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Reverse Cauchy–Schwarz type inequalities in pre-inner product C^*-modules

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Abstract. In the framework of a pre-inner product C^*-module over a unital C^*-algebra, we show several reverse Cauchy–Schwarz type inequalities of additive and multiplicative types, by using some ideas in N. Elezović et al. [Math. Inequal. Appl., 8 (2005), no. 2, 223–231]. We apply our results to give Klamkin-Mclenaghan, Shisha-Mond and Cassels type inequalities. We also present a Grüss type inequality.

Key words: C^*-algebra, reverse Cauchy–Schwarz inequality, pre-inner product C^*-module, Cassels' inequality, operator geometric mean, operator inequality

1. Introduction

A Hilbert C^*-module is a generalization of a Hilbert space in which the inner product takes its values in a C^*-algebra instead of the complex numbers. The theory of Hilbert C^*-modules is different from that of Hilbert spaces, for example, not any bounded linear operator between Hilbert C^*-modules is adjointable and not any closed submodule of a Hilbert C^*-module is complemented, see [10].

The theory of Hilbert C^*-modules over commutative C^*-algebras was first appeared in a work of Kaplansky [8] in 1953. The research on this subject started in 1970's independently by Paschke [16] and Rieffel [17] and since then it has grown rapidly and has played significant roles in the theory of operator algebras and noncommutative geometry.

Let $\mathscr{A}$ be a unital C^*-algebra with the unit element e and the center $\mathcal{Z}(\mathscr{A})$. For $a \in \mathscr{A}$, we denote the real part of a by $\mathrm{Re}\, a = \frac{1}{2}(a + a^*)$. If $a \in \mathscr{A}$ is positive (that is selfadjoint with positive spectrum), then $a^{\frac{1}{2}}$ denotes a unique positive $b \in \mathscr{A}$ such that $b^2 = a$. For $a \in \mathscr{A}$, we denote the absolute value of a by $|a| = (a^*a)^{\frac{1}{2}}$. If $a \in \mathcal{Z}(\mathscr{A})$ is positive, then $a^{\frac{1}{2}} \in \mathcal{Z}(\mathscr{A})$. If $a, b \in \mathscr{A}$ are positive and $ab = ba$, then ab is positive and

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$(ab)^{\frac{1}{2}} = a^{\frac{1}{2}} b^{\frac{1}{2}}$.

Let $\mathscr{X}$ be an algebraic left $\mathscr{A}$-module which is a complex linear space fulfilling $a(\lambda x) = (\lambda a)x = \lambda(ax)$ $(x \in \mathscr{X}, a \in \mathscr{A}, \lambda \in \mathbb{C})$. The space $\mathscr{X}$ is called a *(left) pre-inner product $\mathscr{A}$-module* (or a *pre-inner product C^*-module over the unital C^*-algebra $\mathscr{A}$*) if there exists a mapping $\langle \cdot, \cdot \rangle \colon \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ satisfying

(i) $\langle x, x \rangle \geq 0$,
(ii) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$,
(iii) $\langle ax, y \rangle = a \langle x, y \rangle$,
(iv) $\langle y, x \rangle = \langle x, y \rangle^*$,

for all $x, y, z \in \mathscr{X}$, $a \in \mathscr{A}$, $\lambda \in \mathbb{C}$. Moreover, if

(v) $x = 0$ whenever $\langle x, x \rangle = 0$,

then $\mathscr{X}$ is called an *inner product $\mathscr{A}$-module.* In this case $\|x\| := \sqrt{\|\langle x, x \rangle\|}$, where the latter norm denotes the C^*-norm on $\mathscr{A}$. If this norm is complete, then $\mathscr{X}$ is called a *Hilbert $\mathscr{A}$-module.* Any inner product space is an inner product $\mathbb{C}$-module and any C^*-algebra $\mathscr{A}$ is a Hilbert C^*-module over itself via $\langle a, b \rangle = ab^*$ $(a, b \in \mathscr{A})$. For more details on Hilbert C^*-modules, see [10]. Notice that (iii) and (iv) imply $\langle x, ay \rangle = \langle x, y \rangle a^*$ for all $x, y \in \mathscr{X}$, $a \in \mathscr{A}$.

The Cauchy–Schwarz inequality asserts that

$$\langle x, y \rangle \langle y, x \rangle \leq \|\langle y, y \rangle\| \, \langle x, x \rangle \tag{1.1}$$

in a pre-inner product module $\mathscr{X}$ over $\mathscr{A}$; see [10, Proposition 1.1]. This is a generalization of the classical Cauchy–Schwarz inequality. There have been proved several reverse Cauchy–Schwarz inequalities of additive and multiplicative types in the literature. The reader is refereed to [2], [6], [13], [14], [15] and references therein for more information.

In this paper, as a continuation of [13] and by using some ideas of [4], we investigate complementary Cauchy-Schwarz type inequalities in the framework of pre-inner product C^*-modules over a unital C^*-algebra. We apply our results to present Klamkin-Mclenaghan, Shisha-Mond and Cassels type inequalities. We also present a Grüss type inequality.

