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Algebraic BP-theory and norm varieties

Nobuaki Yagita

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Abstract. Let p be an odd prime and $BP^*(pt) \cong \mathbb{Z}_{(p)}[v_1, v_2, \dots]$ the coefficient ring of the Brown-Peterson cohomology theory $BP^*(-)$ with $|v_i| = -2p^i + 2$. We study $ABP^{*,*'}(-)$ theory, which is the counter part in algebraic geometry of the $BP^*(-)$ theory. Let k be a field with $k \subset \mathbb{C}$ and $K_*^M(k)$ the Milnor K-theory. For a nonzero symbol $a \in K_{n+1}^M(k)/p$, a norm variety V_a is a smooth variety such that $a|_{k(V_a)} = 0 \in K_{n+1}^M(k(V_a))/p$ and $V_a(\mathbb{C}) = v_n$. In particular, we compute $ABP^{*,*'}(M_a)$ for the Rost motive M_a which is a direct summand of the motive $M(V_a)$ of some norm variety V_a.

Key words: algebraic cobordism, BP-theory, norm variety.

1. Introduction

A. Suslin and V. Voevodsky constructed and developed the motivic cohomology theory $H^{*,*'}(X; \mathbb{Z}/p)$ for algebraic sets X (objects of the $\mathbb{A}^1$-homotopy category) over the base field k. This theory is the counter part in algebraic geometry of the usual mod p singular cohomology in algebraic topology. Let $ch(k) = 0$ and fix an embedding $k \subset \mathbb{C}$. As the counter part of the complex cobordism theory $MU^*(X)$, Voevodsky defined the algebraic cobordism theory $MGL^{*,*'}(X)$ and used it in the first proof of the Milnor conjecture [Vo1], [Vo2].

Given a nonzero symbol $a \in K_{n+1}^M(k)/p$, the norm variety V_a is a variety such that $a|_{k(V_a)} = 0 \in K_{n+1}^M(k(V_a))/p$ and $V_a(\mathbb{C}) = v_n$. Here v_n is the $2(p^n - 1)$-dimensional complex manifold generating

$$\mathbb{Z}_{(p)}[v_1, v_2, \dots] \cong BP^*(pt.) \subset MU^*(pt.)_{(p)}$$

the coefficient ring of the $BP^*(-)$ theory in algebraic topology.

For $p = 2$, we can take the norm variety by the smallest neighbor Q_a of the Pfister quadric defined by a. Voevodsky proved [Vo2], [Vo3] the

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Milnor conjecture by studying cohomology operations on $H^{*,*'}(Q_a;\mathbb{Z}/2)$. Moreover $MGL^{2*,*}(Q_a)$ is studied by Vishik and Yagita [Vi-Ya] and applied to determine the multiplicative structure of the Chow rings of excellent quadrics [Ya3].

Recently Rost ([Ro], [Su-Jo]) announced the constructions of the norm variety V_a also for p odd, and Voevodsky ([Vo6]) gives the proof of the Bloch-Kato conjecture (which is the odd prime version of the Milnor conjecture) by studying $H^{*,*'}(V_a;\mathbb{Z}/p)$.

In this paper we study the algebraic $ABP^{*,*'}(X)$-theory, which is an algebraic version of the topological $BP^*(X)$-theory such that

$$ABP^{*,*'}(X) \cong MGL^{*,*'}(X)_{(p)} \otimes_{MU^*} BP^*.$$

For examples, we explicitly study the cohomology operations and Gysin maps in $ABP^{*,*'}$-theory. Moreover, we compute $ABP^{2*,*}(M_a)$ for the (generalized) Rost motive M_a, which is a direct summand of $ABP^{2*,*}(V_a)$. This computation extends the results in [Vi-Ya] to odd p cases. This result can be applied to seek the Chow rings of nontrivial torsors of exceptional groups for $p \geq 3$ [Ya5].

For the above arguments, we use the Atiyah-Hirzebruch spectral sequence (AHss) for $ABP^{*,*'}(-)$ from [Ya2], which is reduced from the result for $MGL^{*,*'}(-)$ by Hopkins-Morel. Hopkins and Morel announced their result more than ten years ago, but the text is still unavailable. We note here that we are using the existence and convergence of AHss for $ABP^{*,*'}(X)$ when X are smooth varieties or Thom spaces of smooth varieties in this paper.

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2. Cohomology operations

Let p be a fixed prime number. Let k be a field with $ch(k) = 0$, which contains a primitive p-th root of unity. In this paper, the mod(p) motivic cohomology $H^m_{Zar}(X;\mathbb{Z}/p(n))$ is written by $H^{m,n}(X;\mathbb{Z}/p)$ for an object X in the $\mathbb{A}^1$-homotopy category. We fix an embedding $k \subset \mathbb{C}$ and denote by

$t_{\mathbb{C}}$ the realization map

$$t_{\mathbb{C}} : H^{*,*'}(X;\mathbb{Z}) \to H^*(X(\mathbb{C});\mathbb{Z})$$

where the right hand side is the usual (singular) cohomology of the complex manifold of $\mathbb{C}$-rational points of X when X is a smooth variety.

In the motivic mod(p) cohomology, we have the Bockstein and the reduced power operations

$$P^i : H^{*,*'}(X;\mathbb{Z}/p) \to H^{*+2(p-1)i,*'+(p-1)i}(X;\mathbb{Z}/p) \tag{2.1}$$

$$\beta P^i : H^{*,*'}(X;\mathbb{Z}/p) \to H^{*+2(p-1)i+1,*'+(p-1)i}(X;\mathbb{Z}/p) \tag{2.2}$$

which are compatible with the usual Bockstein and the reduced powers operations via the realization map $t_{\mathbb{C}}$ ([Vo2], [Vo4]). (We identify $\beta = \beta P^0$ but note that βP^i is not assumed to have the decomposition $\beta \cdot P^i$.)

Let $\tau \in H^{0,1}(pt.;\mathbb{Z}/p) \cong \mathbb{Z}/p$ and $\rho \in H^{1,1}(pt.;\mathbb{Z}/p) \cong k^*/(k^*)^p$ be elements corresponding to the primitive root ζ of unity. Then $\beta(\tau) = \rho$. Reduced power operations have the following properties for all primes (Lemma 9.7, Lemma 9.8 in [Vo4]),

$$P^0 = \text{Identity}, \quad P^n(x) = x^p \quad \text{if } x \in H^{2n,n}(X;\mathbb{Z}/p), \tag{2.3}$$

$$P^i(x) = 0 \quad \text{if } x \in H^{m,n}(X;\mathbb{Z}/p),\ i > m-n \text{ and } i \geq n. \tag{2.4}$$

When $p > 2$, the Cartan formula

$$P^i(xy) = \sum_{0\leq j\leq i} P^j(x)P^{i-j}(y)$$

and the Adem relations are also satisfied as the topological cases. However when $p = 2$ we need some modification for τ and ρ ($P^i = Sq^{2i}$ and $\beta = Sq^1$). For example

$$Sq^{2i}(uv) = \sum_{0\leq i\leq i} Sq^{2j}(u)Sq^{2i-2j}(v) + \tau \sum_{0\leq j\leq i-1} Sq^{2j+1}(u)Sq^{2i-2j-1}(v). \tag{2.5}$$

Moreover we have the Milnor operation

$$Q_i : H^{*,*'}(X;\mathbb{Z}/p) \to H^{*+2p^i-1,*'+p^i-1}(X;\mathbb{Z}/p). \tag{2.6}$$

When $p \geq 3$, we have $Q_0 = \beta$ and $Q_{i+1} = [Q_i, P^{p^i}]$. But for $p = 2$ the above property holds only with $\mathrm{mod}(\rho)$ (see [Vo4] for details). We note $Q_i^2 = 0$ and $Q_iQ_j = -Q_jQ_i$. But Q_i is not a derivation when $\rho \neq 0$ and $p = 2$ (while it is a derivation whenever $p \geq 3$).

For a non zero element x in $H^{m,n}(X;\mathbb{Z}/p)$ or each cohomology operation (or differential in the spectral sequence), we define the weight and the difference by $w(x) = 2n - m$ and $d(x) = m - n$ so that if X is a smooth variety, then

$$w(x) \geq 0, \quad d(x) \leq \dim(X).$$

We also note $w(\beta) = -1$, $w(P^i) = 0$, $w(Q_i) = -1$.

The solution of the Bloch-Kato conjecture by Voevodsky implies

$$H^{*,*'}(X;\mathbb{Z}/p) \cong H^*_{et}(X;\mathbb{Z}/p) \quad \text{for } * \leq *',$$
$$H^{*,*}(pt.;\mathbb{Z}/p) \cong K^M_*(k)/p \cong H^*_{et}(pt.;\mathbb{Z}/p).$$

Since $d(x) \leq 0$ for non zero $x \in H^{*,*'}(pt.;\mathbb{Z}/p)$, we have

Lemma 2.1 *$H^{*,*'}(pt.;\mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \otimes K^M_*(k)/p$.*

Corollary 2.2 *Let $p \geq 3$. For $x \in H^{*,*'}(pt.;\mathbb{Z}/p)$, we see $Q_i(x) = 0$ and $P^j(x) = 0$ for all $i, j \geq 1$.*

Proof. By dimensional reason, $P^n(x) = 0$ for $x \in H^{*,*}(pt.;\mathbb{Z}/p) \cong K^M_*(k)/p$ or $x = \tau$. When $p > 2$, the Cartan formula holds, hence $P^n(x) = 0$ for all $x \in H^{*,*'}(pt;\mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \otimes K^M_*(k)$ and $n > 0$. We see also $Q_n(x) = 0$ for $n > 0$, since Q_n is a derivation, and is trivial on $K^M_*(k)/p$ and on τ. □

Remark However when $p = 2$, in general, $P^n(x) \neq 0$ and $Q_n(x) \neq 0$ for $x \in H^{*,*'}(pt,;\mathbb{Z}/2)$, for example, see [Vo4] or [Ya2].

V. Voevodsky (the main theorem in [Vo7]) showed that the $\mathrm{mod}\, p$ motivic Steenrod algebra $A_p^{*,*'}$ is generated as an $H^{*,*'}(pt,\mathbb{Z}/p)$-module by products of P^i and βP^j. Moreover he also proved

$$A_p^{*,*'} \cong H^{*,*'}(pt;\mathbb{Z}/p) \otimes RP \otimes \Lambda(Q_0, Q_1, \dots) \tag{2.7}$$

where RP is the $\mathbb{Z}/p$-module generated by products of reduced powers $P^{i_1} \dots P^{i_n}$ (without the Bockstein).

3. *ABP* theories

Hereafter, in this paper, we assume that p is an odd prime number. We recall that $MU^*(-)$ is the complex cobordism theory defined on the category of topological spaces and ([Mi], [Ha], [Ra])

$$MU^* = MU^*(pt.) = \mathbb{Z}[x_1, x_2, \dots] \quad |x_i| = -2i.$$

Here each x_i is represented by sum of hypersurfaces of $\dim(x_i) = 2i$ defined by polynomials with the coefficient in $\mathbb{Z}$, in some product of complex projective spaces.

Let $MGL^{*,*'}(-)$ be the motivic cobordism theory defined by Voevodsky. By the Thom isomorphism, it is easily proved that ([Hu-Kr], [Ve]) MGL is cellular and there is an $H^{*,*'}(pt)$-module isomorphism

$$H^{*,*'}(MGL) \cong H^{*,*'}(BGL) \cong H^{*,*'}(pt)[c_1, c_2, \dots] \quad \text{with } \deg(c_i) = (2i, i).$$

This isomorphism induces the $A_p^{*,*'}$-module isomorphism

$$H^{*,*'}(MGL;\mathbb{Z}/p) \cong H^{*,*'} \otimes RP \otimes \mathbb{Z}/p[m_i \mid i \neq p^j - 1]$$

with $H^{*,*'} = H^{*,*'}(pt.;\mathbb{Z}/p)$ and $\deg(m_i) = (2i, i)$. (Here $A_p^{>0,*'}$ acts trivially on m_i. The Cartan formula for P^i and the fact that Q_i is a derivation give the $A_p^{*,*'}$ action on $H^{*,*'}(MGL;\mathbb{Z}/p)$ above.)

Let us write by AMU the spectrum $MGL_{(p)}$ representing the motivic cobordism theory (localized at p), i.e., $MGL^{*,*'}(-)_{(p)} = AMU^{*,*'}(-)$. Since $AMU^{*,*'}(X)$ is a multiplicative cohomology theory, we know it is an $AMU^{*,*'}(pt.)$-algebra. Moreover we can embeds MU^* into $AMU^{2*,*}(pt.)$ ([Vo1]). Hence $AMU^{*,*'}(X)$ is also an $MU^*_{(p)}$-algebra.

Given a regular sequence $S_n = (s_1, \dots, s_n)$ with $s_i \in MU^*_{(p)}$, we can inductively construct the AMU-module spectrum by the cofibering of spectra ([Bo], [Hu], [Ya2])

$$\mathbb{T}^{-1/2|s_i|} \wedge AMU(S_{i-1}) \xrightarrow{\times s_i} AMU(S_{i-1}) \to AMU(S_i) \qquad (3.1)$$

where $\mathbb{T}$ is the Tate object (so that $H^{*+2,*'+1}(\mathbb{T} \wedge X) \cong H^{*,*'}(X)$).

