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Global solutions of the wave-Schrödinger system below L^2

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Abstract. We prove that the 3 dimensional wave-Schrödinger system is globally well-posed for data in $(H^{s_1} \times \dot{H}^{s_2} \times \dot{H}^{s_2-1})(\mathbb{R}^3)$, where both s_1 and s_2 are some negative indices.

Key words: global well-posedness, wave-Schrödinger system.

1. Introduction and main results

In this paper, we consider the Yukawa coupled wave-Schrödinger system in 3 dimensions:

$$\begin{cases} i\partial_t u + \Delta u = 2vu, \\ \partial_t^2 v - \Delta v = -|u|^2, \end{cases} \tag{1.1}$$

where u and v are complex and real valued on $\mathbb{R}^3 \times [0, \infty)$, respectively. This system is a physical model describing an interaction between electrons and phonon, e.g., it describes the superconductivity [16].

We are interested in the global well-posedness of the Cauchy problem for this system, especially, when all initial data are below L^2. Here the notion of global well-posedness includes the global existence, the uniqueness and the continuous dependence of solutions on initial data. In general, to prove global well-posedness, conservation laws such as L^2-norm and Hamiltonian play an important role. But in our case, i.e., the case below L^2, such conservation laws do not make sense.

Global well-posedness below the conservation law is recently developed by J. Bourgain [3, 4] and J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao [6, 7, 8] and several authors [14, 15]. For KdV and Schrödinger equations, sharp results are obtained in the sense that the global well-posedness is proved up to the regularity below which the uniformly continuous dependence of solutions on initial data breaks down, [7, 8]. These results reflect that we understand the structures of these interactions. Indeed, in

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the view of [3], it is important to show that the energy at high frequencies does not move rapidly to low frequencies. Moreover, to obtain better global well-posedness result, we have to treat two types of frequency interactions, that is, the resonant interaction and the coherent interaction. Actually, we make a close investigation of the coherent interaction to prove our main Theorem 1.1 below (see Section 4.2).

In spite of the importance, there are few global well-posedness result for systems such as (1.1) and Zakharov system. For (1.1), H. Pecher proved the global well-posedness for data in $(H^{s_1} \times H^{s_2} \times H^{s_2-1})(\mathbb{R}^3)$ when $1 \geq s_1,\ s_2 > 7/10$ and $s_1 + s_2 > 3/2$ in [15]. Then the result is improved by several authors [22], [1] and we know that (1.1) is globally well-posed when $s_1,\ s_2 > (\sqrt{57}-5)/4$ in [1]. But the results are far from the satisfactory result. Indeed, the scaling argument (cf. [11]) suggests that we expect the well-posedness below L^2, in particular, when $s_1 > -1$ and $s_2 > -1/2$. But there has been no result below L^2. Thus our aim here is to extend the global well-posedness results for (1.1) to the case below L^2.

The system (1.1) is transformed into an equivalent first order system in time via the transformations $\psi = u$, $\phi = v + i|\nabla|^{-1}\partial_t v$ (cf. [11, 15]) and so, in what follows, we consider the following Cauchy problem.

$$
\text{(WS)} \quad \begin{cases} i\partial_t\psi + \Delta\psi = (\phi + \overline{\phi})\psi, & x \in \mathbb{R}^3, \quad t \geq 0, \\ i\partial_t\phi - |\nabla|\phi = |\nabla|^{-1}(|\psi|^2), & x \in \mathbb{R}^3, \quad t \geq 0, \\ \psi(0) = \psi_0, & x \in \mathbb{R}^3, \\ \phi(0) = \phi_0, & x \in \mathbb{R}^3, \end{cases}
$$

where both ψ and ϕ are complex valued, and $|\nabla|$ denotes the Fourier multiplier whose symbol coincides with $|\xi|$.

For (WS), we formally have the L^2 and the Hamiltonian conservation laws:

$$
\|\psi(t)\|_{L^2(\mathbb{R}^3)} = \|\psi_0\|_{L^2(\mathbb{R}^3)}, \tag{1.2}
$$

$$
H(\psi(t), \phi(t)) = H(\psi_0, \phi_0) \tag{1.3}
$$

where

$$
H(f, g) := \|f\|_{\dot{H}^1(\mathbb{R}^d)}^2 + \|g\|_{\dot{H}^1(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d} (g(x) + \overline{g}(x))|f(x)|^2\, dx.
$$

These quantities are important to prove the global well-posedness. So far, without the Hamiltonian conservation law (1.3), we have not been able to control $\dot{H}^s$-norms of ϕ. This is the reason why the known results are far from the case below L^2. But we find that we can control the L^2-norm of ϕ by only L^2 conservation law (1.2) (cf. Section 3 below). This enables us to prove the global well-posedness in L^2. Moreover this motivates to prove the following theorem, which is our main result.

Theorem 1.1 (Global well-posedness below L^2) *There exists $\theta \in (0, 1/10)$ such that, if s_1, $s_2 \le 0$ satisfy that $|s_1| < 1/8$, $|s_2| < \theta$ and*

$$\min\left\{\frac{\theta - 2|s_2|}{4-\theta}, \frac{\theta - 4|s_2|}{4-3\theta}, \frac{\theta - 4|s_1|}{5-\theta}, \frac{\theta - 4|s_1| - 2|s_2|}{5-3\theta}\right\}$$
$$> 4\max\left\{\frac{6|s_1| + |s_2|}{4-\theta}, \frac{8|s_1|}{2-\theta}\right\} > \max\{4|s_1|, |s_2|\},$$

then (WS) is globally well-posed for data $(\psi_0, \phi_0) \in H^{s_1}(\mathbb{R}^3) \times \dot{H}^{s_2}(\mathbb{R}^3)$.

Remark 1 When $s = s_1 = s_2$, the conditions of s_1 and s_2 are reduced to $s > -(2\theta - \theta^2)/(172 - 102\theta)$ if $\theta \le 2/33$ and $s > -(2\theta - \theta^2)/(168 - 36\theta)$ if $\theta \ge 2/33$. For the exponent θ, see Remark 2 in Section 4.1.

To prove the theorem, we use the idea of Bourgain [3]. The main ingredients for the proof are L^2-a priori estimate of ϕ (Proposition 3.1) and bilinear estimates for negative indices (Propositions 4.1, 4.2). The bilinear estimate for the part of Schrödinger equation is very complicated. This is caused by the complicated resonance structure. On the other hand, in (WS), we expect that the coherent interaction does not dominate the bilinear estimate so much. To see this, we decompose the wave part into its free evolution and perturbation terms. Then the structure of the interaction turns to be clearer and we can obtain bilinear estimates for negative indices by the help of the arguments of [5] and [17]. For details, see the Section 4.

This paper is organized as follows. Section 2 is assigned for preliminaries, where we introduce Bourgain's spaces and the integral equations associated to (WS), and give notations following [17], and lemmas used in this paper. In Section 3 we give the proof of Theorem 1.1 for special case $s_1 = s_2 = 0$, i.e. the global well-posedness in L^2. In Section 4, we give and prove the bilinear estimates (Propositions 4.1, 4.2). In Section 5, we prove the Theorem 1.1. In the Appendix, we give well-known Strichartz type estimates. Moreover we give the useful time-gain estimate.

2. Preliminaries

Throughout this paper, we use $\mathcal{F}_x$ and $\mathcal{F}_x^{-1}$ to denote the Fourier and inverse Fourier transforms in a variable x, respectively.

We first introduce Bourgain's spaces. Let h be a real valued function on $\mathbb{R}^d$. Then Bourgain's space associated to h is defined by

$$X_h^{s,b} := \{u \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}) \mid \|u\|_{X_h^{s,b}} < \infty\},$$

where $\mathcal{S}'(\mathbb{R}^d \times \mathbb{R})$ denotes the class of tempered distributions on $\mathbb{R}^d \times \mathbb{R}$ and

$$\begin{aligned} \|u\|_{X_h^{s,b}} &:= \|e^{itH}u\|_{H_t^b H_x^s(\mathbb{R}^d\times\mathbb{R})} \\ &= \|\langle\xi\rangle^s \langle\tau + h(\xi)\rangle^b \mathcal{F}_{x,t}[u]\|_{L^2_{\xi,\tau}(\mathbb{R}^d\times\mathbb{R})}, \end{aligned} \tag{2.1}$$

where H denotes the Fourier multiplier whose symbol coincides h. We also define the homogeneous counterpart of $X_h^{s,b}$ by

$$\dot{X}_h^{s,b} := \{u \in \mathcal{Z}'(\mathbb{R}^d \times \mathbb{R}) \mid \|u\|_{\dot{X}_h^{s,b}} < \infty\},$$

where $\mathcal{Z}'(\mathbb{R}^d \times \mathbb{R})$ denotes the dual space of

$$\mathcal{Z}(\mathbb{R}^d \times \mathbb{R}) := \{f \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}) \mid (D^\alpha \mathcal{F}_x[f])(0) = 0,\ \forall \alpha \in (\mathbb{N} \cup \{0\})^d\}$$

(cf. Section 5 in [20]) and $\|u\|_{\dot{X}_h^{s,b}}$ is defined by replacing $\langle\xi\rangle$ with $|\xi|$ in (2.1).

Because $X_h^{s,b}$ is a time global space, we need the following time localized one: Let $I \subset \mathbb{R}$ be a time interval with $|I| < 1$. Then we define

$$X_h^{s,b}(I) := \{u \colon \mathbb{R}^d \times I \to \mathbb{C} \mid \exists \widetilde{u} \in X_h^{s,b} \text{ s.t. } \widetilde{u}|_I = u\} \tag{2.2}$$

which is equipped with the norm

$$\|u\|_{X_h^{s,b}(I)} := \inf\{\|\widetilde{u}\|_{X_h^{s,b}} \mid \widetilde{u} \in X_h^{s,b} \text{ with } \widetilde{u}|_I = u\} \tag{2.3}$$

Similarly we define the time local version of $\dot{X}_h^{s,b}$ and denote it by $\dot{X}_h^{s,b}(I)$.

Next we introduce the integral equations associated to (WS) on an interval $I = [t_0, t_1]$:

$$\begin{cases} \psi(t) = e^{i(t-t_0)\Delta}\psi(t_0) - iG_1^{(I)}[\psi(\phi + \overline{\phi})](t), & t \in I, \\ \phi(t) = e^{-i(t-t_0)|\nabla|}\phi(t_0) + iG_2^{(I)}[|\nabla|^{-1}|\psi|^2](t), & t \in I, \end{cases} \tag{2.4}$$

where, for all $f \in \mathcal{S}(\mathbb{R}^3 \times I)$, $G_1^{(I)}$ and $G_2^{(I)}$ are represented as follows:

$$G_1^{(I)}[f](t) = \int_{t_0}^{t} e^{i(t-s)\Delta} f(s)\,ds, \quad G_2^{(I)}[f](t) = \int_{t_0}^{t} e^{i(t-s)|\nabla|} f(s)\,ds. \tag{2.5}$$

Moreover, we can extend $G_1^{(I)}$ (Resp. $G_2^{(I)}$) to the bounded linear operator from $X^{s,b-1}_{|\xi|^2}(I)$ to $X^{s,b}_{|\xi|^2}(I)$ (Resp. $\dot{X}^{s,b-1}_{\pm|\xi|}(I)$ to $\dot{X}^{s,b}_{\pm|\xi|}(I)$), if $b > 1/2$ and $|I| < 1$ (cf. Lemma 2.1 (ii) in [11]). We note the following commutator relation: for any $s \in \mathbb{R}$,

$$\big[G_2^{(I)}, |\nabla|^s\big] = 0. \tag{2.6}$$

Finally we introduce the general framework for bilinear estimates following [17]. For any integer $k \geq 2$, we define the hyperplane $\Gamma_k(\mathbb{R}^D)$ by

$$\Gamma_k(\mathbb{R}^D) := \big\{(\eta_1, \dots, \eta_k) \in \mathbb{R}^{kD} \,\big|\, \eta_1 + \cdots + \eta_k = 0\big\}$$

and define the integral on this hyperplane by

$$\int_{\Gamma_k(\mathbb{R}^D)} f(\eta_1, \dots, \eta_{k-1}, \eta_k) := \int_{\mathbb{R}^{(k-1)D}} f(\eta_1, \dots, \eta_{k-1}, -\eta_1 - \cdots - \eta_{k-1})\, d\eta_1 \cdots d\eta_{k-1}.$$

A $[k;\mathbb{R}^D]$-multiplier is a function $m\colon \Gamma_k(\mathbb{R}^D) \to \mathbb{C}$. For $[k;\mathbb{R}^D]$-multiplier m we define $\|m\|_{[k;\mathbb{R}^D]}$ to be the best constant such that the inequality

$$\left| \int_{\Gamma_k(\mathbb{R}^D)} m(\eta_1, \dots, \eta_k) \prod_{j=1}^{k} f_j(\eta_j) \right| \leq C \prod_{j=1}^{k} \|f_j\|_{L^2(\mathbb{R}^D)}$$

holds for all "non-negative" functions $f_1, \dots, f_k \in \mathcal{S}(\mathbb{R}^D)$. $\|m\|_{[k;\mathbb{R}^D]}$ is called a multiplier norm for m.

Now we state the basic properties of multiplier norm without the proofs, all of them are contained in [17]. First of all, we easily see that for all $a \in \mathbb{C}$, $\|am\|_{[k;\mathbb{R}^D]} = |a|\,\|m\|_{[k;\mathbb{R}^D]}$, and for all $[k;\mathbb{R}^D]$-multipliers m_1 and m_2, $\|m_1 + m_2\|_{[k;\mathbb{R}^D]} \leq \|m_1\|_{[k;\mathbb{R}^D]} + \|m_2\|_{[k;\mathbb{R}^D]}$.

Lemma 2.1 (Comparison principle) *Let m_1 and m_2 be $[k;\mathbb{R}^D]$-multipliers with*

$$|m_1(\eta_1, \dots, \eta_k)| \leq |m_2(\eta_1, \dots, \eta_k)|$$

for almost all $(\eta_1,\dots,\eta_k)\in\Gamma_k(\mathbb{R}^D)$*. Then we have* $\|m_1\|_{[k;\mathbb{R}^D]}\le\||m_2|\|_{[k;\mathbb{R}^D]}$.

Lemma 2.2 (Linear transformation) *Let* m *be a* $[k;\mathbb{R}^D]$*-multiplier and* L *a linear transformation in* $\mathbb{R}^D$*. Then* $\|m\circ L\|_{[k;\mathbb{R}^D]}=|\det(L)|^{k/2-1}\|m\|_{[k;\mathbb{R}^D]}$.