2. Reverse Cauchy–Schwarz type inequality I

In a semi-inner product space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$, the classical Cauchy-Schwarz inequality says that $|\langle x,y\rangle|^2 \le \langle x,x\rangle\langle y,y\rangle$ for all $x,y \in \mathscr{H}$. We discuss around Cauchy-Schwarz inequality under a non-commutative situation. In a pre-inner product C^*-module $\mathscr{X}$ over a unital C^*-algebra $\mathscr{A}$, since the product $\langle x,x\rangle\langle y,y\rangle$ is not selfadjoint in general, we would expect that a symmetric form $|\langle x,y\rangle|\langle y,y\rangle^{-1}|\langle x,y\rangle| \le \langle x,x\rangle$ holds for $x,y \in \mathscr{X}$ such that $\langle y,y\rangle$ is invertible. But we have a counterexample. As a matter of fact, let $\mathscr{A} = M_2(\mathbb{C})$ be the C*-albegra of 2×2 matrices with an inner product $\langle x,y\rangle = xy^*$ for $x,y \in \mathscr{A}$. Put $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then we have $|\langle x,y\rangle|\langle y,y\rangle^{-1}|\langle x,y\rangle| \not\le \langle x,x\rangle$. In this section, we present some reverse Cauchy–Schwarz inequalities of additive and multiplicative types which differs from [13, Theorem 3.3]. For this, we need the following lemma:

Lemma 2.1 *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x,y \in \mathscr{X}$ such that $\langle x,y\rangle$ is normal and*

$$\mathrm{Re}\langle Ay - x, x - ay\rangle \ge 0 \tag{2.1}$$

for some $a, A \in \mathcal{Z}(\mathscr{A})$. Then

$$\langle x,x\rangle + \mathrm{Re}(Aa^*)\langle y,y\rangle \le |a+A|\,|\langle x,y\rangle|. \tag{2.2}$$

Proof. Since $\mathrm{Re}\,\langle Ay - x, x - ay\rangle \ge 0$, we have

$$\begin{aligned}
\langle x,x\rangle + \mathrm{Re}(Aa^*)\langle y,y\rangle &\le \mathrm{Re}(A\langle x,y\rangle^* + a^*\langle x,y\rangle) \\
&= \mathrm{Re}(A^*\langle x,y\rangle + a^*\langle x,y\rangle) = \mathrm{Re}((A^* + a^*)\langle x,y\rangle) \\
&\le |(A^* + a^*)\langle x,y\rangle| \quad \text{by the normality of } (A^* + a^*)\langle x,y\rangle \\
&= |A + a|\,|\langle x,y\rangle|.
\end{aligned}$$

□

Theorem 2.2 *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x,y \in \mathscr{X}$ such that $\langle x,y\rangle$ is normal,* $\mathrm{Re}(Aa^*)$ *is a positive invertible operator for $A, a \in \mathcal{Z}(\mathscr{A})$ and* (2.1) *holds. If $\langle y,y\rangle$ is invertible, then*

$$\text{(i)} \quad \langle x,x\rangle \le \frac{1}{4}\mathrm{Re}(Aa^*)^{-1}|A+a|^2|\langle x,y\rangle|\langle y,y\rangle^{-1}|\langle x,y\rangle|,$$

$$\begin{aligned}\text{(ii)}\quad & \langle x,x\rangle - |\langle x,y\rangle|\langle y,y\rangle^{-1}|\langle x,y\rangle| \\ & \le \frac{1}{4}\mathrm{Re}(Aa^*)^{-1}|A-a|^2|\langle x,y\rangle|\langle y,y\rangle^{-1}|\langle x,y\rangle|.\end{aligned}$$

Proof. For (i), it follows from Lemma 2.1 that

$$\begin{aligned}\langle x,x\rangle &\le |A+a||\langle x,y\rangle| - \mathrm{Re}(Aa^*)\langle y,y\rangle \\ &= \frac{1}{4}\mathrm{Re}(Aa^*)^{-1}|A+a|^2|\langle x,y\rangle|\langle y,y\rangle^{-1}|\langle x,y\rangle| - X^*X,\end{aligned}$$

where $X = \mathrm{Re}(Aa^*)^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}} - \frac{1}{2}\mathrm{Re}(Aa^*)^{-\frac{1}{2}}|A+a|\langle y,y\rangle^{-\frac{1}{2}}|\langle x,y\rangle|$ and hence we get (i). For (ii), it follows from (i) that

$$\begin{aligned}&\langle x,x\rangle - |\langle x,y\rangle|\langle y,y\rangle^{-1}|\langle x,y\rangle| \\ &\quad\le \frac{1}{4}\mathrm{Re}(Aa^*)^{-1}|A+a|^2|\langle x,y\rangle|\langle y,y\rangle^{-1}|\langle x,y\rangle| - |\langle x,y\rangle|\langle y,y\rangle^{-1}|\langle x,y\rangle| \\ &\quad= \frac{1}{4}\mathrm{Re}(Aa^*)^{-1}(|A+a|^2 - 4\mathrm{Re}(Aa^*))|\langle x,y\rangle|\langle y,y\rangle^{-1}|\langle x,y\rangle| \\ &\quad= \frac{1}{4}\mathrm{Re}(Aa^*)^{-1}|A-a|^2|\langle x,y\rangle|\langle y,y\rangle^{-1}|\langle x,y\rangle|.\end{aligned}$$ □

The next result is a generalization of both Klamkin–Mclenaghan's inequality and Shisha–Mond's inequality [4, Theorem 2].

