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Two necessary and sufficient conditions for uniform domains

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Abstract. Let D be a proper subdomain of Euclidean n-space R^n $(n \geq 2)$. The following two necessary and sufficient conditions for uniform domains are obtained in this paper: (1) D is a uniform domain if and only if there exists a constant $m = m(D)$ such that $k_D(x_1, x_2) \leq m j_D(x_1, x_2)$ for any x_1, $x_2 \in D$, where k_D is the quasi-hyperbolic metric in D, $j_D(x_1, x_2) = (1/2) \log\left[\left(|x_1 - x_2|/d(x_1, \partial D) + 1\right)\left(|x_1 - x_2|/d(x_2, \partial D) + 1\right)\right]$. (2) D is a uniform domain if and only if there exists a constant $M = M(D)$ such that each pair of points x_1, $x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ which satisfies

$$\frac{1}{(c_2^\alpha - c_1^\alpha)} \int_{\gamma_{j,[c_1,c_2]}} d(x, \partial D)^{\alpha - 1}\, \mathrm{d}s \leq \frac{M}{\alpha} |x_1 - x_2|^\alpha$$

for any $0 < \alpha \leq 1$ and $0 \leq c_1 < c_2 \leq 1/2$, $j = 1, 2$, where $\gamma_{j,[c_1,c_2]}$ denotes the subarc between $\gamma_j(c_1 l(\gamma))$ and $\gamma_j(c_2 l(\gamma))$, γ_j is the arc γ which starts from x_j and use arc length s as parameter, $l(\gamma)$ is the Euclidean length of γ.

Key words: uniform domain, quasi-hyperbolic metric, rectifiable arc.

1. Introduction

We shall assume through this paper that D is a proper subdomain of Euclidean n-space R^n $(n \geq 2)$.

We say that D is a uniform domain if there exists a constant $a \geq 1$ such that each pair of points x_1, $x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which

$$\begin{cases} l(\gamma) \leq a|x_1 - x_2|, \\ \min\limits_{j=1,2} l(\gamma(x_j, x)) \leq a d(x, \partial D) \quad \text{for all} \quad x \in \gamma, \end{cases} \tag{1.1}$$

where $l(\gamma)$ denotes the Euclidean length of γ, $\gamma(x_j, x)$ is the part of γ between x_j and x, and $d(x, \partial D)$ the Euclidean distance from x to ∂D.

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Next for each x_1, $x_2 \in D$, we set

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} d(x, \partial D)^{-1}\, \mathrm{d}s, \tag{1.2}$$

where the infimum is taken over all rectifiable arcs γ joining x_1 and x_2 in D. We call k_D the quasi-hyperbolic metric in D [1]. From Lemma 2.1 in [1] it follows that

$$\begin{cases} \left|\log \dfrac{d(x_1, \partial D)}{d(x_2, \partial D)}\right| \le k_D(x_1, x_2), \\ \log\left(\dfrac{|x_1 - x_2|}{d(x_j, \partial D)} + 1\right) \le k_D(x_j, x), \qquad j = 1,\, 2, \end{cases} \tag{1.3}$$

for all x_1, $x_2 \in D$. Hence

$$j_D(x_1, x_2) \le k_D(x_1, x_2), \tag{1.4}$$

where

$$j_D(x_1, x_2) = \frac{1}{2} \log\left(\frac{|x_1 - x_2|}{d(x_1, \partial D)} + 1\right)\left(\frac{|x_1 - x_2|}{d(x_2, \partial D)} + 1\right).$$

Uniform domains were first introduced in [2] and [3] by O. Martio and J. Sarvas in connection with approximation and injectivity properties of functions defined in R^n. P. W. Jones studied in [4] the domains D for which there exist constants c and d such that

$$k_D(x_1, x_2) \le c j_D(x_1, x_2) + d, \tag{1.5}$$

for all x_1, $x_2 \in D$; it is precisely this class of domains D for which each function u with bounded mean oscillation in D has an extension v with bounded mean oscillation in R^n. F. W. Gehring and B. G. Osgood in [5] proved that a domain D is a uniform domain if and only if it satisfies (1.5) for some constants c and d. Hence the two classes of domains mentioned in the above paragraph are identical. When D is a unit ball, it is easy to verify that $k_D(x_1, x_2) \le 2j_D(x_1, x_2)$ for any x_1, $x_2 \in D$, so it is natural to ask whether the constant d would be zero in (1.5) when D is a uniform domain. In this paper we shall affirm and prove this conjecture and obtain the following Theorem 1.