Lemma 2.3 *Let* A *and* B *be subsets of* $\mathbb{R}^d$ *and let* χ_A *and* χ_B *be their indicator functions, respectively. Let* $h,\,h'\colon\mathbb{R}^d\to\mathbb{R}$*,* $L,\,L'>0$ *and set* $\lambda_1:=\tau_1+h(\xi_1)$*,* $\lambda_2:=\tau_2+h'(\xi_2)$*. Then*

$$\begin{aligned}&\big\|\chi_A(\xi_1)\chi_B(\xi_2)\chi_{|\lambda_1|\lesssim L}(\xi_1,\tau_1)\chi_{|\lambda_2|\lesssim L'}(\xi_2,\tau_2)\big\|_{[3;\mathbb{R}^D]}\\&\quad\lesssim\min\{L,L'\}^{1/2}\sup_{(\widetilde{\xi},\widetilde{\tau})\in\mathbb{R}^{d+1}}\bigg|\Big\{\xi\in A\,\Big|\,\widetilde{\xi}-\xi\in B,\\&\qquad\qquad\qquad\big|h(\xi)+h'(\widetilde{\xi}-\xi)+\widetilde{\tau}\big|\lesssim\max\{L,L'\}\Big\}\bigg|^{1/2}.\end{aligned}$$

Similar statements hold if we permute the indices 1, 2, 3.

Lemma 2.4 (Schur's test) *Let* $\{m_\lambda\}_{\lambda\in\Lambda}$ *be a collection of* $[3;\mathbb{R}^D]$*-multipliers with*

$$\sup_{\eta\in\mathbb{R}^D}\sharp\big\{\lambda\in\Lambda\,\big|\,\eta\in\pi_j(\operatorname{supp}(m_\lambda))\big\}\le C_j,$$

where C_j *is some constant and* π_j *denotes the projection onto* η_j*-plane* $\mathbb{R}^D$ $(j=1,\,2)$*. Then we have*

$$\bigg\|\sum_{\lambda\in\Lambda}m_\lambda\bigg\|_{[3;\mathbb{R}^D]}\le(C_1C_2)^{1/2}\sup_{\lambda\in\Lambda}\|m_\lambda\|_{[3;\mathbb{R}^D]}.$$

Similar statements hold if we permute the indices 1, 2, 3.

Now we recall a box covering of $\mathbb{R}^D$, which is a partition of $\mathbb{R}^D$ into sets $\{R+v\}_{v\in\Sigma}$, where R, Σ and $R+v$ $(v\in\Sigma)$ are called fundamental domain, tiling lattice and box, respectively. The fundamental domain R is a subset of $\mathbb{R}^D$ containing the origin and symmetric around the origin. Σ is a discrete subgroup of $\mathbb{R}^D$ such that $R+R$ can be covered by $O(1)$ boxes and the overlap of boxes is bounded by a uniform constant. Throughout this paper we use $A\lesssim B$ to denote the estimate $A\le CB$, where C is a universal constant.

Lemma 2.5 (Box localization) *Let* $\{R+v\}_{v\in\Sigma}$ *be a box covering of* $\mathbb{R}^D$ *and let* m *be* $[3;\mathbb{R}^D]$*-multiplier with* $\pi_1(\operatorname{supp}(m))\subset R+v_1$ *for some* $v_1\in\Sigma$.

Then

$$\|m\|_{[3;\mathbb{R}^D]} \lesssim \sup_{v_2,v_3\in\Sigma} \|m\chi_{R+v_2}(\eta_2)\chi_{R+v_3}(\eta_3)\|_{[3;\mathbb{R}^D]}.$$

Similar statements hold if we permute the indices 1, , 2, 3.

Lemma 2.6 (Transverse interactions) *Let E_1, E_2 be open subsets of $\mathbb{R}^d$, and let η be a unit vector in $\mathbb{R}^d$. Let $\theta > 0$, and let $h_1\colon E_1 \to \mathbb{R}$, $h_2\colon E_2 \to \mathbb{R}$ be smooth functions which satisfy the transversality condition*

$$|D_\eta h_1(\xi_1) - D_\eta h_2(\xi_2)| \gtrsim \theta$$

for all $\xi_1 \in E_1$, $\xi_2 \in E_2$, where D_η is the directional derivative in the direction η. Then for any L_1, $L_2 > 0$ we have

$$\begin{aligned}\Big\|\prod_{j=1}^{2}\chi_{E_j}(\xi_j)\chi_{|\lambda|\lesssim L_j}\Big\|_{[3;\mathbb{R}^d\times\mathbb{R}]}&\\ \lesssim L_1^{1/2}L_2^{1/2}\theta^{-1/2}\min\{|\pi_\eta^\perp(E_1)|, |\pi_\eta^\perp(E_2)|\}^{1/2},&\end{aligned}$$

where $|\pi_\eta^\perp(E)|$ is the $d-1$ dimensional measure of the projection of E onto the orthogonal complement of η.

3. Global well-posedness in L^2

We know that (WS) is locally well-posed in L^2, i.e. for data $(\psi_0, \phi_0) \in L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ (see, e.g. Theorem 1.1 in [1]). Moreover the existence time depends only on the size of L^2-norm of the initial data. Since we have the conservation law (1.2), to prove the global well-posedness in L^2, it suffices to show a priori bound of $\|\phi(t)\|_{L^2}$.

Proposition 3.1 *Let (ψ, ϕ) be the L^2-solution of (WS) on $[0,T]$. Then we have*

$$\begin{aligned}&\|\phi\|_{L_t^\infty L_x^2([0,T])}\\ &\quad\le \|\phi_0\|_{L^2} + C\big(T^{5/8}\|\psi_0\|_{L^2}^2 + T^{3/4}\|\psi_0\|_{L^2}^2\|\phi_0\|_{L^2}^{1/2} + T^{3/2}\|\psi_0\|_{L^2}^4\big),\end{aligned}$$

where C is some universal constant, and

$$\|\psi\|_{L_t^2L_x^6([0,T])} \lesssim \|\psi_0\|_{L^2} + T^{1/2}\|\psi_0\|_{L^2}\|\phi_0\|_{L^2}^2 + T^{7/2}\|\psi_0\|_{L^2}^9.$$

Proof of Proposition 3.1. In this proof, we use C to denote a universal constant, which may vary from line to line.

Let (ψ, ϕ) be the L^2-solution of (WS) on $[0,T]$. Then we know that $\psi \in C([0,T]; L^2) \cap L^2([0,T]; L^6)$. By the integral equation (2.4) and the Strichartz estimates (cf. Proposition 3.1 in [10]), we have

$$\|\phi(t)\|_{L^2} \leq \|\phi_0\|_{L^2} + C\|\psi\|^2_{L_t^{8/3} L_x^{24/11}([0,T])}. \tag{3.1}$$

By the interpolation,

$$\begin{aligned}\text{R.H.S. of (3.1)} &\leq \|\phi_0\|_{L^2} + C\|\psi\|^{7/4}_{L_t^{14/5} L_x^2([0,T])} \|\psi\|^{1/4}_{L_t^2 L_x^6([0,T])} \\ &\leq \|\phi_0\|_{L^2} + CT^{5/8}\|\psi\|^{7/4}_{L_t^\infty L_x^2([0,T])} \|\psi\|^{1/4}_{L_t^2 L_x^6([0,T])}.\end{aligned} \tag{3.2}$$

We apply the L^2-conservation law (1.2), which yields

$$\|\phi\|_{L_t^\infty L_x^2([0,T])} \leq \|\phi_0\|_{L^2} + CT^{5/8}\|\psi_0\|^{7/4}_{L^2} \|\psi\|^{1/4}_{L_t^2 L_x^6([0,T])}. \tag{3.3}$$

Here we need to consider the term $\|\psi\|_{L_t^2 L_x^6([0,T])}$. By the integral equation (2.4) and the Strichartz estimate, we have

$$\begin{aligned}\|\psi\|_{L_t^2 L_x^6([0,T])} &\lesssim \|\psi_0\|_{L^2} + \|\psi(\phi + \overline{\phi})\|_{L_t^2 L_x^{6/5}([0,T])} \\ &\lesssim \|\psi_0\|_{L^2} + \|\phi\|_{L_t^\infty L_x^2([0,T])} \|\psi\|_{L_t^2 L_x^3([0,T])}.\end{aligned} \tag{3.4}$$

By the interpolation, Hölder's inequality and (1.2), we have

$$\|\psi\|_{L_t^2 L_x^6([0,T])} \lesssim \|\psi_0\|_{L^2} + T^{1/4}\|\phi\|_{L_t^\infty L_x^2([0,T])} \|\psi_0\|^{1/2}_{L^2} \|\psi\|^{1/2}_{L_t^2 L_x^6([0,T])}. \tag{3.5}$$

We insert (3.3) into the second term in R.H.S. of (3.5) and obtain

$$\begin{aligned}\|\psi\|_{L_t^2 L_x^6([0,T])} \lesssim\ & \|\psi_0\|_{L^2} + T^{1/4}\|\phi_0\|_{L^2} \|\psi_0\|^{1/2}_{L^2} \|\psi\|^{1/2}_{L_t^2 L_x^6([0,T])} \\ & + T^{7/8}\|\psi_0\|^{9/4}_{L^2} \|\psi\|^{3/4}_{L_t^2 L_x^6([0,T])}.\end{aligned}$$

Applying Young's inequality ($ab \leq a^p/p + b^q/q$ for $1/p + 1/q = 1$), we have

$$\|\psi\|_{L_t^2 L_x^6([0,T])} \lesssim \|\psi_0\|_{L^2} + T^{1/2}\|\phi_0\|^2_{L^2} \|\psi_0\|_{L^2} + T^{7/2}\|\psi_0\|^9_{L^2}, \tag{3.6}$$

which proves the second claim of the proposition.

We insert (3.6) into (3.3) and obtain the result. □

4. Bilinear estimates

For both local and global well-posednesses, we usually consider the following estimates (cf. (2.4)):

$$\left\|G_1^{(I)}[\psi(\phi+\overline{\phi})]\right\|_{X^{s,1/2+}_{|\xi|^2}(I)} \lesssim \|\psi\|_{X^{s_1,b_1}_{|\xi|^2}(I)}\|\phi\|_{\dot{X}^{s_2,b_2}_{|\xi|}(I)}, \tag{4.1}$$

$$\left\|G_2^{(I)}[|\nabla|^{-1}|\psi|^2]\right\|_{X^{s,1/2+}_{\pm|\xi|}(I)} \lesssim \|\psi\|^2_{X^{s_1,b_1}_{|\xi|^2}(I)}, \tag{4.2}$$

where $1/2+$, more generally, for $a \in \mathbb{R}$, $a+$ (Resp. $a-$) denotes a number greater (Resp. less) than a and sufficiently close to a. As stated above (cf. Section 2), $G_1^{(I)}$ and $G_2^{(I)}$ are bounded linear operators from $X^{s,b-1}_{|\xi|^2}(I)$ to $X^{s,b}_{|\xi|^2}(I)$ and from $\dot{X}^{s,b-1}_{\pm|\xi|}(I)$ to $\dot{X}^{s,b}_{\pm|\xi|}(I)$, respectively, for any $b > 1/2$ and any time interval I with $|I| \le 1$. Thus we can reduce (4.1) and (4.2) to the following bilinear estimates:

$$\|\psi(\phi+\overline{\phi})\|_{X^{s,-1/2+}_{|\xi|^2}} \lesssim \|\psi\|_{X^{s_1,b_1}_{|\xi|^2}}\|\phi\|_{\dot{X}^{s_2,b_2}_{|\xi|}}, \tag{4.3}$$

$$\||\nabla|^{-1}|\psi|^2\|_{X^{s,-1/2+}_{|\xi|}} \lesssim \|\psi\|^2_{X^{s_1,b_1}_{|\xi|^2}}. \tag{4.4}$$

The estimate (4.3) is open for any s_1, $s_2 < 0$ and $s \ge s_1$, s_2.

For any given ψ, from the integral equation (2.4), ϕ is represented as follows:

$$\phi = e^{-it|\nabla|}\phi_0 + G_2^{(I)}[|\nabla|^{-1}|\psi|^2].$$

This representation of ϕ enables us to prove Theorem 1.1 without (4.3). Indeed, we have the following estimate for the L.H.S. of (4.3):

$$\begin{aligned}\|\psi(\phi+\overline{\phi})\|_{X^{s,-b_3}_{|\xi|^2}} \le{}& \|\psi(e^{-it|\nabla|}\phi_0)\|_{X^{s,-b_3}_{|\xi|^2}} + \|\psi\overline{(e^{-it|\nabla|}\phi_0)}\|_{X^{s,-b_3}_{|\xi|^2}} \\ &+ \left\|\psi G_2^{(I)}[|\nabla|^{-1}|\psi|^2]\right\|_{X^{s,-b_3}_{|\xi|^2}} \\ &+ \left\|\psi\overline{G_2^{(I)}[|\nabla|^{-1}|\psi|^2]}\right\|_{X^{s,-b_3}_{|\xi|^2}}. \end{aligned}\tag{4.5}$$

The first and second terms in the R.H.S. of (4.5) seem to be easier than the estimate (4.3). Indeed, they are estimated by using the result in Section 11 of [5]. On the other hand, the third and fourth terms in the R.H.S. of (4.5), and the L.H.S. of (4.4) are good thanks to the factor $|\nabla|^{-1}$ and treated similarly. The details appear in the following two subsections.

4.1. Estimate I

We consider the first and the second terms in the R.H.S. of (4.5). Then we have the following proposition:

Proposition 4.1 *Let s, s_1, $s_2 \le 0$ and let b_1 satisfy $b_1 > 1/4 + |s_1|/2 + |s_2|/2$. Then there exists $\theta > 0$ such that if $b_1 > 1/2 - \theta/2 + |s_1|/2 + |s_2|/2 - |s|/2$, then, taking $b_3 < 1/2$ sufficiently close to $1/2$, we have the following estimates*:

(i) $\left\|\psi(e^{-it|\nabla|}\phi_0)\right\|_{X^{s,-b_3}_{|\xi|^2}} \lesssim \|\psi\|_{X^{s_1,b_1}_{|\xi|^2}} \|\phi_0\|_{\dot{H}^{s_2}}$,

(ii) $\left\|\psi\overline{(e^{-it|\nabla|}\phi_0)}\right\|_{X^{s,-b_3}_{|\xi|^2}} \lesssim \|\psi\|_{X^{s_1,b_1}_{|\xi|^2}} \|\phi_0\|_{\dot{H}^{s_2}}$,

where the implicit constants depending only on s, s_1, s_2, b_1 and b_3.

Proof of Proposition 4.1. Most of the proof follows [5]. Since the proof of (ii) is similar to (i), we only give the proof of (i).