Theorem 2.3 *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x,y\in\mathscr{X}$ such that $\langle x,y\rangle$ is normal and invertible, $\langle y,y\rangle$ is invertible and $A,a\in\mathscr{Z}(\mathscr{A})$ satisfy* $\mathrm{Re}(Aa^*)\ge 0$ *and* (2.1). *Then*

$$\begin{aligned}&|\langle x,y\rangle|^{-\frac{1}{2}}\langle x,x\rangle|\langle x,y\rangle|^{-\frac{1}{2}} - |\langle x,y\rangle|^{\frac{1}{2}}\langle y,y\rangle^{-1}|\langle x,y\rangle|^{\frac{1}{2}} \\ &\quad\le |A+a| - 2\mathrm{Re}(Aa^*)^{\frac{1}{2}}.\end{aligned}$$

Proof. It follows from Lemma 2.1 that

$$\begin{aligned}&|\langle x,y\rangle|^{-\frac{1}{2}}\langle x,x\rangle|\langle x,y\rangle|^{-\frac{1}{2}} - |\langle x,y\rangle|^{\frac{1}{2}}\langle y,y\rangle^{-1}|\langle x,y\rangle|^{\frac{1}{2}} \\ &\quad\le |A+a| - \mathrm{Re}(Aa^*)|\langle x,y\rangle|^{-\frac{1}{2}}\langle y,y\rangle|\langle x,y\rangle|^{-\frac{1}{2}} - |\langle x,y\rangle|^{\frac{1}{2}}\langle y,y\rangle^{-1}|\langle x,y\rangle|^{\frac{1}{2}}\end{aligned}$$

$$= |A+a| - 2\mathrm{Re}(Aa^*)^{\frac{1}{2}} - \big(\mathrm{Re}(Aa^*)^{\frac{1}{2}}(|\langle x,y\rangle|^{-\frac{1}{2}}\langle y,y\rangle|\langle x,y\rangle|^{-\frac{1}{2}})^{\frac{1}{2}} - (|\langle x,y\rangle|^{\frac{1}{2}}\langle y,y\rangle^{-1}|\langle x,y\rangle|^{\frac{1}{2}})^{\frac{1}{2}}\big)^2$$

$$\leq |A+a| - 2\mathrm{Re}(Aa^*)^{\frac{1}{2}}. \qquad \square$$

The next result is an integral version of Klamkin–Mclenaghan's inequality.

Corollary 2.4 *Let (X,μ) be a probability space and $f,g \in L^\infty(\mu)$ with $mg \leq f \leq Mg$ for some scalars $M > m > 0$. Then*

$$\frac{\int_X |f|^2 d\mu}{\left|\int_X fg d\mu\right|} - \frac{\left|\int_X fg d\mu\right|}{\int_X |g|^2 d\mu} \leq \left(\sqrt{M} - \sqrt{m}\right)^2. \tag{2.3}$$

Proof. $\mathscr{X} = L^\infty(X,\mu)$ is regarded as a subspace of $L^2(X,\mu)$ via $\langle f,g\rangle = \int_X f\overline{g}d\mu \quad (f,g \in \mathscr{X})$. Then Theorem 2.3 implies the desired inequality since $\langle Mg-f, f-mg\rangle \geq 0$. $\square$

Considering $\mathbb{C}^n$ equipped with the natural inner product defined with weights $(w_1,\ldots,w_n)$ or, equivalently, starting with a weighted counting measure $\mu = \sum_{i=1}^n w_i\delta_i$, where w_i's are positive numbers and δ_i's are the Dirac delta functions, a discrete version of the above is a weighted Shisha–Mond's inequality as follows:

Corollary 2.5 *If $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are sequences of positive real numbers satisfying the condition $0 < m_1 \leq y_i \leq M_1 < \infty$ and $0 < m_2 \leq x_i \leq M_2 < \infty$, then*

$$\frac{\sum_{i=1}^n w_i x_i^2}{\sum_{i=1}^n w_i x_i y_i} - \frac{\sum_{i=1}^n w_i x_i y_i}{\sum_{i=1}^n w_i y_i^2} \leq \left(\sqrt{M_2/m_1} - \sqrt{m_2/M_1}\right)^2.$$

Now we give an additive reverse Cauchy–Schwarz inequality, which seems to be nicer than [13, Theorem 3.1].

Theorem 2.6 *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x,y \in \mathscr{X}$ such that $\langle x,y\rangle$ is normal, and $A,a \in \mathscr{Z}(\mathscr{A})$ such that $|A+a|$ is invertible and (2.1) holds. Then*

(i) $$\mathrm{Re}\big(\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}}\big) - |\langle x,y\rangle| \leq \frac{1}{4}|A-a|^2|A+a|^{-1}\langle y,y\rangle.$$

If moreover $\mathrm{Re}(Aa^*)$ *is positive invertible, then*

(ii) $\mathrm{Re}\big(\langle x,x\rangle^{\frac12}\langle y,y\rangle^{\frac12}\big) - |\langle x,y\rangle| \le \frac14 |A-a|^2|A+a|^{-1}\mathrm{Re}(Aa^*)^{-1}\langle x,x\rangle.$

Proof. For (i), by Lemma 2.1, we have

$$\begin{aligned}
&\mathrm{Re}\big(\langle x,x\rangle^{\frac12}\langle y,y\rangle^{\frac12}\big) - |\langle x,y\rangle| \\
&\quad\le \mathrm{Re}\big(\langle x,x\rangle^{\frac12}\langle y,y\rangle^{\frac12}\big) - |A+a|^{-1}\langle x,x\rangle - |A+a|^{-1}\mathrm{Re}(Aa^*)\langle y,y\rangle \\
&\quad= \left[\frac14|A+a| - \mathrm{Re}(Aa^*)|A+a|^{-1}\right]\langle y,y\rangle \\
&\qquad - |A+a|^{-1}\left(\langle x,x\rangle^{\frac12} - \frac12|A+a|\,\langle y,y\rangle^{\frac12}\right)^2 \\
&\quad\le \frac14\big[|A+a|^2 - 4\mathrm{Re}(Aa^*)\big]|A+a|^{-1}\langle y,y\rangle \\
&\quad= \frac14|A-a|^2|A+a|^{-1}\langle y,y\rangle.
\end{aligned}$$

