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# On the fine spectrum of the generalized difference operator $B(r, s)$ over the sequence spaces $\ell_1$ and $bv$

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**Abstract.** The main purpose of this paper is to determine the fine spectrum of the operator $B(r, s)$ defined by a band matrix over the sequence spaces $\ell_1$ and $bv$. Since the operator $B(r, s)$ is reduced to the right-shift, difference and Zweier matrices in special cases $(r, s) = (0, 1)$, $(r, s) = (1, -1)$ and $(r, s) = (r, 1 - r)$, respectively, our results are more general than the corresponding results in the existing literature.

*Key words*: spectrum of an operator, generalized difference operator and the sequence spaces $\ell_1$ and $bv$.

## 1. Preliminaries, background and notation

Let $X$ and $Y$ be the Banach spaces and $T\colon X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of $T$, i.e.,

$$R(T) = \{y \in Y : y = Tx,\ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on $X$ into itself. If $X$ is any Banach space and $T \in B(X)$ then the adjoint $T^*$ of $T$ is a bounded linear operator on the dual $X^*$ of $X$ defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a complex normed space and $T\colon \mathcal{D}(T) \to X$ also be a linear operator with domain $\mathcal{D}(T) \subset X$. With $T$, we associate the operator

$$T_\alpha = T - \alpha I,$$

where $\alpha$ is a complex number and $I$ is the identity operator on $\mathcal{D}(T)$. If $T_\alpha$ has an inverse, which is linear, we denote it by $T_\alpha^{-1}$, that is

$$T_\alpha^{-1} = (T - \alpha I)^{-1}$$

and call it the *resolvent operator* of $T$.

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The name *resolvent* is appropriate, since $T_\alpha^{-1}$ helps to solve the equation $T_\alpha x = y$. Thus, $x = T_\alpha^{-1} y$ provided $T_\alpha^{-1}$ exists. More important, the investigation of properties of $T_\alpha^{-1}$ will be basic for an understanding of the operator $T$ itself. Naturally, many properties of $T_\alpha$ and $T_\alpha^{-1}$ depend on $\alpha$, and spectral theory is concerned with those properties. For instance, we shall be interested in the set of all $\alpha$ in the complex plane such that $T_\alpha^{-1}$ exists. Boundedness of $T_\alpha^{-1}$ is another property that will be essential. We shall also ask for what $\alpha$'s the domain of $T_\alpha^{-1}$ is dense in $X$, to name just a few aspects. For our investigation of $T$, $T_\alpha$ and $T_\alpha^{-1}$, we need some basic concepts in spectral theory which are given as follows (see [6, pp. 370–371]):

The *resolvent set* $\rho(T)$ of $T$ is the set of all regular values $\alpha$ of $T$. Its complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ in the complex plane $\mathbb{C}$ is called the *spectrum* of $T$. Furthermore, the spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows:

The *point* (*discrete*) *spectrum* $\sigma_p(T)$ is the set such that $T_\alpha^{-1}$ does not exist. An $\alpha \in \sigma_p(T)$ is called an *eigenvalue* of $T$.

The *continuous spectrum* $\sigma_c(T)$ is the set such that $T_\alpha^{-1}$ exists and is defined on a set which is dense in $X$ but $T_\alpha^{-1}$ is unbounded.

The *residual spectrum* $\sigma_r(T)$ is the set such that $T_\alpha^{-1}$ exists (and may be bounded or not) but the domain of $T_\alpha^{-1}$ is not dense in $X$.

To avoid trivial misunderstandings, let us say that some of the sets defined above, may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T) = \sigma_r(T) = \emptyset$ and the spectrum $\sigma(T)$ consists of only the set $\sigma_p(T)$ in the finite dimensional case.