By duality we easily find that (i) is reduced to the following estimate:

$$\begin{aligned} &\bigg|\int_{\Gamma_3(\mathbb{R}^{3+1})} \frac{\langle\xi_1\rangle^{-s_1}|\xi_2|^{-s_2}\langle\xi_3\rangle^{s}}{\langle\lambda_1\rangle^{b_1}\langle\lambda_3\rangle^{b_3}}\,\psi_1(\xi_1,\tau_1) \\ &\times \mathcal{F}_{x,t}\big[e^{\mp it|\nabla|}|\nabla|^{s_2}\phi_0\big](\xi_2,\tau_2)\omega_3(\xi_3,\tau_3)\bigg| \\ &\quad\lesssim \|\psi_1\|_{L^2_\tau L^2_\xi}\||\nabla|^{s_2}\phi_0\|_{L^2}\|\omega_3\|_{L^2_\tau L^2_\xi}, \end{aligned} \tag{4.6}$$

where $\lambda_1 := \tau_1 + |\xi_1|^2$ and $\lambda_3 := \tau_3 - |\xi_3|^2$.

Since $\mathcal{F}_{x,t}[e^{\mp it|\nabla|}|\nabla|^{s_2}\phi_0](\xi_2,\tau_2) = \delta(\tau_2 \pm |\xi_2|)\mathcal{F}_x[|\nabla|^{s_2}\phi_0](\xi_2)$, the L.H.S. of (4.6) is estimated by

$$\begin{aligned} &\int_{\Gamma_3(\mathbb{R}^{3+1})} \delta(\tau_2 \pm |\xi_2|)\frac{\langle\xi_1\rangle^{-s_1}|\xi_2|^{-s_2}\langle\xi_3\rangle^{s}}{\langle\lambda_1\rangle^{b_1}\langle\lambda_3\rangle^{b_3}}\,|\psi_1(\xi_1,\tau_1)| \\ &\quad\times |\mathcal{F}_x[|\nabla|^{s_2}\phi_0](\xi_2)|\,|\omega_3(\xi_3,\tau_3)| \end{aligned} \tag{4.7}$$

In (4.7), we divide the region of the integral into two cases: Case 1: $\max\{|\lambda_1|, |\lambda_3|\} \gtrsim |\xi_1|^2$, and Case 2: $\max\{|\lambda_1|, |\lambda_3|\} \ll |\xi_1|^2$.

Estimate of Case 1: In this case, we can estimate (4.7) only by the Strichartz estimates (Lemmas 6.1 and 6.2). We will divide the integral according to the size of the symbols λ_1, λ_3 and then estimate the factor of wave equation in $L^\infty_t L^2_x$, the factor of Schrödinger equation with the bigger symbol in $L^2_t L^3_x$ and the one with smaller symbol in $L^2_t L^6_x$.

Since $\big||\xi_1|^2 \mp |\xi_2| - |\xi_3|^2\big| = |\lambda_1 + \lambda_3| \le 2\max\{|\lambda_1|, |\lambda_3|\}$ on $\{\tau_2 \pm |\xi_2|\}$, if $|\xi_1| \ll |\xi_3|$, then $|\xi_2|^2 \sim |\xi_3|^2 \lesssim \max\{|\lambda_1|, |\lambda_3|\}$. Thus in Case 1, we have $\max\{\langle\xi_1\rangle, |\xi_2|, \langle\xi_3\rangle\}^2 \lesssim \max\{\langle\lambda_1\rangle, \langle\lambda_3\rangle\}$.

We first consider the case where $|\lambda_1| \gtrsim |\lambda_3|$. Then the integral (4.7) over $|\lambda_1| \gtrsim |\lambda_3|$ is estimated by

$$\begin{aligned}&\int_{\Gamma_3(\mathbb{R}^{3+1})} \delta(\tau_2 \pm |\xi_2|)\frac{1}{\langle\lambda_1\rangle^{b_1+s_1/2+s_2/2}\langle\lambda_3\rangle^{b_3}}|\psi_1(\xi_1,\tau_1)| \\ &\quad \times |\mathcal{F}_x[|\nabla|^{s_2}\phi_0](\xi_2)|\,|\omega_3(\xi_3,\tau_3)|, \end{aligned} \tag{4.8}$$

where we have used the fact that $s \le 0$ and therefore $\langle\xi_3\rangle^s \le 1$. By the assumption $b_1 + s_1/2 + s_2/2 > 1/4$. We set $b_1 + s_1/2 + s_2/2 = 1/4 + \varepsilon_0$ for some $\varepsilon_0 > 0$. Then, e.g., taking $b_3 = 1/2 - \varepsilon_0/2$, we estimate (4.8) by

$$\begin{aligned}&\int_{\Gamma_3(\mathbb{R}^{3+1})} \delta(\tau_2 \pm |\xi_2|)\frac{1}{\langle\lambda_1\rangle^{1/4+}\langle\lambda_3\rangle^{1/2+}}|\psi_1(\xi_1,\tau_1)| \\ &\quad \times |\mathcal{F}_x[|\nabla|^{s_2}\phi_0](\xi_2)|\,|\omega_3(\xi_3,\tau_3)|. \end{aligned} \tag{4.9}$$

By Plancherel's theorem, (4.9) is equal to

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}} \mathcal{F}^{-1}_{\xi,\tau}\big[\langle\tau + |\xi|^2\rangle^{-(1/4+)}|\psi_1|\big](x,t) \\ &\quad \times \mathcal{F}^{-1}_{\xi,\tau}\big[\delta(\tau \pm |\xi|)\,|\mathcal{F}_x[|\nabla|^{s_2}\phi_0]|\big](x,t) \\ &\quad \times \mathcal{F}^{-1}_{\xi,\tau}\big[\langle\xi_3\rangle^s\langle\tau - |\xi|^2\rangle^{-(1/2+)}|\omega_3|\big](x,t)\,dx\,dt \\ &= \int_{\mathbb{R}^3\times\mathbb{R}} \mathcal{F}^{-1}_{\xi,\tau}\big[\langle\tau + |\xi|^2\rangle^{-(1/4+)}|\psi_1|\big](x,t) \\ &\quad \times \mathcal{F}^{-1}_{\xi}\big[e^{\mp it|\xi|}\,|\mathcal{F}_x[|\nabla|^{s_2}\phi_0]|\big](x) \\ &\quad \times \mathcal{F}^{-1}_{\xi,\tau}\big[\langle\xi_3\rangle^s\langle\tau - |\xi|^2\rangle^{-(1/2+)}|\omega_3|\big](x,t)\,dx\,dt. \end{aligned} \tag{4.10}$$

By Hölder's inequality,

$$\begin{aligned}&\text{R.H.S. of (4.10)} \\ &\le \big\|\mathcal{F}^{-1}_{\xi,\tau}\big[\langle\tau + |\xi|^2\rangle^{-(1/4+)}|\psi_1|\big]\big\|_{L^2_tL^3_x} \\ &\quad \times \big\|\mathcal{F}^{-1}_{\xi}\big[e^{\mp it|\xi|}\,|\mathcal{F}_x[|\nabla|^{s_2}\phi_0]|\big]\big\|_{L^\infty_tL^2_x} \\ &\quad \times \big\|\mathcal{F}^{-1}_{\xi,\tau}\big[\langle\tau - |\xi|^2\rangle^{-(1/2+)}|\omega_3|\big]\big\|_{L^2_tL^6_x}. \end{aligned} \tag{4.11}$$

Applying the Strichartz type estimate (Lemma 6.1), we have

$$\text{R.H.S. of (4.11)} \lesssim \|\psi_1\|_{L^2_\tau L^2_\xi} \||\nabla|^{s_2}\phi_0\|_{L^2} \|\omega_3\|_{L^2_\tau L^2_\xi}, \tag{4.12}$$

hence the claim follows in this case.

Next we consider the case where $|\lambda_3| \geq |\lambda_1|$. In this case, we also take $b_3 = 1/2 - \varepsilon_0/2$. Then the claim follows by a similar way to the above.

Estimate of Case 2: We apply the result given in pp. 541–544 of [5] (see also Remark 2 below). Thus we find that there exists $\theta > 0$ such that, for any $s_1,\ s_2 \leq 0$ with $\theta > |s_1| + |s_2| - |s| + 2 - 2b_1 - 2b_3$, the integral (4.7) for Case 2 is estimated by $\|\psi_1\|_{L^2_\tau L^2_\xi} \||\nabla|^{s_2}\phi_0\|_{L^2} \|\omega_3\|_{L^2_\tau L^2_\xi}$. The condition $\theta > |s_1| + |s_2| - |s| + 2 - 2b_1 - 2b_3$ is reduced to $b_1 > 1/2 - \theta/2 + |s_1|/2 + |s_2|/2 - |s|/2$, if we take b_3 sufficiently close to 1/2.

Completion of the proof: From the estimates of Cases 1 and 2, we have proved (4.6) and therefore Proposition 4.1. □

Remark 2 We state a remark on the exponent θ which appears in the estimate of Case 2. The key inequality in [5] is the following: let σ be an invariant measure on S^2, the unit sphere in $\mathbb{R}^3$. For any $\varepsilon > 0$, let ρ_ε be a localizing function vanishing on ε-neighborhoods of 0 and the sphere with radius 2. Then there is some $p < 2$ and $C > 0$ such that

$$\big\|\mathcal{F}[\rho_\varepsilon(f_1\, d\sigma * f_2\, d\sigma)]\big\|_{L^p(\mathbb{R}^3)} \lesssim \varepsilon^{-C} \|f_1\|_{L^2(S^2;d\sigma)} \|f_2\|_{L^2(S^2;d\sigma)} \tag{4.13}$$

for all $f_1,\ f_2 \in L^2(S^2; d\sigma)$.

From the above inequality, we can choose

$$\theta = \frac{1}{2C+1}\left(\frac{1}{p} - \frac{1}{2}\right).$$

In view of the Fourier restriction estimate, the lower bound of p is 5/3 and thus the upper bound of θ is 1/10 (cf. [18]). The estimate (4.13) plays an important role to analyze the transversality. On the other hand, the parallel interaction part is estimated by the Tomas-Stein restriction estimate.

4.2. Estimate II

In this section, we consider (4.4) and the third and fourth terms in the R.H.S. of (4.5). We state a strategy to estimate these terms. First we apply the Littlewood-Paley decomposition and so we suppose that the frequency support of ψ is similar to N_1 and one of $G_2[|\psi|^2]$ is similar to N_2, where N_1 and N_2 vary dyadically. Then, thanks to $|\nabla|^{-1} \sim N_2^{-1}$, the diagonal ($N_1 \sim N_2$) and low-high ($N_1 \ll N_2$) interaction cases are harmless. Moreover mild high-low ($N_2 \ll N_1 \lesssim N_2^2$) interaction case is controllable. On the other hand, in the very high-low ($N_1 \gg N_2^2$) interaction case, we will find that the transversality of the characteristic surfaces $\tau_1 = |\xi_1|^2$ and $\tau_2 = \pm|\xi_2|$ works very well. Besides, these principal interactions, $|\nabla|^{-1}$ causes the mild singularity case ($N_1 \ll 1/N_2$). But this case is easily treated. This strategy is accomplished below. Besides the notations $a+$ and $a-$, we also use $\infty-$ to denote a sufficiently large finite number.

The estimate (4.4) and the third and fourth terms in the R.H.S. of (4.5) are uniformly treated. Indeed, we want to bound the third and fourth terms in (4.5) by

$$\|\psi\|_{X^{s_1,b_1}_{|\xi|^2}} \left\|G_2[|\nabla|^{-1}|\psi|^2]\right\|_{\dot{X}^{s_2',1/2+}_{|\xi|}}, \tag{4.14}$$

$$\|\psi\|_{X^{s_1,b_1}_{|\xi|^2}} \left\|\overline{G_2[|\nabla|^{-1}|\psi|^2]}\right\|_{\dot{X}^{s_2',1/2+}_{-|\xi|}}, \tag{4.15}$$

respectively. These estimates are reduced to the boundedness of the $[3;\mathbb{R}^{3+1}]$-multiplier

$$\left\|\frac{\langle\xi_1\rangle^{s_1}|\xi_2|^{s_2'}\langle\xi_3\rangle^{s}}{\langle\lambda_1\rangle^{b_1}\langle\lambda_2\rangle^{1/2-}\langle\lambda_3\rangle^{b_3}}\right\|_{[3;\mathbb{R}^{3+1}]}, \tag{4.16}$$

where $\lambda_1 := \tau_1 + |\xi_1|^2$, $\lambda_2 := \tau_2 \pm |\xi_2|$ and $\lambda_3 := \tau_3 - |\xi_3|^2$.

Moreover, by the boundedness of G_2, the second factors in (4.14) and (4.15) are estimated by

$$\||\nabla|^{-1}|\psi|^2\|_{\dot{X}^{s_2',-1/2+}_{|\xi|}}, \tag{4.17}$$

which has a similar form to the L.H.S. of (4.4). We want to bound the L.H.S. of (4.4) and (4.17) by $\|\psi\|^2_{X^{s_1,b_1}_{|\xi|^2}}$. Then we easily see that these estimates are reduced to the boundedness of type of (4.16).

Our main proposition in this subsection is the following:

Proposition 4.2 *Let α, $\gamma \geq 0$ be such that $\alpha + \gamma < 1/4$ and let $-5/4 < \beta \leq -1/2$. Let b_1, $b_3 > \max\{1/4, 3/8 + (\alpha+\beta+\gamma)/4\}$ with $b_1 + b_3 > \max\{1/2 + (\alpha+\gamma), 1 - |\beta|/2\}$, and $b_2 < 1/2$ be sufficiently close to $1/2$.*

Then

$$\left\| \frac{\langle \xi_1 \rangle^{\alpha} |\xi_2|^{\beta} \langle \xi_3 \rangle^{\gamma}}{\langle \lambda_1 \rangle^{b_1} \langle \lambda_2 \rangle^{b_2} \langle \lambda_3 \rangle^{b_3}} \right\|_{[3;\mathbb{R}^{3+1}]} \tag{4.18}$$

is finite, where $\lambda_1 := \tau_1 + |\xi_1|^2$*,* $\lambda_2 := \tau_2 \pm |\xi_2|$ *and* $\lambda_3 := \tau_3 - |\xi_3|^2$*.*

Remark 3 We will encounter the case $\gamma < 0$ in the Section 5.3 below. At that time, we will use the crude estimate $\langle \xi \rangle^{\gamma} \le 1$.

Proof of Proposition 4.2. By the dyadic decomposition of variables ξ_j and τ_j, (4.18) is written as follows:

$$\Bigg\| \sum_{\ell_1,\ell_2,\ell_3 \in \mathbb{N}_0} \sum_{\substack{k_1,k_3 \in \mathbb{N}_0 \\ k_2 \in \mathbb{Z}}} \frac{m(\xi_1,\xi_2,\xi_3)}{\prod_{j=1}^3 \langle \lambda_j \rangle^{b_j}} \chi_{k_1}(\xi_1) \chi_{|\xi_2| \sim 2^{k_2}}(\xi_2) \chi_{k_3}(\xi_3) \times \prod_{j=1}^3 \chi_{\ell_j}(\xi_j,\tau_j) \Bigg\|_{[3;\mathbb{R}^{3+1}]}, \tag{4.19}$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $m(\xi_1,\xi_2,\xi_3) := \langle \xi_1 \rangle^{\alpha} |\xi_2|^{\beta} \langle \xi_3 \rangle^{\gamma}$ and $\chi_{k_j} := \chi_{|\xi_j| \lesssim 1}$ if $k_j = 0$, $:= \chi_{|\xi_j| \sim 2^{k_j}}$ if $k_j \ge 1$ ($j = 1, 3$). The function χ_{ℓ_j} is similarly defined for the modulation λ_j.