For (ii), it similarly follows from

$$\begin{aligned}
&\mathrm{Re}\big(\langle x,x\rangle^{\frac12}\langle y,y\rangle^{\frac12}\big) - |\langle x,y\rangle| \\
&\quad\le \frac14|A-a|^2|A+a|^{-1}\mathrm{Re}(Aa^*)^{-1}\langle x,x\rangle \\
&\qquad - \mathrm{Re}(Aa^*)|A+a|^{-1}\left(\langle y,y\rangle^{\frac12} - \frac12|A+a|\mathrm{Re}(Aa^*)^{-1}\langle x,x\rangle^{\frac12}\right)^2. \qquad \square
\end{aligned}$$

Corollary 2.7 *Let* φ *be a positive linear functional on a* C^**-algebra* $\mathscr{A}$ *and let* $x, y \in \mathscr{A}$ *be such that*

$$\mathrm{Re}\,\varphi((\Lambda y - x)^*(x-\lambda y)) \ge 0$$

for some $\lambda, \Lambda \in \mathbb{C}$. *Then*

(i) $\varphi(x^*x)^{1/2}\varphi(y^*y)^{1/2} \le \dfrac{|\lambda+\Lambda|}{2\sqrt{\mathrm{Re}(\overline{\lambda}\Lambda)}}|\varphi(y^*x)|.$

$$\text{(ii)}\quad \varphi(x^*x)^{1/2}\varphi(y^*y)^{1/2} - |\varphi(y^*x)| \le \frac{|\Lambda-\lambda|^2}{4|\Lambda+\lambda|}\min\{\varphi(y^*y), \varphi(x^*x)\}.$$

Proof. The C^*-algebra $\mathscr{A}$ can be regarded as a pre-inner product module over $\mathbb{C}$ via $\langle x, y\rangle = \varphi(y^*x)$. Now (i) and (ii) follow from Theorem 2.2 and Theorem 2.6 and an obvious symmetry argument, respectively. □

Remark 2.8 Let $\mathscr{A}$ be a C^*-algebra, $x, y \in \mathscr{A}$ such that $xy = yx$, $m_1 \le x \le M_1$, $m_2 \le y \le M_2$ and φ is a positive linear functional on $\mathscr{A}$. Setting $\lambda = m_1/M_2$ and $\Lambda = M_1/m_2$, we observe that $x - \lambda y \ge 0$ and $\Lambda y - x \ge 0$, whence

$$\varphi((\Lambda y - x)(x - \lambda y)^*) \ge 0.$$

Thus the requirements of Theorems 2.2 and 2.6 are fulfilled.

Considering the C^*-algebra $\mathscr{A} = \mathbb{B}(\mathscr{H})$ of all bounded linear operators on a Hilbert space $\mathscr{H}$ and the positive linear functional $\varphi(R) = \sum_{i=1}^n \langle Re_i, e_i\rangle$, where $e_1, \dots, e_n \in \mathscr{H}$ we deduce the following result from (i) and (ii) of Corollary 2.7.

Corollary 2.9 *Let $\mathscr{H}$ be a Hilbert space, $e_1, \dots, e_n \in \mathscr{H}$, $T, S \in \mathbb{B}(\mathscr{H})$ with $TS = ST$ and $mS \le T \le MS$ for some scalars $M > m > 0$. Then*

$$\text{(i)}\quad \left(\sum_{i=1}^n \|Te_i\|^2\right)^{1/2}\left(\sum_{i=1}^n \|Se_i\|^2\right)^{1/2} \le \frac{M+m}{2\sqrt{Mm}}\left|\sum_{i=1}^n \langle Te_i, Se_i\rangle\right|.$$

$$\text{(ii)}\quad \left(\sum_{i=1}^n \|Te_i\|^2\right)^{1/2}\left(\sum_{i=1}^n \|Se_i\|^2\right)^{1/2} - \left|\sum_{i=1}^n \langle Te_i, Se_i\rangle\right|$$
$$\le \frac{(M-m)^2}{4(M+m)}\min\left\{\sum_{i=1}^n \|Se_i\|^2, \sum_{i=1}^n \|Te_i\|^2\right\}.$$

3. Reverse Cauchy–Schwarz type inequality II

In [6], Ilisević and Varošanec sharpened (1.1) in a restricted case: If $x, y \in \mathscr{X}$ and $\langle x, x\rangle \in \mathscr{Z}(\mathscr{A})$, then

$$|\langle x, y\rangle|^2 \le \langle x, x\rangle\langle y, y\rangle, \tag{3.1}$$

which implies

$$|\langle x,y\rangle| \le \langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}}. \tag{3.2}$$

We present another version of the Cauchy–Schwarz inequality in a pre-inner product C*-module, in which we assume the invertibility of $\langle y,y\rangle$ instead of $\langle x,x\rangle \in \mathcal{Z}(\mathscr{A})$:

Proposition 3.1 *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x,y \in \mathscr{X}$ such that $\langle y,y\rangle$ is invertible. Then*

$$\langle x,y\rangle\langle y,y\rangle^{-1}\langle x,y\rangle^* \le \langle x,x\rangle. \tag{3.3}$$

Proof. By the module properties and the Cauthy–Schwarz inequality (1.1), we have

$$\begin{aligned}\langle x,y\rangle\langle y,y\rangle^{-1}\langle y,x\rangle &= \big\langle x,\langle y,y\rangle^{-\frac{1}{2}}y\big\rangle\big\langle\langle y,y\rangle^{-\frac{1}{2}}y,x\big\rangle\\ &\le \big\|\big\langle\langle y,y\rangle^{-\frac{1}{2}}y,\langle y,y\rangle^{-\frac{1}{2}}y\big\rangle\big\|\langle x,x\rangle\\ &= \langle x,x\rangle.\end{aligned}$$