From Goldberg [4, pp. 58–71], if $X$ is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and $T^{-1}$:

**(I)** $R(T) = X$,
**(II)** $R(T) \neq \overline{R(T)} = X$,
**(III)** $\overline{R(T)} \neq X$

and

**(1)** $T^{-1}$ exists and is continuous,
**(2)** $T^{-1}$ exists but is discontinuous,
**(3)** $T^{-1}$ does not exist.

Applying Goldberg's classification to $T_\alpha = T - \alpha I$, where $\alpha$ is a complex number we have three possibilities for $T_\alpha$ and $T_\alpha^{-1}$;

**(I)** $T_\alpha$ is surjective,
**(II)** $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,

**(III)** $\overline{R(T_\alpha)} \neq X$

and

**(1)** $T_\alpha$ is injective and $T_\alpha^{-1}$ is continuous,

**(2)** $T_\alpha$ is injective and $T_\alpha^{-1}$ is discontinuous,

**(3)** $T_\alpha$ is not injective.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $I_1$, $I_2$, $I_3$, $II_1$, $II_2$, $II_3$, $III_1$, $III_2$ and $III_3$. If $\alpha$ is a complex number such that $T_\alpha \in I_1$ or $T_\alpha \in II_1$ then $\alpha$ is in the resolvent set $\rho(T, X)$ of $T$ is the set of all regular values $\alpha$ of $T$ on $X$. The other classification gives rise to the fine spectrum of $T$. If an operator is in state $II_2$ for example, then $R(T) \neq \overline{R(T)} = X$ and $T^{-1}$ exists but is discontinuous and we write $\alpha \in II_2\sigma(T, X)$.

By $w$, we shall denote the space of all real valued sequences. Any vector subspace of $w$ is called as a *sequence space*. We shall write $\ell_\infty$, $c$, $c_0$ and $bv$ for the spaces of all bounded, convergent, null and bounded variation sequences, respectively. Also by $\ell_1$ and $bs$, we denote the space of all absolutely convergent and bounded series, respectively.

Let $\lambda$ and $\mu$ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers $a_{nk}$, where $n,\ k \in \mathbb{N} = \{0,\ 1,\ 2, \ldots\}$. Then, we say that $A$ defines a matrix mapping from $\lambda$ into $\mu$, and we denote it by writing $A\colon \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the $A$-transform of $x$, is in $\mu$; where

$$(Ax)_n = \sum_k a_{nk} x_k, \qquad (n \in \mathbb{N}). \tag{1.1}$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to $\infty$. By $(\lambda, \mu)$, we denote the class of all matrices $A$ such that $A\colon \lambda \to \mu$. Thus, $A \in (\lambda, \mu)$ if and only if the series on the right side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \mu$ for all $x \in \lambda$.

Let us consider the generalized difference operator $B(r, s)$ represented by the band matrix with $r, s \in \mathbb{R}$ and $s \neq 0$ that

$$B(r, s) = \begin{bmatrix} r & 0 & 0 & \ldots \\ s & r & 0 & \ldots \\ 0 & s & r & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Now, we may give:

**Lemma 1.1** *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from $\ell_1$ to itself if and only if the supremum of $\ell_1$ norms of the columns of $A$ is bounded.*

**Corollary 1.2** $B(r,s)\colon \ell_1 \to \ell_1$ *is a bounded linear operator and* $\|B(r,s)\|_{(\ell_1,\ell_1)} = |r| + |s|$.

**Lemma 1.3** *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(bv)$ from $bv$ to itself if and only if*

$$\sup_l \sum_n \left| \sum_{k=l}^{\infty} a_{nk} - a_{n-1,k} \right| < \infty.$$

**Corollary 1.4** $B(r,s)\colon bv \to bv$ *is a bounded linear operator and* $\|B(r,s)\|_{(\ell_1,\ell_1)} = \|B(r,s)\|_{(bv,bv)}$.