It is also immediate that $t_{\mathbb{C}}(AMU(S_n)) \cong MU(S_n)$ with

$$MU(S_n)^* = MU^*/(\mathrm{Ideal}(S_n)).$$

Recall that the Brown-Peterson theory $BP^*(X)$ is defined ([Ra], [Ha], [No], [Ya1]) by

$$BP^*(X) = MU(x_i \mid i \neq p^j - 1)^*(X)_{(p)}$$

so that $BP^* \cong \mathbb{Z}_{(p)}[v_1, \dots]$ with identifying $v_i = x_{p^i-1}$. For $S = (v_{i_1}, \dots, v_{i_n})$, let us write

$$ABP(S) = AMU(S \cup \{x_i \mid i \neq p^j - 1\}) \qquad (3.2)$$

so that $t_{\mathbb{C}}(ABP(S)) = BP(S)$ with $BP(S)^* = BP^*/(S)$. By using the long exact sequence induced from (3.1), we have

Lemma 3.1 ([Bo], Lemma 3.1 in [Ya2]) *Let $S = (v_{i_1}, \dots, v_{i_n})$. Then*

$$\begin{aligned} H^{*,*'}(ABP(S); \mathbb{Z}/p) &\cong H^{*,*'}(pt.; \mathbb{Z}/p) \otimes H^*(BP(S); \mathbb{Z}/p) \\ &\cong H^{*,*'}(pt.; \mathbb{Z}/p) \otimes RP \otimes \Lambda(Q_{i_1}, \dots, Q_{i_n}). \end{aligned}$$

For each $ABP(S)^{*,*'}(X)$-theory, we can construct the Atiyah-Hirzebruch spectral sequence (AHss).

Theorem 3.2 (Theorem 3.5 in [Ya2]) *Let $Ah = ABP(S)$ for $S = (v_{i_1}, v_{i_2}, \dots)$. Then there is AHss (the Atiyah-Hirzebruch spectral sequence)*

$$E(Ah)_2^{(m,n,2n')} \cong H^{m,n}(X; h^{2n'}) \Longrightarrow Ah^{m+2n',n+n'}(X)$$

with the differential $d_{2r+1} : E_{2r+1}^{(m,n,2n')} \to E_{2r+1}^{(m+2r+1,n+r,2n'-2r)}$.

From the above theorem and dimensional reason (Corollary 3.8 in [Ya2]), we see

$$ABP(S)^{2*,*}(pt) \cong BP(S)^* = BP^*/(S). \qquad (3.3)$$

The above AHss is the spectral sequence of $h^* \cong BP^*/S$-algebras. When $p \notin S$, we have for smooth X (Corollary 3.9 in [Ya2]),

$$ABP(S)^{2*,*}(X) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong H^{2*,*}(X)_{(p)} \cong CH^*(X)_{(p)}. \tag{3.4}$$

We also note the following lemma (the motivic version of the main theorem in [Ya1]).

Lemma 3.3 *If $\sum v_i y_i = 0 \in ABP^{*,*'}(X)$, then there is $x \in H^{*,*'}(X; \mathbb{Z}/p)$ such that $Q_i(x) = \rho(y_i)$ where $\rho : ABP \to AH\mathbb{Z}/p$ is the natural (Thom) map.*

Proof. (This proof is a motivic version of the argument of Tamanoi [Ta].) Define the map κ by the following composition map

$$\Pi \mathbb{T}^{p^i-1} ABP \xrightarrow{\vee v_i} \bigvee ABP \xrightarrow{\text{folding}} ABP$$

so that $\kappa_*(b_0, b_1, \dots) = \sum v_i b_i$ for $b_i \in ABP^{*,*'}(X)$. Let AL be the spectrum and Πq_i, θ be maps defined by the following cofiber sequence

$$S_s^{-1} AL \xrightarrow{\Pi q_i} \Pi \mathbb{T}^{p^i-1} ABP \xrightarrow{\kappa} ABP \xrightarrow{\theta} AL$$

Since $v_n^* = 0$ (from $v_n^*(1) = 0$) on $H^{*,*'}(ABP; \mathbb{Z}/p)$, we see $\kappa^* = 0$ on $H^{*,*'}(ABP; \mathbb{Z}/p)$. Hence we have

$$\begin{aligned} 0 \to\ & H^{*-1,*'}(\Pi \mathbb{T}^{p^i-1} ABP; \mathbb{Z}/p) \\ & \xrightarrow{\Pi q_i^*} H^{*,*'}(AL; \mathbb{Z}/p) \to H^{*,*'}(ABP; \mathbb{Z}/p) \to 0. \end{aligned}$$

Recall $H^{*,*'}(ABP; \mathbb{Z}/p) \cong H^{*,*'}(pt.; \mathbb{Z}/p) \otimes RP$ (see Lemma 3.1). Hence the mod p cohomology is easily computed

$$H^{*,*'}(AL; \mathbb{Z}/p) \cong H^{*,*'}(pt.; \mathbb{Z}/p) \otimes RP \otimes \{1, q_0^*(1_0), q_1^*(1_1), \dots\}$$

where $1_i \in H^{2p^i-1, p^i-1}(\mathbb{T}^{p^i-1} ABP; \mathbb{Z}/p) \cong \mathbb{Z}/p$ which is represented by $\rho : ABP \to H\mathbb{Z}/p$, and where $1 \in H^{0,0}(AL; \mathbb{Z}/p)$ is $(\theta^*)^{-1}(1)$ for $H^{0,0}(AL; \mathbb{Z}/p) \overset{\theta^*}{\cong} H^{0,0}(ABP; \mathbb{Z}/p)$ since $H^{-1,0}(\Pi \mathbb{T}^{p^i-1} ABP; \mathbb{Z}/p) = 0$.

Here we can prove that

$$q_i^*(1_i) = Q_i(1) \quad \text{for } 1 \in H^{0,0}(AL;\mathbb{Z}/p).$$

Because this holds for topological case (see [Ta] for details), and $H^{2*+1,*}(AL;\mathbb{Z}/p)$ is isomorphic to $RP \otimes \{q_0^*(1_0), q_1^*(1_1), \dots\}$ which maps injectivity to (the topological) $H^{2*+1}(AL;\mathbb{Z}/p)$ by the realization map $t_{\mathbb{C}}$.

Let $\eta : AL \to H\mathbb{Z}/p$ be the map of spectra representing $1 \in H^{0,0}(AL;\mathbb{Z}/p)$. The above equation $q_i^*(1_i) = Q_i(1)$ means

$$\rho q_i = Q_i\eta : AL \to S^{2p^i-1,p^i-1}H\mathbb{Z}/p$$

as homotopy maps.

Suppose $\sum v_i y_i = 0 \in ABP^{*,*'}(X)$. Then $\kappa(\Pi(y_i)) = 0$. So there is $z \in AL^{*-1,*}(X)$ with $\Pi(q_i(z)) = \Pi(y_i)$. Take $x = \eta(z)$ and we get

$$\rho(y_i) = \rho q_i(z) = Q_i\eta(z) = Q_i(x). \qquad \square$$

Corollary 3.4 *Let $z \in E_\infty^{*,*',0} \subset H^{*,*'}(X;\mathbb{Z}/p)$ in AHss converging to $ABP^{*,*'}(X)$ such that $v_n z = 0 \in E_\infty^{*,*',*''}$ for some $n \geq 0$. Then there is $x \in H^{*,*'}(X;\mathbb{Z}/p)$ such that $\sum_{i\geq 0} v_i y_i = 0$ in $ABP^{*,*}(X)$ with $\rho(y_i) = Q_i(x)$ for all $i \geq n$ and $z = \rho(y_n)$.*

Proof. Let F_* be the filtration of $ABP^{*+*'',*'+1/2*''}(X)$ such that $E_\infty^{*,*',*''} \cong F_*/F_{*+1}$. Then $v_n z = 0 \in E_\infty^{*,*',*''}$ means that $v_n z = 0 \bmod(F_{*+1})$ in $ABP^{*,*'}(X)$. So there is a relation in $ABP^{*,*'}(X)$ such that

$$v_n y_n' + v_{n+1}y_{n+1}' + \cdots = 0 \quad \bmod(p, v_1, v_2, \dots)^2$$

with $\rho(y_n') = z$. Taking $y_i = y_i' \bmod(p, v_1, v_2, \dots)$ (so $y_i = 0 \bmod(p, v_1, \dots)$ for $i < n$), we have

$$py_0 + v_1 y_1 + \cdots + v_n y_n + v_{n+1}y_{n+1} + \cdots = 0.$$

Since $\rho(y_i') = \rho(y_i)$, from the preceding lemma, we have the corollary. $\square$

4. Cohomology operations in $ABP^{*,*'}(-)$-theory

Recall that

$$H^{*,*'}(MGL) \cong H^{*,*'}(pt.;\mathbb{Z}) \otimes H^*(MU) \tag{4.1}$$

where additively $H^*(MU) \cong H^*(BU) \cong \mathbb{Z}[c_1, \dots]$ and where c_i is the i-th Chern class with $\deg(c_i) = (2i, i)$. It is known ([Hu-Kr], [Ve], [Bo]) that

$$MGL^{*,*'}(MGL) \cong MGL^{*,*'}(pt) \otimes H^*(MU).$$

Consider AHss for X

$$E(X)_2^{*,*',*''} = H^{*,*'}(X, \mathbb{Z}) \otimes MU^{*''} \Longrightarrow MGL^{*,*'}(X).$$

Since each element in $H^*(MU)$ is a permanent cycle, we have the isomorphism for all $r \geq 2$,

$$E(MGL)_r^{*,*',*''} \cong E(pt.)_r^{*,*',*''} \otimes H^*(MU).$$

The Steenrod algebra of MGL-theory is isomorphic to $MGL^{*,*'}(MGL)$. Hence for each $MGL^{*,*'}$-basis $\{\bar{c}_\beta\}$, we can take (not canonically) an cohomology operation $\bar{s}_\beta$ corresponding $\bar{c}_\beta$. In particular, given $\alpha = (\alpha_1, \dots, \alpha_s)$, with $\alpha_i \geq 0$, take a base c_α as the symmetrization of $x_1^{\beta_1} x_2^{\beta_2} \dots$ where $\alpha_j = \sharp(i|\beta_i = j)$ (identifying c_i is the i-th elementary symmetric function of $x_1, x_2, \dots$). Let us write by S_α the corresponding operation in $ABP^{*,*'}(-)$ and call it the Landweber-Novikov operation ([No], [Ra], [Ha]).

By using these Landweber-Novikov operations, we can define [Ya2] (see also [No] for the topological case) the projector $\Phi : MGL_{(p)} \to ABP$. Hence $ABP^{*,*'}(X)$ is a direct summand of $MGL^{*,*'}(X)_{(p)}$.

Lemma 4.1 (Lemma 4.1 in [Ya2]) *The theory $ABP^{*,*'}(-)$ is a multiplicative theory and there exists a map $ABP \to AMGL_{(p)}$ which induces the natural BP^*-algebra isomorphism*

$$ABP^{*,*'}(X) \cong MGL^{*,*'}(X)_{(p)} \otimes_{MU^*_{(p)}} BP^*,$$

and the natural $MU^*_{(p)}$-algebra isomorphism

$$MGL^{*,*'}(X)_{(p)} \cong ABP^{*,*'}(X) \otimes_{BP^*} MU^*_{(p)}$$

(*identifying* $v_i \in BP^*$ *with* $x_{p^i-1} \in MU^*_{(p)}$).

Proposition 4.2 (Proposition 4.2 in [Ya2]) *Let us write*

$$\tilde{RP} = \mathbb{Z}_{(p)}\{r_\alpha | \alpha = (\alpha_1, \alpha_2, \dots), \alpha_i \geq 0\}$$

with $\deg(r_\alpha) = (2\sum \alpha_i(p^i-1), \sum \alpha_i(p^i-1))$. *Then there are* $ABP^{*,*'}(pt.)$*-module isomorphisms*

$$ABP^{*,*'}(ABP) \cong ABP^{*,*'}(pt.) \otimes H^*(BP) \cong ABP^{*,*'}(pt) \otimes \tilde{RP}.$$

Since $ABP^{2*,*} \cong BP^{2*}$ from (3.3), we have the isomorphism

$$ABP^{2*,*}(ABP) \cong BP^{2*}(BP).$$

Hence for each cohomology operation in $BP^*(-)$ theory, there is a unique operation in $ABP^{*,*'}(-)$-theory. The Steenrod algebra of BP-theory is generated as an BP^*-module by the Quillen operation r_α for $\alpha = (\alpha_1, \dots)$ with $|r_\alpha| = 2\sum \alpha_i(p^i-1)$. Hence $ABP^{*,*'}(ABP)$ is also generated by (ABP-)Quillen operation r_α as an $ABP^{*,*'}$-module.

Remark The Landweber-Novikov operation S_α is also defined as the cohomology operations in $ABP^{*.*'}(-)$ theory by

$$ABP \to MGL_{(p)} \xrightarrow{S_\alpha} MGL_{(p)} \to ABP.$$

We use the same letter S_α for this operation in $ABP^{*,*'}(-)$. Then for each sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ such that $\alpha_i = 0$ if $i \neq p^k - 1$ for each k, the Landweber-Novikov operation S_α generates $ABP^{*,*'}(ABP)$ as an $ABP^{*,*'}(pt.)$-module.

Each multiplicative operation $o(-)$ in $BP^*(X)$ theory is determined by an element (see [Ha], [Ra])

$$o(y) \in (BP^*[[y]])^2 \cong BP^2(\mathbb{C}P^\infty), \quad |y| = 2.$$

The total Quillen operation r_t (resp. S_t) in $BP^*(-)[t_1, t_2, \dots]$ theory ($|t_i| =$

$2(p^i - 1))$ is defined by

$$o(y) = r_t(y) = \sum^{F_{BP}} t_n y^{p^n} \quad \left(\text{resp. } S_t(y) = \sum t_n y^{p^n}\right)$$

where $\sum^{F_{BP}}$ means sum of the formal group law of $BP^*(-)$ theory. Then the operation $r_\alpha(-)$ is defined from the total operation

$$r_t(x) = \sum r_\alpha(x) t^\alpha \quad \text{with } t^\alpha = t_1^{\alpha_1} \dots$$

and S_α is defined similarly.