□

To obtain reverse inequalities of additive and multiplicative types to the Cauchy-Schwarz one (3.3), we need the following lemma which differs from Lemma 2.1:

Lemma 3.2 *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x,y \in \mathscr{X}$ such that*

$$\langle Ay-x,x-ay\rangle \ge 0 \tag{3.4}$$

for some positive invertible elements $a,A \in \mathcal{Z}(\mathscr{A})$. Then

$$\langle x,x\rangle \le (A+a)\mathrm{Re}\langle x,y\rangle - Aa\langle y,y\rangle. \tag{3.5}$$

Proof. The assumption (3.4) implies

$$A\langle y,x\rangle - A\langle y,y\rangle a - \langle x,x\rangle + \langle x,y\rangle a \ge 0. \tag{3.6}$$

Taking the adjoint in (3.6),

$$\langle y,x\rangle^* A - a\langle y,y\rangle A - \langle x,x\rangle + a\langle x,y\rangle^* \geq 0. \tag{3.7}$$

Combining with (3.6) and (3.7), since $a, A \in \mathscr{Z}(\mathscr{A})$ are positive, we have the desired inequality (3.5). $\square$

Theorem 3.3 *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x, y \in \mathscr{X}$ such that $\langle y,y\rangle$ is invertible and* (3.4) *holds for some positive invertible elements $a, A \in \mathscr{Z}(\mathscr{A})$. Then*

(i) $\langle x,x\rangle \leq \dfrac{1}{4}(Aa)^{-1}(A+a)^2\langle x,y\rangle\langle y,y\rangle^{-1}\langle x,y\rangle^*.$

(ii) $\langle x,x\rangle - \langle x,y\rangle\langle y,y\rangle^{-1}\langle x,y\rangle^* \leq \big(A^{\frac{1}{2}} - a^{\frac{1}{2}}\big)^2 \mathrm{Re}\langle x,y\rangle.$

Proof. For (i), it follows from Lemma 3.2 that

$$\begin{aligned}\langle x,x\rangle &\leq (A+a)\mathrm{Re}\langle x,y\rangle - Aa\langle y,y\rangle \\ &= \frac{1}{4}(Aa)^{-1}(A+a)^2\langle x,y\rangle\langle y,y\rangle^{-1}\langle x,y\rangle^* - X^*X,\end{aligned}$$

where $X = (Aa)^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}} - \frac{1}{2}(Aa)^{-\frac{1}{2}}(A+a)\langle y,y\rangle^{-\frac{1}{2}}\langle x,y\rangle^*$ and hence we have (i).

For (ii), by using Lemma 3.2 again, we have

$$\begin{aligned}&\langle x,x\rangle - \langle x,y\rangle\langle y,y\rangle^{-1}\langle x,y\rangle^* \\ &\quad\leq (A+a)\mathrm{Re}\langle x,y\rangle - Aa\langle y,y\rangle - \langle x,y\rangle\langle y,y\rangle^{-1}\langle x,y\rangle^* \\ &\quad= \big(A+a-2(Aa)^{\frac{1}{2}}\big)\mathrm{Re}\langle x,y\rangle \\ &\qquad - \big((Aa)^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}} - \langle x,y\rangle\langle y,y\rangle^{-frac12}\big)\big((Aa)^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}} - \langle x,y\rangle\langle y,y\rangle^{-\frac{1}{2}}\big)^* \\ &\quad\leq \big(A^{\frac{1}{2}} - a^{\frac{1}{2}}\big)^2\mathrm{Re}\langle x,y\rangle.\end{aligned}$$ $\square$

We can also obtain the following reverse Cauchy-Schwarz type inqualities related to (3.2):

Theorem 3.4 *Let $\mathscr{X}$ be a pre-inner product C^*-module over $\mathscr{A}$. Suppose that $x, y \in \mathscr{X}$ such that* (3.4) *holds for some positive invertible elements $A, a \in \mathscr{Z}(\mathscr{A})$. Then*

(i) $\mathrm{Re}\big(\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}}\big) \le \dfrac{1}{2}(Aa)^{-\frac{1}{2}}(A+a)\mathrm{Re}\langle x,y\rangle.$

(ii) $\mathrm{Re}\big(\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}} - \langle x,y\rangle\big) \le \dfrac{1}{4}(A-a)^2(A+a)^{-1}\langle y,y\rangle.$

(iii) $\mathrm{Re}\big(\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}} - \langle x,y\rangle\big) \le \dfrac{1}{4}(A-a)^2(A+a)^{-1}(Aa)^{-1}\langle x,x\rangle.$

Proof. For (i), by Lemma 3.2, we have

$$
\begin{aligned}
(A+a)\mathrm{Re}\langle x,y\rangle &\ge \langle x,x\rangle + Aa\langle y,y\rangle \\
&= \big(\langle x,x\rangle^{\frac{1}{2}} - (Aa)^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}}\big)^2 + 2(Aa)^{\frac{1}{2}}\mathrm{Re}\big(\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}}\big) \\
&\ge 2(Aa)^{\frac{1}{2}}\mathrm{Re}\big(\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}}\big).
\end{aligned}
$$

For (ii), it follows from Lemma 3.2 that $\langle x,x\rangle \le (A+a)\mathrm{Re}\langle x,y\rangle - Aa\langle y,y\rangle$ and since $A+a$ is invertible,

$$(A+a)^{-1}\langle x,x\rangle + Aa(A+a)^{-1}\langle y,y\rangle \le \mathrm{Re}\langle x,y\rangle.$$