The fine spectrum of the Cesàro operator on the sequence space $\ell_p$ has been studied by Gonzàlez [5], where $1 < p < \infty$. Also, weighted mean matrices of operators on $\ell_p$ have been investigated by Cartlidge [3]. The spectrum of the Cesàro operator on the sequence spaces $bv_0$ and $bv$ have also been investigated by Okutoyi [7] and Okutoyi [8], respectively. More recently, the fine spectra of the difference operator $\Delta$ over the sequence space $\ell_p$ has been determined by Akhmedov and Başar [1], where $1 \le p < \infty$. In this work, our purpose is to determine the fine spectrum of the generalized difference operator $B(r,s)$ on the sequence spaces $\ell_1$ and $bv$, which is a natural continuation of Altay and Başar [2].

## 2. The spectrum of the operator $B(r,s)$ on the sequence spaces $\ell_1$ and $bv$

In this section, we compute the spectrum, the point spectrum, the continuous spectrum and residual spectrum of the operator $B(r,s)$ on the sequence spaces $\ell_1$ and $bv$.

**Theorem 2.1** $\sigma(B(r,s),\ell_1) = \{\alpha \in \mathbb{C} : |\alpha - r| \le |s|\}$.

*Proof.* First, we prove that $(B(r,s) - \alpha I)^{-1}$ exists and in $(\ell_1 : \ell_1)$ for $\alpha \notin \{\alpha \in \mathbb{C} : |\alpha - r| \le |s|\}$ and nextly the operator $(B(r,s) - \alpha I)$ is not invertible for $\alpha \in \{\alpha \in \mathbb{C} : |\alpha - r| \le |s|\}$.

Let $\alpha \notin \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$, i.e. we have $|\alpha - r| > |s|$. Since $s \neq 0$ we have $r \neq \alpha$, therefore $B(r,s) - \alpha I$ is triangle and has an inverse. Solving the equation $(B(r,s) - \alpha I)x = y$ for $x$ in terms of $y$ gives the matrix of $(B(r,s) - \alpha I)^{-1}$. The $n^{th}$ row turns out to be

$$\frac{(-s)^{n-k}}{(r-\alpha)^{n-k+1}}$$

in the $k^{th}$ place for $k \leq n$ and zero otherwise.

$$(B(r,s) - \alpha I)^{-1} = \begin{bmatrix} \frac{1}{(r-\alpha)} & 0 & 0 & \cdots \\ \frac{-s}{(r-\alpha)^2} & \frac{1}{(r-\alpha)} & 0 & \cdots \\ \frac{s^2}{(r-\alpha)^3} & \frac{-s}{(r-\alpha)^2} & \frac{1}{(r-\alpha)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Thus, we observe that

$$\begin{aligned} \|(B(r,s) - \alpha I)^{-1}\|_{(\ell_1,\ell_1)} &= \sup_k \sum_{n=k}^{\infty} \left|\frac{s}{r-\alpha}\right|^{n-k} \left|\frac{1}{r-\alpha}\right| \\ &= \left|\frac{1}{r-\alpha}\right| \sum_{n=0}^{\infty} \left|\frac{s}{r-\alpha}\right|^{n} < \infty, \end{aligned} \tag{2.1}$$

i.e., $(B(r,s) - \alpha I)^{-1} \in (\ell_1, \ell_1)$. This shows that $\sigma(B(r,s), \ell_1) \subseteq \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$.

Now let $\alpha \in \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$ and $\alpha \neq r$. Since $B(r,s) - \alpha I$ is triangle, $(B(r,s) - \alpha I)^{-1}$ exists but one can see by (2.1) that

$$\|(B(r,s) - \alpha I)^{-1}\|_{(\ell_1,\ell_1)} = \infty$$

whenever $\alpha \in \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$, i.e., $(B(r,s) - \alpha I)^{-1}$ is not in $B(\ell_1)$. If $\alpha = r$, then the operator $B(r,s) - \alpha I = B(0,s)$ is represented by the matrix

$$B(0,s) = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ s & 0 & 0 & \cdots \\ 0 & s & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since $B(0,s)x = \theta$ implies $x = \theta$, $B(0,s)\colon \ell_1 \to \ell_1$ is injective but not onto. Hence, $B(0,s)$ has not a right inverse, i.e., $B(0,s)$ is not invertible. This shows that $\{\alpha \in \mathbb{C} : |\alpha - r| \le |s|\} \subseteq \sigma(B(r,s), \ell_1)$.

