A lower bound for the class number of $P^n(C)$ and $P^n(H)$

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Abstract. We obtain new lower bounds on the codimension of local isometric imbeddings of complex and quaternion projective spaces. We show that any open set of the complex projective space $P^n(C)$ (resp. quaternion projective space $P^n(H)$) cannot be locally isometrically imbedded into the euclidean space of dimension $4n - 3$ (resp. $8n - 4$). These estimates improve the previously known results obtained in [2] and [7].

Key words: curvature invariant, isometric imbedding, complex projective space, quaternion projective space, root space decomposition.

1. Introduction

Let $M$ be a Riemannian manifold. As is known, $M$ can be locally or globally isometrically imbedded into a euclidean space of sufficiently large dimension (see Janet [19], Cartan [14], Nash [24], Greene-Jacobowitz [16], Gromov-Rokhlin [17]). It is a natural and interesting question to ask the least dimension of euclidean spaces into which $M$ can be locally or globally isometrically imbedded. In this paper we will investigate the problem of local isometric imbeddings of the projective spaces $P^n(C)$ and $P^n(H)$ and give a new estimate on the least dimension of the ambient euclidean spaces.

Let $x \in M$. Assume that there is a neighborhood $U$ of $x$ in $M$ such that $U$ is isometrically imbedded into a euclidean space $R^D$. If any neighborhood of $x$ cannot be isometrically imbedded into $R^{D-1}$, then the codimension $D-\dim M$ is called the class number of $M$ at $x$ and is denoted by $\text{class}(M)_x$.

Let $G/K$ be a Riemannian symmetric space. By homogeneity, the class number of $G/K$ is constant everywhere on $G/K$, which is denoted by $\text{class}(G/K)$. In Agaoka-Kaneda [4], [5], [7], [8], [9] and [10] we have tried to estimate $\text{class}(G/K)$ from below. In doing this we mainly used the following inequality

$$\text{class}(G/K) \geq \dim G/K - p(G/K),$$

where $p(G/K)$ is the pseudo-nullity of $G/K$ (see §2 below or [4]).

the following Riemannian symmetric spaces $G/K$ our estimates just hit class($G/K$), i.e., class($G/K$) = dim$G/K - p(G/K)$:

a) The sphere $S^n$ ($n \geq 2$);
b) $CI$: $Sp(n)/U(n)$ ($n \geq 1$) (see [4]);
c) The symplectic group $Sp(n)$ ($n \geq 1$) (see [5]).

As for the class numbers of the projective spaces such as the complex projective space $P^n(C)$, the quaternion projective space $P^n(H)$ and the Cayley projective plane $P^2(Cay)$, the following are known:

1. class($P^n(C)$) $\geq \max\{n + 1, \lceil \frac{6}{n} \rceil \}$ ($n \geq 2$) (see [2] and [7]);
2. class($P^n(H)$) $\geq \min\{4n - 3, 3n + 1\}$ ($n \geq 3$) (see [7]);
3. class($P^n(C)$) $\leq n^2$ ($n \geq 2$); class($P^n(H)$) $\leq 2n^2 - n$ ($n \geq 2$) (see [22]);
4. class($P^2(Cay)$) = 6; class($P^2(Cay)$) = 10 (see [8] and [22]).

It should be noted that any local isometric imbedding of $P^2(H)$ (resp. $P^2(Cay)$) into the euclidean space $R^{14}$ (resp. $R^{26}$) is rigid in the strongest sense (see [9] and [10]).

In this paper we will propose a new type of estimate and by applying it we will prove

**Theorem 1** Let $G/K$ denote the complex projective space $P^n(C)$ ($n \geq 3$) or the quaternion projective space $P^n(H)$ ($n \geq 3$). Define an integer $q(G/K)$ by

$$q(G/K) = \begin{cases} 4n - 2, & \text{if } G/K = P^n(C) \ (n \geq 3); \\ 8n - 3, & \text{if } G/K = P^n(H) \ (n \geq 3). \end{cases}$$

Then, any open set of $G/K$ cannot be isometrically imbedded into the euclidean space $R^D$ with $D \leq q(G/K) - 1$. In other words,

- class($P^n(C)$) $\geq 2n - 2$ ($n \geq 3$);
- class($P^n(H)$) $\geq 4n - 3$ ($n \geq 3$).

It is clearly seen that Theorem 1 improves the estimates (1) and (2) stated above. However, we have to recognize a large gap between our estimate and the upper bound stated in (3), which cannot be filled at present.

Throughout this paper we will assume the differentiability of class $C^\infty$.

For the notations of Lie algebras and Riemannian symmetric spaces, see Helgason [18].
2. The Gauss equation

Let $M$ be a Riemannian manifold and $g$ be the Riemannian metric of $M$. We denote by $R$ the Riemannian curvature tensor of type $(1,3)$ with respect to $g$.

