lambda|}{\sqrt{2}} \leqq \mu|1+2\lambda|$, *then* $f(z)$ *is starlike in* $|z| < \sqrt{\frac{|1+2\lambda|}{\sqrt{2}\mu}}$;

(b) *If* $\frac{|1+2\lambda|}{2} \leqq \mu|1+2\lambda|$, *then* $\mathrm{Re}\, f'(z) > 0$ *for* $|z| < \sqrt{\frac{|1+2\lambda|}{2\mu}}$.

Proof. We follow the technique by Yang and Liu in [5]. For $\mathrm{Re}(\lambda) \geqq 0$ and $0 < \mu \leqq |1+2\lambda|$, the inequality (5) in the proof of Theorem 1 gives us that

$$\left| \arg \frac{z^2 f'(z)}{f(z)^2} \right| \leqq \arcsin\left(\frac{\mu}{|1+2\lambda|}|z|^2 \right) \qquad (z \in \mathbb{E}). \tag{16}$$

Also it follows from (10) in Theorem 3 with $a_2 = 0$ that

$$\left|\arg\left(\frac{z}{f(z)}\right)\right| \leqq \arcsin\left(\frac{\mu}{|1+2\lambda|}|z|^2\right) \qquad (z \in \mathbb{E}). \tag{17}$$

(a) If $\frac{|1+2\lambda|}{\sqrt{2}} \leqq \mu \leqq |1+2\lambda|$, then from (16) and (17), we obtain

$$\begin{aligned}\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| &\leqq \left|\arg\left(\frac{z^2 f'(z)}{f(z)^2}\right)\right| + \left|\arg\left(\frac{z}{f(z)}\right)\right| \\ &\leqq 2\arcsin\left(\frac{\mu}{|1+2\lambda|}|z|^2\right) < \frac{\pi}{2}\end{aligned}$$

for $|z| < r_1 = \sqrt{\frac{|1+2\lambda|}{\sqrt{2}\mu}} \leqq 1$. This shows that $f(z)$ is starlike in $|z| < r_1$.

(b) If $\frac{|1+2\lambda|}{2} \leqq \mu \leqq |1+2\lambda|$, then it follows from (16) and (17) that

$$\begin{aligned}|\arg f'(z)| &\leqq \left|\arg\left(\frac{z^2 f'(z)}{f(z)^2}\right)\right| + 2\left|\arg\left(\frac{z}{f(z)}\right)\right| \\ &\leqq 3\arcsin\left(\frac{\mu}{|1+2\lambda|}|z|^2\right) < \frac{\pi}{2}\end{aligned}$$

for $|z| < r_2 = \sqrt{\frac{|1+2\lambda|}{2\mu}} \leqq 1$. This implies that $\operatorname{Re} f'(z) > 0$ for $|z| < r_2$.

□

Remark 4 Letting $\lambda = 0$ and $\mu = 1$ in Theorem 5, we see that (a) is the same as in [3] and (b) is the improvement of a result in [3].

Theorem 5 gives us the following corollary.

Corollary 3 *Let* $\operatorname{Re}(\lambda) \geqq 0$ *and* $f(z) \in \mathcal{H}_0(\lambda, \mu)$.

(a) *If* $0 < \mu \leqq \frac{|1+2\lambda|}{\sqrt{2}}$, *then* $f(z)$ *is starlike in* $\mathbb{E}$;

(b) *If* $0 < \mu \leqq \frac{|1+2\lambda|}{2}$, *then* $\operatorname{Re} f'(z) > 0$ *for* $z \in \mathbb{E}$.

Finally we derive

Theorem 6 *Let*

$$f(z) = \frac{z}{1 + \sum_{n=2}^{\infty} b_n z^n}, \tag{18}$$

$f_1(z) = z$ and

$$f_m(z) = \frac{z}{1+\sum_{n=2}^{m} b_n z^n}.$$

If $\mathrm{Re}(\lambda) \geqq 0$, $0 < \mu \leqq |1+2\lambda|$ *and*

$$\sum_{n=2}^{\infty}(n-1)|1+n\lambda||b_n| \leqq \mu, \tag{19}$$

then we have

(a) $f(z) \in \mathcal{H}_0(\lambda,\mu) \subset \mathcal{S}$:

(b) *for* $z \in \mathbb{E}$,

$$\mathrm{Re}\left(\frac{f_m(z)}{f(z)}\right) > 1 - \frac{\mu}{m|1+(m+1)\lambda|} \tag{20}$$

and

$$\mathrm{Re}\left(\frac{f(z)}{f_m(z)}\right) > \frac{m|1+(m+1)\lambda|}{m|1+(m+1)\lambda|+\mu}. \tag{21}$$

The results are sharp for each $m \in \mathbb{N}$.