This completes the proof. □

**Theorem 2.2** $\sigma_p(B(r,s), \ell_1) = \emptyset$.

*Proof.* Suppose $B(r,s)x = \alpha x$ for $x \neq \theta = (0,0,0,\dots)$ in $\ell_1$. Then by solving the system of linear equations

$$\left.\begin{aligned} rx_0 &= \alpha x_0 \\ sx_0 + rx_1 &= \alpha x_1 \\ sx_1 + rx_2 &= \alpha x_2 \\ &\vdots \end{aligned}\right\}$$

we find that if $x_t$ is the first non-zero entry of the sequence $x = (x_n)$, then $\alpha = r$ and from the equation

$$sx_t + rx_{t+1} = \alpha x_{t+1}$$

we get $sx_t = 0$. Since $s \neq 0$ we must have $x_t = 0$, contradicting the fact that $x_t \neq 0$. This completes the proof. □

If $T\colon \ell_1 \to \ell_1$ is a bounded linear operator with matrix $A$, then it is known that the adjoint operator $T^*\colon \ell_1^* \to \ell_1^*$ is defined by the transpose of the matrix $A$. The dual space of $\ell_1$ is isomorphic to $\ell_\infty$, the space of bounded sequences, with the norm $\|x\| = \sup_k |x_k|$.

**Theorem 2.3** $\sigma_p(B(r,s)^*, \ell_1^*) = \{\alpha \in \mathbb{C} : |\alpha - r| \le |s|\}$.

*Proof.* Suppose $B(r,s)^*x = \alpha x$ for $x \neq \theta \in \ell_1^* \cong \ell_\infty$. Then by solving the system of linear equations

$$\left.\begin{aligned} rx_0 + sx_1 &= \alpha x_0 \\ rx_1 + sx_2 &= \alpha x_1 \\ rx_2 + sx_3 &= \alpha x_2 \\ &\vdots \end{aligned}\right\}$$

we obtain that

$$x_n = \left(\frac{\alpha - r}{s}\right)^n x_0.$$

This shows that $x = (x_k) \in \ell_1^*$ if and only if $|\alpha - r| \leq |s|$. This completes the proof. $\square$

Now we give the following lemma required in the proof of next theorem.

**Lemma 2.4** ([4, p. 59]) *$T$ has a dense range if and only if $T^*$ is one to one, where $T^*$ denotes the adjoint operator of the operator $T$.*

**Lemma 2.5** ([4, p. 60]) *The adjoint operator $T^*$ of $T$ is onto if and only if $T$ has a bounded inverse.*

**Theorem 2.6** $\sigma_r(B(r,s), \ell_1) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$.

*Proof.* We show that the operator $B(r,s) - \alpha I$ has an inverse and $\overline{R(B(r,s) - \alpha I)} \neq \ell_1$ for $\alpha$ satisfying $|\alpha - r| \leq |s|$. For $\alpha \neq r$ the operator $B(r,s) - \alpha I$ is triangle and has an inverse. For $\alpha = r$, the operator $B(r,s) - \alpha I$ is one to one hence has at least two left inverse. But $B(r,s)^* - \alpha I$ is not one to one by Theorem 2.3. Now Lemma 2.4 yields the fact that the range of the operator $B(r,s) - \alpha I$ is not dense in $\ell_1$ and this step completes the proof. $\square$

**Theorem 2.7** *If $\alpha = r$, then $\alpha \in III_1\sigma(B(r,s), \ell_1)$.*

*Proof.* By Theorem 2.3 and Lemma 2.4, $B(r,s) - \alpha I \in III$ whenever $\alpha = r$. On the other hand, $\alpha = r$ is not in $\sigma_p(B(r,s), \ell_1)$ by Theorem 2.2. Hence $B(r,s) - rI$ has an inverse. Then, $B(r,s) - rI \in 1 \cup 2$.