We divide the region into three cases. Case 1: $2^{k_1} \sim 2^{k_3} \gg \max\{2^{2k_2}, 2^{\ell_{\max}}, 1\}$ and $2^{k_2} \gtrsim 1/2^{k_1}$, Case 2: $2^{k_1} \sim 2^{k_3} \gg \max\{2^{2k_2}, 2^{\ell_{\max}}, 1\}$ and $2^{k_2} \ll 1/2^{k_1}$, Case 3: Otherwise, i.e. $2^{k_1} \gg 2^{k_3}$ or $2^{k_1} \ll 2^{k_3}$ or $2^{k_1} \sim 2^{k_3} \lesssim \max\{2^{2k_2}, 2^{\ell_{\max}}, 1\}$.

We consider Cases 1 and 2. In these cases, since the range of k_3 is restricted by $2^{k_1} \sim 2^{k_3}$, we estimate (4.19) with Case 1 or Case 2 by

$$\sum_{\ell_1,\ell_2,\ell_3 \in \mathbb{N}_0} \sum_{k_2 \in \mathbb{Z}} \frac{1}{\prod_{j=1}^3 \langle 2^{\ell_j} \rangle^{b_j}} \Bigg\| \sum_{\substack{k_1 \in \mathbb{N}_0 \\ \text{Case 1 or Case 2}}} m \chi_{k_1,k_2} \prod_{j=1}^3 \chi_{\ell_j} \Bigg\|_{[3;\mathbb{R}^{3+1}]}, \tag{4.20}$$

where $\chi_{k_1,k_2} = \chi_{k_1,k_2}(\xi_1,\xi_2,\xi_3) := \chi_{|\xi_1| \sim 2^{k_1}}(\xi_1) \chi_{|\xi_2| \sim 2^{k_2}}(\xi_2) \chi_{|\xi_3| \sim 2^{k_1}}(\xi_3)$.

Since $\sup_{\xi \in \mathbb{R}^3} \sharp\{k_1 \in \mathbb{N}_0 \,|\, \xi \in \pi_j(\chi_{k_1,k_2})\} \lesssim 1$ ($j = 1, 3$), by Schur's test (Lemma 2.4), (4.20) is estimated by

$$\sum_{\ell_1,\ell_2,\ell_3\in\mathbb{N}_0}\sum_{k_2\in\mathbb{Z}}\frac{1}{\prod_{j=1}^{3}\langle 2^{\ell_j}\rangle^{b_j}}\sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 1}\\ \text{or Case 2}}}2^{(\alpha+\gamma)k_1}2^{\beta k_2}\left\|\chi_{k_1,k_2}\prod_{j=1}^{3}\chi_{\ell_j}\right\|_{[3;\mathbb{R}^{3+1}]}. \tag{4.21}$$

Breaking up the annulus $\{|\xi_2|\sim 2^{k_2}\}$ into $O(1)$ sections of size $\ll 2^{k_2}$, we estimate (4.21) by

$$\sum_{\ell_1,\ell_2,\ell_3\in\mathbb{N}_0}\sum_{k_2\in\mathbb{Z}}\frac{1}{\prod_{j=1}^{3}\langle 2^{\ell_j}\rangle^{b_j}}\sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 1 or Case 2}}}2^{(\alpha+\gamma)k_1}2^{\beta k_2} \times\left\|\sum_{v_2\in V}\chi_{|\xi_2-v_2|\ll 2^{k_2}}\chi_{k_1,k_2}\prod_{j=1}^{3}\chi_{\ell_j}\right\|_{[3;\mathbb{R}^{3+1}]}, \tag{4.22}$$

where V is the set of centers of the section induced by the partition of the annulus. The cardinal number of V is independent of 2^{k_2}. Moreover, since $\sharp V\lesssim 1$, (4.22) is estimated by

$$\sum_{\ell_1,\ell_2,\ell_3\in\mathbb{N}_0}\sum_{k_2\in\mathbb{Z}}\frac{1}{\prod_{j=1}^{3}\langle 2^{\ell_j}\rangle^{b_j}}\sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 1 or Case 2}}}2^{(\alpha+\gamma)k_1}2^{\beta k_2}\sup_{v_2\in V} \times\left\|\chi_{|\xi_2-v_2|\ll 2^{k_2}}\chi_{k_1,k_2}\prod_{j=1}^{3}\chi_{\ell_j}\right\|_{[3;\mathbb{R}^{3+1}]}. \tag{4.23}$$

By the Box localization (Lemma 2.5), (4.23) is estimated by

$$\sum_{\ell_1,\ell_2,\ell_3\in\mathbb{N}_0}\sum_{k_2\in\mathbb{Z}}\frac{1}{\prod_{j=1}^{3}\langle 2^{\ell_j}\rangle^{b_j}}\sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 1 or Case 2}}}2^{(\alpha+\gamma)k_1}2^{\beta k_2} \times\sup_{\substack{v_2\in V\\ v_1,v_3\in\Sigma}}\left\|\chi_{k_1,k_2}\prod_{j=1}^{3}\chi_{|\xi_j-v_j|\ll 2^{k_2}}\chi_{\ell_j}\right\|_{[3;\mathbb{R}^{3+1}]}, \tag{4.24}$$

where Σ is a tiling lattice induced by V. The term (4.24) is comparable to

$$\sum_{\ell_1,\ell_2,\ell_3\in\mathbb{N}_0}\sum_{k_2\in\mathbb{Z}}\frac{1}{\prod_{j=1}^{3}\langle 2^{\ell_j}\rangle^{b_j}}\sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 1 or Case 2}}}2^{(\alpha+\gamma)k_1}2^{\beta k_2} \times\sup_{\substack{v_1,v_3\in\Sigma,v_2\in V\\ |v_1|\sim|v_3|\sim 2^{k_1}}}\left\|\prod_{j=1}^{3}\chi_{|\xi_j-v_j|\ll 2^{k_2}}\chi_{\ell_j}\right\|_{[3;\mathbb{R}^{3+1}]}. \tag{4.25}$$

In (4.25), we divide the range of ℓ_1 and ℓ_3 into $\ell_1 \le \ell_3$ and $\ell_3 \le \ell_1$.

First we consider the case $\ell_1 \le \ell_3$. Then by the Comparison principle (Lemma 2.1), (4.25) is estimated by

$$\sum_{\substack{\ell_1,\ell_2,\ell_3\in\mathbb{N}_0\\ \ell_1\le\ell_3}} \sum_{k_2\in\mathbb{Z}} \frac{1}{\prod_{j=1}^3 \langle 2^{\ell_j}\rangle^{b_j}} \sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 1 or Case 2}}} 2^{(\alpha+\gamma)k_1} 2^{\beta k_2} \times \sup_{\substack{v_1,v_3\in\Sigma, v_2\in V\\ |v_1|\sim|v_3|\sim 2^{k_1}}} \left\| \prod_{j=1}^{2} \chi_{|\xi_j - v_j|\ll 2^{k_2}} \chi_{\ell_j} \right\|_{[3;\mathbb{R}^{3+1}]}. \tag{4.26}$$

Now we consider the Case 1. Note that, since $|v_1| \sim 2^{k_1} \gtrsim 2^{k_2}$ and $2^{k_1} \gg 1$, from $|\xi_1 - v_1| \ll 2^{k_2}$ it follows that $|\xi_1| \sim 2^{k_1} \gg 1$. Thus we have

$$\left| D_{v_1}|\xi_1|^2 - D_{v_1}(\pm|\xi_2|) \right| \gtrsim |\xi_1| \sim 2^{k_1}$$

for all $\xi_1 \in \{|\xi| \sim 2^{k_1}\}$ and $\xi_2 \in \{|\xi| \sim 2^{k_2}\}$, and therefore by the transverse interaction (Lemma 2.6), (4.26) with Case 1 is estimated by

$$\begin{aligned}
&\sum_{\substack{\ell_1,\ell_2,\ell_3\in\mathbb{N}_0\\ \ell_1\le\ell_3}} \sum_{k_2\in\mathbb{Z}} \frac{1}{2^{(b_1+b_3-)\ell_1} 2^{b_2\ell_2} 2^{(0+)\ell_3}} \\
&\qquad \times \sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 1}}} 2^{(\alpha+\gamma)k_1} 2^{\beta k_2} 2^{\ell_1/2} 2^{\ell_2/2} 2^{-k_1/2} 2^{k_2} \\
&\lesssim \sum_{\ell_1,\ell_2\in\mathbb{N}_0} \sum_{k_2\ge 0} \sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 1}}} 2^{(1/2-b_1-b_3+)\ell_1} 2^{(1/2-b_2)\ell_2} \\
&\qquad \times 2^{(\alpha+\gamma-1/2+\delta)k_1} 2^{(\beta+1-2\delta)k_2} \\
&\quad + \sum_{\ell_1,\ell_2\in\mathbb{N}_0} \sum_{k_2\le -1} \sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 1}}} 2^{(1/2-b_1-b_3+)\ell_1} 2^{(1/2-b_2)\ell_2} \\
&\qquad \times 2^{(\alpha+\gamma-1/2+\delta')k_1} 2^{(\beta+1+\delta')k_2},
\end{aligned} \tag{4.27}$$

where we have used the conditions $2^{k_1} \gg 2^{2k_2}$ and $2^{k_2} \gtrsim 1/2^{k_1}$ to derive the first and second term in the R.H.S. of (4.27), respectively. We choose $\delta = (1+\beta)/2$, and $\delta' = |\beta+1|+$ if $\beta \le -1$, $\delta' = 0$ if $\beta > -1$. Then, in the first term above, $\alpha+\gamma-1/2+\delta = \alpha+\gamma+\beta/2 < 0$ and $\beta+1-2\delta = 0$, and, in the second term above, $\alpha+\gamma-1/2+\delta' < 0$ and $\beta+1+\delta' > 0$. Since $2^{k_1} \gg 2^{\ell_{\max}}$ and $1/2 - b_1 - b_3+ < 0$ (if "+" denotes a sufficiently small number), if we take b_2 so close to $1/2$ that $1/2 - b_2 < |\alpha+\gamma-1/2+$

$\delta|/2$, $|\alpha+\gamma-1/2+\delta'|/2$, the first and second terms in the R.H.S. of (4.27) converge.

Next we consider Case 2 with $\ell_1 \le \ell_3$. Then by Lemma 2.3, (4.26) with Case 2 is estimated by

$$\begin{aligned}
&\sum_{\substack{\ell_1,\ell_2,\ell_3\in\mathbb{N}_0\\ \ell_1\le\ell_3}} \sum_{k_2<0} \frac{1}{\prod_{j=1}^3 \langle 2^{\ell_j}\rangle^{b_j}} \sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 2}}} 2^{(\alpha+\gamma)k_1} 2^{\beta k_2} \\
&\qquad\qquad \times \sup_{|v_1|\sim 2^{k_1}} \min_{j=1,2} 2^{\ell_j/2} \big|\{\xi\in\mathbb{R}^3 \,\big|\, |\xi-v_1| \ll 2^{k_2}\}\big|^{1/2} \\
&\lesssim \sum_{\substack{\ell_1,\ell_2,\ell_3\in\mathbb{N}_0\\ \ell_1\le\ell_3}} \sum_{k_2<0} \frac{1}{2^{(b_1+b_3-)\ell_1} 2^{b_2\ell_2} 2^{(0+)\ell_3}} \\
&\qquad\qquad \times \sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 2}}} 2^{(\alpha+\gamma)k_1} 2^{\beta k_2} \min_{j=1,2} 2^{\ell_j/2} 2^{3/2 k_2} \\
&\lesssim \sum_{\substack{\ell_1,\ell_2\in\mathbb{N}_0\\ \ell_1\le\ell_3}} \sum_{k_2<0} \frac{\min_{j=1,2} 2^{\ell_j/2}}{2^{(b_1+b_3-)\ell_1} 2^{b_2\ell_2}} \sup_{\substack{k_1\in\mathbb{N}_0\\ \text{Case 2}}} 2^{(\alpha+\gamma-\delta'')k_1} 2^{(3/2+\beta-\delta'')k_2},
\end{aligned} \tag{4.28}$$

where we have used the condition $2^{k_2} \ll 1/2^{k_1}$ to derive the final inequality. We choose $\delta'' = 1/4$. Then since $\alpha+\gamma < 1/4$ and $1/4 < 3/2+\beta \le 1$, we find that the R.H.S. of (4.28) converges and thus the claim follows for Cases 1 and 2 with $\ell_1 \le \ell_3$.

The case $\ell_1 \ge \ell_3$ follows by changing the role of indices 1, 3 in the above estimate.

Finally we consider Case 3. First suppose that $2^{k_1} \gg 2^{k_3}$. Then the range of k_2 is restricted to $2^{k_2} \sim 2^{k_1}$. Moreover we suppose that $k_1 \gg 1$. The case $k_1 \lesssim 1$ is treated by a similar way to $k_1 \gg 1$, in spite of the singularity $|\xi_2|^\beta$ ($\beta > -5/4$). In this case (i.e. Case 3 with $2^{k_1} \gg 2^{k_3}$ and $k_1 \gg 1$), we have $|\xi_1|^2 \lesssim \max_{1\le j\le 3} |\lambda_j|$. Let $|\lambda_1| = \max_{1\le j\le 3} |\lambda_j|$. Then, since $b_1 - (\alpha+\gamma) > 0$, by the Comparison principle (Lemma 2.1), we estimate (4.19) with this case by

$$\left\| \sum_{\ell_1,\ell_2,\ell_3\in\mathbb{N}_0} \sum_{\substack{k_1,k_3\in\mathbb{N}_0\\ k_2\in\mathbb{Z}}} \frac{|\xi_2|^\beta}{\langle\lambda_1\rangle^{0+}\langle\lambda_2\rangle^{b_2}\langle\lambda_3\rangle^{b_3+b_1-(\alpha+\gamma)-}} \right.$$

$$\times \chi_{k_1}(\xi)\chi_{|\xi_2|\sim 2^{k_2}}(\xi_2)\chi_{k_3}(\xi_3)\prod_{j=1}^{3}\chi_{\ell_j}(\xi_j,\tau_j)\Bigg\|_{[3;\mathbb{R}^{3+1}]}, \quad (4.29)$$

where we have only used the fact that $|\xi_1|, |\xi_3| \lesssim |\lambda_1|$ in order that the estimate in this case is also valid for the case $2^{k_1} \sim 2^{k_3} \lesssim 2^{\ell_{\max}} = \max\{2^{2k_2}, 2^{\ell_{\max}}, 1\}$ below. To estimate (4.29) above, we return to the integral form. Thus our aim is to show that

$$\int_{\Gamma_3(\mathbb{R}^{3+1})} \frac{|\xi_2|^\beta}{\langle\lambda_1\rangle^{0+}\langle\lambda_2\rangle^{b_2}\langle\lambda_3\rangle^{b_3+b_1-(\alpha+\gamma)-}} \prod_{j=1}^{3} f_j(\xi_j,\tau_j) \lesssim \prod_{j=1}^{3} \|f_j\|_{L^2_\tau L^2_\xi} \quad (4.30)$$

for any non-negative functions f_1, f_2, $f_3 \in \mathcal{S}(\mathbb{R}^{3+1})$. By Plancherel's theorem and Hölder's inequality, the L.H.S. of (4.30) is estimated by

$$\begin{aligned}&\left\|\mathcal{F}^{-1}_{\xi,\tau}\left[\langle\tau+|\xi|^2\rangle^{0-}f_1\right]\right\|_{L^{2+}_t L^2_x}\left\|\mathcal{F}^{-1}_{\xi,\tau}\left[|\xi|^\beta\langle\tau\pm|\xi|\rangle^{-b_2}f_2\right]\right\|_{L^{\infty-}_t L^{6/(3+2\beta)}_x}\\ &\quad\times\left\|\mathcal{F}^{-1}_{\xi,\tau}\left[\langle\tau-|\xi|^2\rangle^{-(b_3+b_1-(\alpha+\gamma)-)}f_3\right]\right\|_{L^2_t L^{3/(-\beta)}_x}. \quad (4.31)\end{aligned}$$

Since $b_3 + b_1 - (\alpha+\gamma) > 1/2$ and $12/5 < 3/(-\beta) \le 6$, by the Strichartz type estimate (Lemma 6.1), Sobolev's embedding and the energy estimate (Lemma 6.2), we obtain the desired result.