The motivic r_α in $ABP^{*,*'}(-)$ is defined just by the inverse image of the topological r_α from the isomorphism $ABP^{2*,*}(ABP) \cong BP^{2*}(BP)$. (This does not means that $r_t(x)$ is multiplicative.) However, we see that the motivic r_t is also multiplicative, indeed, the Quillen operation r_α (and the Landweber-Novikov operation S_α) satisfies the Cartan formula also in $ABP^{*,*'}(-)$-theory.

Lemma 4.3 *In $ABP^{*,*'}(X)$, we have the Cartan formula, i.e.,*

$$r_\alpha(xy) = \sum_{\alpha=\alpha'+\alpha''} r_{\alpha'}(x) r_{\alpha''}(y).$$

Proof. The Cartan formula holds if

$$\mu^*(r_\alpha) = \sum_{\alpha=\alpha'+\alpha''} r_{\alpha'} \otimes r_{\alpha''} \tag{$*$}$$

for the coproduct map $\mu^* : ABP^{*,*'}(ABP) \to ABP^{*,*'}(ABP \wedge ABP)$. Here note for $X = ABP, ABP \wedge ABP$ (from Proposition 4.2), we have

$$ABP^{*,*'}(X) \cong ABP^{*,*'}(pt) \otimes H^*(t_{\mathbb{C}}(X))_{(p)}.$$

In particular

$$ABP^{2*,*}(X) \cong BP^{2*}(t_{\mathbb{C}}(X)).$$

The Cartan formular holds in $BP^*(-)$ theory and the formula $(*)$ holds in BP^*-theory and so does in $ABP^{2*,*}(ABP \wedge ABP)$, indeed $r_\alpha \in ABP^{2*,*}(ABP)$. $\square$

As a BP^*-algebra, we have $ABP^{2*,*}(ABP) \cong BP^{2*}(BP)$. Of course $ABP^{*,*'}(X)$ is a $ABP^{*,*'}(ABP)$-module.

Lemma 4.4 *$ABP^{*,*'}(ABP)$ is a $BP^*(BP)$-module.*

Recall that $H_*(BP) \cong \mathbb{Z}_{(p)}[m_1, m_2, \dots]$ where $m_i = 1/(p^i)\mathbb{C}P^{p^i-1}$ where $\mathbb{C}P^m$ is the m-dimensional complex projective space ([Ha], [Ra]). The Quillen operation r_α on m_n is explicitly written.

Lemma 4.5 (Quillen [Ha], [Ra])

$$r_\alpha(m_n) = \begin{cases} m_i & \textit{if } \alpha = p^i\Delta_{n-i} \textit{ for } \Delta_{n-i} = (0, \dots, 0, \overset{n-i}{1}, 0, \dots) \\ 0 & \textit{otherwise.} \end{cases}$$

Hazewinkel showed the following expression of v_n by m_i ([Ha])

$$v_n = pm_n - \sum_{1 \le i \le n-1} m_i v_{n-i}^{p^i}$$

identifying $\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, \dots] \subset H_*(BP) = \mathbb{Z}_{(p)}[m_1, \dots]$.

Let us write by I_n the ideal in BP^* generated by $(v_0, \dots, v_{n-1})$. (Let $v_0 = p$.) One of important properties of r_α is;

Lemma 4.6 (Hazewinkel [Ha], [Ra])

$$r_\alpha(v_n) = \begin{cases} v_i \mod(I_i^2) & \textit{if } \alpha = p^i\Delta_{n-i} \\ 0 \mod(I_n^2) & \textit{otherwise.} \end{cases}$$

An Ideal J in BP^* is called invariant if it is so under the Quillen (or Landweber-Novikov) operations, i.e., $r_\alpha(J) \subset J$ for all α.

Lemma 4.7 (prime invariant ideal theorem [La]) *If for $a \in BP^*$, the ideal $J = (I_n, a)$ is invariant, then $a = \lambda v_n^s \mod(I_n)$ for $\lambda \in \mathbb{Z}/p$ and $s \ge 1$. In particular, prime invariant ideals are written as I_m for $m \ge 1$ or I_∞.*

One of examples of invariant ideals is following. For AHss converging $ABP^{*,*'}(X)$, we recall the filtration of the infinite term $E_\infty^{s,*',*''} \cong$

$F_s(X)/F_{s+1}(X)$.

Corollary 4.8 *If $x \in E_\infty^{*,*',0}$ and $BP^*/J\{x\} \subset E_\infty^{*,*',*''}$ for some ideal J, then this ideal J is invariant.*

Proof. Let us write $x' \in ABP^{*,*'}(X)$ a corresponding element to $x \in E_\infty^{n,*',0}$. Let $a \in J$ so that $ax' = 0 \bmod(F_{n+1})$. Then

$$0 = r_\alpha(ax') = \sum_{\alpha=\alpha'+\alpha''} r_{\alpha'}(a) r_{\alpha''}(x') = r_\alpha(a)x' \quad \mathrm{mod}(F_{n+1})$$

since $r_{\alpha''}(x') \in F_{n+|r_{\alpha''}|} \subset F_{n+1}$ for $\alpha'' \neq 0$. (Of course $ABP^{s,*'}(X) \subset F_s$.) Hence $r_\alpha(a)$ is also in J. □

5. Gysin maps

First we recall the Thom isomorphism. Let V be an m-dimensional vector bundle over X and $Th_X(V)$ be the induced Thom space. Then it is well known that there is the Thom isomorphism (for details, see [Vo1], [Vo2], [Pa], [Ne], [St], [Ra])

$$Th : H^{*,*'}(X;\mathbb{Z}) \cong \tilde{H}^{*+2m,*'+m}(Th_X(V);\mathbb{Z}).$$

The element $Th(1) \in H^{2m,m}(Th_X(V))$ is called its Thom class and the above isomorphism is that of $H^{*,*'}(X;\mathbb{Z})$-modules. The right hand module is a free $H^{*,*'}(X;\mathbb{Z})$-module generated by the Thom class $Th(1)$ (by the diagonal map $Th_X(V) \to Th_X(V) \wedge X$).

Lemma 5.1 *The Thom isomorphism also holds in $ABP^{*,*'}(X)$ for smooth X*

$$Th : ABP^{*,*'}(X) \cong ABP^{*+2m,*'+m}(Th_X(V)).$$

Proof. Consider the AHss $E(Th_X(V))_r$ (resp. $E(X)_r$) converging to $ABP^{*,*'}(Th_X(V))$ (resp. $ABP^{*,*'}(X)$). Since $w(Th(1)) = 0$, we see that the Thom class $Th(1)$ is a permanent cycle in $E(Th_X(V))_r$. Then we see inductively that $E(TH_X(V))_r$ is the free $E(X)_r$-module generated by $Th(1)$. Hence we get the lemma. □

For a projective map $f : Y \to X$ of smooth projective varieties such that $c = \mathrm{codim}_X(Y)$ is constant, we will define the Gysin map

$$f_* : ABP^{*,*'}(Y) \to ABP^{*+2c,*'+c}(X).$$

(For more algebraic treatments of the Gysin map, see Nenashev [Ne], and for topological one see [St].)

By definition, the projective map is factored as

$$f : Y \xrightarrow{i} \mathbb{P}^m \times X \xrightarrow{p} X$$

where i is a closed embedding to the product $\mathbb{P}^m \times X$ and p is the projection.

For a close regular embedding $i : Y \to Z$ of $\mathrm{codim}_Z(Y) = c$, we define the Gysin map i_* by

$$i_* : ABP^{*,*'}(Y) \cong ABP^{*+2c,*'+c}(Th_Y(N_{Z/Y})) \xrightarrow{q^*} ABP^{*+2c,*'+c}(Z)$$

where $N_{Z/Y}$ is the normal bundle of Y in Z and $q : Z \to Th_Y(N_{Z/Y})$ is the quotient map.

For $p : Z \times X \to X$, the Gysin map p_* is defined as follows. There is an m dimensional vector bundle V on Z with $\dim(Z) = d$ (Theorem 2.11 [Vo3]) such that there is a map $i : \mathbb{T}^{m+d} \to Th_Z(V)$ having the property that the composition of maps

$$\begin{aligned} H^{2d,d}(Z) &\cong H^{2(m+d),m+d}(Th_Z(V)) \\ &\xrightarrow{i^*} H^{2(m+d),m+d}(\mathbb{T}^{m+d}) \cong H^{0,0}(pt.) = \mathbb{Z} \end{aligned}$$

coincides with the degree map. Then we can define the Gysin map

$$\begin{aligned} p_* : ABP^{*,*'}(Z \times X) &\cong ABP^{*+2m,*'+m}(Th_Z(V) \times X) \\ &\xrightarrow{i^*} ABP^{*+2m,*'+m}(\mathbb{T}^{m+d} \times X) \cong ABP^{*-2d,*'-d}(X). \end{aligned}$$

Of course for a projective map f, we define the Gysin map by $f_* = p_* i_*$. (We can see that the above is well defined by considering the embedding to $\mathbb{P}^M \times X$ for a sufficient large $M > 0$, see Nenashev [Ne] for example.)

In particular, $ABP^{2*,*}(X)$ is closed under f_* and f^*, that is an oriented cohomology theory in Panin's sense [Pa]. Recall here the algebraic cobordism theory $\Omega^*(X)$ defined by Levine and Morel [Le-Mo1], [Le-Mo2]. This theory is the universal oriented theory of which elements are represented by $f : Y \to X$ such that $\mathrm{codim}_X(Y)$ is constant and f is projective. By the universality of $\Omega^*(X)$, we can define the natural map

$$\rho_{BP} : \Omega^*(X)_{(p)} \otimes_{MU^*} BP^* \to ABP^{2*,*}(X)$$

by $\rho_{BP}([f : Y \to X]) = f_*(1_X)$. By the recent result by Levine [Le], the natural map $\Omega^*(X) \to MGL^{2*,*}(X)$ is an isomorphism. This implies that the above map ρ_{BP} is also an isomorphism. Therefore, each element $x \in ABP^{2*,*}(X)$ is represented by $f_*(1_Y) = [f : Y \to X]$ such that $\mathrm{codim}_X(Y)$ is constant and f is projective.

Recall that $S_t = \sum S_\alpha t^\alpha$ (resp. $c_t = \sum c^\alpha t^\alpha$, this c^α is that defined just before Lemma 4.1) is the total Landweber-Novikov operation (resp. total Chern class). Let us write

$$\nu_f = -f^*(T_X) + T_Y \in K(Y)$$

for the tangent bundles T_X and T_Y. Then on $ABP^{2*,*}(X)$, we can define the operations s_t by

$$s_t(f_*(1_Y)) = f_*(c_t(\nu_f)) \tag{1}$$

such that $t_{\mathbb{C}} s_t = S_t t_{\mathbb{C}}$ (see [Qu], [No] for MU^*-case).

Remark In Section 4 we defined the Landweber-Novikov operation S_t for all $ABP^{*,*'}(-)$. The author does not prove yet that $S_t|ABP^{2*,*}(X) = s_t$ while $t_{\mathbb{C}}(S_t) = t_{\mathbb{C}}(s_t)$.

Example Consider the inclusion $i : \mathbb{P}^d \to \mathbb{P}^{d+1}$. Then the total Chern class of the normal bundle ν_i is

$$c_t(\nu_i) = \Big(\sum t_n y^{p^n - 1}\Big) \quad \text{with } e(\nu_i) = y,$$

in fact, $c_{\Delta_i}(L) = e(L)^{p^i - 1}$ for line bundles L. (Here $\Delta_i = (0, \ldots, 0, \overset{i}{1}, 0, \ldots, 0)$ and see the explanation before Lemma 4.1 for the definition c_α.) On the other hand, the total Landweber-Novikov operation is

$$S_t(y) = \sum t_n y^{p^n}$$

from the definition of S_t (see the explanation before Lemma 4.3). Indeed, we show (1)

$$i_*(c_t(\nu_i)) = i_*\Big(\sum t_n y^{p^n-1}\Big) = \sum t_n y^{p^n} = S_t(i_*(1))$$

since $i^*i_*(1) = e(\nu_i) = y$.

Lemma 5.2 (ABP-version of a theorem of [Qu], [Ka-Me], the Riemann-Roch theorem in Panin [Pa]) *Let $x \in ABP^{2*,*}(Y)$ and $f : Y \to X$ be projective. Then $s_t(f_*(x)) = f_*(c_t(\nu_f)s_t(x))$.*

Proof. Let $x = [g : Z \to Y]$. By the definition

$$\nu_{fg} = -g^*f^*T_X + T_Z = g^*(-f^*T_X + T_Y) - g^*T_Y + T_Z = g^*\nu_f + \nu_g.$$

This implies $c_t(\nu_{fg}) = g^*(c_t(\nu_f))c_t(\nu_g)$. Hence we have from (1)

$$\begin{aligned} s_t(f_*x) &= s_t(f_*g_*(1)) = f_*g_*(c_t(\nu_{fg})) = f_*g_*(g^*(c_t(\nu_f)c_t(\nu_g)) \\ &= f_*(c_t(\nu_f)g_*(c_t(\nu_g)) = f_*(c_t(\nu_f)s_t(x)). \end{aligned}$$

□

Let $\pi : X \to pt.$ be the projection. Let us write

$$I(X) = \pi_* ABP^{2*,*}(X) \subset ABP^{2*,*}(pt.) = BP^{2*}.$$

From Quillen's lemma, it is immediate

Lemma 5.3 *The ideal $I(X)$ is generated by elements x with $-2\dim(X) \leq |x| \leq 0$ as a BP^*-module. Moreover $I(X)$ is an invariant ideal of BP^*.*

Proof. Since $ABP^{2*,*}(X)$ is generated as a BP^* module by elements y with $0 \leq |y| \leq 2\dim(X)$, we have the first statement. If $a \in I(X)$, then $a = \pi_*(x)$ for some $x \in ABP^{2*,*}(X)$. Then $s_t(a) = \pi_*(c_t(\nu_\pi)s_t(x)) \in I(X)[t]$.