To show that $B(r,s) - rI \in 1$, it is enough to show by Lemma 2.5 that $B(r,s)^* - rI = B(0,s)^*$ is onto. Given $y = (y_n) \in \ell_\infty$ we must find $x = (x_n) \in \ell_\infty$ such that $B(0,s)^* x = y$. Then, a direct calculation gives that

$$x_n = \frac{1}{s}\, y_{n-1}.$$

This shows that $B(0,s)^*$ is onto. $\square$

**Theorem 2.8** *If $\alpha \neq r$ and $\alpha \in \sigma_r(B(r,s), \ell_1)$, then $\alpha \in III_2\sigma(B(r,s), \ell_1)$.*

*Proof.* Since $\alpha \neq r$, the operator $B(r,s) - \alpha I$ is triangle, hence it has an inverse. By (2.1), the inverse of the operator $B(r,s) - \alpha I$ is discontinuous. Therefore, $B(r,s) - \alpha I \in 2$.

By Theorem 2.3, $B(r,s)^* - \alpha I$ is not one to one. By Lemma 2.4 $B(r,s) - \alpha I$ does not have a dense range. Therefore $B(r,s) - \alpha I \in III$. This completes the proof. $\square$

**Theorem 2.9** $\sigma_c(B(r,s),\ell_1)=\emptyset$.

*Proof.* Since $\sigma_p(B(r,s),\ell_1)=\emptyset$ and $\sigma(B(r,s),\ell_1)$ is the disjoint union of the parts $\sigma_p(B(r,s),\ell_1)$, $\sigma_r(B(r,s),\ell_1)$ and $\sigma_c(B(r,s),\ell_1)$ we must have $\sigma_c(B(r,s),\ell_1)=\emptyset$. □

**Theorem 2.10** *If $\alpha$ satisfies $|\alpha-r|>|s|$, then $B(r,s)-\alpha I\in I_1$.*

*Proof.* We prove that the operator $B(r,s)-\alpha I$ is bijective and has a continuous inverse for $\alpha$ with $|\alpha-r|>|s|$. Since $s\neq 0$, $\alpha\neq r$ and $B(r,s)-\alpha I$ is triangle, hence it has an inverse. By (2.1), the inverse of $B(r,s)-\alpha I$ exists and is continuous.

Let $y\in\ell_1$ and define $x=(x_n)$ by

$$x_n=\frac{1}{r-\alpha}\sum_{k=0}^{n}\Big(\frac{-s}{r-\alpha}\Big)^{n-k}y_k=\big((B(r,s)-\alpha I)^{-1}y\big)_n$$

for all $n\in\mathbb{N}$. Since $(B(r,s)-\alpha I)^{-1}$ is in $(\ell_1,\ell_1)$, we observe by (2.1) that $x\in\ell_1$. Hence, $B(r,s)-\alpha I$ is onto. □

Now we may investigate the spectrum of $B(r,s)$ over the sequence space $bv$.

**Theorem 2.11** $\sigma(B(r,s),bv)=\{\alpha\in\mathbb{C}:|\alpha-r|\le|s|\}$.

*Proof.* Let $\alpha\notin\{\alpha\in\mathbb{C}:|\alpha-r|\le|s|\}$, i.e. we have $|\alpha-r|>|s|$. Since $s\neq 0$ we have $r\neq\alpha$, therefore $B(r,s)-\alpha I$ is triangle and has an inverse. Solving the equation

$$(B(r,s)-\alpha I)x=y \tag{2.2}$$

for $x$ in terms of $y\in bv$, we can formally derive the matrix in the proof of Theorem 2.1.