The case where $|\lambda_3| = \max_{1\le j\le 3}|\lambda_j|$ follows by replacing the role of indices 1, 3. Thus the case where $|\lambda_2| = \max_{1\le j\le 3}|\lambda_j|$ remains. This case is also similar. In particular, we estimate the corresponding integral by

$$\begin{aligned}&\left\|\mathcal{F}^{-1}_{\xi,\tau}\left[\langle\tau+|\xi|^2\rangle^{-b_1}f_1\right]\right\|_{L^{8/3}_t L^{12/(7-8b_1)-}_x}\\ &\quad\times\left\|\mathcal{F}^{-1}_{\xi,\tau}\left[|\xi|^\beta\langle\tau\pm|\xi|\rangle^{-(b_2-(\alpha+\gamma))}f_2\right]\right\|_{L^4_t L^{6/(3+2\beta)}_x}\\ &\quad\times\left\|\mathcal{F}^{-1}_{\xi,\tau}\left[\langle\tau-|\xi|^2\rangle^{-b_3}f_3\right]\right\|_{L^{8/3}_t L^{12/(8b_1+4|\beta|-1)+}_x}. \quad (4.32)\end{aligned}$$

Note that $b_2-(\alpha+\gamma) > 1/4$ if we take $b_2 < 1/2$ sufficiently close to 1/2. To employ the Strichartz type estimate (Lemma 6.1), we require the condition $b_3 > 1 - b_1 - |\beta|/2$. Thus, by Sobolev's embedding, the Strichartz type estimate (Lemma 6.1) and the energy estimate (Lemma 6.2), we easily see that the claim follows in this case.

The case where $2^{k_1} \ll 2^{k_3}$ is similar to the above. Thus we may assume that $2^{k_1} \sim 2^{k_3} \lesssim \max\{2^{2k_2}, 2^{\ell_{\max}}, 1\}$. When $\max\{2^{2k_2}, 2^{\ell_{\max}}, 1\} = 1$ or $2^{\ell_{\max}}$, we can obtain the desired result by a similar way to the above estimate. Thus we assume that $2^{k_1} \sim 2^{k_3} \lesssim 2^{2k_2}$. The case where $2^{k_2} \ll 1$ is similar to the above and easier, and therefore we omit this case. We assume that $2^{k_2} \gtrsim 1$. Then we need to consider the following multiplier norm:

$$\Bigg\| \sum_{\ell_1,\ell_2,\ell_3\in\mathbb{N}_0} \sum_{\substack{k_1,k_3\in\mathbb{N}_0\\ k_2\in\mathbb{Z}}} \frac{|\xi_2|^{\alpha+\gamma+\beta}}{\langle\lambda_1\rangle^{b_1}\langle\lambda_2\rangle^{b_2}\langle\lambda_3\rangle^{b_3}} \times \chi_{k_1}(\xi_1)\chi_{|\xi_2|\sim 2^{k_2}}(\xi_2)\chi_{k_3}(\xi_3)\prod_{j=1}^{3}\chi_{\ell_j}(\xi_j,\tau_j)\Bigg\|_{[3;\mathbb{R}^{3+1}]}. \quad (4.33)$$

Now we set $\beta' := \alpha+\beta+\gamma$. Note that $-5/4 < \beta' < -1/4$. Then the corresponding integral form is estimated by

$$\Big\|\mathcal{F}^{-1}_{\xi,\tau}\big[\langle\tau+|\xi|^2\rangle^{-b_1}f_1\big]\Big\|_{L_t^{2+}L_x^{12/(3-2\beta')}} \times \Big\|\mathcal{F}^{-1}_{\xi,\tau}\big[|\xi|^{\beta'}\langle\tau\pm|\xi|\rangle^{-b_2}f_2\big]\Big\|_{L_t^{\infty-}L_x^{6/(3+2\beta')}} \times \Big\|\mathcal{F}^{-1}_{\xi,\tau}\big[\langle\tau-|\xi|^2\rangle^{-b_3}f_3\big]\Big\|_{L_t^{2}L_x^{12/(3-2\beta')}}.$$

Note that $24/11 < 12/(3-2\beta') < 24/7$. To employ the Strichartz type estimate (Lemma 6.1), we require the condition $b_1,\ b_3 > 3/8+(\alpha+\beta+\gamma)/4$. Then, by Sobolev's embedding, the Strichartz type estimate (Lemma 6.1) and the energy estimate (Lemma 6.2), we obtain the desired result. □

5. Proof of Theorem 1.1

We employ the idea of Bourgain (cf. [3, 15]). Let θ be as in Proposition 4.1. We may assume that $\theta \le 1/10$ (cf. Remark 2).

Let (ψ_0,ϕ_0) be initial data of (WS). We reserve the letters s_1 and s_2 for the regularity of the initial data, so $(\psi_0,\phi_0) \in H^{s_1}\times \dot{H}^{s_2}$.

5.1. Decomposition of the equation

Decompose the initial data of (WS) as follows: $\psi_0 = \psi_0^L + \psi_0^H$ and $\phi_0 = \phi_0^L + \phi_0^H$, where $\mathcal{F}_x[\psi_0^L] = \chi_{|\xi|\le N}\mathcal{F}_x[\psi_0]$ and $\mathcal{F}_x[\phi_0^L] = \chi_{|\xi|\le N}\mathcal{F}_x[\phi_0]$.

Then we easily see that

$$\|\psi_0^L\|_{L^2} \le (2N)^{|s_1|}\|\psi_0\|_{H^{s_1}}, \tag{5.1}$$

$$\|\phi_0^L\|_{L^2} \le N^{|s_2|}\|\phi_0\|_{\dot H^{s_2}}. \tag{5.2}$$

Now we consider the following systems:

$$(L_1)\begin{cases} i\partial_t\psi^{L_1} + \Delta\psi^{L_1} = (\phi^{L_1} + \overline{\phi^{L_1}})\psi^{L_1}, & x \in \mathbb{R}^3, \quad t \ge 0, \\ i\partial_t\phi^{L_1} - |\nabla|\phi^{L_1} = |\nabla|^{-1}(|\psi^{L_1}|^2), & x \in \mathbb{R}^3, \quad t \ge 0, \\ \psi^{L_1}(0) = \psi_0^L, & x \in \mathbb{R}^3, \\ \phi^{L_1}(0) = \phi_0^L, & x \in \mathbb{R}^3, \end{cases}$$

and

$$(H_1)\begin{cases} i\partial_t\psi^{H_1} + \Delta\psi^{H_1} = f_1^{H_1} + f_2^{H_1} + f_3^{H_1}, & x \in \mathbb{R}^3, \quad t \ge 0, \\ i\partial_t\phi^{H_1} - |\nabla|\phi^{H_1} = |\nabla|^{-1}g_1^{H_1} + |\nabla|^{-1}g_2^{H_1}, & x \in \mathbb{R}^3, \quad t \ge 0, \\ \psi^{H_1}(0) = \psi_0^H, & x \in \mathbb{R}^3, \\ \phi^{H_1}(0) = \phi_0^H, & x \in \mathbb{R}^3. \end{cases}$$

where $f_1^{H_1} := (\phi^{H_1} + \overline{\phi^{H_1}})\psi^{H_1}$, $f_2^{H_1} := (\phi^{H_1} + \overline{\phi^{H_1}})\psi^{L_1}$, $f_3^{H_1} := (\phi^{L_1} + \overline{\phi^{L_1}})\psi^{H_1}$ and $g_1^{H_1} := |\psi^{H_1}|^2$, $g_2^{H_1} := 2\Re[\psi^{H_1}\overline{\psi^{L_1}}] = \psi^{H_1}\overline{\psi^{L_1}} + \overline{\psi^{H_1}}\psi^{L_1}$. This problem is equivalent to the original system (WS). Indeed, when (ψ^{L_1}, ϕ^{L_1}) and (ψ^{H_1}, ϕ^{H_1}) are solutions of (L_1) and (H_1), respectively, then $(\psi, \phi) := (\psi^{L_1} + \psi^{H_1}, \phi^{L_1} + \phi^{H_1})$ is a solution of (WS).

Note that since ψ_0^H and ϕ_0^H do not contain the low frequency part $|\xi| \le N$, we have for $s_1 > s_1'$ and $s_2 > s_2'$,

$$\|\psi_0^H\|_{H^{s_1'}} \le N^{-(s_1 - s_1')}\|\psi_0\|_{H^{s_1}}, \tag{5.3}$$

$$\|\phi_0^H\|_{\dot H^{s_2'}} \le N^{-(s_2 - s_2')}\|\phi_0\|_{\dot H^{s_2}}, \tag{5.4}$$

and thus $\psi_0^H \in H^{s_1'}$ and $\phi_0^H \in \dot H^{s_2'}$.

5.2. Regular part (L_1)

We consider the system (L_1). Since the initial data of (L_1) belong to $L^2(\mathbb{R}^3)$, the global well-posedness of (L_1) in L^2 follows (cf. Section 3).

We give the refinement of Proposition 4.1.

Lemma 5.1 *Let $-1/2 < s \le s_1 \le 0$ and $b_1 > 1/4$. Then, taking $b_3 < 1/2$ sufficiently close to $1/2$, we have the following estimates*:

(i) $\left\|\psi(e^{-it|\nabla|}\phi_0)\right\|_{X^{s,-b_3}_{|\xi|^2}} \lesssim \|\psi\|_{X^{s_1,b_1}_{|\xi|^2}}\|\phi_0\|_{L^2}$,

(ii) $\big\|\psi\overline{(e^{-it|\nabla|}\phi_0)}\big\|_{X^{s,-b_3}_{|\xi|^2}} \lesssim \|\psi\|_{X^{s_1,b_1}_{|\xi|^2}}\|\phi_0\|_{L^2}$,
where the implicit constants depending only on s, s_1 and b_1, b_3.

Proof of Lemma 5.1. We do not use the result of [5] (cf. Case 2 in the proof of Proposition 4.1). Since the proof of (ii) is similar to (i), we only give the proof of (i). The result for smaller b_1 implies bigger one and therefore we may assume that $b_1 = 1/4+$. By the same way as the proof of Proposition 4.1 until the formula (4.7), it suffices to show that

$$\int_{\Gamma_3(\mathbb{R}^{3+1})} \delta(\tau_2 \pm |\xi_2|)\frac{\langle\xi_1\rangle^{-s_1}\langle\xi_3\rangle^{s}}{\langle\lambda_1\rangle^{b_1}\langle\lambda_3\rangle^{b_3}}|\psi_1(\xi_1,\tau_1)||\mathcal{F}_x[\phi_0](\xi_2)||\omega_3(\xi_3,\tau_3)| \lesssim \|\psi_1\|_{L^2_\tau L^2_\xi}\|\phi_0\|_{L^2}\|\omega_3\|_{L^2_\tau L^2_\xi}, \tag{5.5}$$

where $\lambda_1 := \tau_1 + |\xi_1|^2$ and $\lambda_3 := \tau_3 - |\xi_3|^2$.

In the case where $|\xi_1| \lesssim |\xi_3|$, we have $\langle\xi_1\rangle^{-s_1}\langle\xi_3\rangle^{s} \lesssim 1$ and thus, by a similar way to Case 1 in the proof of Proposition 4.1 with $s_1 = s_2 = 0$, we obtain the result. So it remains the case where $|\xi_1| \gg |\xi_3|$. In this case, we have $|\xi_1|^2 \lesssim \max\{|\lambda_1|, |\lambda_3|\}$. We first consider the case $|\lambda_1| \le |\lambda_3|$. Then the L.H.S. of (5.5) over the region $|\xi_1| \gg |\xi_3|$ and $|\lambda_1| \le |\lambda_3|$ is estimated by

$$\int_{\Gamma_3(\mathbb{R}^{3+1})} \delta(\tau_2 \pm |\xi_2|)\frac{\langle\xi_3\rangle^{-|s|}}{\langle\lambda_1\rangle^{b_3+}\langle\lambda_3\rangle^{b_1-|s_1|/2-}}|\psi_1(\xi_1,\tau_1)| \times |\mathcal{F}_x[\phi_0](\xi_2)||\omega_3(\xi_3,\tau_3)|, \tag{5.6}$$

where we have used the fact that

$$\frac{1}{\langle\lambda_1\rangle^{b_1}\langle\lambda_3\rangle^{b_3-|s_1|/2}} \le \frac{1}{\langle\lambda_1\rangle^{b_3+}\langle\lambda_3\rangle^{b_1-|s_1|/2-}}$$

on $\{|\lambda_1| \le |\lambda_3|\}$. By Plancherel's theorem and Hölder's inequality (cf. (4.10) and (4.11)), the R.H.S. of (5.6) is estimated by

$$\Big\|\mathcal{F}^{-1}_{\xi,\tau}\big[\langle\tau+|\xi|^2\rangle^{-(b_3+)}|\psi_1|\big]\Big\|_{L^2_tL^6_x}\Big\|\mathcal{F}^{-1}_{\xi}\big[e^{\mp it|\xi|}|\mathcal{F}_x[\phi_0]|\big]\Big\|_{L^\infty_tL^2_x} \times \Big\|\mathcal{F}^{-1}_{\xi,\tau}\big[\langle\tau-|\xi|^2\rangle^{-(b_1-|s_1|/2-)}|\omega_3|\big]\Big\|_{L^2_tL^{3/(1+|s|)}_x}, \tag{5.7}$$

where we have used Sobolev's embedding $L^{3/(1+|s|)} \hookrightarrow \dot{H}^{-|s|,3}$. Taking b_3 sufficiently close to $1/2$, we have $b_3+ > 1/2$ and $b_1 - |s_1|/2- > 1/4+|s_1|/2-|s|/2$. Thus applying Lemmas 6.1 and 6.2 to (5.7), we obtain the result for the case where $|\lambda_1| \le |\lambda_3|$.