For each $x \in M$ we denote by $T_x(M)$ (resp. $T_x^*(M)$) the tangent (resp. cotangent) vector space of $M$ at $x \in M$. Let $r$ be a non-negative integer. We define a quadratic equation with respect to an unknown $\Psi \in S^2 T_x^*(M) \otimes R^r$ by

\begin{equation}
- g(R(X,Y)Z,W) = \langle \Psi(X,Z),\Psi(Y,W) \rangle - \langle \Psi(X,W),\Psi(Y,Z) \rangle,
\end{equation}

where $X, Y, Z, W \in T_x(M)$ and $\langle , \rangle$ is the standard inner product of $R^r$. We call (2.1) the Gauss equation in codimension $r$ at $x$. It is well-known that for a sufficiently large $r$ the Gauss equation (2.1) in codimension $r$ admits a solution (see Berger [12], Berger-Bryant-Griffiths [13]). On the other hand, in general, for a small $r$ (2.1) does not admit any solution. By $\text{Crank}(M)_x$ we denote the least value of $r$ with which (2.1) admits a solution and call it the curvature rank of $M$ at $x$. It should be noted that $\text{Crank}(M)_x$ is a curvature invariant, i.e., it can be determined only by the curvature $R$ of $M$ at $x$.

As is well-known, if there is an isometric immersion $f$ of $M$ into $R^D$, then the second fundamental form of $f$ at $x$ satisfies the Gauss equation in codimension $r = D - \dim M$. Therefore, we have

**Lemma 2** $\text{class}(M)_x \geq \text{Crank}(M)_x$ holds for any $x \in M$.

In the following, we assume that $\Psi \in S^2 T_x^*(M) \otimes R^r$ is a solution of the Gauss equation in codimension $r$. Let $X \in T_x(M)$. We define a linear mapping $\Psi_X : T_x(M) \rightarrow R^r$ by $\Psi_X(Y) = \Psi(X,Y)$ ($Y \in T_x(M)$). The kernel of this map $\Psi_X$ is denoted by $\text{Ker}(\Psi_X)$. Then we can easily show the following

**Lemma 3** Let $X \in T_x(M)$. Then $R(\text{Ker}(\Psi_X),\text{Ker}(\Psi_X))X = 0$.

For the proof, see [4]. By this lemma we can get the following estimate for $\text{Crank}(M)_x$: Let $X \in T_x(M)$. By $d(X)$ we denote the maximum value of the dimensions of those subspaces $V \subset T_x(M)$ such that $R(V,V)X = 0$. Then by Lemma 3 it is easily seen that $d(X) \geq \dim \text{Ker}(\Psi_X) \geq \dim M - r$. Set $p_M(x) = \min\{d(X) \mid X \in T_x(M)\}$. Then $p_M(x) \geq \dim M - r$, i.e.,
r ≥ \dim M - \imath M(x). The number \imath M(x) thus defined is also a curvature invariant, which is called the pseudo-nullity of \( M \) at \( x \). By the above discussion we have

**Lemma 4** \( \text{Crank}(M)_x \geq \dim M - \imath M(x) \).

In the case where \( M \) is a Riemannian homogeneous space \( G/K \), the class number, the curvature rank and the pseudo-nullity of \( G/K \) are constant everywhere on \( G/K \), which are denoted by \( \text{class}(G/K) \), \( \text{Crank}(G/K) \) and \( \imath(G/K) \), respectively. Combining Lemma 4 with Lemma 2, we obtain

**Proposition 5** Let \( G/K \) be a Riemannian homogeneous space. Then:

\[
\text{class}(G/K) \geq \dim G/K - \imath(G/K).
\]

This is a fundamental tool in our works [5] and [7] to estimate the class numbers of Riemannian symmetric spaces from below.

Now, we show a new type of estimate:

**Theorem 6** Let \( \Psi \in \mathcal{S}^{2T_x^*(M)} \otimes \mathbb{R}^r \) be a solution of the Gauss equation in codimension \( r \). Assume that there are tangent vectors \( X, Y \in T_x(M) \) and a subspace \( U \) of \( T_x(M) \) satisfying

1. \( \Psi(X, Y) = 0; \)
2. \( U \supset \text{Ker}(\Psi_X) \) and \( R(U, Y)X = 0 \).