□

6. I_{n+1}-torsion spaces

Recall that $I_{n+1}=(p,v_1,\dots,v_n)$. In this section, we consider I_{n+1}-torsion spaces and their applications according to V. Voevodsky. Recall that $BP\langle n\rangle^*(X)$ is the cohomology theory with the coefficient $BP\langle n\rangle^*=\mathbb{Z}_{(p)}[v_1,\dots,v_n]$ so that $BP\langle -1\rangle^*(X)=H^*(X;\mathbb{Z}/p)$ and $BP\langle\infty\rangle^*(X)=BP^*(X)$.

Lemma 6.1 (Lemma 5.2 in [Ya2]) *Let $E_r^{*,*',*''}$ be the AHss for $ABP^{*,*'}(X)$. If $x=Q_n\dots Q_1Q_0x'$ in $H^{*,*'}(X;\mathbb{Z}/p)$, then $x\in E_{2p^n}^{*,*',0}$ and x is I_{n+1}-torsion in $E_{2p^n}^{*,*',*''}$.*

Proof. For each $k\geq 1$, there is a cofiber sequence of spectra (3.1)

$$\mathbb{T}^{p^k-1}\wedge ABP\langle k\rangle \xrightarrow{v_k} ABP\langle k\rangle \xrightarrow{\rho_k} ABP\langle k-1\rangle.$$

Consider the Baas-Sullivan exact sequence, namely, the long exact sequence induced from the above cofiber map

$$\to ABP\langle k\rangle^{*+2p^k-2,*'+p^k-1}(X) \xrightarrow{v_k} ABP\langle k\rangle^{*,*'}(X) \xrightarrow{\rho_k}$$
$$ABP\langle k-1\rangle^{*,*'}(X) \xrightarrow{\delta_k} ABP\langle k\rangle^{*+2p^k-1,*'+p^k-1}(X)\to\cdots$$

The induced map

$$\mathrm{Im}(ABP\langle n-1\rangle^{*,*'}(X)\to H^{*,*'}(X;\mathbb{Z}/p))\to H^{*,*'}(X:\mathbb{Z}/p)$$

defined by $\rho_0\dots\rho_{k-1}(y)\mapsto \rho_0\dots\rho_k\delta_k(y)$ for $y\in ABP\langle n-1\rangle^{*,*'}(X)$ represents Q_k $\mathrm{mod}(P^IQ_J||J|\geq 2)$ from the topological case [Ya1] and (2.7). In particular, $x\mapsto \rho_0\dots\rho_n\delta_n\delta_{n-1}\dots\delta_0(x)$ for $x\in H^{*,*'}(X;\mathbb{Z}/p)$ represents exactly the operation $Q_n\dots Q_0(x)$ (using $Q_i^2(x)=0$).

By the Baas-Sullivan exact sequence, we can see that $x''=\delta_n\dots\delta_0(x')\in ABP\langle n\rangle^{*,*'}(X)$ is I_{n+1}-torsion since the map δ_i is a map of ABP-module spectra. In particular,

$$x=Q_n\dots Q_0(x')=\rho_0\dots\rho_n(x'')$$

is a permanent cycle in the spectral sequence

$$E(ABP\langle n\rangle)_2^{*,*',*''} = H^{*,*'}\big(X; BP\langle n\rangle^{*''}\big) \Longrightarrow ABP\langle n\rangle^{*,*'}(X).$$

Compare the above spectral sequence with the ABP-spectral sequence

$$E(ABP)_2^{*,*',*''} \cong H^{*,*'}\big(X; BP^{*''}\big) \Longrightarrow ABP^{*,*'}(X).$$

Since $BP^* \cong BP\langle n\rangle^*$ for $* > -2p^{n+1}+2$, we can see that x exists in $E(ABP)_{2p^n}^{*,*',*''}$ and x is I_{n+1}-torsion. □

Let us write by $Q(n)$ the exterior algebra $\Lambda(Q_0, \dots, Q_n)$.

Lemma 6.2 (Lemma 5.3 in [Ya2]) *If $ABP\langle k\rangle^{*,*'}(X)$ is I_{k+1}-torsion for all $k \le n$, then $H^{*,*'}(X; \mathbb{Z}/p)$ is a free $Q(n)$-module.*

Proof. Consider the Baas and Sullivan exact sequence in the proof of the preceding lemma. Here $v_k = 0$ in our case, so we have

$$ABP\langle k-1\rangle^{*,*'}(X) \cong \{\rho_k, \delta_k^{-1}\} ABP\langle k\rangle^{*,*'}(X).$$

Hence by induction on n, we have the isomorphism

$$H^{*,*'}(X; \mathbb{Z}/p) \cong ABP\langle -1\rangle^{*,*'}(X) \cong Q(n) \otimes \{\delta_0^{-1} \dots \delta_n^{-1} ABP\langle n\rangle^{*,*'}(X)\},$$

which is of course a $Q(n)$-free module. □

Let the *Čech* complex $\check{C}(X)$ be the simplicial scheme such that $\check{C}(X)_n = X^{n+1}$ and the faces and degeneracy maps are given by partial projections and diagonals respectively ([Vo1], [Vo2]). One of the important properties of $\check{C}(X)$ is the following.

Lemma 6.3 ([Vo1], [Vo2], [Vo3]) *Let X, Y be smooth schemes such that*

$$\mathrm{Hom}(Y, X) \neq \emptyset.$$

Then the projection $\check{C}(X) \times Y \to Y$ is an equivalence in the $\mathbb{A}^1$-homotopy category.

In the *stable* $\mathbb{A}^1$ homotopy category, define $\tilde{C}(X)$ by the following cofiber sequence

$$\tilde{C}(X) \to \check{C}(X) \to \mathrm{Spec}(k). \tag{6.1}$$

Lemma 6.4 ([Vo1], [Ya2]) *Let $\pi : Y \to pt.$ be the projection and $\pi_*([Y]) = y$ in BP^*. Let $Ah = ABP(S_n)$ for some regular sequence S_n in BP^*. If* $\mathrm{Hom}(Y, X) \neq \emptyset$, *then $Ah^{*,*'}(\tilde{C}(X))$ is y-torsion.*

Proof. Let $p : \tilde{C}(X) \times Y \to \tilde{C}(X)$ be the projection, and consider the composition map

$$p_*p^* : Ah^{*,*'}(\tilde{C}(X)) \to Ah^{*,*'}(\tilde{C}(X) \times Y) \to Ah^{*+|y|,*'+1/2|y|}(\tilde{C}(X)).$$

Here $p_*p^*(x) = yx$, indeed,

$$p_*p^*(x) = p_*\big(1_{\tilde{C}(X)\times Y} \cdot p^*(x)\big) = p_*\big(1_{\tilde{C}(X)\times Y}\big) \cdot x = \big(y1_{\tilde{C}(X)}\big) \cdot x.$$

But $Ah^*(\tilde{C}(X) \times Y) \cong 0$ since $Ah^{*,*'}(\check{C}(X) \times Y) \cong Ah^{*,*'}(Y)$ from the cofibering $\tilde{C}(X) \times Y \to \check{C}(X) \times Y \to Y$. Hence $yx = p_*p^*(x) = 0$. $\square$

Recall that $I(X) = \pi_*(ABP^{2*,*}(X))$ for $\pi : X \to pt.$

Corollary 6.5 *If $v_n \in I(X)$, then $H^{*,*'}(\tilde{C}(X); \mathbb{Z}/p)$ is a free $Q(n)$-module.*

Proof. If there are maps $V_i \to X$ such that $t_{\mathbb{C}}(\pi_*[V_i]) = v_i$ for all $i \leq n$, then we have the result. From Lemma 4.6, we know $r_{p^i\Delta_{n-i}}(v_n) = v_i$ mod(I_i). Since $I(X)$ is invariant ideal, we see that $v_i \in I(X)$ for all $i \leq n$. This means the existence of V_i and above maps. $\square$

7. Chow motive

For smooth X_1 and X_2, an element $\theta \in CH^{\dim(X_2)}(X_1 \times X_2)$ can be viewed as a correspondence from X_1 to X_2. More generally, for an element $\theta \in CH^*(X_1 \times X_2)$, we have a homomorphism

$$f_\theta : H^{*,*'}(X_1; \mathbb{Z}/p) \to H^{*,*'}(X_2; \mathbb{Z}/p) \quad \text{by } f_\theta(x) = pr_{2*}(pr_1^*(x) \cup \theta)$$

where pr_i are projections of $X_1 \times X_2$ onto X_i.

For $\theta \in CH^{\dim X}(X \times X)$, the morphism $p_\theta = f_\theta$ is called a projector if $p_\theta \circ p_\theta = p_\theta$. The objects of $Chow^{eff}(k)$ are pairs (X, p), where X are smooth varieties and $p \in CH^{\dim(X)}(X \times X)$ are projectors. Morphisms in $Chow^{eff}(k)$ are defined by morphisms f_θ. (Namely, the category

$Chow^{eff}(k)$ of Chow motive is the pseudo abelian envelop of the category of correspondences). (See [Vi] for details.) Objects (X,p) are simply called motives M which are direct summand of $M(X)=(X,id_X)$, and $H^{*,*'}(M;\mathbb{Z}/p)$ are defined as $\mathrm{Im}(p)$.

Lemma 7.1 *Let $\theta\in CH^*(X\times X)$. Then f_θ commutes with Q_i. In particular, for a direct summand M of $M(X)$, its p_θ commutes with Q_i. Hence $H^{*,*'}(M;\mathbb{Z}/p)$ has the natural $Q(\infty)$-module structure.*

Proof. For $\theta\in CH^*(X\times X)$, we have

$$f_\theta(Q_i(x))=pr_{2*}(pr_1^*(Q_i(x))\cdot\theta)=pr_{2*}(Q_i(pr_1^*(x)\cdot\theta)).$$

The last equation follows from $Q_i(\theta)=0$ since $w(\theta)=0$. Hence we have the desired result if $pr_{2*}Q_i=Q_ipr_{2*}$.

By definition of the Gysin map (recall Section 7), we know

$$pr_{2*}(x)=i^*(Th_X(1)\cdot x)$$

where $Th_X(1)\in H^{2m,m}(Th_X(V);\mathbb{Z}/p)$ is the Thom class for some bundle V over X and $i:\mathbb{T}^m\times X\subset Th_X(V)\times X$. Since $w(Th_X(1))=0$, we see $Q_i(Th_X(1)\cdot x)=Th_X(1)\cdot Q_i(x)$. Therefore we see that pr_{2*} commutes with Q_i. (Indeed, Q_i commutes with the Gysin maps.) □

Remark The reduced powers P^i do not act naturally on $H^{*,*'}(M;\mathbb{Z}/p)$, see Lemma 7.3 bellow.

Let $A^*(X)$ be an oriented generalized cohomology theory on the category of smooth varieties X over k, in the sense of Panin [Pa]. The theories $CH^*(X)$ and $ABP^{2*,*}(X)$ are oriented generalized cohomology theories.

We can define the category of A-motive $M_A(k)$ as a pseudo abelian envelop of the category of A-correspondences Cor_A (of degree 0). Here objects in Cor_A are classes $[X]$ of smooth varieties and its morphisms are given by

$$Mor_{Cor_A}([X],[Y])=A^{\dim(X)}(X\times Y).$$

Theorem 7.2 ([Vi-Ya]) *Let $\rho^A:A^*(X)\to CH^*(X)$ be a map of oriented cohomology theories such that ρ^A are epic and $\mathrm{Ker}(\rho^A)$ are nilpotent for all X. Then ρ^A induces the natural 1 to 1 correspondence between the set of isomorphism classes of objects in $M_A(k)$ and $M_{CH}(k)$.*

The theory $ABP^{2*,*}(X)$ satisfies the assumption of the above theorem (with localized at p) from the fact that BP^* is generated by non positive degree elements.