At this situation,

$$\|(B(r,s)-\alpha I)^{-1}\|_{(bv,bv)}=\left|\frac{1}{r-\alpha}\right|\sum_n\left|\frac{s}{r-\alpha}\right|^n<\infty, \tag{2.3}$$

since $|s|<|r-\alpha|$. This shows that $\sigma(B(r,s),bv)\subseteq\{\alpha\in\mathbb{C}:|\alpha-r|\le|s|\}$.

Now let $\alpha\in\{\alpha\in\mathbb{C}:|\alpha-r|\le|s|\}$. If $\alpha\neq r$, then $B(r,s)-\alpha I$ is triangle and has an inverse. On the other hand we have from (2.3) that

$$\|(B(r,s)-\alpha I)^{-1}\|_{(bv,bv)}=\infty.$$

i.e. $(B(r,s)-\alpha I)^{-1}$ is not in $B(bv)$.

If $\alpha = r$ then similar arguments in the proof of Theorem 2.1 show that the operator $B(r,s)-\alpha I = B(0,s)$ is not invertible. This shows that $\{\alpha \in \mathbb{C} : |\alpha - r| \le |s|\} \subseteq \sigma(B(r,s), bv)$.

This completes the proof. □

**Theorem 2.12** $\sigma_p(B(r,s), bv) = \emptyset$.

*Proof.* The proof is similar to that of Theorem 2.2. □

If $T\colon bv \to bv$ is a bounded matrix operator with matrix $A = (a_{nk})$, then $T^*\colon bv^* \to bv^*$ acting on $\mathbb{C} \oplus bs$ has matrix of the form

$$\begin{bmatrix} \overline{\chi} & v_0 - \overline{\chi} & v_1 - \overline{\chi} & v_2 - \overline{\chi} & \dots \\ a_0 & a_{00} - a_0 & a_{10} - a_0 & a_{20} - a_0 & \dots \\ a_1 & a_{01} - a_1 & a_{11} - a_1 & a_{21} - a_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix};$$

where

$$\overline{\chi} = \lim_{n\to\infty} \sum_{v=0}^{\infty} a_{nv}, \qquad a_n = \lim_{k\to\infty} a_{kn}, \qquad v_k = P_k(T(\delta)),$$

where $\delta = (1,1,1,\dots)$ and $P_k$ is the $k^{th}$ coordinate function for each $k \in \mathbb{N}$ [8]. For $B(r,s)\colon bv \to bv$, the matrix $B(r,s)^* \in B(\mathbb{C} \oplus bs)$ is in the form,

$$B(r,s)^* = \begin{bmatrix} r+s & -s & 0 & 0 & 0 & \dots \\ 0 & r & s & 0 & 0 & \dots \\ 0 & 0 & r & s & 0 & \dots \\ 0 & 0 & 0 & r & s & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

**Theorem 2.13** $\sigma_p(B(r,s)^*, bv^*) = \{\alpha \in \mathbb{C} : |\alpha - r| \le |s|\}$.

*Proof.* Suppose $B(r,s)^* x = \alpha x$ for $x \ne \theta \in bv^* = \mathbb{C} \oplus bs$. Then by solving the system of linear equations

$$\left.\begin{aligned} (r+s)x_0 - sx_1 &= \alpha x_0 \\ rx_1 + sx_2 &= \alpha x_1 \\ rx_2 + sx_3 &= \alpha x_2 \\ &\vdots \end{aligned}\right\} \tag{2.4}$$

we obtain that

$$x_n = \Big(\frac{\alpha - r}{s}\Big)^{n-1}\Big(\frac{r-\alpha}{s}+1\Big)x_0; \qquad (n \geq 1). \tag{2.5}$$