Therefore we have

$$
\begin{aligned}
&\mathrm{Re}\big(\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}} - \langle x,y\rangle\big) \\
&\quad\le \mathrm{Re}\big(\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}}\big) - (A+a)^{-1}\langle x,x\rangle - Aa(A+a)^{-1}\langle y,y\rangle \\
&\quad= \frac{1}{4}(A+a)^{-1}(A-a)^2\langle y,y\rangle - (A+a)^{-1}\left(\langle x,x\rangle^{\frac{1}{2}} - \frac{1}{2}(A+a)\langle y,y\rangle^{\frac{1}{2}}\right)^2 \\
&\quad\le \frac{1}{4}(A-a)^2(A+a)^{-1}\langle y,y\rangle.
\end{aligned}
$$

For (iii), it similarly follows from

$$
\begin{aligned}
&\mathrm{Re}\big(\langle x,x\rangle^{\frac{1}{2}}\langle y,y\rangle^{\frac{1}{2}} - \langle x,y\rangle\big) \\
&\quad\le \frac{1}{4}(A-a)^2(A+a)^{-1}(Aa)^{-1}\langle x,x\rangle \\
&\qquad - Aa(A+a)^{-1}\left(\langle y,y\rangle^{\frac{1}{2}} - \frac{1}{2}(A+a)(Aa)^{-1}\langle x,x\rangle^{\frac{1}{2}}\right)^2. \qquad \square
\end{aligned}
$$

Remark 3.5 Theorem 3.4 is also a non-commutative version of the following results in [3, Theorem 2.2] and [4, Theorem 4]: Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product over a complex number field $\mathbb{C}$. If $x, y \in H$ and $c, C \in \mathbb{C}$ such that $\mathrm{Re}\langle Cy - x, x - cy\rangle \geq 0$ and $\mathrm{Re}(C\overline{c}) > 0$, then

$$\frac{\sqrt{\langle x,x\rangle\langle y,y\rangle}}{|\langle x,y\rangle|} \leq \frac{|C+c|}{2\sqrt{\mathrm{Re}(C\overline{c})}} \quad \text{and} \quad \sqrt{\langle x,x\rangle\langle y,y\rangle} - |\langle x,y\rangle| \leq \frac{|C-c|^2}{4|C+c|}\langle y,y\rangle.$$

4. Cassels type inequalities

In 1952 Cassels (see [18] and [15]) established that if for some real numbers m, M the positive $n-$tuples $(a_1,\dots,a_n)$ and $(b_1,\dots,b_n)$ satisfy $0 < m \leq \frac{a_k}{b_k} \leq M < \infty$ $(1 \leq k \leq n)$ for some scalars $M > m > 0$, then

$$\sum_{k=1}^{n} w_k a_k^2 \sum_{k=1}^{n} w_k b_k^2 \leq \frac{(M+m)^2}{4mM}\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2 \tag{4.1}$$

for any weight $(w_1,\dots,w_n)$.

In this section, we consider Cassels type inequalities by using the geometric mean of $\langle x,x\rangle$ and $\langle y,y\rangle$. We recall that the geometric mean of two positive elements $a, b \in \mathscr{A}$ is defined by

$$a \sharp b = a^{\frac{1}{2}}\left(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^{\frac{1}{2}} a^{\frac{1}{2}}$$

if a is invertible, also see [9]. We notice that if a and b commute, then $a \sharp b = a^{\frac{1}{2}} b^{\frac{1}{2}}$. Unfortunately, the following Cauchy-Schwarz type inequality $\mathrm{Re}\langle x,y\rangle \leq \langle x,x\rangle \sharp \langle y,y\rangle$ does not hold in general. As a matter of fact, let $\mathscr{A} = M_2(\mathbb{C})$ be the C*-albegra of 2×2 matrices with an inner product $\langle x,y\rangle = xy^*$ for $x, y \in \mathscr{A}$. Put $x = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Then we have $\mathrm{Re}\langle x,y\rangle \not\leq \langle x,x\rangle \sharp \langle y,y\rangle$. However, we can obtain Cassels type inequalities by virtue of Lemma 3.2 again:

Theorem 4.1 *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x, y \in \mathscr{X}$ such that* (3.4) *holds for some positive invertible elements $a, A \in \mathcal{Z}(\mathscr{A})$. Then*

(i) $\langle x,x\rangle \sharp \langle y,y\rangle \leq \dfrac{1}{2}(Aa)^{-\frac{1}{2}}(A+a)\mathrm{Re}\langle x,y\rangle.$

(ii) $\langle x,x\rangle \sharp \langle y,y\rangle - \mathrm{Re}\langle x,y\rangle \leq \frac{1}{4}(Aa)^{-1}(A+a)^{-1}(A-a)^2\langle x,x\rangle$.

(iii) $\langle y,y\rangle \sharp \langle x,x\rangle - \mathrm{Re}\langle x,y\rangle \leq \frac{1}{4}(A+a)^{-1}(A-a)^2\langle y,y\rangle$.

Proof. For any $\varepsilon > 0$, since $\langle x,x\rangle + \varepsilon e$ is invertible, it follows from the arithmetic-geometric mean inequality and Lemma 3.2 that

$$\begin{aligned}(Aa)^{\frac{1}{2}}(\langle x,x\rangle + \varepsilon e) \sharp \langle y,y\rangle &= (\langle x,x\rangle + \varepsilon e) \sharp (Aa\langle y,y\rangle)\\ &\leq \frac{1}{2}(\langle x,x\rangle + \varepsilon e + Aa\langle y,y\rangle)\\ &\leq \frac{1}{2}((A+a)\mathrm{Re}\langle x,y\rangle + \varepsilon e).\end{aligned}$$

As $\varepsilon \downarrow 0$, we get (i).