For an element $\theta \in ABP(S)^{2\dim(X_2),\dim(X_2)}(X_1 \times X_2)$, as the case of Chow rings, we can define the homomorphism

$$f_\theta : ABP(S)^{*,*'}(X_1) \to ABP(S)^{*,*'}(X_2) \quad \text{by } f_\theta(x) = pr_{2_*}(pr_1^*(x) \cap \theta),$$
$$\text{and } ABP(S)^{*,*'}(M) = p_\theta ABP(S)^{*,*'}(X) \quad \text{for } M = (X, p_\theta).$$

Lemma 7.3 (ABP-version of a theorem in [Ka-Me]) *For $x \in ABP^{2*,*}(X)$ and $\theta \in ABP^{2d,d}(X \times X)$, $d = \dim(X)$, we have*

$$s_t(f_\theta(x)) = f_{s_t(\theta)}(s_t(x)c_t(\nu_X)).$$

Proof. From Lemma 5.2, we have

$$s_t(f_\theta(x)) = s_t\big(pr_{2*}(pr_1^*(x)\cdot\theta)\big) = pr_{2*}\big(s_t(pr_1^*(x)\theta)c_t(\nu_{pr_2})\big).$$

Here $c_t(\nu_{pr_2}) = pr_1^*(c_t(\nu_X))$. Hence the above element is

$$pr_{2*}\big(s_t(pr_1^*(x))s_t(\theta)pr_1^*(c_t(\nu_X)\big) = pr_{2*}\big(pr_1^*(s_t(x)c_t(\nu_X))s_t(\theta))\big),$$

which is $f_{s_t(\theta)}(s_t(x)c_t(\nu_X))$. $\square$

8. Norm Variety

Recently, Voevodsky announced the proof of the Bloch-Kato conjecture for all odd primes [Vo6]. For non zero $a = \{a_0, \ldots, a_n\} \in K_{n+1}^M(k)/p$, Rost ([Ro]) constructed the (smooth projective) norm variety V_a such that

$$(1) \qquad \pi_*[1_{V_a}] = V_a(\mathbb{C}) = v_n, \quad a|_{k(V_a)} = 0 \in K_{n+1}^M(k(V_a))/p$$

(2) the following sequence is exact

$$H_{-1,-1}(V_a \times V_a, \mathbb{Z}) \xrightarrow{pr_1 - pr_2} H_{-1,-1}(V_a;\mathbb{Z}) \to k^*.$$

Let us write $\chi_a = \check{C}(V_a)$ and $\tilde{\chi}_a = \tilde{C}(V_a)$. By the solution of Bloch-Kato conjecture, we see the exact sequence

$$0 \to H^{*+1,*}(\chi_a;\mathbb{Z}/p) \xrightarrow{\times\tau} K^M_{*+1}(k)/p \to K^M_{*+1}(k(V_a))/p \tag{8.1}$$

identifying $H^{*+1,*+1}(\chi_a;\mathbb{Z}/p) \cong K^M_{*+1}(k)/p$. (For $p=2$ case, see the proof of Proposition 2.3 in [Or-Vi-Vo].) Since $a|_{k(V_a)} = 0 \in K^M_{n+1}(k(V_a))/p$, there is a unique element $a' \in H^{n+1,n}(\chi_a;\mathbb{Z}/p)$ such that $\tau a' = a$.

Let M_a be the object in DM^{eff}_- defined by the following distinguished triangle

$$\begin{aligned} &M(\chi_a)(b_n)[2b_n] \to M_a \to M(\chi_a) \\ &\xrightarrow{\delta_a = Q_0 \ldots Q_{n-1}(a')} M(\chi_a)(b_n)[2b_n+1] \end{aligned} \tag{8.2}$$

where $b_n = (p^n-1)/(p-1) = p^{n-1}+\cdots+p+1$ so that $\deg(\delta_a) = (2b_n+1, b_n)$. For $i<p$, define the symmetric powers

$$M_a^i = S^i(M_a) = q_i\big(M_a^{\otimes i}\big) \subset M_a^{\otimes i}$$

where $q_i = (1/i!)\sum_{\sigma\in S_i}\sigma$ and $\sigma : M_a^{\otimes i} \to M_a^{\otimes i}$ is the motivic endomorphism given by the permutation. One of the important results in [Vo6] Voevodsky proved is that M_a^{p-1} is a direct summand of a motive of V_a (for details see [Vo6]). Moreover, there are distinguished triangles (for details, see (5.5), (5.6) in [Vo6])

$$M_a^{i-1}(b_n)[2b_n] \to M_a^i \to M(\chi_a) \xrightarrow{s_i} M_a^{i-1}(b_n)[2b_n+1] \tag{8.3}$$

$$M(\chi_a)(b_n i)[2b_n i] \to M_a^i \to M_a^{i-1} \xrightarrow{r_i} M(\chi_a)(b_n i)[2b_n i+1]. \tag{8.4}$$

Then we have the diagram

$$\begin{tikzcd}
 & H^{*,*'}(\chi_a;\mathbb{Z}/p) \arrow[d, "r^*_{p-1}"'] & \\
H^{\sharp,\sharp'}(\chi_a;\mathbb{Z}/p) & H^{\natural,\natural'}(M_a^{p-2};\mathbb{Z}/p) \arrow[l, "s^*_{p-1}"'] \arrow[d] & H^{\sharp-1,\sharp'}(M_a^{p-1};\mathbb{Z}/p) \arrow[l] \\
 & H^{\natural,\natural'}(M_a^{p-1};\mathbb{Z}/p) &
\end{tikzcd}$$

where

$$(\sharp,\sharp')=(*+2(p^n+b_n),*'+p^n+b_n-1)=(*+2b_{n+1},*'+b_{n+1}-1),$$
$$(\natural,\natural')=(*+2p^n-1,*'+p^n-1),$$

and the vertical and horizontal arrows are exact. From the result of Voevodsky, we know (Appendix in [Su-Jo])

Lemma 8.1 ([Vo6]) *For $x\in H^{*,*'}(\chi_a;\mathbb{Z}/p)$, we have*

$$s_{p-1}^*r_{p-1}^*(x)=\lambda Q_0Q_1\dots Q_n(a')\cup x\quad \lambda\neq 0\in\mathbb{Z}/p.$$

Corollary 8.2 *The following map*

$$Q_0\dots Q_n(a')\cup - : H^{*,*'}(\chi_a;\mathbb{Z}/p)\to H^{\sharp,\sharp'}(\chi_a;\mathbb{Z}/p)$$

is surjective (resp. isomorphic) if the difference $*-*'\geq 0$ (resp. $*-*'>0$) i.e., $\sharp-\sharp'\geq b_{n+1}-1$ (resp. $\sharp-\sharp'>b_{n+1}-1$).

Proof. Let the difference $*-*'\geq 0$. Note that M_a^{p-1} is a direct summand of the motive of V_a. Hence we see

$$H^{\sharp,\sharp'}\big(M_a^{p-1};\mathbb{Z}/p\big)=0,\quad H^{\natural,\natural'}\big(M_a^{p-1};\mathbb{Z}/p\big)=0$$

since their difference is larger than $p^n-1=\dim(V_a)$. Hence we know the surjectivity of $s_{p-1}^*r_{p-1}^*$. When the difference $*-*'>0$, we get moreover

$$H^{\sharp-1,\sharp'}\big(M_a^{p-1};\mathbb{Z}/p\big)=0,\quad H^{\natural-1,\natural'}\big(M_a^{p-1};\mathbb{Z}/p\big)=0,$$

by the same reasons. Thus we see the injectivity. □

Denote by $k(V_a)$ the function field of V_a and by $(V_a)_0$ the set of closed points of V_a. One of the main theorems of the paper ([Or-Vi-Vo]) by Orlov, Vishik and Voevodsky is the $p=2$ case of the following theorem.

Theorem 8.3 *For any $a=\{a_0,\dots,a_n\}\in K_*^M(k)/p$, the following sequence is exact*

$$\amalg_{x\in(V_a)_0}K_*^M(k(x))/p\xrightarrow{Tr_{k(x)/k}}K_*^M(k)/p$$
$$\xrightarrow{a}K_{*+n+1}^M(k)/p\to K_{*+n+1}^M(k(V_a))/p.$$

Outline of proof (See for the case $*=1$, A.1 in [Su-Jo]). The exactness of the last part in the above sequence follows from (8.1). The first part is also just an odd prime p version of the arguments of the proof by Orlov, Vishik and Voevodsky. From arguments by Voevodsky ([Vo6], the main theorem in Appendix in [Su-Jo]), we see the exact sequence

$$\amalg_{x\in(V_a)_0} K_*^M(k(x))/p \xrightarrow{Tr_{k(x)/k}} K_*^M(k)/p \\ \xrightarrow{\delta_a} H^{*+2b_n+1,*+b_n}(\chi_a;\mathbb{Z}/p). \tag{8.5}$$

The last map δ_a is epic by the following reason. Consider the composition

$$K_*^M(k)/p \xrightarrow{\delta_a} H^{*+2b_n+1,*+b_n}(\chi_a;\mathbb{Z}/p) \xrightarrow{Q_n} H^{2pb_n+2,pb_n}(\chi_a;\mathbb{Z}/p).$$

Since $Q_n\delta_a = Q_n\ldots Q_0(a')$, we see that $Q_n\delta_a$ is epic from the preceding lemma. Since $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)$ is $\Lambda(Q_n)$-free (from Corollary 6.5), we see that Q_n above is monic. (Note that $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)\cong H^{*,*'}(\chi_a;\mathbb{Z}/p)$ when $*>*'$.) Thus we show that the map δ_a in the above sequence is epic.

We also know that the map

$$K_*^M(k)/p \xrightarrow{\delta_a} H^{*+2b_n+1,*+b_n}(\chi_a;\mathbb{Z}/p) \\ \xrightarrow{(Q_{n-1}\ldots Q_0)^{-1}} H^{*+n+1,*+n}(\chi_a;\mathbb{Z}/p) \xrightarrow{\times\tau} K_{*+n+1}^M(k)/p \tag{8.6}$$

is the multiplication with a because V_a is a splitting variety of a and the maps are those of $K_*^M(k)/p$-modules. Here we also use (8.1) and the fact that $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)$ is $Q(n)$-free from Corollary 6.5, in fact the above map $Q_{n-1}\ldots Q_0$ is an isomorphism. Thus we get the exact sequence. $\square$

Corollary 8.4 (For $p=2$, this is Theorem 2.10 in [Or-Vi-Vo]) *For each $0\neq h\in K_n^M(k)/p$, there is a field E/k and a nonzero pure symbol $a\in K_n^M(k)/p$ such that $0\neq h|_E = a|_E$ in $K_n^M(E)$.*

Proof. Let $h=b_1+\cdots+b_l$ and each b_i a pure symbol for $1\le i\le l$. Let V_{b_i} be the norm varieties and $E_i=k(V_{b_1}\times\cdots\times V_{b_i})$. Then of course $h|_{E_l}=0$. Take i such that $h|_{E_{i-1}}\neq 0$ but $h|_{E_i}=0$. Then from the above theorem,

$$\mathrm{Ker}\left(K_n^M(E_{i-1})/p\to K_n^M(E_i)/p)\right) = b_iK_0^M(E_{i-1})/p.$$

Hence $h|E_{i-1} = \lambda b_i|E_{i-1}$ for $\lambda \neq 0 \in \mathbb{Z}/p$. Then $a = b_i$ and $E = E_{i-1}$ satisfy the desired result. $\square$

Let us write by K_a the quotient algebra of $K_*^M(k)/p$ by the annihilator ideal of a (which is the ideal generated by x with $ax = 0 \in K_*^M(k)/p$), so that

$$K_*^M(k)/p \supset K_*^M(k)(a) \cong K_a\{a\}.$$

Namely, $K_*^M(k)/p$-module generated by a in $K_*^M(k)/p$ is isomorphic to the free K_a-module generated by a.

Theorem 8.5 (For $p = 2$, this is Theorem 5.8 in [Ya2]) *Let $0 \neq a = (a_0, \dots, a_n) \in K_{n+1}^M(k)/p$. Then there is a $K_*^M(k) \otimes Q(n)$-module isomorphism*

$$H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/p) \cong K_a \otimes Q(n) \otimes \mathbb{Z}/p[\xi_a]\{a'\}$$

where $\xi_a = Q_n Q_{n-1} \dots Q_0(a')$ and $\deg(a') = (n+1, n)$.

Proof. Recall the difference $d(x) = * - *'$ for $x \in H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/p)$. Hence if $0 \neq x \in H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/p)$, then $d(x) > 0$. From (8.6) we already know that

$$K_*^M(k)/p \xrightarrow{\delta_a} H^{*+2b_n+1,*+b_n+1}(\chi_a; \mathbb{Z}/p) \xrightarrow{(Q_{n-1}\dots Q_0)^{-1}} H^{*+n+1,*+n}(\chi_a; \mathbb{Z}/p)$$

is an epimorphism, indeed, the map δ_a is epic from Corollary 8.2 and the map $Q_{n-1} \dots Q_0$ is isomorphic since $H^{*,*'}(\tilde{\chi}_a; \mathbb{Z}/2)$ is $Q(n)$-free. The composition of the above map with

$$\tau : H^{*+n+1,*+n}(\chi_a; \mathbb{Z}/p) \to K_{*+n+1}^M(k)/2$$

is multiplying by a from (8.6). Since the last map τ is monic from (8.1), we see that

$$\begin{aligned} &H^{*+n+1,*+n}(\tilde{\chi}_a; \mathbb{Z}/p) \\ &\quad \cong \tau H^{*+n+1,*+n}(\tilde{\chi}_a; \mathbb{Z}/p) \cong \tau(Q_0 \dots Q_{n-1})^{-1}\delta_a K_*^M(k)/p \\ &\quad \cong K_*^M(k)/p(a) \cong K_a\{a\} \subset K_{*+n+1}^M(k)/p. \end{aligned}$$

Thus we get the case $d(x) = 1$.