In a way similar to the above, we obtain the result for the case where $|\lambda_1| \ge |\lambda_3|$. Hence we have proved the lemma. □

Moreover, we give the following bilinear estimates:

Lemma 5.2 *Let* $-1/2 < s \le s_1 \le 0$ *and* $b_1 > 1/4$*. Then, taking* $b_3 < 1/2$ *sufficiently close to* $1/2$*, we have*

(i) $\|\psi\phi\|_{X^{s,-b_3}_{|\xi|^2}} \lesssim \|\psi\|_{X^{s_1,b_1}_{|\xi|^2}} \|\phi\|_{\dot X^{0,1/2+}_{\pm|\xi|}}$,

(ii) $\|\psi\overline{\phi}\|_{X^{s,-b_3}_{|\xi|^2}} \lesssim \|\psi\|_{X^{s_1,b_1}_{|\xi|^2}} \|\phi\|_{\dot X^{0,1/2+}_{\pm|\xi|}}$.

Moreover, let I *be a time interval with* $|I| < 1$*. If* $0 < b < 1/2$*, then we have*

(iii) $\big\|G_2^{(I)}[|\nabla|^{-1}|\psi|^2]\big\|_{\dot X^{0,1/2+b}_{\pm|\xi|}(I)} \lesssim \|\psi\|^2_{X^{0,1/8+b/2+}_{|\xi|^2}(I)}$.

Remark 4 From Lemma 5.2, we find that the solution $(\psi^{L_1}, \phi^{L_1}) \in X^{0,1/2+}_{|\xi|^2}(I) \times \dot X^{0,1-}_{\pm|\xi|}(I)$.

Proof of Lemma 5.2. The proofs of (i) and (ii) are the same as Lemma 5.1 except for the difference ϕ and $e^{\mp t|\nabla|}\phi_0$. So we omit the proofs.

Next we consider (iii). By Lemma 2.1 (ii) in [11], the L.H.S. of (iii) is estimated by

$$\||\nabla|^{-1}|\psi|^2\|_{\dot X^{0,-1/2+b}_{\pm|\xi|}(I)}.$$

Then, to obtain the desired result, it suffices to show that

$$\begin{aligned}\int_{\Gamma_3(\mathbb{R}^{3+1})} & \frac{|\xi_2|^{-1}}{\langle\lambda_1\rangle^{1/8+b/2}\langle\lambda_2\rangle^{1/2-b}\langle\lambda_3\rangle^{1/8+b/2}} \\ & \times |\psi_1(\xi_1,\tau_1)|\,|\omega_2(\xi_2,\tau_2)|\,|\psi_3(\xi_3,\tau_3)| \\ & \quad \lesssim \|\psi_1\|_{L^2_\tau L^2_\xi}\|\omega_2\|_{L^2_\tau L^2_\xi}\|\psi_3\|_{L^2_\tau L^2_\xi}, \end{aligned} \tag{5.8}$$

where $\lambda_1 := \tau_1 + |\xi_1|^2$, $\lambda_1 := \tau_2 \pm |\xi_2|$ and $\lambda_3 := \tau_3 - |\xi_3|^2$.

By Plancherel's theorem and Hölder's inequality (cf. (4.10) and (4.11)), we have

$$\begin{aligned}&\Big\|\mathcal{F}^{-1}_{\xi,\tau}\big[\langle\tau+|\xi|^2\rangle^{-(1/8+b/2)}|\psi_1|\big]\Big\|_{L_t^{2/(1-b)+}L_x^{12/5}} \\ &\quad \times \Big\|\mathcal{F}^{-1}_{\xi}\big[\langle\tau\pm|\xi|\rangle^{-(1/2-b)}|\mathcal{F}_x[\omega_2]|\big]\Big\|_{L_t^{1/b-}L_x^2}\end{aligned}$$

$$\times \left\| \mathcal{F}^{-1}_{\xi,\tau} \big[\langle \tau - |\xi|^2 \rangle^{-(1/8+b/2-)} |\psi_3| \big] \right\|_{L_t^{2/(1-b)+} L_x^{12/5}}, \tag{5.9}$$

where we have used Sobolev's embedding $L^2 \hookrightarrow \dot{H}^{-1,6}$ in the second factor. Applying Lemmas 6.1 and 6.2 to (5.9), we obtain the result. □

Now let (ψ^{L_1}, ϕ^{L_1}) be the solution of (L_1).

We set

$$\delta_{L_1} := \min\big\{ [C_1(\|\phi_0^L\|_{L^2} + \|\psi_0^L\|_{L^2})]^{-2-}, 1 \big\} \tag{5.10}$$

where C_1 is some large universal constant. Then we show that

$$\|\phi^{L_1}\|_{X^{0,3/4}_{\pm|\xi|}([0,\delta_{L_1}])} \lesssim \|\psi_0^L\|_{L^2} + \|\phi_0^L\|_{L^2}. \tag{5.11}$$

We prove (5.11). Applying Lemma 5.2 (i) and (iii) to the integral equation (2.4), we have

$$\begin{aligned} &\|\psi^{L_1}\|_{X^{0,1/2+}_{|\xi|^2}([0,\delta_{L_1}])} + \|\phi^{L_1}\|_{X^{0,3/4}_{\pm|\xi|}([0,\delta_{L_1}])} \\ &\quad \lesssim \|\psi_0^L\|_{L^2} + \|\phi_0^L\|_{L^2} + \|\psi^{L_1}\|_{X^{0,1/4+}_{|\xi|^2}([0,\delta_{L_1}])} \|\phi^{L_1}\|_{X^{0,1/2+}_{\pm|\xi|}([0,\delta_{L_1}])} \\ &\qquad + \|\psi^{L_1}\|^2_{X^{0,1/4+}_{|\xi|^2}([0,\delta_{L_1}])}. \end{aligned} \tag{5.12}$$

By Lemma 6.3, the R.H.S. of (5.12) is estimated by

$$\begin{aligned} &\|\psi_0^L\|_{L^2} + \|\phi_0^L\|_{L^2} \\ &\quad + \delta_{L_1}^{1/2-} \left(\|\psi^{L_1}\|_{X^{0,1/2+}_{|\xi|^2}([0,\delta_{L_1}])} + \|\phi^{L_1}\|_{X^{0,3/4+}_{\pm|\xi|}([0,\delta_{L_1}])} \right)^2 \end{aligned} \tag{5.13}$$

Considering a quadratic inequality $x \lesssim a + \delta^{1/2-} x^2$, we obtain the claim (5.11).

Moreover, taking C_1 larger, we have

$$\|\psi^{L_1}\|_{X^{0,1/2+}_{|\xi|^2}([0,\delta_{L_1}])} \lesssim \|\psi_0^L\|_{L^2}. \tag{5.14}$$

Indeed, by Lemma 5.1 (i),

$$\begin{aligned} &\|\psi^{L_1}\|_{X^{0,1/2+}_{|\xi|^2}([0,\delta_{L_1}])} \\ &\quad \lesssim \|\psi_0^L\|_{L^2} + \delta_{L_1}^{1/2-} \|\phi^{L_1}\|_{X^{0,3/4+}_{\pm|\xi|}([0,\delta_{L_1}])} \|\psi^{L_1}\|_{X^{0,1/2+}_{|\xi|^2}([0,\delta_{L_1}])}. \end{aligned} \tag{5.15}$$

By (5.11) and (5.10), taking larger C_1 if necessary, we find that the second term in the R.H.S. of (5.15) is $\ll \|\psi^{L_1}\|_{X^{0,1/2+}_{|\xi|^2}([0,\delta_{L_1}])}$. Hence we obtain the claim (5.14).

5.3. Rough part (H_1) and local well-posedness below L^2

We consider (H_1). As stated above, if (ψ^{L_1},ϕ^{L_1}) and (ψ^{H_1},ϕ^{H_1}) are solutions of (L_1) and (H_1), respectively, then $(\psi,\phi) := (\psi^{L_1}+\psi^{H_1},\phi^{L_1}+\phi^{H_1})$ is the solution of (WS). In the previous subsection, we have already proved the local well-posedness of (L_1) and therefore the local well-posedness of (H_1) implies one of (WS).

Let (ψ^{L_1},ϕ^{L_1}) be the solution of (L_1) (cf. Section 5.2). We use I to denote a time interval with $|I|<1$.

We first consider the part of wave equation. Take any $\psi \in X^{s_1,1/2+}_{|\xi|^2}(I)$. We set ϕ_ψ as follows:

$$\phi_\psi = e^{-it|\nabla|}\phi_0^H + iG_2^{(I)}[|\nabla|^{-1}|\psi|^2] + 2iG_2^{(I)}[|\nabla|^{-1}\Re[\psi\overline{\psi^{L_1}}]],$$

which corresponds to the integral equation for the part of wave equation in (H_1) (cf. (2.4) and (2.5)). If $s_1 > -1/8$ and $s_2 > -1/4$, then by Proposition 4.2 we have

$$\begin{aligned}\|\phi_\psi\|_{\dot X^{s_2,1/2+}_{|\xi|}(I)} \lesssim \|\phi_0^H\|_{\dot H^{s_2}} + \|\psi\|^2_{X^{s_1,3/8+}_{|\xi|^2}(I)} \\ + \|\psi^{L_1}\|_{X^{0,3/8+}_{|\xi|^2}(I)}\|\psi\|_{X^{s_1,3/8+}_{|\xi|^2}(I)}.\end{aligned} \tag{5.16}$$

Thus we find that $\phi_\psi \in \dot X^{s_2,1/2+}_{|\xi|}(I)$ for any $\psi \in X^{s_1,1/2+}_{|\xi|^2}(I)$. Moreover, we show that the interaction term of the integral equation associated to the wave part of (H_1) belongs to $\dot X^{0,1/2+}_{|\xi|}(I)$ and thus $C_tL^2_x(I)$. If $s_1' > -1/8$, then by the boundedness of G_2, Proposition 4.2 with $\alpha=\gamma=|s_1'|$, $\beta=-1$, $b_1=b_3=1/4+|s_1'|+$, and Lemma 6.3 we have

$$\big\|G_2^{(I)}[|\nabla|^{-1}|\psi|^2]\big\|_{\dot X^{0,1/2+}_{|\xi|}(I)} \lesssim |I|^{\frac12-2|s_1'|-}\|\psi\|^2_{X^{s_1',1/2+}_{|\xi|^2}(I)}. \tag{5.17}$$

Similarly we have, for $s_1' > -1/8$,

$$\begin{aligned}&\big\|G_2^{(I)}[|\nabla|^{-1}\Re[\psi\overline{\psi^{L_1}}]]\big\|_{\dot X^{0,1/2+}_{|\xi|^2}(I)} \\ &\qquad \lesssim |I|^{1/2-|s_1'|-}\|\psi^{L_1}\|_{X^{0,1/2+}_{|\xi|^2}(I)}\|\psi\|_{X^{s_1',1/2+}_{|\xi|^2}(I)}.\end{aligned} \tag{5.18}$$

Next we consider the part of Schrödinger equation. We work in a general setting. For given ψ, we set $f_1 := (\phi_\psi + \overline{\phi_\psi})\psi$, $f_2 := (\phi_\psi + \overline{\phi_\psi})\psi^{L_1}$, $f_3 := (\phi^{L_1} + \overline{\phi^{L_1}})\psi$. Then, by Propositions 4.1, 4.2 and Lemma 6.3, for s_1', s_1'' and s_2' with $0 \geq s_1'$, $s_1'' > -1/8$ and $\theta + |s_1'| > |s_1''| + |s_2'|$, we have

$$\begin{aligned}\|f_1\|_{X^{s_1',-\frac{1}{2}+}_{|\xi|^2}(I)} &\lesssim |I|^{\theta/2+|s_1'|/2-|s_1''|/2-|s_2'|/2-}\|\psi\|_{X^{s_1'',1/2+}_{|\xi|^2}(I)}\|\phi_0^H\|_{\dot{H}^{s_2'}}\\ &+|I|^{1/2-|s_1''|/4-}\|\psi\|^3_{X^{s_1'',1/2+}_{|\xi|^2}(I)}\\ &+|I|^{1/2-|s_1''|/4-}\|\psi\|^2_{X^{s_1'',1/2+}_{|\xi|^2}(I)}\|\psi^{L_1}\|_{X^{0,1/2+}_{|\xi|^2}(I)},\end{aligned} \tag{5.19}$$

$$\begin{aligned}\|f_2\|_{X^{s_1',-1/2+}_{|\xi|^2}(I)} &\lesssim |I|^{\theta/2+|s_1'|/2-|s_2'|/2-}\|\psi^{L_1}\|_{X^{0,1/2+}_{|\xi|^2}(I)}\|\phi_0^H\|_{\dot{H}^{s_2'}}\\ &+|I|^{1/2-}\|\psi\|^2_{X^{s_1'',1/2+}_{|\xi|^2}(I)}\|\psi^{L_1}\|_{X^{0,1/2+}_{|\xi|^2}(I)}\\ &+|I|^{1/2-}\|\psi\|_{X^{s_1'',1/2+}_{|\xi|^2}(I)}\|\psi^{L_1}\|^2_{X^{0,1/2+}_{|\xi|^2}(I)},\end{aligned} \tag{5.20}$$

and, if $s_1' \geq s_1''$, from $\phi^{L_1} = e^{-it|\nabla|}\phi_0^L + iG_2^{(I)}[|\nabla|^{-1}|\psi^{L_1}|]$,

$$\begin{aligned}\|f_3\|_{X^{s_1',-1/2+}_{|\xi|^2}(I)} &\lesssim |I|^{\theta/2+|s_1'|/2-|s_1''|/2-}\|\psi\|_{X^{s_1'',1/2+}_{|\xi|^2}(I)}\|\phi_0^L\|_{L^2}\\ &+|I|^{1/2-|s_1''|/4-}\|\psi\|_{X^{s_1'',1/2+}_{|\xi|^2}(I)}\|\psi^{L_1}\|^2_{X^{0,1/2+}_{|\xi|^2}(I)}\end{aligned} \tag{5.21}$$

and if $s_1' \leq s_1''$, by Lemma 5.2 and Lemma 6.3,

$$\|f_3\|_{X^{s_1',-1/2+}_{|\xi|^2}(I)} \lesssim |I|^{1/2-}\|\psi\|_{X^{s_1'',1/2+}_{|\xi|^2}(I)}\|\phi^{L_1}\|_{\dot{X}^{0,3/4}_{\pm}(I)}, \tag{5.22}$$

where we can take the implicit constants independent of s_1', s_1'', s_2' in (5.19), (5.20), (5.21) and (5.22). Hence, from (5.19), (5.20) and (5.21) with $s_1' = s_1'' = s_1$ and $s_2' = s_2$, by (5.16) and the standard contraction argument, local well-posedness of (H_1) follows and therefore one of (WS) holds for s_1, $s_2 \leq 0$ with $|s_1| < 1/8$ and $|s_2| < \theta$.