Theorem 1 *D is a uniform domain if and only if there exists a constant $m = m(D)$ such that $k_D(x_1, x_2) \le m j_D(x_1, x_2)$ for all x_1, $x_2 \in D$.*

Uniform domains were studied and applied extensively in quasiconformal mappings theory and Heisenberg group theory [see 6–11]. In this paper, we also obtain the following Therorem 2.

Theorem 2 *D is a uniform domain if and only if there exists a constant $M = M(D)$ such that each pair of points x_1, $x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ which satisfies*

$$\frac{1}{c_2^\alpha - c_1^\alpha}\int_{\gamma_{j,[c_1,c_2]}} d(x,\partial D)^{\alpha-1}\,\mathrm{d}s \le \frac{M}{\alpha}|x_1 - x_2|^\alpha$$

for any $0 < \alpha \le 1$ and $0 \le c_1 < c_2 \le 1/2$, $j = 1, 2$, where $\gamma_{j,[c_1,c_2]}$ denotes the subarc between $\gamma_j(c_1 l(\gamma))$ and $\gamma_j(c_2 l(\gamma))$, γ_j is the arc γ which starts from x_j and use arc length s as parameter, $l(\gamma)$ is the Euclidean length of γ.

2. Proof of Theorem 1 and Theorem 2

To prove Theorem 1, we shall first give the following two lemmas.

In [12], Anderson, Vamanamurthy and Vuorinen proved the following Lemma 1.

Lemma 1 *If D is a proper subdomain of R^n, then*

$$k_D(x_1, x_2) \le \log\left(\frac{|x_1 - x_2|}{d(x_1,\partial D) - |x_1 - x_2|} + 1\right) \tag{2.1}$$

for $|x_1 - x_2| < d(x_1, \partial D)$.

Lemma 2 $\log(1 + tx) \le t\log(1 + x)$ *for all $t \ge 1$ and $x \ge 0$.*

Proof of Lemma 2. If let $f(x) = t\log(1 + x) - \log(1 + tx)$, $t \ge 1$, $x \ge 0$, then $f'(x) = \frac{tx(t-1)}{(1+x)(1+tx)} \ge 0$. Hence $f(x)$ is a monotone increasing function in $[0,\infty)$ for $t \ge 1$, $f(x) \ge f(0) = 0$, $\log(1 + tx) \le t\log(1 + x)$. □

Next we can prove our Theorem 1 and Theorem 2 by using Lemma 1 and Lemma 2.

Proof Theorem 1. **The sufficiency.** If there exists a constant $m = m(D)$ such that $k_D(x_1, x_2) \le m j_D(x_1, x_2)$ for all x_1, $x_2 \in D$, then D must be a uniform domain by [5].

The necessity. If D is a uniform domain, then there exists a constant $a \ge 1$ such that each pair of points x_1, $x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which (1.1) holds.

Next we consider the following two cases to prove the necessity.

(1) If

$$\min_{j=1,2}\left\{\frac{|x_1-x_2|}{d(x_1,\partial D)},\frac{|x_1-x_2|}{d(x_2,\partial D)}\right\}<\frac{1}{2a},$$

without loss of generality, we may assume that $|x_1-x_2|/d(x_1,\partial D)<1/2a$ $|x_1-x_2|<d(x_1,\partial D)$. Then since $a\geq 1$, using Lemma 1 and Lemma 2 we can obtain

$$\begin{aligned} k_D(x_1,x_2) &\leq \log\left(1+\frac{|x_1-x_2|}{d(x_1,\partial D)-|x_1-x_2|}\right)\\ &\leq \log\left(1+\frac{2a}{2a-1}\frac{|x_1-x_2|}{d(x_1,\partial D)}\right)\\ &\leq \frac{2a}{2a-1}\log\left(1+\frac{|x_1-x_2|}{d(x_1,\partial D)}\right)\\ &\leq \frac{4a}{2a-1}\frac{1}{2}\log\left(1+\frac{|x_1-x_2|}{d(x_1,\partial D)}\right)\left(1+\frac{|x_1-x_2|}{d(x_2,\partial D)}\right)\\ &= \frac{4a}{2a-1}j_D(x_1,x_2)<8a(a+1)j_D(x_1,x_2). \end{aligned} \tag{2.2}$$

(2) If

$$\min_{j=1,2}\left\{\frac{|x_1-x_2|}{d(x_1,\partial D)},\frac{|x_1-x_2|}{d(x_2,\partial D)}\right\}\geq\frac{1}{2a},$$

then choose $x_0\in\gamma$ such that $l(\gamma(x_1,x_0))=l(\gamma(x_2,x_0))$.