Since $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)$ is $Q(n)$-free (from Corollary 6.5), its contains

$$Q(n)(K_a\{a'\}) \cong K_a \otimes Q(n)\{a'\}, \quad \text{with } \tau a' = a.$$

Moreover the multiplying by ξ_a is isomorphic (from Lemma 8.1 and Corollary 8.2) in $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)$, and we have the injection

$$K_a \otimes Q(n) \otimes \mathbb{Z}/p[\xi_a]\{a'\} \subset H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p).$$

We will prove that the above injection is an epimorphism also. Since the multiplying by ξ_a is isomorphic and $d(\xi_a) = b_{n+1}$, it is sufficient to prove that there are no additional $K_*^M(k) \otimes Q(n)[\xi_a]$-module generators x for $d(x) \leq b_{n+1}+1$. Suppose that t is such a generator with $1 < d(t) \leq b_{n+1}+1$. Then we see

$$d(Q_0 \dots Q_n t) = p^n + p^{n-1} + \cdots + 1 + d(t) > b_{n+1} + 1.$$

From Corollary 8.2, there is an element $y \in H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)$ such that

$$Q_0 \dots Q_n(t) = s_{p-1}^* r_{p-1}^*(y) = \xi_a \cup y.$$

Since Q_i is a derivation, we have

$$\begin{aligned} \xi_a \cup Q_i(y) &= Q_0 \dots Q_n(a') \cup Q_i(y) \\ &= Q_i(Q_0 \dots Q_n(a') \cup y) = Q_i(Q_0 \dots Q_n(t)) = 0 \end{aligned}$$

for $i \leq n$. Since the map multiplying by ξ_a is injective (indeed, isomorphic) for $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)$ from Corollary 8.2, we see

$$Q_i(y) = 0 \quad \text{for all } 0 \leq i \leq n.$$

Then the fact that $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p)$ is $Q(n)$-free implies that

$$y = Q_n Q_{n-1} \dots Q_0(y') \quad \text{for some } y' \in H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p).$$

Of course $d(y') = d(t) - d(\xi_a) \leq 0$. This is a contradiction. $\square$

Remark The poof of this theorem was first given in the preprint [Ya4] in 2006, while Merkerjev and Suslin [Me-Su] also gave its proof in the recent paper [Me-Su].

Lemma 8.6 *If* $* < 4b_n$*, then* $H^{*,*'}(M_a;\mathbb{Z}/p) \cong H^{*,*'}(M_a^{p-1};\mathbb{Z}/p)$*. Moreover if* $* < 2b_n$*, then* $H^{*,*'}(M_a;\mathbb{Z}/p) \cong H^{*,*'}(\chi_a;\mathbb{Z}/p)$*.*

Proof. Since $H^{*,*'}(\chi_a;\mathbb{Z}/p) \cong 0$ for $* < 0$, we get this lemma from (8.2), (8.4). For example, (8.4) induces the long exact sequence

$$\leftarrow H^{*-2b_ni,*'-b_ni}(\chi_a;\mathbb{Z}/p) \leftarrow H^{*,*'}\big(M_a^i;\mathbb{Z}/p\big) \leftarrow H^{*,*'}\big(M_a^{i-1};\mathbb{Z}/p\big) \leftarrow \cdots,$$

which induces the isomorphism $H^{*,*'}(M_a^i;\mathbb{Z}/p) \cong H^{*,*'}(M_a^{i-1};\mathbb{Z}/p)$ for the first degree $* < 2b_ni$. □

Let us consider the following triangular domain generated by bidegree

$$D_i = \big\{ \deg(x) \mid w(x) \geq 0,\ \text{first.}\deg(x) < 2b_ni,\ d(x) \geq b_n(i-1)\big\}$$

and $D = \bigcup_{j=1}^{p-1} D_j$. Note that all bidegree $\deg(x)$ of $w(x) \leq 1$ (indeed, $w(\tau) = 2$), are contained in D.

Lemma 8.7 *Let us write* $K = K_*^M(k)/p$ *and* ${}_aK = \{x \in K; ax = 0\}$ *the annihilator ideal by* a *so that* $K_a \cong K/{}_aK$*. For bidegree* $(*,*') \in D$ *defined above, we have the* K*-module* (*but not ring*) *isomorphism,*

$$H^{*,*'}\big(M_a^{p-1};\mathbb{Z}/p\big) \cong \big(K_a \otimes Q(n-1)\{a'\} \oplus {}_aK\{t\}\big)[t]/(t^{p-1}) \oplus K\{1\}$$

where $\deg(t) = (2b_n, b_n)$*.*

Proof. Consider the exact sequence induced from (8.3)

$$\begin{aligned}&\leftarrow H^{*,*'}\big(M_a^{i-1}(b_n)[2b_n];\mathbb{Z}/p\big)\\ &\qquad \xleftarrow{j_1} H^{*,*'}\big(M_a^i;\mathbb{Z}/p\big) \xleftarrow{j_2} H^{*,*'}(\chi_a;\mathbb{Z}/p) \xleftarrow{s_i} \cdots\end{aligned}$$

Note $|Q_na'| = 2p^n + n$ and recall that $H^{*,*'}(\tilde{\chi}_a;\mathbb{Z}/p) \cong H^{*,*'}(\chi_a;\mathbb{Z}/p)$ for $* > *'$. Hence for $* < 2p^n + n$ we have the isomorphism

$$H^{*,*'}(\chi_a;\mathbb{Z}/p) \cong K_a \otimes Q(n-1)\{a'\} \oplus K\{1\}.$$

In particular, it is isomorphic to $K_a\{c_0 = Q_1 \dots Q_{n-1}(a')\}$ when $(*,*') \in \bigcup_{j=2}^{i} D_j$. Hence j_2 is zero and j_1 is surjective in $\bigcup_{j=2}^{i} D_j$.

By induction we assume for $(*,*') \in \bigcup_{j=1}^{i-1} D_j$

$$H^{*,*'}(M_a^{i-1};\mathbb{Z}/p) \cong (K_a \otimes Q(n-1)\{a'\} \oplus {}_aK\{t\})[t]/(t^{i-1}) \oplus K\{1\}.$$

Then for $(*,*') \in \bigcup_{j=2}^{i} D_j$, we see

$$\begin{aligned} &H^{*,*'}\big(M_a^{i-1}(b_n)[2b_n];\mathbb{Z}/p\big) \\ &\quad \cong \big((K_a \otimes Q(n-1)\{a'\} \oplus {}_aK\{t\})[t]/(t^{i-1}) \oplus K\{1\}\big) \otimes \{t\}. \end{aligned}$$

In particular, both sides of the above are zero if $(*,*') \in D_1$. Hence j_2 is injective when $(*,*') \in D$.

We consider the exact sequence (8.3) again

$$\begin{aligned} &\xleftarrow{j_2} H^{*+1,*'}(\chi_a;\mathbb{Z}/p) \\ &\qquad \xleftarrow{s_i} H^{*+2b_n,*'+b_n}(M_a^{i-1}(b_n)[2b_n];\mathbb{Z}/p) \xleftarrow{j_1} H^{*,*'}(M_a^i;\mathbb{Z}/p) \end{aligned}$$

Here $s_i(t) = \delta_a$ from (8.2). Hence we can see

$$\mathrm{Ker}(s_i) \cong \big((K_a \otimes Q(n-1)\{a'\} \oplus {}_aK\{t\})[t]/(t^{i-1}) \oplus {}_aK\{1\}\big) \otimes \{t\}.$$

(The map j_1 is not injective, but the degree $\mathrm{Im}(s_i) = K_a\{\delta\}$ does not contained in D, and hence j_1 is injective when we restricted degree in D.) Thus we can prove the lemma. $\square$

Remark The isomorphism in the above theorem is that of $K \otimes Q(n)$-modules. This fact is proved in Lemma 9.5 in the next section.

Corollary 8.8 *Let $c_i = Q_0 \dots \hat{Q}_i \dots Q_{n-1}(a')$ (hence $|c_i| = 2(b_n - p^i + 1)$). Then there is an additive (not ring) isomorphism*

$$\begin{aligned} &CH^*\big(M_a^{p-1}\big)/p \cong \mathbb{Z}/p\{1\} \oplus \mathbb{Z}/p[t]/(t^{p-1})\{c_0, \dots, c_{n-1}\}, \\ &CH^*\big(M_a^{p-1}\big)_{(p)} \cong \mathbb{Z}_{(p)} \oplus \big(\mathbb{Z}_{(p)}\{c_0\} \oplus \mathbb{Z}/p\{c_1, \dots, c_{n-1}\}\big)[t]/(t^{p-1}). \end{aligned}$$

Note for $i \geq 1$, $c_i \in \mathrm{Im}(Q_0)$ and it is just a p-torsion in the integral

Chow ring.

9. Rost's basic correspondence

We recall the arguments of Rost in [Ro]. Let $q : V_a \to \chi_a$ be the natural map and $K^{*,*'}$ be the kernel of the induced map $q^* : H^{*,*'}(\chi_a;\mathbb{Z}/p) \to H^{*,*'}(V_a;\mathbb{Z}/p)$. Then the natural filtration on the simplicial scheme χ_a gives the map

$$proj : K^{*,*'} \to H^1\big(V_a; H^{*-1,*'}/p\big)$$

where $H^1(X;F)$ is an abbreviation for the homology of complex

$$F(X) \xrightarrow{p_1^*-p_2^*} F(X\times X) \xrightarrow{p_1^*-p_2^*+p_3^*} F(X\times X\times X) \to \cdots$$

Since $H^{2*+1,*}(Y;\mathbb{Z}/p)=0$ and $H^{2*,*}(Y;\mathbb{Z}/p)=CH^*(Y)/p$ for any smooth variety Y, we see $K^{2b_n+1,b_n}\cong H^{2b_n+1,b_n}(\chi_a;\mathbb{Z}/p)$ and

$$H^1\big(V_a; H^{2b_n,b_n}/p\big) \cong CH^{b_n}(V_a\times V_a)/(p, p_1^*-p_2^*).$$

Hence the map *proj* is written as

$$\mathrm{proj} : H^{2b_n+1,b_n}(\chi_a;\mathbb{Z}/p) \to CH^{b_n}(V_a\times V_a)/(p,p_1^*-p_2^*).$$

Define

$$\rho = \mathrm{proj}(Q_{n-1}\dots Q_0(a')) \in CH^{b_n}(V_a\times V_a) \quad \mathrm{mod}(p, p_1^*-p_2^*),$$

and also define $c(V_a) = p_{1*}(\rho^{p-1}) \in CH^d(V_a)/p \cong \mathbb{Z}/p$. In [Ro] Rost constructs a norm variety such that $c(V_a)\neq 0 \in \mathbb{Z}/p$. Moreover he shows that we can take a projector $\theta \in CH^d(V_a\times V_a)$ such that

$$\theta = 1/c(V_a)(\rho^{p-1}) \in CH^d(V_a\times V_a)/p.$$

Thus we can define the motive $M_a^R = f_\theta(M(V_a))$. We will see that M_a^R is isomorphic to M_a^{p-1} as motives in Corollary 9.3 below.

Hereafter this paper, the variety V_a always *means* a norm variety which has this property

$$c(V_a) \neq 0 \in \mathbb{Z}/p$$

and has the projector f_θ.

Let $\bar{k}$ be the algebraic closure of k and $X|_{\bar{k}} = X \otimes_k \bar{k}$. Let $i_{\bar{k}} : H^{*,*'}(X;\mathbb{Z}/p) \to H^{*,*'}(X|_{\bar{k}};\mathbb{Z}/p)$ be the induced map. From the exact sequence induced from (8.3) or (8.4), we have the isomorphism of modules (for the Voevodsky motive M_a^{p-1})

$$H^{*,*'}\big(M_a^{p-1}|_{\bar{k}};\mathbb{Z}_{(p)}\big) \cong \mathbb{Z}_{(p)}[\bar{t}]/(\bar{t}^p)$$

for $\deg(\bar{t}) = (2b_n, b_n)$.

Recall that the Rost motive M_a^R is defined by $M_a^R = f_{\rho^{p-1}}(M(V_a))$. We will study $\rho|_{\bar{k}}$. Let $j : V_a \times \mathrm{Spec}(\bar{k}) \subset (V_a \times V_a)|_{\bar{k}}$ and let $\bar{t} = j^*(\rho|_{\bar{k}})$. Then (Lemma 5.2 in [Ro])

$$\rho|_{\bar{k}} = \bar{t} \otimes 1 - 1 \otimes \bar{t} \in CH^{b_n}(V_a|_{\bar{k}} \times V_a|_{\bar{k}})/p$$

from the cocycle condition $p_1^* - p_2^* + p_3^* = 0$. Thus we get

$$(\rho|_{\bar{k}})^{p-1} = \sum \bar{t}^{p-1-i} \otimes \bar{t}^i \in CH^d(V_a|_{\bar{k}} \times V_a|_{\bar{k}})/p. \qquad (*)$$

The fact $0 \neq c(V_a) = p_{1*}((\rho|_{\bar{k}})^{p-1})$ implies that $\bar{t}^{p-1} \neq 0 \in CH^d(V_a|_{\bar{k}})/p$. Moreover the fact that $p_{2*}(\bar{t}^i \otimes \bar{t}^i) = \delta_{p-1,j}\bar{t}^i$ implies that $f_\theta(\bar{t}^i) = \bar{t}^i$ for all $0 \leq i \leq p-1$. By the definition of the map f_θ and $(*)$, we see

$$f_\theta(V_a|_{\bar{k}}) = \mathbb{Z}_{(p)}\{1, \bar{t}, \dots, \bar{t}^{p-1}\},$$

that is, $CH^*(M_a^R|_{\bar{k}})_{(p)} \cong \mathbb{Z}_{(p)}[\bar{t}]/(\bar{t}^p)$. Thus we have the ring epimorphism

$$CH^*(V_a|_{\bar{k}})_{(p)} \to CH^*\big(M_a^R|_{\bar{k}}\big)_{(p)} \cong \mathbb{Z}_{(p)}[\bar{t}]/(\bar{t}^p).$$

Note also $CH^*(M_a^R|_{\bar{k}}) \cong CH^*(M_a^{p-1}|_{\bar{k}})$.