Then the following inequality holds:

\[
r \geq \dim M + \dim U - \dim \text{Ker}(\Psi_X) - \dim \text{Ker}(\Psi_Y) - \dim \text{Ker}(\Psi_Y).
\]

**Proof.** Let \( Z \) be an arbitrary element of \( T_x(M) \). Then by the Gauss equation (2.1) it follows that

\[
0 = -g(R(U, Y)X, Z) = \langle \Psi(X, Z), \Psi(Y, X) \rangle - \langle \Psi(U, Z), \Psi(Y, X) \rangle = \langle \Psi_X(U), \Psi_Y(Z) \rangle.
\]

Hence, we have \( \langle \Psi_X(U), \Psi_Y(Z) \rangle = 0 \). This implies that the image of \( T_x(M) \) via the map \( \Psi_Y \) is included in the orthogonal complement of \( \Psi_X(U) \). Since \( \dim \Psi_X(U) = \dim U - \dim \text{Ker}(\Psi_X) \), we have \( \dim \Psi_Y(T_x(M)) \leq r - \dim U + \dim \text{Ker}(\Psi_X) \). Moreover, since \( \dim \Psi_Y(T_x(M)) = \dim M - \dim \text{Ker}(\Psi_Y) \), we immediately obtain the inequality (2.2). \( \square \)

As is easily seen, the right side of the inequality (2.2) heavily depends
on tangent vectors $X$, $Y$ and $\Psi$. Accordingly, only by (2.2) we cannot obtain an estimate for $\text{Crank}(M)_x$. In the following sections, by applying Theorem 6 to the complex and quaternion projective spaces we will show Theorem 1.

3. Projective spaces $P^n(C)$ and $P^n(H)$

In this section we make several preparations that are needed in the succeeding sections. Hereafter, $G/K$ denotes one of the following projective spaces:

1. The complex projective spaces $P^n(C) = SU(n+1)/S(U(n) \times U(1))$ ($n \geq 2$).
2. The quaternion projective spaces $P^n(H) = Sp(n+1)/Sp(n) \times Sp(1)$ ($n \geq 2$).

Let $g$ (resp. $\mathfrak{k}$) be the Lie algebra of $G$ (resp. $K$) and let $g = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $g$ associated with the Riemannian symmetric pair $(G, K)$. Let $(\ , \ )$ be the inner product of $g$ given by the $(-1)$-multiple of the Killing form of $g$. We define a $G$-invariant Riemannian metric $\tilde{g}$ of $G/K$ by

$$\tilde{g}(X, Y) = (X, Y) \quad (X, Y \in \mathfrak{m}),$$

where we identify $\mathfrak{m}$ with the tangent space $T_o(G/K)$ at the origin $o = \{K\} \in G/K$. Since the curvature at $o$ is given by $R(X, Y)Z = -[[[X, Y], Z], X]$ ($X, Y, Z \in \mathfrak{m}$) (see Helgason [18]), the Gauss equation (2.1) in codimension $r$ at $o$ can be written as follows:

$$\langle [[X, Y], Z], W \rangle = \langle \Psi(X, Z), \Psi(Y, W) \rangle - \langle \Psi(X, W), \Psi(Y, Z) \rangle,$$

(3.1)

where $\Psi \in S^2\mathfrak{m}^* \otimes \mathbb{R}^r$, $X, Y, Z$ and $W \in \mathfrak{m}$.

Let us take and fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$. Then, since $\text{rank}(G/K) = 1$, we have $\dim \mathfrak{a} = 1$. We call an element $\lambda \in \mathfrak{a}$ a restricted root when the subspaces $\mathfrak{k}(\lambda)$ ($\subset \mathfrak{k}$) and $\mathfrak{m}(\lambda)$ ($\subset \mathfrak{m}$) defined below are not non-trivial:

$$\mathfrak{k}(\lambda) = \{ X \in \mathfrak{k} \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X, \forall H \in \mathfrak{a} \},$$

$$\mathfrak{m}(\lambda) = \{ Y \in \mathfrak{m} \mid [H, [H, Y]] = -\langle \lambda, H \rangle^2 Y, \forall H \in \mathfrak{a} \}. $$

As is known, by use of a non-zero restricted root $\mu$ the set of non-zero restricted roots $\Sigma$ can be written as $\Sigma = \{ \pm \mu, \pm 2\mu \}$. Further, we have the following orthogonal decompositions:

$$\mathfrak{k} = \mathfrak{k}(0) + \mathfrak{k}(\mu) + \mathfrak{k}(2\mu) \quad (\text{orthogonal direct sum}),$$

$$\mathfrak{m} = \mathfrak{m}(0) + \mathfrak{m}(\mu) + \mathfrak{m}(2\mu) \quad (\text{orthogonal direct sum}),$$

$$\mathfrak{a} = \mathfrak{a}(0) + \mathfrak{a}(\mu) + \mathfrak{a}(2\mu) \quad (\text{orthogonal direct sum}).$$
\[ m = m(0) + m(\mu) + m(2\mu) \quad \text{(orthogonal direct sum)}, \]

where \( m(0) = a = R\mu \) (see §5 of [7]).