(i) If $l(\gamma(x_1,x_0))\leq\big(a/(a+1)\big)d(x,\partial D)$, then for any $x\in\gamma(x_1,x_0)$ we have

$$\begin{aligned} d(x,\partial D) &\geq d(x_1,\partial D)-l(\gamma(x_1,x))\\ &\geq d(x_1,\partial D)-l(\gamma(x_1,x_0))\\ &\geq \frac{1}{a+1}\,d(x,\partial D). \end{aligned} \tag{2.3}$$

Hence we can obtain

$$\begin{aligned} k_D(x_1,x_0) &\leq \int_{\gamma(x_1,x_0)} d(x_1,\partial D)^{-1}\,\mathrm{d}s \leq (a+1)\frac{l(\gamma(x_1,x_0))}{d(x_1,\partial D)}\leq a\\ &= a\,\frac{\log\big(1+1/(2a)\big)}{\log\big(1+1/(2a)\big)}\leq\frac{a}{\log\big(1+1/(2a)\big)}\log\left(1+\frac{|x_1-x_2|}{d(x_1,\partial D)}\right) \end{aligned} \tag{2.4}$$

by (1.2), (2.3) and the above hypotheses.

(ii) If $l(\gamma(x_1,x_0)) \geq \big(a/(a+1)\big)d(x,\partial D)$, then choose $y_1 \in \gamma(x_1,x_0)$ such that $l(\gamma(x_1,y_1)) = \big(a/(a+1)\big)d(x_1,\partial D)$, we can prove that

$$k_D(x_1,y_1) \leq \frac{a}{\log\big(1+1/(2a)\big)} \log\left(1+\frac{|x_1-x_2|}{d(x_1,\partial D)}\right) \tag{2.5}$$

as in (2.4).

If let $x \in \gamma(y_1,x_0)$, then

$$d(x,\partial D) \geq \frac{1}{a}\, l(\gamma(x_1,x)) \tag{2.6}$$

by (1.1).

Hence

$$\begin{aligned} k_D(y_1,x_0) &\leq \int_{\gamma(y_1,x_0)} d(x,\partial D)^{-1}\,\mathrm{d}s \leq a \int_{\gamma(y_1,x_0)} \frac{\mathrm{d}s}{l(\gamma(x_1,x))} \\ &= a\log\frac{l(\gamma(x_1,x_0))}{l(\gamma(x_1,y_1))} = a\log\left[\frac{a+1}{a}\frac{l(\gamma(x_1,x_0))}{d(x_1,\partial D)}\right] \\ &\leq a\log\left(\frac{a+1}{a}\frac{a|x_1-x_2|}{d(x_1,\partial D)}+1\right) \\ &\leq a(a+1)\log\left(1+\frac{|x_1-x_2|}{d(x_1,\partial D)}\right) \end{aligned} \tag{2.7}$$

by (1.2), (2.6), (1.1) and Lemma 2.

Using triangle inequality, (2.6) and (2.7) we have

$$\begin{aligned} k_D(x_1,x_0) &\leq k_D(x_1,y_1) + k_D(y_1,x_0) \\ &\leq \left[\frac{a}{\log\big(1+1/(2a)\big)} + a(a+1)\right] \log\left(1+\frac{|x_1-x_2|}{d(x_1,\partial D)}\right) \\ &< 4a(a+1)\log\left(1+\frac{|x_1-x_2|}{d(x_1,\partial D)}\right). \end{aligned} \tag{2.8}$$

By using the same method we can obtain

$$k_D(x_2,x_0) < 4a(a+1)\log\left(1+\frac{|x_1-x_2|}{d(x_2,\partial D)}\right). \tag{2.9}$$

Consequently

$$k_D(x_1,x_2) \leq k_D(x_1,x_0) + k_D(x_2,x_0)$$

$$
\begin{aligned}
&< 4a(a+1)\log\left(1+\frac{|x_1-x_2|}{d(x_1,\partial D)}\right)\left(1+\frac{|x_1-x_2|}{d(x_2,\partial D)}\right)\\
&= 8a(a+1)j_D(x_1,x_2) \qquad (2.10)
\end{aligned}
$$

by the triangle inequality, (2.8) and (2.9).