Lemma 9.1 *We have for* $0 \leq s \leq p-2$,

$$i_{\bar{k}}(c_0 \otimes t_s) = \lambda_s p \bar{t}^{s+1} \in CH^*(V_a|_{\bar{k}})_{(p)} \quad \lambda_s \neq 0 \mod(p).$$

Proof. First we prove $i_{\bar{k}}(c_0 \otimes t^{p-2}) = p\bar{t}^{p-1}$. Here $c_0 \otimes t^{p-2}$ (resp. $\bar{t}^{p-1}$)

generates $\mathbb{Z}_{(p)} \subset CH^d(V_a)$ (resp. $CH^d(V_a|_{\bar{k}}) \cong \mathbb{Z}_{(p)}$) where $d = \dim(V_a) = p^n - 1$.

Let us write $\deg(X) = \pi_*(CH^{\dim(X)}(X))$ for $\pi : X \to pt$. Since V_a has no k-rational points,

$$\deg(V_a) \subset pCH^0(\mathrm{Spec}(k)) = p\mathbb{Z}_{(p)}.$$

On the other hand, the fact $t_{\mathbb{C}}(V_a) = v_n$ implies that

$$\pi_*(r_{\Delta_n}(-T_{V_a})) = \pi_*(r_{\Delta_n}(-T_{V_a|\bar{k}})) = p \mod(p^2).$$

Hence we have $\deg(V_a) = p\mathbb{Z}_{(p)}$, while $\deg(V_a|_{\bar{k}}) = \mathbb{Z}_{(p)}$. Since $\deg = \deg \cdot i_{\bar{k}}$, we see that $i_{\bar{k}}(c_0 \otimes t^{p-2}) = p\bar{t}^{p-1}$.

From (8.3), we have the following commutative diagram

$$\begin{tikzcd}
H^{2*-2b_n,*-b_n}(M_a^{p-2}|_{\bar{k}}) & H^{2*,*}(M_a^{p-1}|_{\bar{k}}) \arrow[l, "j_1|\bar{k}"'] & H^{2*,*}(\chi_a|_{\bar{k}}) \arrow[l] \\
H^{2*-2b_n,*-b_n}(M_a^{p-2}) \arrow[u, "i_{\bar{k}}"] & H^{2*,*}(M_a^{p-1}) \arrow[l, "j_1"'] \arrow[u, "i_{\bar{k}}"] & H^{2*,*}(\chi_a). \arrow[l] \arrow[u, "i_{\bar{k}}"]
\end{tikzcd}$$

When $(*, *') = (2b_n i, b_n i)$, we see that $H^{*,*'}(\chi_a) = H^{*,*'}(\chi_a|_{\bar{k}}) = 0$ from Theorem 8.5. Hence $j_1|_{\bar{k}}$ and j_1 are isomorphism for these degree. (Note $H^{*,*}(\chi_a(pb_n)[2pb_n]) = 0$.) Moreover

$$j_1|_{\bar{k}}(\bar{t}^i) = \lambda\bar{t}^{i-1} \quad \text{for } \lambda \neq 0 \in \mathbb{Z}/p$$

and $j_1(c_0 \otimes t^i) = \lambda c_0 \otimes t^{i-1}$. By induction starting $i_{\bar{k}}(c_0 \otimes t^{p-2}) = p\bar{t}^{p-1}$, we have the desired result $i_{\bar{k}}(c_0 \otimes t^{i-2}) = \lambda p\bar{t}^{i-1}$, from the above diagram. □

Now we recall properties of generically split over X. We say that a k-motive M which is a direct summand of $M(X)$ is generically split over X if $M|_{k(X)}$ splits as a sum of Tate motives $\mathbb{T}^{\otimes i}$. Vishik and Zainoulline prove [Vi-Za] that if motives N, M are generically split over X and a map $f : N \to M$ of k-motives is generically split over X (i.e., $f|_{k(X)}$ is a split $k(X)$-epimorphism), then f itself splits (i.e., f is a split k-epimorphism). Moreover, Vishik and Zainoulline prove the Rost nilpotent theorem for $k(X)/k$.

The following lemma and corollary are suggested by the referee.

Lemma 9.2 *Let $N \to M(V_a)$ be a map of k-motives over V_a such that $CH^d(N|_{k(V_a)})_{(p)} \to CH^d(V_a|_{k(V_a)})_{(p)} \cong \mathbb{Z}_{(p)}$ is an epimorphism. Moreover let N be generically split over V_a. Then N contains M_a^R as a k-motive direct summand.*

Proof. Let p_N be a projector for N. Then from the assumption of the epimorphism ($p_N(\bar{t}^{p-1}) = \bar{t}^{p-1}$), we have

$$i_{\bar{k}}(p_N) = 1 \otimes \bar{t}^{p-1} + a\bar{t} \otimes \bar{t}^{p-2} + \cdots \in CH^*(V_a \times V_a|_{\bar{k}})/p.$$

For the projector $p_M = f_\theta$ (for M_a^R) we know (with mod(p))

$$i_{\bar{k}}(p_M) = 1 \otimes \bar{t}^{p-1} + \bar{t} \otimes \bar{t}^{p-2} + \cdots + \bar{t}^{p-1} \otimes 1.$$

If they are the same in $CH^*(M_a^R \times M_a^R|_{\bar{k}})/p$, then from Vishik and Zainoulline results, we see that N contains M_a^R also as k-motives.

Suppose that for $0 < i < p-1$

$$i_{\bar{k}}(p_N - p_M) = \bar{t}^i \otimes \bar{t}^{p-1-i} + a'\bar{t}^{i+1} \otimes \bar{t}^{p-i-2} + \cdots$$

Then we show

$$i_{\bar{k}}\big((p_N - p_M)\rho^i)\big) = \bar{t}^i \otimes \bar{t}^{p-1} + a''\bar{t}^{i+1} \otimes \bar{t}^{p-2} + \cdots$$

Therefore we can compute

$$i_{\bar{k}}\big(pr_{1*}((p_N - p_M)\rho^i)\big) = pr_{1*}\big(\bar{t}^i \otimes \bar{t}^{p-1} + a''\bar{t}^{i+1} \otimes \bar{t}^{p-2} + \cdots\big) = \bar{t}^i.$$

Hence $\bar{t}^i \in \mathrm{Im}(i_{\bar{k}})$, and this contradicts to the preceding lemma. □

From the nilpotent theorem for $k(V_a)/k$ by Vishik-Zainoulline and the fact that M_a^{p-1} is indecomposable (over k), we have the following corollary.

Corollary 9.3 *The motives M_a^{p-1} and M_a^R are isomorphic as k-motives. Hence the generalized Rost motive M_a (which is an indecomposable summand of $M(V_a)$ with $\deg(M_a) = p\mathbb{Z}_{(p)}$) is uniquely determined by the norm variety V_a.*

Lemma 9.4 *There is $\xi \in CH^{(p-2)b_n}(M_a^R \times M_a^R)$ such that $\xi|_{\bar{k}} = \rho^{p-2}|_{\bar{k}}$ and for $(*,*') \in \bigcup_{j=2}^{n-1} D_j - \{(2b_n, b_n)\}$, we have the isomorphism*

$$f_\xi : H^{*,*'}\big(M_a^R; \mathbb{Z}/p\big) \cong H^{*-2b_n,*'-b_n}\big(M_a^R; \mathbb{Z}/p\big).$$

Proof. We consider the following diagram in the derived category DM_-^{eff} ([Vo1], Section 9 in [Fr-Vo])

$$\begin{tikzcd}
M_a^R(b_n)[2b_n] \arrow[r, "(1)"] & M_a^{p-2}(b_n)[2b_n] \arrow[r, "(2)"] \arrow[d, "="'] & M(\chi_a)(pb_n)[2pb_n+1] & \text{(I)} \\
M(\chi_a)(0)[-1] \arrow[r, "(3)"] & M_a^{p-2}(b_n)[2b_n] \arrow[r, "(4)"] & M_a^R & \text{(II).}
\end{tikzcd}$$

Here (I) and (II) are distinguished triangles from (8.4) and (8.3). Let

$$\xi = (4)\cdot(1) \in \mathrm{Hom}\big(M_a^R(b_n)[2b_n], M_a^R\big) \cong CH^{(p-2)b_n}\big(M_a^R \times M_a^R\big).$$

We note that

$$f_\xi|_{\bar{k}}(\bar{t}^{p-1}) = \bar{t}^{p-2} = f_{\rho^{p-2}}|_{\bar{k}}(\bar{t}^{p-1}).$$

Here the first equation follows from (8.3) and Lemma 9.1. The second equation follows from $\rho^{p-2}|_{\bar{k}} = 1 \otimes \bar{t}^{p-2} + \dots$. Using arguments in the proof of Lemma 9.2 (considering $pr_{1*}(\rho^i(\xi - \rho^{p-2}))$ for some i), we see $\xi|_{\bar{k}} = \rho^{p-2}|_{\bar{k}}$.

Next we consider the induced exact sequences

$$\begin{tikzcd}
H^{*-2b_n,*'-b_n}(M_a^R;\mathbb{Z}/p) & H^{*-2b_n,*'-b_n}(M_a^{p-2};\mathbb{Z}/p) \arrow[l, "(1)^*"'] & H^{*-2pb_n-1,*'-pb_n}(\chi_a;\mathbb{Z}/p) \arrow[l, "(2)^*"'] \\
H^{*+1,*'}(\chi_a;\mathbb{Z}/p) & H^{*-2b_n,*'-b_n}(M_a^{p-2};\mathbb{Z}/p) \arrow[l, "(3)^*"'] \arrow[u, "="] & H^{*,*'}(M_a^R;\mathbb{Z}/p). \arrow[l, "(4)^*"']
\end{tikzcd}$$

For $(*,*') \in \bigcup_{j=2}^{n-1} D_j - \{(2b_n, b_n)\}$, we see

$$H^{*+1,*'}(\chi_a;\mathbb{Z}/p) \cong H^{*-2pb_n-1,*-pb_n}(\chi_a;\mathbb{Z}/p) \cong 0,$$

Hence we have the exact sequence

$$\begin{aligned} &\leftarrow H^{*-2pb_n,*'-pb_n}(\chi_a;\mathbb{Z}/p) \leftarrow H^{*-2b_n,*'-b_n}\big(M_a^R;\mathbb{Z}/p\big) \\ &\xleftarrow{f_\xi} H^{*,*'}\big(M_a^R;\mathbb{Z}/p\big) \leftarrow H^{*,*'}(\chi_a;\mathbb{Z}/p) \leftarrow . \end{aligned}$$

For $(*,*') \in \bigcup_{j=2}^{n-1} D_j$, we see

$$H^{*,*'}(\chi_a;\mathbb{Z}/p) \cong H^{*-2pb_n,*-pb_n}(\chi_a;\mathbb{Z}/p) \cong 0,$$

Thus we have the isomorphism in the lemma. □

Remark By arguments just before Remark 2.6 in [Ro], Rost showed an exact quadrangle

$$M(\chi_a)(pb_n)[2pb_n] \to M^R(b_n)[2b_n] \xrightarrow{\rho^{p-2}} M^R \to M(\chi_a).$$

The proof of the above lemma is suggested by this exact quadrangle.

Lemma 9.5 *For bidegree $(*,*') \in D$, we have the $K \otimes Q(n-1)$-module (but not ring) isomorphism,*

$$H^{*,*'}\big(M_a^R;\mathbb{Z}/p\big) \cong \big(K_a \otimes Q(n-1)\{a'\} \oplus {}_aK\{t\}\big)[t]/(t^{p-1}) \oplus K\{1\}$$

namely, $Q_i(a't^s) = Q_i(a')t^s$ for $0 \le i \le n-1$, $0 \le s \le p-2$.

Proof. From Lemma 8.7 and the preceding corollary, we have the K-module isomorphism in this lemma. For $(*,*') \in \bigcup_{j=2}^{n-1} D_j - \{(2b_n, b_n)\}$, we see that $f_\xi|H^{*,*'}(M_a^R)$ is an isomorphism. Let $a_{p-1} = a' \otimes t^{p-2}$. For each $1 \le j \le p-2$, define $a_j \in H^{*,*'}(M_a^R;\mathbb{Z}/p)$ by $f_\xi(a_{j+1}) = a_j$. By dimensional reason, we note

$$a_j = a' \otimes t^{j-1} \mod \big(K_+^M(k)/p\big).$$

Recall that from Lemma 7.1, f_ξ commute with Q_i. By the induction on

j, we assume $Q_1 \dots Q_{n-1}(a_j) = \lambda_j c_0 t^{j-1}$ for $\lambda_j \neq 0$. Then

$$\begin{aligned} f_\xi(Q_1 \dots Q_{n-1}(a_{j+1})) &= Q_1 \dots Q_{n-1}(f_\xi(a_{j+1})) = Q_1 \dots Q_{n-1}(a_j) \\ &= \lambda'_j c_0 t^{j-1} = \lambda''_j f_\xi(c_0 t^j). \end{aligned}$$

Hence we see that

$$Q_1 \dots Q_{n-1}(a_{j+1}) = \lambda''_j c_0 t^j.$$

Thus we show for all $0 \leq i_1 < \dots < i_s \leq n-1$,

$$Q_{i_1} \dots Q_{i_s}(a') \otimes t^{j-1} = Q_{i_1} \dots Q_{i_s}(a_j) \mod \big(K_+^M(k)/p\big).$$

Take $Q_{i_1} \dots Q_{i_s}(a_j)$ (rewrite it $Q_{i_1} \dots Q_{i_s}(a') \otimes t^{j-1}$) as a basis of the K_a-free submodule of $H^{*,*'}(M_a^R; \mathbb{Z}/p)$. Then we see that the isomorphism in this lemma is that of $Q(n-1)$-modules. □

10. $ABP^{2*,*}(V_a)$ for the norm varieties V_a.