5.4. The iteration process and global well-posedness below L^2

From the preceding subsections, we find that, if s_1, $s_2 < 0$ satisfy that $s_1 > -1/8$ and $\theta > |s_2|$, then there exist solutions of (L_1) and (H_1) on some time interval $[0, \delta_0]$.

We set $I_j := [(j-1)\delta, j\delta]$ and $\delta_j := j\delta$, $j \in \mathbb{N}$, for some $0 < \delta < 1$, which is determined below. Note that $|I_j| = \delta$ for all $j \in \mathbb{N}$. Then, for $j \geq 2$ ($j \in \mathbb{N}$), we consider the following systems:

$$
(L_j) \begin{cases} i\partial_t \psi^{L_j} + \Delta \psi^{L_j} = (\phi^{L_j} + \overline{\phi^{L_j}})\psi^{L_j}, & x \in \mathbb{R}^3, \quad t \in I_j, \\ i\partial_t \phi^{L_j} - |\nabla|\phi^{L_j} = |\nabla|^{-1}(|\psi^{L_j}|^2), & x \in \mathbb{R}^3, \quad t \in I_j, \\ \psi^{L_j}(\delta_{j-1}) = \psi^{L_{j-1}}(\delta_{j-1}) \\ \qquad\qquad + G_1^{(I_{j-1})}\big[f_1^{H_{j-1}} + f_2^{H_{j-1}} + f_3^{H_{j-1}}\big](\delta_{j-1}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^3, \\ \phi^{L_j}(\delta_{j-1}) = \phi^{L_{j-1}}(\delta_{j-1}) + G_2^{(I_{j-1})}\big[g_1^{H_{j-1}} + g_2^{H_{j-1}}\big](\delta_{j-1}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x \in \mathbb{R}^3, \end{cases}
$$

and

$$
(H_j) \begin{cases} i\partial_t \psi^{H_j} + \Delta \psi^{H_j} = f_1^{H_j} + f_2^{H_j} + f_3^{H_j}, & x \in \mathbb{R}^d, \quad t \in I_j, \\ i\partial_t \phi^{H_j} - |\nabla|\phi^{H_j} = |\nabla|^{-1} g_1^{H_j} + |\nabla|^{-1} g_2^{H_j}, & x \in \mathbb{R}^d, \quad t \in I_j, \\ \psi^{H_1}(\delta_{j-1}) = e^{i\delta_{j-1}\Delta}\psi_0^H, & x \in \mathbb{R}^3, \\ \phi^{H_1}(\delta_{j-1}) = e^{-i\delta_{j-1}|\nabla|}\phi_0^H, & x \in \mathbb{R}^3. \end{cases}
$$

where $f_1^{H_j} := (\phi^{H_j} + \overline{\phi^{H_j}})\psi^{H_j}$, $f_2^{H_j} := (\phi^{H_j} + \overline{\phi^{H_j}})\psi^{L_j}$, $f_3^{H_j} := (\phi^{L_j} + \overline{\phi^{L_j}})\psi^{H_j}$ and $g_1^{H_j} := |\psi^{H_j}|^2$, $g_2^{H_j} := 2\Re[\psi^{H_j}\overline{\psi^{L_j}}] = \psi^{H_j}\overline{\psi^{L_j}} + \overline{\psi^{H_j}}\psi^{L_j}$ and $G_1^{(I_{j-1})}$, $G_2^{(I_{j-1})}$ are the integral operators defined in (2.5).

For the solutions (ψ^{L_j}, ϕ^{L_j}), (ψ^{H_j}, ϕ^{H_j}), $(\psi, \phi) := (\psi^{L_j} + \psi^{H_j}, \phi^{L_j} + \phi^{H_j})$ is a solution of (WS) on I_j.

Note that, in (H_j), H^{s_j}-norm, $j = 1, 2$, of the initial data remain unchanged at each step, in particular we have

$$
\|\psi^{H_j}(\delta_{j-1})\|_{H^{s_1'}} = \|\psi_0^H\|_{H^{s_1'}}, \quad \|\phi^{H_j}(\delta_{j-1})\|_{\dot{H}^{s_2'}} = \|\phi_0^H\|_{\dot{H}^{s_2'}} \tag{5.23}
$$

for all $s_1 \geq s_1'$ and $s_2 \geq s_2'$.

Suppose that the solutions of (L_j) and (H_j) exist on I_j for all $j \in \mathbb{N}$. Then from (5.19), (5.20), (5.22), where replace L_1 with L_j, and from (5.23), we have, for $0 \geq s_1', s_2', s_2''$ with $s_1' > -1/8$, $\theta > |s_2'|, |s_2''|$,

$$
\begin{aligned} &\|\psi^{H_j}\|_{X^{s_1', 1/2+}_{|\xi|^2}(I_j)} \\ &\quad \lesssim \|\psi_0^H\|_{H^{s_1'}} + \delta^{\theta/2 + |s_1'|/2 - |s_2'|/2 -}\|\psi^{L_j}\|_{X^{0,1/2+}_{|\xi|^2}(I_j)}\|\phi_0^H\|_{\dot{H}^{s_2'}} \end{aligned}
$$

$$
\begin{aligned}
&+ \Big(\delta^{\theta/2-|s_2''|/2-} \|\phi_0^H\|_{\dot H^{s_2''}} + \delta^{1/2-} \|\psi^{L_j}\|^2_{X^{0,1/2+}_{|\xi|^2}(I_j)} \\
&\quad + \delta^{1/2-} \|\phi^{L_j}\|_{\dot X^{0,3/4}_{\pm|\xi|}(I_j)} \Big) \|\psi^{H_j}\|_{X^{s_1',1/2+}_{|\xi|^2}(I_j)} \\
&+ \delta^{1/2-|s_1'|/4-} \|\psi^{L_j}\|_{X^{0,1/2+}_{|\xi|^2}(I_j)} \|\psi^{H_j}\|^2_{X^{s_1',1/2+}_{|\xi|^2}(I_j)} \\
&+ \delta^{1/2-|s_1'|/4-} \|\psi^{H_j}\|^3_{X^{s_1',1/2+}_{|\xi|^2}(I_j)},
\end{aligned} \tag{5.24}
$$

where we can take the implicit constant independent of s_1', s_2', s_2''.

In what follows, we may assume that s_1 and s_2 are not zero. Indeed, by the regularity result, we will obtain the global well-posedness for the case where s_1 or s_2 is zero.

We use notations $D := \|\psi_0\|_{H^{s_1}} + \|\phi_0\|_{\dot H^{s_2}}$ and $D_1 := \max\{D, 1\}$. Now take any $T > 1$. Since the initial data for rough parts remain unchanged, to continue the solution of (WS) until T, it suffices to show that, under the suitable choice of N and δ, we have

$$
\begin{aligned}
&\|\psi^{L_j}(\delta_{j-1})\|_{L^2} \le \mathscr{C} D_1 N^{|s_1|}, \\
&\|\phi^{L_j}(\delta_{j-1})\|_{L^2} \le \mathscr{C}^3 D_1^2 \big(N^{|s_2|} + \delta^{-1/4} N^{2|s_1|}\big)
\end{aligned} \tag{5.25}
$$

for all $T/\delta \ge j \ge 2$ and some constant $\mathscr{C} > 1$ which is determined below.

To show the uniform bound (5.25), we first consider the increment of L^2-norms of initial data at each step under the assumption such that (5.25) holds for all $j \ge 2$ (cf. [1]). After the consideration, we determined the parameters N, δ and $\mathscr{C}$ such that (5.25) holds for $T/\delta \ge j \ge 2$.

Now we assume that (5.25) holds for all $j \ge 2$. Then we obtain the global solutions of (L_j) and therefore (5.24) still holds. Now we set $A := \mathscr{C}^3 D_1^2 (N^{|s_2|} + \delta^{-1/4} N^{2|s_1|})$. Note that $\|\phi^{L_j}(\delta_{j-1})\|_{L^2} \le A$ and $\|\psi^{L_j}(\delta_{j-1})\|_{L^2} + \|\phi^{L_j}(\delta_{j-1})\|_{L^2} \lesssim A$. Then, by the same way as the derivation of (5.10), (5.11) and (5.14), we have

$$
\|\psi^{L_j}\|_{X^{0,1/2+}_{|\xi|^2}(I_j)} \lesssim \mathscr{C} D_1 N^{|s_1|}, \quad \|\phi^{L_j}\|_{\dot X^{0,3/4}_{\pm|\xi|}(I_j)} \lesssim A \tag{5.26}
$$

for all $j \ge 1$, if $|I_j| = \delta$ is so small that

$$
\delta \le (C_2 A)^{-2-}, \tag{5.27}
$$

where C_2 is some large universal constant. For (5.27), it is sufficient to take

$$\delta \le \delta_L(s_1, s_2, c) := c \min\{N^{-8|s_1|-}, N^{-2|s_2|-}\} \tag{5.28}$$

for any $c \le (2C_2 \mathscr{C}^3 D_1^2)^{-2-}$. Now we take

$$c_2 \ll (2C_2 \mathscr{C}^3 D_1^2)^{-2/(\theta-|s_2|)-} \tag{5.29}$$

and $|I_j| = \delta \le \delta_L(s_1, s_2, c_2)$. From (5.24) replacing s_1' with s_1 and s_2', s_2'' with s_2, and from the assumption and (5.26), it follows that, for $0 \ge s_1$, s_2 with $|s_1| < 1/8$ and $|s_2| < \theta$,

$$\begin{aligned}
&\|\psi^{H_j}\|_{X^{s_1,1/2+}(I_j)} \\
&\quad \lesssim D + \delta^{\theta/2+|s_1|/2-|s_2|/2-} \mathscr{C} D_1^2 N^{|s_1|} \\
&\qquad + \Big(\delta^{\theta/2-|s_2|/2-} D + \delta^{1/2-} \mathscr{C}^2 D_1^2 N^{2|s_1|} + \delta^{1/2-} A\Big) \\
&\qquad\quad \times \|\psi^{H_j}\|_{X^{s_1,1/2+}_{|\xi|^2}(I_j)} \\
&\qquad + \delta^{1/2-|s_1|/4-} \mathscr{C} D_1 N^{|s_1|} \|\psi^{H_j}\|^2_{X^{s_1,1/2+}_{|\xi|^2}(I_j)} \\
&\qquad + \delta^{1/2-|s_1|/4-} \|\psi^{H_j}\|^3_{X^{s_1,1/2+}_{|\xi|^2}(I_j)},
\end{aligned} \tag{5.30}$$

where we have used the fact that $\|\psi_0^H\|_{H^{s_1}}$, $\|\phi_0^H\|_{\dot{H}^{s_2}} \le D$. By the choice of δ, the first factor of the third term $\ll 1$ and thus we have

$$\begin{aligned}
&\|\psi^{H_j}\|_{X^{s_1,1/2+}(I_j)} \\
&\quad \lesssim D + \mathscr{C} D_1^2 \delta^{\theta/2+|s_1|/2-|s_2|/2-} N^{|s_1|} \\
&\qquad + \mathscr{C} D_1 \delta^{1/2-|s_1|/4-} N^{|s_1|} \|\psi^{H_j}\|^2_{X^{s_1,1/2+}_{|\xi|^2}(I_j)} \\
&\qquad + \delta^{1/2-|s_1|/4-} \|\psi^{H_j}\|^3_{X^{s_1,1/2+}_{|\xi|^2}(I_j)}.
\end{aligned} \tag{5.31}$$

By Young's inequality and (5.28), the third term is bounded by

$$\delta^{1/4-|s_1|/4-} \mathscr{C} D_1 N^{|s_1|} + \delta^{1/2-|s_1|/4-} \|\psi^{H_j}\|^3_{X^{s_1,1/2+}_{|\xi|^2}(I_j)}.$$

Since $\theta < 1/10$, we have $\delta^{1/4-|s_1|/4} \le \delta^{\theta/2+|s_1|/2-|s_2|/2} \le \mathscr{C}^{-3} D_1^{-2}$ (cf. (5.29)) and thus the R.H.S. of (5.31) is estimated by

$$N^{|s_1|} + \delta^{1/2-|s_1|/4-} \|\psi^{H_j}\|^3_{X^{s_1,1/2+}_{|\xi|^2}(I_j)}. \tag{5.32}$$

Considering a cubic inequality $x \lesssim a + \delta^{1/2-|s_1|/4-} x^3$, we find that, if $|I_j| = \delta \le \delta_L(s_1, s_2, c_2)$, then for $0 \ge s_1 > -1/8$ and $0 \ge s_2 > -\theta$ we have

$$\|\psi^{H_j}\|_{X^{s_1,1/2+}_{|\xi|^2}(I_j)} \lesssim N^{|s_1|}. \tag{5.33}$$

Now we show that, for $0 \ge s_1,\ s_2 > -\theta/2$ and $|I_j| = \delta \le \delta_L(s_1, s_2, c_2)$,

$$\begin{aligned}
&\|\psi^{H_j}\|_{X^{-\theta/2+,1/2+}_{|\xi|^2}(I_j)} \\
&\quad \lesssim DN^{-\eta_1+} + \mathscr{C} D_1^2 \delta^{\theta/2-} N^{-\eta_2+|s_1|+} \\
&\qquad + \mathscr{C} D_1^3 \delta^{\eta-} N^{-2\eta_1+|s_1|+} + \mathscr{C}^3 D_1^5 \delta^{\eta+\theta-} N^{-2\eta_2+3|s_1|+} \\
&\qquad + D^3 \delta^{\eta-} N^{-3\eta_1+} + \mathscr{C}^3 D_1^6 \delta^{\eta+3\theta/2-} N^{-3\eta_2+3|s_1|+} \\
&\qquad + (\mathscr{C} D_1)^3 \delta^{3\eta-} N^{7|s_1|},
\end{aligned} \tag{5.34}$$

where $\eta := 1/2 - \theta/8$, $\eta_1 := \theta/2 - |s_1|$ and $\eta_2 := \theta/2 - |s_2|$.