At last, combining the two cases of (1) and (2) we conclude that

$$
k_D(x_1,x_2) \le m j_D(x_1,x_2),
$$

where $m = m(D) = 8a(a+1)$, this complete the proof of the necessity. □

Proof of Theorem 2. **The necessity.** If D is a uniform domain, then there exists a constant $a \ge 1$ such that each pair of points $x_1,\ x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ for which (1.1) holds.

Next for any $0 < \alpha \le 1$ and $0 \le c_1 < c_2 \le 1/2$, we have

$$
\begin{aligned}
\int_{\gamma_{j,[c_1,c_2]}} d(x,\partial D)^{\alpha-1}\,\mathrm{d}s &\le a^{1-\alpha}\int_{\gamma_{j,[c_1,c_2]}} l(\gamma_j(x_j,x))^{\alpha-1}\,\mathrm{d}s\\
&= a^{1-\alpha}\int_{c_1 l(\gamma)}^{c_2 l(\gamma)} s^{\alpha-1}\,\mathrm{d}s\\
&= \frac{a^{1-\alpha}(c_2^\alpha - c_1^\alpha)}{\alpha}\, l(\gamma)^\alpha\\
&\le \frac{a}{\alpha}(c_2^\alpha - c_1^\alpha)|x_2-x_1|^\alpha \qquad (2.11)
\end{aligned}
$$

by (1.1).

This implies that

$$
\frac{1}{c_2^\alpha - c_1^\alpha}\int_{\gamma_{j,[c_1,c_2]}} d(x,\partial D)^{\alpha-1}\,\mathrm{d}s \le \frac{M}{\alpha}|x_1-x_2|^\alpha, \qquad (2.12)
$$

where $M = a$.

The sufficiency. If there exists a constant $M = M(D)$ such that each pair of points $x_1,\ x_2 \in D$ can be joined by a rectifiable arc $\gamma \subset D$ which satisfies (2.12) for any $0 < \alpha \le 1$ and $0 \le c_1 < c_2 \le 1/2$. Then we first take $\alpha = 1$, $c_1 = 0$ and $c_2 = 1/2$ in (2.12) and obtain

$$
l(\gamma) = 2\int_{\gamma_{j,[0,1/2]}} \mathrm{d}s \le M|x_1-x_2|. \qquad (2.13)
$$

Next by using (2.12) again we get

$$\begin{aligned}\frac{1}{c_2^\alpha - c_1^\alpha}\int_{\gamma_{j,[c_1,c_2]}} d(x,\partial D)^{\alpha-1}\,\mathrm{d}s &\le \frac{M}{\alpha}|x_1 - x_2|^\alpha \\ &\le \frac{M}{\alpha}\, l(\gamma)^\alpha \\ &= \frac{M}{c_2^\alpha - c_1^\alpha}\int_{\gamma_{j,[c_1,c_2]}} l(\gamma(x_j,x))^{\alpha-1}\,\mathrm{d}s.\end{aligned} \tag{2.14}$$

We conclude that

$$d(x,\partial D)^{\alpha-1} \le Ml(\gamma(x_j,x))^{\alpha-1}, \qquad x_j \in \gamma_j \tag{2.15}$$

by the continuity of $d(x,\partial D)$ and $l(\gamma(x_j,x))$ and the arbitrariness of $0 \le c_1 < c_2 \le 1/2$.

Let $\alpha \to 0$ in (2.15), we have

$$\begin{aligned}&l(\gamma(x_j,x)) \le Md(x,\partial D), \qquad x \in \gamma_j,\\ &\min\nolimits_{j=1,2} l(\gamma(x_j,x)) \le Md(x,\partial D).\end{aligned} \tag{2.16}$$

D is a uniform domain by (2.13) and (2.16), this complete the proof of the sufficiency. □

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