Similarly we may assume that $\langle x,x\rangle$ and $\langle y,y\rangle$ are invertible to prove (ii) and (iii).

For (ii), set $X := \langle x,x\rangle^{-\frac{1}{2}}\langle y,y\rangle\langle x,x\rangle^{-\frac{1}{2}}$. Then it follows from Lemma 3.2 and invertibility of $A+a$ that

$$\begin{aligned}&\langle x,x\rangle \sharp \langle y,y\rangle - \mathrm{Re}\langle x,y\rangle\\ &\leq \langle x,x\rangle^{\frac{1}{2}}X^{\frac{1}{2}}\langle x,x\rangle^{\frac{1}{2}} - (A+a)^{-1}\langle x,x\rangle - Aa(A+a)^{-1}\langle y,y\rangle\\ &= \langle x,x\rangle^{\frac{1}{2}}\big(X^{\frac{1}{2}} - (A+a)^{-1} - Aa(A+a)^{-1}X\big)\langle x,x\rangle^{\frac{1}{2}}\\ &= \langle x,x\rangle^{\frac{1}{2}}\Bigg(\frac{(Aa(A+a))^{-1}(A-a)^2}{4}\\ &\qquad - Aa(A+a)^{-1}\left(X^{\frac{1}{2}} - \frac{(Aa)^{-1}(A+a)}{2}\right)^2\Bigg)\langle x,x\rangle^{\frac{1}{2}}\\ &\leq \frac{1}{4}(Aa(A+a))^{-1}(A-a)^2\langle x,x\rangle.\end{aligned}$$

For (iii), set $Y := \langle y,y\rangle^{-\frac{1}{2}}\langle x,x\rangle\langle y,y\rangle^{-\frac{1}{2}}$ as in (ii). Then it follows that

$$\begin{aligned}&\langle y,y\rangle \sharp \langle x,x\rangle - \mathrm{Re}\langle x,y\rangle\\ &\leq \langle y,y\rangle^{\frac{1}{2}}\big(Y^{\frac{1}{2}} - (A+a)^{-1}Y - Aa(A+a)^{-1}\big)\langle y,y\rangle^{\frac{1}{2}}\end{aligned}$$

$$= \langle y, y\rangle^{\frac{1}{2}} \left(\frac{(A+a)^{-1}(A-a)^2}{4} - (A+a)^{-1}\left(Y^{\frac{1}{2}} - \frac{(A+a)}{2}\right)^2 \right)\langle y, y\rangle^{\frac{1}{2}}$$

$$\leq \frac{1}{4}(A+a)^{-1}(A-a)^2\langle y, y\rangle. \qquad \square$$

The next result is an integral version of the Cassels inequality:

Corollary 4.2 *Let (X, μ) be a probability space and $f, g \in L^\infty(\mu)$ with $mg \leq f \leq Mg$. Then*

$$\int_X |f|^2 d\mu \int_X |g|^2 d\mu \leq \frac{(M+m)^2}{4Mm} \left| \int_X fg d\mu \right|^2.$$

Proof. $\mathscr{X} = L^\infty(X, \mu)$ is regarded as a subspace of $L^2(X, \mu)$ via $\langle f, g\rangle = \int_X f\overline{g} d\mu$ $(f, g \in \mathscr{X})$ and use Theorem 4.1 since $\langle Mg - f, f - mg\rangle \geq 0$. $\square$

Considering $\mathbb{C}^n$ equipped with the natural inner product defined with weights $(w_1, \ldots, w_n)$ we obtain the Cassels inequality (4.1).

5. A Grüss type inequality

In order to establish a complement of Chebyshev's inequality, Grüss [5] proved the following inequality: If f and g are integrable real functions on $[a, b]$ such that $C \leq f(x) \leq D$ and $E \leq g(x) \leq F$ for some real constants C, D, E, F and for all $x \in [a, b]$, then

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{1}{4}(D-C)(F-E); \tag{5.1}$$

and the constant $1/4$ is the best possible, see [3], [11], [12] and references therein.

In the final section, we show a Grüss type inequality in a pre-inner product C^*-module. Some norm inequalities of Grüss type have been obtained in [1], [7]. First, we state the following lemma by using some ideas of [7, Lemma 2.4].

Lemma 5.1 *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x, h \in \mathscr{X}$ such that $\langle h, h\rangle$ is the unit element e of $\mathscr{A}$ and* (3.4) *holds for some positive invertible elements $a, A \in \mathcal{Z}(\mathscr{A})$. Then*

$$0 \le \langle x,x\rangle - |\langle h,x\rangle|^2 \le \frac{1}{4}(A-a)^2. \tag{5.2}$$

Proof. By the module properties, we have

$$\begin{aligned}
0 &\le \langle x - \langle x,h\rangle h, x - \langle x,h\rangle h\rangle \\
&= \langle x,x\rangle - \langle x,h\rangle\langle h,x\rangle - \langle x,h\rangle\langle h,x\rangle + \langle x,h\rangle\langle h,h\rangle\langle h,x\rangle \\
&= \langle x,x\rangle - \langle x,h\rangle\langle h,x\rangle - \langle x,h\rangle\langle h,x\rangle + \langle x,h\rangle e\langle h,x\rangle \\
&= \langle x,x\rangle - \langle x,h\rangle\langle h,x\rangle \\
&= \langle x,x\rangle - |\langle h,x\rangle|^2.
\end{aligned}$$

Second, it follows from Lemma 3.2 and $\langle h,h\rangle = e$ that

$$\begin{aligned}
\langle x,x\rangle - |\langle h,x\rangle|^2 &\le (A+a)\mathrm{Re}\langle x,h\rangle - Aa - \langle x,h\rangle\langle h,x\rangle \\
&= -\left(\langle x,h\rangle - \frac{A+a}{2}\right)\left(\langle x,h\rangle - \frac{A+a}{2}\right)^* + \frac{(A-a)^2}{4} \\
&\le \frac{(A-a)^2}{4}.
\end{aligned}$$

□

By utilizing Lemma 5.1, we show the following Grüss type inequality in a pre-inner product C^*-module.