Indeed, setting $X_j := X^{-\theta/2+,1/2+}_{|\xi|^2}(I_j)$, by (5.24) replacing s_1', s_2' with $-\theta/2+$ and s_2'' with s_2, and by (5.3), (5.4), (5.26) and the assumption,

$$\begin{aligned}
\|\psi^{H_j}\|_{X_j} \lesssim\, & N^{-(\theta/2-|s_1|)+} D + \delta^{\theta/2-} N^{-(\theta/2-|s_2|)+} \mathscr{C} D_1^2 N^{|s_1|} \\
& + \big(\delta^{\theta/2-|s_2|/2-} D + \delta^{1/2-} \mathscr{C}^2 D_1^2 N^{2|s_1|} + \delta^{1/2-} A\big) \|\psi^{H_j}\|_{X_j} \\
& + \mathscr{C} D_1 \delta^{1/2-\theta/8-} N^{|s_1|} \|\psi^{H_j}\|_{X_j}^2 + \delta^{1/2-\theta/8-} \|\psi^{H_j}\|_{X_j}^3.
\end{aligned} \tag{5.35}$$

Set $\eta := 1/2 - \theta/8$, $\eta_1 := \theta/2 - |s_1|$ and $\eta_2 := \theta/2 - |s_2|$. By the choice of δ (see (5.28) and (5.29)), we find that the first factor of third term in the R.H.S. of (5.35) $\ll 1$. Thus we have

$$\begin{aligned}
\|\psi^{H_j}\|_{X_j} \lesssim\, & DN^{-\eta_1+} + \mathscr{C} D_1^2 \delta^{\theta/2-} N^{-\eta_2+|s_1|+} \\
& + \mathscr{C} D_1 \delta^{\eta-} N^{|s_1|} \|\psi^{H_j}\|_{X_j}^2 + \delta^{\eta-} \|\psi^{H_j}\|_{X_j}^3.
\end{aligned} \tag{5.36}$$

Inserting the R.H.S. of (5.36) into $\|\psi^{H_j}\|_{X_j}$ in the R.H.S. of (5.36) and using $\|\psi^{H_j}\|_{X_j} \le \|\psi^{H_j}\|_{X^{s_1,\frac12+}_{|\xi|^2}(I_j)} \lesssim N^{|s_1|}$ (cf. (5.33)) and $\delta^\eta \le N^{-2|s_1|}$ (cf. (5.28)), we obtain the claim (5.34).

Now we consider the increment of L^2-norm under the assumption. We take $\delta \sim \delta_L(s_1, s_2, c_2)$.

By the L^2-conservation law, we have

$$\|\psi^{L_{j+1}}(\delta_j)\|_{L^2} - \|\psi^{L_j}(\delta_{j-1})\|_{L^2} \leq \left\| G_1^{(I_j)}\big[f_1^{H_1} + f_2^{H_1} + f_3^{H_1}\big](\delta_{j-1})\right\|_{L^2}. \tag{5.37}$$

Applying (5.19) with $s_1' = 0$, $s_1'' = -\theta/2+$, $s_2' = s_2$, (5.20) with $s_1' = 0$, $s_1'' = -\theta/2+$, $s_2' = -\theta/2+$, (5.21) with $s_1' = 0$, $s_1'' = -\theta/2+$, (5.26) and the assumption to R.H.S. of (5.37), if $0 \geq s_1$, $s_2 > -\theta/2$, for all $j \geq 1$, we have

$$\|\psi^{L_{j+1}}(\delta_j)\|_{L^2} - \|\psi^{L_j}(\delta_{j-1})\|_{L^2} \leq C_1^*(\mathscr{C} D_1)^2(P_0 + P_1 + P_2 + P_3), \tag{5.38}$$

where, C_1^* is some universal constant and, letting B be the R.H.S. of (5.34), $\eta := 1/2 - \theta/8-$ and $\eta_2 := \theta/2 - |s_2|$,

$$P_0 := \delta^{\theta/4-} N^{-\eta_2 + |s_1|+}, \tag{5.39}$$

$$P_1 := \big(\delta^{\theta/4-|s_2|/2-} + \delta^{\eta-} N^{2|s_1|} + \delta^{\theta/4-} A\big)B, \tag{5.40}$$

$$P_2 := \delta^{\eta-} N^{|s_1|} B^2, \tag{5.41}$$

$$P_3 := \delta^{\eta-} B^3. \tag{5.42}$$

On the other hand, we have

$$\|\phi^{L_{j+1}}(\delta_j)\|_{L^2} \leq \|\phi^{L_j}(\delta_j)\|_{L^2} + \left\| G_2^{(I_j)}\big[g_1^{H_j} + g_2^{H_j}\big](\delta_j)\right\|_{L^2}. \tag{5.43}$$

By (5.17) with $s_1' = -\theta/2+$, (5.18) with $s_1' = -\theta/2+$, and (5.26), R.H.S. of (5.43) is bounded by

$$\begin{aligned} &\|\phi^{L_j}(\delta_j)\|_{L^2} + C\delta^{1/2-\theta-} \|\psi^{H_j}\|^2_{X^{-\theta/2+,1/2+}_{|\xi|^2}(I_j)} \\ &\quad + C\mathscr{C} D_1 \delta^{1/2-\theta/2-} N^{|s_1|} \|\psi^{H_j}\|_{X^{-\theta/2+,1/2+}_{|\xi|^2}(I_j)}, \end{aligned} \tag{5.44}$$

where C is some universal constant. Moreover, applying the Strichartz estimate (Proposition 3.1 in [10]) to the integral equation, by Sobolev's embedding $L^{6/5} \hookrightarrow \dot{H}^{-1,2}$ we bound the first term of (5.44) by

$$\begin{aligned} &\|\phi^{L_j}(\delta_{j-1})\|_{L^2} + C\|\psi\|^2_{L_t^2 L_x^{12/5}(I_j)} \\ &\quad \leq \|\phi^{L_j}(\delta_{j-1})\|_{L^2} + C\delta^{3/4}\|\psi\|^2_{L_t^8 L_x^{12/5}(I_j)}. \end{aligned} \tag{5.45}$$

By the embedding $X^{0,1/2+}_{|\xi|^2}(I_j) \hookrightarrow L_t^8 L_x^{12/5}(I_j)$, R.H.S. of (5.45) is bounded by

$$\|\phi^{L_j}(\delta_{j-1})\|_{L^2} + C\delta^{3/4}\|\psi\|^2_{X^{0,1/2+}_{|\xi|^2}(I_j)}. \tag{5.46}$$

Consequently, by (5.46) and (5.26), the first term of (5.44) is bounded as follows:

$$\|\phi^{L_j}(\delta_j)\|_{L^2} \leq \|\phi^{L_j}(\delta_{j-1})\|_{L^2} + C\mathscr{C}^2 D_1^2 \delta^{3/4} N^{2|s_1|}, \tag{5.47}$$

where C is some universal constant. Combining (5.44) with (5.47), we have

$$\|\phi^{L_{j+1}}(\delta_j)\|_{L^2} - \|\phi^{L_j}(\delta_{j-1})\|_{L^2} \leq C_2^*(\mathscr{C}D_1)^2(Q_0 + Q_1 + Q_2), \tag{5.48}$$

where C_2^* is some universal constant and

$$Q_0 := \delta^{3/4}N^{2|s_1|}, \tag{5.49}$$

$$Q_1 := \delta^{1/2-\theta/2-}N^{|s_1|}B, \tag{5.50}$$

$$Q_2 := \delta^{1/2-\theta-}B^2. \tag{5.51}$$

Now we consider the conditions for the uniform bound (5.25). We set $\kappa := T/\delta$. Taking δ suitably, we may assume that $\kappa \in \mathbb{N}$. Then

$$\begin{aligned} \|\psi^{L_\kappa}(\delta_{\kappa-1})\|_{L^2} = {} & \|\psi^{L_\kappa}(\delta_{\kappa-1})\|_{L^2} - \|\psi^{L_{\kappa-1}}(\delta_{\kappa-2})\|_{L^2} \\ & + \cdots \\ & + \|\psi^{L_2}(\delta_1)\|_{L^2} - \|\psi_0^L\|_{L^2} \\ & + \|\psi_0^L\|_{L^2}. \end{aligned} \tag{5.52}$$

Hence, by (5.38), we have

$$\|\psi^{L_\kappa}(\delta_{\kappa-1})\|_{L^2} \leq D_1 N^{|s_1|} + T\delta^{-1}C_1^*(\mathscr{C}D_1)^2(P_0 + P_1 + P_2 + P_3). \tag{5.53}$$

Similarly, we have

$$\|\phi^{L_\kappa}(\delta_{\kappa-1})\|_{L^2} \leq D_1 N^{|s_2|} + T\delta^{-1}C_2^*(\mathscr{C}D_1)^2(Q_0 + Q_1 + Q_2). \tag{5.54}$$

Set $C^* := \max\{C_1^*, C_2^*\}$. From (5.53) and (5.54), in order to get uniform control of the initial data (5.25), we have to ensure that $T\delta^{-1}C^*(\mathscr{C}D_1)^2(P_0 + P_1 + P_2 + P_3) \leq D_1 N^{|s_1|}$ and $T\delta^{-1}C^*(\mathscr{C}D_1)^2(Q_0 + Q_1 + Q_2) \leq \mathscr{C}^3 D_1^2 \delta^{-1/4} N^{2|s_1|}$. For this, since $\delta^{1/4}N^{-2|s_1|}Q_1 \leq N^{-|s_1|}P_1$ and

$\delta^{1/4}N^{-2|s_1|}Q_2 \le N^{-|s_1|}P_2$, it suffices to consider that $T\delta^{-1}C^*\mathscr{C}^2 D_1(P_0 + P_1 + P_2 + P_3) \le N^{|s_1|}$ and $T\delta^{-1}C^*Q_0 \le \mathscr{C}\delta^{-1/4}N^{2|s_1|}$. We take $\mathscr{C} = \max\{TC^*, 100\}$. Then it clearly holds that $T\delta^{-1}C^*Q_0 \le \mathscr{C}\delta^{-1/4}N^{2|s_1|}$. Thus it remains to consider the condition for $\delta^{-1}(P_0 + P_1 + P_2 + P_3) \le (TC^*\mathscr{C}^2 D_1)^{-1}N^{|s_1|}$. Since $\delta \le \delta_L(s_1, s_2, c_2) \le (\mathscr{C}^3 D_1^2)^{-10}N^{-8|s_1|-}$, we have

$$B \lesssim (\mathscr{C}D_1)^3 B', \tag{5.55}$$

where $B' := N^{-\eta_1+} + \delta^{\theta/2-}N^{-\eta_2+|s_1|+} + \delta^{3\eta-}N^{7|s_1|}$. Since $\delta^{\theta/2}N^{-\eta_2+|s_1|+} \le B \lesssim (\mathscr{C}D_1)^3 N^{|s_1|}$, we easily see that

$$\begin{aligned}\delta^{-1}(P_0 + P_1 + P_2 + P_3) &\lesssim \delta^{-1}(\mathscr{C}D_1)^3 P_1 \\ &\lesssim \delta^{-1+\theta/4-}(\mathscr{C}D_1)^6 AB' \\ &\le \delta^{-1+\theta/4-}(\mathscr{C}D_1)^9 (N^{|s_2|} + \delta^{-1/4}N^{2|s_1|})B'.\end{aligned}$$

Thus in order that $\delta^{-1}(P_0 + P_1 + P_2 + P_3) \le (TC^*\mathscr{C}^2 D_1)^{-1}N^{|s_1|}$, it suffices to show that

$$\delta^{-1+\theta/4-}N^{-|s_1|}(N^{|s_2|} + \delta^{-1/4}N^{2|s_1|})B' \le (TC^*)^{-1}(\mathscr{C}D_1)^{-20},$$

where $\mathscr{C} = \max\{TC^*, 100\}$. For this we take N so large that

$$(TC^*)(\mathscr{C}D_1)^{20} \ll N^{\varepsilon} \tag{5.56}$$

for sufficiently small $\varepsilon > 0$, and then take δ such that

$$\delta^{-1+\theta/4-}N^{-|s_1|}(N^{|s_2|} + \delta^{-1/4}N^{2|s_1|})B' \lesssim N^{-\varepsilon}. \tag{5.57}$$

In (5.57), taking s_1, $s_2 < 0$ sufficiently close to 0, we can take positive ε. For (5.57), it suffices to take δ such that

$$\begin{aligned}\delta \ge \max\Big\{&N^{-2((\theta-2|s_2|-2\varepsilon)/(4-\theta))+}N^{-2((\theta-4|s_2|-2\varepsilon)/(4-3\theta))+}, \\ &N^{-2((\theta-4|s_1|-2\varepsilon)/(5-\theta))+}, N^{-2((\theta-4|s_1|-2|s_2|-2\varepsilon)/(5-3\theta))+}\Big\},\end{aligned}$$

and

$$\begin{aligned}\delta &\le \min\Big\{N^{-8((6|s_1|+|s_2|+\varepsilon)/(4-\theta))-}, N^{-8((8|s_1|+\varepsilon)/(2-\theta))-}\Big\} \\ &\le \min\big\{N^{-8|s_1|-}, N^{-2|s_2|-}\big\}.\end{aligned}$$

The condition of Theorem 1.1 ensures the existence of δ satisfying the above conditions for sufficiently small $\varepsilon > 0$.

From the above argument, we find that, taking N and δ as in (5.56) and (5.57) respectively, we actually obtain the uniform bound (5.25) without the assumption. Thus we have completed the proof. □

6. Appendix

In this section, we collect lemmas used in this paper.

Lemma 6.1 (Strichartz type estimate for Schrödinger equation) *Let $d \geq 3$ be spatial dimension. Assume that $b_0 > 1/2$, $0 \leq b \leq b_0$, $0 \leq \theta \leq 1$ and (p, r) is such that $d\big(1/2 - 1/p\big) = (1-\theta)(b/b_0)$ and $2/r = 1 - \theta(b/b_0)$. Then we have*

$$\|u\|_{L_t^r L_x^p} \leq C \|u\|_{X^{0,b}_{|\xi|^2}}$$

where C is a constant depending only on d, b_0, b, r. We may replace $X^{0,b}_{|\xi|^2}$ with $X^{0,b}_{-|\xi|^2}$.

Lemma 6.2 (Energy estimate for wave equation) *Let $d \geq 1$, $s \in \mathbb{R}$, $b_0 > 1/2$, $0 \leq b \leq b_0$ and let r be such that $2/r = 1 - b/b_0$. Then we have*

$$\||\nabla|^s v\|_{L_t^r L_x^2} \leq C \|v\|_{\dot{X}^{s,b}_{\pm|\xi|}}$$

where C is a constant depending only on d, b_0, b.

Lemma 6.3 *Let b, b' with $1/2 > b + b' \geq b' > -1/2$, $s \in \mathbb{R}$ and let I be an interval in $\mathbb{R}$ with $|I| \leq 1$. Then we have*

$$\|u\|_{X^{s,b'}_{|\xi|^2}(I)} \leq C |I|^b \|u\|_{X^{s,b'+b}_{|\xi|^2}(I)},$$

where C is a constant independent of I. The same result holds for the space $\dot{X}^{s,b}_{\pm|\xi|}(I)$.

See [19], Lemma 2.11.

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