We consider AHss

$$E(V_a)_2^{*,*',*''} = H^{*,*'}(V_a : BP^{*''}) \Longrightarrow ABP^{*,*'}(V_a).$$

From Corollary 8.8 and Lemma 9.5, we still show

$$c_i \otimes t^s = Q_0 \dots \hat{Q}_i \dots Q_{n-1}(a' \otimes t^s), \quad 0 \leq s \leq p-2.$$

The Chow ring $CH^*(V_a)_{(p)}$ contains the $\mathbb{Z}_{(p)}$-module

$$\mathbb{Z}_{(p)} \oplus \big(\mathbb{Z}_{(p)}\{c_0\} \oplus \mathbb{Z}/p\{c_1, \dots, c_{n-1}\}\big) \otimes \mathbb{Z}[t]/(t^{p-1}).$$

Lemma 10.1 *In $grABP^{2*,*}(V_a) \cong E_\infty^{*,*',*''}$, the element $c_i \otimes t^s$ is $I_i = (p, v_1, \dots, v_{i-1})$-torsion.*

Proof. Since $c_i \otimes t^s \in \mathrm{Im}(Q_0 \dots Q_{i-1})$, it is I_i-torsion in $E_{2p^i-1}^{*,*',*''}$ from Lemma 6.1. Of course each element in $E_2^{2*,*,0} \cong CH^*(V_a)_{(p)}$ is permanent and we have the result. □

Lemma 10.2 *Let us write by $c_i(s) \in ABP^{2*,*}(V_a)$ a lift of*

$$c_i \otimes t^s \in E(V_a)_\infty^{2*,*,0} \subset grABP^{2*,*}(V_a).$$

Then $c_i(s)$ generates $ABP^{2*,*}(M_a^R)$ as a BP^*-module, and there are relations in $ABP^{2*,*}(M_a^R)$ for $k < i$

$$v_k c_i(s) = 0 \quad \mathrm{mod}\,\big(BP^*\{c_{i'}(s')|i' < i \textit{ and } s = s', \textit{ or } s' > s\}\big),$$
$$v_k c_i(s) - v_i c_k(s) = 0 \quad \mathrm{mod}(I_\infty^2).$$

Proof. Since $c_i \otimes t^s$ generates $CH^*(M_a^R)$, $c_i(s)$ generates $ABP^{2*,*}(M_a^R)$ as a BP^*-module, from $ABP^{2*,*}(X) \otimes_{BP^*} \mathbb{Z}_{(p)} \cong CH^*(X)_{(p)}$.

Since $c_i \otimes t^s$ is a I_i-torsion in $grABP^{2*,*}(V_a)$, it is a v_k-torsion for $k < i$. This means that in $ABP^{2*,*}(V_a)$,

$$v_k c_i(s) = 0 \quad \mathrm{mod}\,\big(BP^{*'} \otimes CH^*(V_a)|2* > |c_i(s)|\big)$$

where $|c_i(s)|$ is the first degree of $c_i(s)$. Hence from Theorem 7.2, we have in $ABP^{2*,*}(M_a^R)$,

$$v_k c_i(s) = 0 \quad \mathrm{mod}\,\big(BP^{*'} \otimes c_{i'}(s')||c_{i'}(s')| > |c_i(s)|\big),$$

which shows the first equality.

From Corollary 3.4, there is z such that

$$Q_k(z) = c_i \otimes t^s \quad \text{in } H^{*,*'}(V_a;\mathbb{Z}/p).$$

From Lemma 7.1, such element also exists in $H^{*,*'}(M_a^R;\mathbb{Z}/p)$ (since $c_i \otimes t^s \in H^{*,*'}(M_a^R;\mathbb{Z}/p)$). Moreover, this z is uniquely written in $H^{*,*'}(M_a^R;\mathbb{Z}/p)$ as

$$z = Q_0 \dots \hat{Q}_k \dots \hat{Q}_i \dots Q_{n-1}(\alpha' \otimes t^s)$$

from Lemma 9.5 by using $w(z) = 1$. (Note that elements of this bidegree are generated by only one element as a $K_*^M(k)/p$-modules.) Then

$$Q_i(z) = -c_k \otimes t^s,$$

and moreover $Q_j(z) = 0$ for $j \neq k$, $j \neq i$ in $H^{*,*'}(M_a^R;\mathbb{Z}/p)$. (Note that for $m \geq n$, $Q_m(z) = 0$ since $d(Q_m z) = p^m - 1 + d(z) > \dim(V_a)$.) From Corollary 3.4, we get the relation $v_k c_i(s) - v_i c_k(s) = 0 \bmod (I_\infty^2)$. $\square$

Remark The results of preceding lemma is more clearly given if we use the AHss for pure motives. In fact, p_θ commutes with the differential d_r of AHss by the same reason as the proof of Lemma 7.1.

By using (8.3) and (8.4), we have the isomorphism of BP^*-modules

$$ABP^{2*,*}\big(M_a^R|_{\bar{k}}\big) \cong BP^* \otimes CH^*\big(M_a^R|_{\bar{k}}\big) \cong BP^*[\bar{t}]/(\bar{t}^p)$$

for $\deg(\bar{t}) = (2b_n, b_n)$. (Note that $\bar{t} \in ABP^{2*,*}(M_a^R|_{\bar{k}})$ is decided only $\bmod(I_\infty)$.) We have the BP^*-algebra epimorphism

$$ABP^{*,*'}(V_a|_{\bar{k}}) \to ABP^{*,*}\big(M_a^R|_{\bar{k}}\big) \cong BP^*[\bar{t}]/(\bar{t}^p).$$

Lemma 10.3 *For $0 \leq s \leq p-2$. we have in $ABP^{2*,*}(V_a|_{\bar{k}})$*

$$i_{\bar{k}}(c_0(s)) = p\bar{t}^{s+1} \quad \bmod(I_\infty^2).$$

Proof. From the isomorphism $ABP^{2*,*}(M_a^R|_{\bar{k}}) \cong BP^* \otimes H^{2*,*}(M_a^R|_{\bar{k}}; \mathbb{Z}_{(2)})$, and Lemma 9.1, we have

$$i_{\bar{k}}(c_0(s)) = p\bar{t}^{s+1} + \sum_{s<s',j<n} \lambda_{j,s'} v_j \bar{t}^{s'+1} \quad \bmod(I_\infty^2).$$

Moreover by dimensional reason such as $|\bar{t}| = 2b_n$ and $-b_n < |v_j| < 0$, we see $\lambda_{j,s'} = 0$. $\square$

Lemma 10.4 *In $ABP^{2*,*}(M_a^R|_{\bar{k}})$, we have*

$$i_{\bar{k}}(c_i(s)) = v_i \bar{t}^{s+1} \quad \bmod\big(I_\infty^2\{\bar{t}^{s+1}, \bar{t}^{s+2}, \dots\}\big).$$

Proof. From Lemma 10.2, we see that

$$(v_i c_0(s) - p c_i(s)) \in I_\infty^2\big\{c_{i'}(s') | i' < i \text{ and } s' = s, \text{ or } s' > s\big\}.$$

Let us write by $I_\infty^r(s)$ the ideal $I_\infty^r\{\bar{t}^{s+1}, \bar{t}^{s+2}, \dots\}$ in $BP^*[\bar{t}]/(\bar{t}^p)$. By induction on i, we assume that $i_{\bar{k}}(c_{i'}(s')) = v_{i'}\bar{t}^{s'+1} \bmod(I^2_\infty(s'))$. In particular $i_{\bar{k}}(c_{i'}(s')) \in I_\infty(s')$. Hence we have

$$i_{\bar{k}}(v_i c_0(s) - pc_i(s)) = 0 \quad \bmod(I^3_\infty(s)).$$

From the preceding lemma, $i_{\bar{k}}(v_i c_0(s)) = pv_i\bar{t}^{s+1} \bmod(I^3_\infty(s))$. Therefore, we have

$$p\big(v_i\bar{t}^{s+1} - i_{\bar{k}}(c_i(s))\big) \quad \bmod(I^3_\infty(s)).$$

Hence $i_{\bar{k}}(c_i(s)) = v_i\bar{t}^{s+1} \bmod(I^2_\infty(s))$, since $\bar{t}^s$ generates a free BP^*-module, indeed, BP^* is a polynomial algebra over $\mathbb{Z}_{(p)}$. □

Corollary 10.5 *Let $i_{\bar{k}} : ABP^{2*,*}(M^R_a) \to ABP^{2*,*}(M^R_a|_{\bar{k}})$ be the restriction map. Then*

$$\mathrm{Im}(i_{\bar{k}}) = BP^*\{1\} \oplus I_n[\bar{t}]^+/(\bar{t}^p) \subset BP^*[\bar{t}]/(\bar{t}^p).$$

Proof. From the preceding lemma, we have

$$i_{\bar{k}}(c_i(s)) = v_i\bar{t}^{s+1} + a\bar{t}^{s+1} + \sum_{s'>s} b_{s'}\bar{t}^{s'+1}$$

with $a, b_{s'} \in I^2_\infty$. By dimensional reason such as $|v_n| = -2\dim(V_a)$, we see $a \in (p, \dots, v_{i-1})^2 = I^2_i$ and $b_{s'} \in (p, v_1, \dots, v_{n-1})^2 = I^2_n$.

We consider the filtration F_i of $ABP^{2*,*}(M^R_a)$ by

$$F_i = BP^{*'} \otimes \big\{CH^*(M^R_a)|2* \geq i\big\} \subset ABP^{2*,*}(M^R_a).$$

By descending induction on j for F_j, we assume that

$$i_{\bar{k}}(F_{|c_i(s)|+1}) = (p, \dots, v_{i-1})\bar{t}^{s+1} \oplus \bigoplus_{s'>s} I_n\{\bar{t}^{s'+1}\}.$$

Then $a\bar{t}^{s+1} + \sum_{s'>s} b_{s'}\bar{t}^{s'+1} \in i_{\bar{k}}(F_{|c_i(s)|+1})$ and hence

$$i_{\bar{k}}(F_{|c_i(s)|}) = (p, \dots, v_i)\bar{t}^{s+1} \oplus \bigoplus_{s'>s} I_n\{\bar{t}^{s'+1}\}.$$

Therefore we have

$$i_{\bar{k}}\big(F_{|c_0(s-1)|+1}/F_{|c_0(s)|+1}\big) \cong I_n\{\bar{t}^{s+1}\},$$

which induces the equality in this lemma. $\square$

Theorem 10.6 (For $p=2$, this is the main theorem in [Vi-Ya]) *Let M_a^R be the Rost motive for a nonzero symbol $a \in K_{n+1}^M(k)/p$. Then the restriction map $i_{\bar{k}} : ABP^{2*,*}(M_a^R) \to ABP^{2*,*}(M_a^R|_{\bar{k}})$ is injective and*

$$ABP^{2*,*}(M_a^R) \cong \mathrm{Im}(i_{\bar{k}}) = BP^*\{1\} \oplus I_n[\bar{t}]^+/(\bar{t}^p).$$

Proof. Recall the filtration F_i of $ABP^{2*,*}(M_a^R)$ defined by

$$F_i = BP^{*'} \otimes \big\{CH^*(M_a^R)|2* \ge i\big\} \subset ABP^{2*,*}\big(M_a^R\big),$$

and induced graded ring $grABP^{2*,*}(M_a^R)$. From the first equation of Lemma 10.2, $F_{|c_i(s)|}/F_{|c_i(s)|+1}$ is generated by one element $c_i(s)$ as a BP^*-module, which is I_i-torsion. Hence there is an epimorphism

$$f_1 : BP^* \oplus \bigoplus_{s=0}^{p-2}\bigoplus_{i=0}^{n-1} BP^*/I_i\{c_i(s)'\} \to grABP^{2*,*}(M_a)$$

by $c_i(s)' \mapsto c_i(s)$.

Next we consider the filration $\bar{F}_i$ of $BP^* \oplus I_n[\bar{t}]^+/(\bar{t}^p)$, by $\bar{F}_i = i_{\bar{k}}(F_i)$, e.g.,

$$\bar{F}_{|c_i(s)|} = BP^*\big\{v_{i'}\bar{t}^{s'+1}|i' \le i \text{ and } s'=s, \text{ or } s'>s\big\}$$

so that we can define the map from the preceding lemma

$$f_2 : grABP^{2*,*}\big(M_a^R\big) \to gr\big(BP^* \oplus I_n[\bar{t}]^+/(\bar{t}^p)\big).$$

We note here

$$I_n = BP^*(p, v_1, \dots, v_{n-1}) \cong \bigoplus_{i=0}^{n-1} BP^*\{c_i\}/(v_ic_j = v_jc_i)$$

by $v_i \mapsto c_i$. Hence we get the isomorphisms

$$grI_n \cong \bigoplus_{i=0}^{n-1} BP^*/I_i\{c_i\} \cong \bigoplus_{i=0}^{n-1} BP^*/I_i\{v_i\}.$$

Therefore we have the isomorphism

$$gr\big(BP^*\{1\} \oplus I_n[\bar{t}]^+/(\bar{t}^p)\big) \cong BP^* \oplus \bigoplus_{s=0}^{p-2}\bigoplus_{i=0}^{n-1} BP^*/I_i\{v_i\bar{t}^{s+1}\}.$$

The composition f_2f_1 is clearly isomorphism by $c_i(s)' \mapsto v_i\bar{t}^{s+1}$. Recall that f_1 is an epimorphism. So f_2 is an isomorphism. Therefore $i_{\bar{k}}$ itself is also an isomorphism. □

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Department of Mathematics
Faculty of Education
Ibaraki University
Mito, Ibaraki, Japan
E-mail: yagita@mx.ibaraki.ac.jp