Theorem 5.2 *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x, y, h \in \mathscr{X}$ such that $\langle h,h\rangle$ is the unit element e of $\mathscr{A}$, $\langle y,y\rangle - |\langle h,y\rangle|^2$ is invertible and*

$$\langle Ah - x, x - ah\rangle \ge 0 \quad \text{and} \quad \langle Bh - y, y - bh\rangle \ge 0$$

hold for some positive invertible elements $a, A, b, B \in \mathcal{Z}(\mathscr{A})$. Then

$$|\langle y,x\rangle - \langle y,h\rangle\langle h,x\rangle| \le \frac{1}{4}|A-a||B-b|. \tag{5.3}$$

Proof. It follows from

$$0 \le \langle x - \langle x,h\rangle h, x - \langle x,h\rangle h\rangle = \langle x,x\rangle - |\langle h,x\rangle|^2$$

that $[x,y]_h := \langle x,y\rangle - \langle x,h\rangle\langle h,y\rangle$ is a pre-inner product $\mathscr{A}$-module. Utilizing

Proposition 3.1 for $[\cdot,\cdot]_h$ we get

$$\begin{aligned}&(\langle x,y\rangle-\langle x,h\rangle\langle h,y\rangle)(\langle y,y\rangle-|\langle h,y\rangle|^2)^{-1}(\langle x,y\rangle-\langle x,h\rangle\langle h,y\rangle)^*\\&\quad\le\langle x,x\rangle-|\langle h,x\rangle|^2.\end{aligned}$$

By Lemma 5.1 and the invertibility of $\langle y,y\rangle-|\langle h,y\rangle|^2$, we have

$$4(B-b)^{-2}\le(\langle y,y\rangle-|\langle h,y\rangle|^2)^{-1}$$

and hence

$$4(B-b)^{-2}|\langle y,x\rangle-\langle y,h\rangle\langle h,x\rangle|^2\le\frac{1}{4}(A-a)^2.$$

This implies the desired inequality. □

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References

[1] Banić S., Ilišević D. and Varošanec S., *Bessel- and Grüss-type inequalities in inner product modules.* Proc. Edinb. Math. Soc. (2), **50 (1)** (2007), 23–36.

[2] Dragomir S. S., *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces.* Nova Science Publishers, New York, 2005.

[3] Dragomir S. S., *Reverses of Schwarz, triangle and bessel inequalities in inner product spaces.* J. Inequal. Pure Appl. Math., **5**, Issue 3, Article 76, 2004.

[4] Elezović N., Marangunić Lj. and Pečarić J. E., *Unified treatment of complemented Schwarz and Grüss inequalities in inner product spaces.* Math. Inequal. Appl., **8 (2)** (2005), 223–231.

[5] Grüss G., *Über das Maximum des absoluten Betrages von* $\frac{1}{b-a}\int_a^b f(x)\cdot g(x)dx-\frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx$. Math. Z., **39** (1935), 215–226.

[6] Ilisević D. and Varošanec S., *On the Cauchy–Schwarz inequality and its reverse in semi-inner product C^*-modules.* Banach J. Math. Anal., **1** (2007), 78–84.

[7] Ilišević D. and Varošanec S., *Grüss type inequalities in inner product modules.* Proc. Amer. Math. Soc., **133 (11)** (2005), 3271–3280.

[8] Kaplansky I., *Modules over operator algebras*. Amer. J. Math., **75** (1953), 839–858.

[9] Kubo F. and Ando T., *Means of positive linear operators*. Math. Ann., **246** (1980), 205–224.

[10] Lance E. C., *Hilbert C^*-Modules*. London Math. Soc. Lecture Note Series 210, Cambridge Univ. Press, 1995.

[11] Mercer A. Mc. D. and Mercer P. R., *New proofs of the Grüss inequality*. Aust. J. Math. Anal. Appl., **1 (2)** (2004), Art. 12, 6 pp.

[12] Mitrinović D. S., Pečarić J. E. and Fink A. M., *Classical and New Inequalities in Analysis*. Kluwer Academic, Dordrecht, 1993.

[13] Moslehian M. S. and Persson L.-E., *Reverse Cauchy–Schwarz inequalities for positive C^*-valued sesquilinear forms*. Math. Inequal. Appl., **4 (12)** (2009), 701–709.

[14] Niculescu C. P., *Converses of the Cauchy–Schwarz inequality in the C^*-framework*. An. Univ. Craiova Ser. Mat. Inform., **26** (1999), 22–28.

[15] Niezgoda M., *Accretive operators and Cassels inequality*. Linear Algebra Appl., **433 (1)** (2009), 136–142.

[16] Paschke W. L., *Inner product modules over B^*-algebras*. Trans. Amer. Math. Soc., **182** (1973), 443–468.

[17] Rieffel M. A., *Induced representations of C^*-algebras*. Advances in Math., **13** (1974), 176–257.

[18] Watson G. S., *Serial correlation in regression analysis I*. Biometrika, **42** (1955), 327–342.

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