If $(\alpha - r)/s = 1$, i.e. $\alpha = r + s$, then since $s \neq 0, x_1 = 0,\ x_2 = 0, \ldots,$ $x_n = 0, \ldots.$ so $x = (x_0, 0, 0, \ldots)$ is an eigenvector corresponding to $\alpha = r + s$. If $(\alpha - r)/s \neq 1$, then $x \in \mathbb{C} \oplus bs$ if and only if

$$\Big|\frac{r-\alpha}{s}+1\Big| \sup_n \frac{1 - \Big(\dfrac{\alpha - r}{s}\Big)^n}{1 - \Big(\dfrac{\alpha - r}{s}\Big)} < \infty$$

which leads us to the desired consequence that $|\alpha - r| \leq |s|$. □

**Theorem 2.14** $\sigma_r(B(r,s), bv) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$.

*Proof.* The proof obtained by the analogy with the proof of Theorem 2.6. □

**Theorem 2.15** $\sigma_c(B(r,s), bv) = \emptyset$.

*Proof.* The proof may be obtained by proceeding as in proving Theorem 2.9. □

In this section, we compute the fine spectrum of the operator $B(r,s)$ on the sequence spaces $\ell_1$ and $bv$. We start with a theorem determining the class of the resolvent set of the operator $B(r,s)$.

Now, a similar argument can be carried out over the space $bv$.

**Theorem 2.16** *If $\alpha$ satisfies $|\alpha - r| > |s|$, then $B(r,s) - \alpha I \in I_1$.*

*Proof.* We prove that the operator $B(r,s) - \alpha I$ is bijective and has a continuous inverse for $\alpha$ with $|\alpha - r| > |s|$. Since $s \neq 0$, $\alpha \neq r$ and $B(r,s) - \alpha I$ is triangle, it has an inverse. By (2.3), the inverse of $B(r,s) - \alpha I$ exists and is continuous.

Let $y \in bv$ and define $x = (x_n)$ by

$$x_n = \frac{1}{r-\alpha}\sum_{k=0}^{n}\Big(\frac{-s}{r-\alpha}\Big)^{n-k} y_k = \big((B(r,s) - \alpha I)^{-1} y\big)_n$$

for all $n \in \mathbb{N}$. Since $(B(r,s) - \alpha I)^{-1}$ is in $(bv, bv)$, we observe by (2.3) that $x \in bv$. Hence, $B(r,s) - \alpha I$ is onto. □

**Theorem 2.17** *If $\alpha = r$, then $\alpha \in III_1\sigma(B(r,s), bv)$.*

*Proof.* By Theorem 2.13 and Lemma 2.4, $B(r,s) - rI \in III$ whenever $\alpha = r$. On the other hand, $\alpha = r$ is not in $\sigma_p(B(r,s), bv)$ by Theorem 2.12. Hence $B(r,s) - rI$ has an inverse. Then, $B(r,s) - rI \in 1 \cup 2$.

To show that $B(r,s) - rI \in 1$, it is enough to show by Lemma 2.5 that $B(r,s)^* - rI = B(0,s)^*$ is onto. Given $y = (y_n) \in \mathbb{C} \oplus bs$ we must find $x = (x_n) \in \mathbb{C} \oplus bs$ such that $B(0,s)^* x = y$. Direct calculation gives that $x_1$ can be taken as zero, then $x_0 = (1/s)y_0$ and

$$x_n = \frac{1}{s}\, y_{n-1} \qquad (n \geq 2).$$

This shows that $B(0,s)^*$ is onto. □

**Theorem 2.18** *If $\alpha \neq r$ and $\alpha \in \sigma_r(B(r,s), bv)$, then $\alpha \in III_2\sigma(B(r,s), bv)$.*

*Proof.* Since $\alpha \neq r$, the operator $B(r,s) - \alpha I$ is triangle, hence it has an inverse. By (2.3), the inverse of the operator $B(r,s) - \alpha I$ is discontinuous. Therefore, $B(r,s) - \alpha I \in 2$.

By Theorem 2.13, $B(r,s)^* - \alpha I$ is not one to one. By Lemma 2.4 $B(r,s) - \alpha I$ does not have a dense range. Therefore $B(r,s) - \alpha I \in III$. This completes the proof. □

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