Proof. Let $c_n = \frac{(n-1)|1+n\lambda|}{\mu}$ $(n \geqq 2)$. Then

$$c_{n+1} > c_n \geqq 1 \qquad (n \geqq 2) \tag{22}$$

for $\mathrm{Re}(\lambda) \geqq 0$ and $0 < \mu \leqq |1+2\lambda|$. From (19) and (22), we deduce that the function $f(z)$ given by (18) is analytic in $\mathbb{E}$ and

$$\sum_{n=2}^{m} |b_n| + c_{m+1} \sum_{n=m+1}^{\infty} |b_n| \leqq \sum_{n=2}^{\infty} c_{n=2}^{\infty} c_n |b_n| \leqq 1 \qquad (m \geqq 2). \tag{23}$$

(a) Noting that $f(z) \in \mathcal{A}$ and $f''(0) = 0$, from the proof of Corollary 2, we see that $f(z) \in \mathcal{H}_0(\lambda,\mu) \subset \mathcal{S}$.

(b) Let us define the function $p_1(z)$ by

$$p_1(z) = c_{m+1}\left\{\frac{f_m(z)}{f(z)} - \left(1 - \frac{1}{c_{m+1}}\right)\right\}.$$

Then

$$p_1(z) = 1 + \frac{c_{m+1}\sum_{n=m+1}^{\infty} b_n z^n}{1+\sum_{n=2}^{m} b_n z^n}$$

and from (23) we deduce that

$$\begin{aligned}\left|\frac{p_1(z)-1}{p_1(z)+1}\right| &= \left|\frac{c_{m+1}\sum_{n=m+1}^{\infty} b_n z^n}{2\left(1+\sum_{n=2}^{m} b_n z^n\right)+c_{m+1}\sum_{n=m+1}^{\infty} b_n z^n}\right| \\ &\leqq \frac{c_{m+1}\sum_{n=m+1}^{\infty} |b_n|}{2-2\sum_{n=2}^{m}|b_n|-c_{m+1}\sum_{n=m+1}^{\infty}|b_n|} \\ &\leqq 1 \qquad (z\in\mathbb{E}).\end{aligned}$$

Hence we conclude that

$$\operatorname{Re}\left(\frac{f_m(z)}{f(z)}\right) > 1-\frac{1}{c_{m+1}}$$

for $z \in \mathbb{E}$. This proves the inequality (20) for $m \geqq 2$.

If we take the function $f(z)$ defined by

$$f(z)=\frac{z}{1+\frac{\mu}{m|1+(m+1)\lambda|}z^{m+1}}, \tag{24}$$

then $f_m(z)=z$ and

$$\frac{f_m(z)}{f(z)} \to 1-\frac{\mu}{m|1+(m+1)\lambda|} \qquad (z\to e^{\frac{i\pi}{m+1}}).$$

Hence, the bound in (20) is best possible for each $m \geqq 2$.

Similarly, if we put

$$p_2(z)=(1+c_{m+1})\left(\frac{f(z)}{f_m(z)}-\frac{c_{m+1}}{1+c_{m+1}}\right),$$

then it follows from (23) that

$$\begin{aligned}\left|\frac{p_2(z)-1}{p_2(z)+1}\right| &= \left|\frac{-(1+c_{m+1}\sum_{n=m+1}^{\infty} b_n z^n}{2\left(1+\sum_{n=2}^{m} b_n z^n\right)-(c_{m+1}-1)\sum_{n=m+1}^{\infty} b_n z^n}\right| \\ &\leqq \frac{(1+c_{m+1})\sum_{n=m+1}^{\infty} |b_n|}{2-2\sum_{n=2}^{m}|b_n|(c_{m+1}-1)\sum_{n=m+1}^{\infty}|b_n|} \\ &\leqq 1 \qquad (z\in\mathbb{E}).\end{aligned}$$

Now we easily show the inequality (21) for $m \geqq 2$ and the bound in (21) is sharp for the function $f(z)$ given by (24). □

Finally, the coefficient inequality (23) becomes

$$c_2 \sum_{n=2}^{\infty} |b_n| \leqq \sum_{n=2}^{\infty} c_n |b_n| \leqq 1$$

when $m = 1$. By using the same way as in the above, the proof of the theorem is completed.

References

[1] Miller S.S. and Mocanu P.T., *Differential subordinations and univalent functions.* Michigan Math. J. **28** (1981), 157–171.

[2] Nunokawa M., Obradović M. and Owa S., *One criterion for univalency.* Proc. Amer. Math. Soc. **106** (1989), 1035–1037.

[3] Obradović M., Pascu N.N. and Radamir I., *A class of univalent functions.* Math. Japon. **44** (1996), 565–568.

[4] Ozaki S. and Nunokawa M., *The Schwarzian derivative and univalent functions.* Proc. Amer. Math. Soc. **33** (1972), 392–394.

[5] Yang D. and Liu J., *On a class of univalent functions.* Internat. J. Math. Math. Sci. **22** (1999), 605–810.

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