For convenience, in the following we set \( \mathfrak{t}_i = \mathfrak{t}(\mu|\mu) \), \( m_i = m(\mu|\mu) \) \((|i| \leq 2)\) and \( \mathfrak{t}_i = m_i = 0 \ (|i| > 2)\) for any integer \( i \). Then for \( i, j = 0, 1, 2 \) we have a formula:

\[ [\mathfrak{t}_i, \mathfrak{t}_j] \subset \mathfrak{t}_{i+j} + \mathfrak{t}_{i-j}, \quad [m_i, m_j] \subset \mathfrak{t}_{i+j} + \mathfrak{t}_{i-j}, \quad [\mathfrak{t}_i, m_j] \subset m_{i+j} + m_{i-j}. \]

We summarize in the following table the basic data for the spaces \( P^n(C) \) and \( P^n(H) \) (see [18], [7]):

<table>
<thead>
<tr>
<th>( G/K )</th>
<th>( \dim m_1 )</th>
<th>( \dim m_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P^n(C) ) ((n \geq 2))</td>
<td>( 2(n-1) )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( P^n(H) ) ((n \geq 2))</td>
<td>( 4(n-1) )</td>
<td>( 3 )</td>
</tr>
</tbody>
</table>

As is known, each non-zero element of \( m \) is conjugate to a scalar multiple of \( \mu \) under the action of the isotropy group \( \text{Ad}(K) \), because \( \text{rank}(P^n(C)) = \text{rank}(P^n(H)) = 1 \). More precisely we can show the following

**Proposition 7** Let \( Y_i \in m_i \ (i = 0, 1, 2) \). Assume that \( Y_i \neq 0 \). Then there is an element \( k_i \in K \) such that \( \text{Ad}(k_i^{\pm 1})\mu \in RY_i \).

**Proof.** In the case \( i = 0 \) we have only to set \( k_0 = e \), where \( e \) is the identity element of \( K \).

Now assume \( i = 1 \) or \( 2 \). Set \( X_i = [\mu, Y_i] \). Then we have \( X_i \in \mathfrak{t}_i \).

Further, we have \( [X_i, [X_i, \mu]] \in a \), because \( [X_i, [X_i, \mu]] \in m \) and \( [\mu, [X_i, [X_i, \mu]]] = ([\mu, X_i], [X_i, \mu]) + [X_i, [\mu, [X_i, \mu]]] = 0 \). Since

\[ (\mu, [X_i, [X_i, \mu]]) = ([\mu, X_i], [X_i, \mu]) = ([\mu, [\mu, X_i]], X_i) = -i^2(\mu, \mu)^2(X_i, X_i), \]

we have \( [X_i, [X_i, \mu]] = -i^2(\mu, \mu)(X_i, X_i)\mu \). By this equality and the fact \( [X_i, \mu] = [[\mu, Y_i], \mu] = i^2(\mu, \mu)^2Y_i \) we have

\[ \text{Ad}(\exp(tX_i))\mu = \cos(|\mu||X_i|t)\mu + \frac{1}{i|\mu||X_i|} \sin(|\mu||X_i|t)[X_i, \mu], \quad \forall t \in R. \]

Define \( t_i \in R \) by \( i|\mu||X_i|t_i = \pi/2 \). Then, by setting \( k_i = \exp(t_iX_i) \in K \), we easily get \( \text{Ad}(k_i^{\pm 1})\mu \in RY_i \). \( \square \)
4. Pseudo-abelian subspaces

Let $G/K = P^n(C)$ or $P^n(H)$. We say that a subspace $V$ of $\mathfrak{m}$ is pseudo-abelian if $[V, V] \subset \mathfrak{t}_0$. It is easily seen that a subspace $V$ of $\mathfrak{m}$ is pseudo-abelian if and only if $[[V, V], \mu] = 0$, because $\text{rank}(G/K) = 1$. We note that the pseudo-nulility $p(G/K)$ coincides with the maximum dimension of pseudo-abelian subspaces in $\mathfrak{m}$ (see [4]). In [7] we have determined the pseudo-nullities for $P^n(C)$ and $P^n(H)$: $p(P^n(C)) = \max\{n-1, 2\}$ $(n \geq 2)$; $p(P^n(H)) = \max\{n-1, 3\}$ $(n \geq 2)$ (see Theorem 5.1 of [7]).

For later use, we here study more detailed facts about pseudo-abelian subspaces in $\mathfrak{m}$ for $P^n(C)$ and $P^n(H)$. We first prove

Lemma 8 Let $V \subset \mathfrak{m}$ be a pseudo-abelian subspace of $\mathfrak{m}$. If $V \cap \mathfrak{m}_i \neq 0$ for some $\mathfrak{m}_i$ ($i = 0, 1, 2$), then $V \subset \mathfrak{m}_i$.

Proof. Assume that $V \cap \mathfrak{m}_i \neq 0$. Take a non-zero element $Y_i^0 \in V \cap \mathfrak{m}_i$. Let $Y = Y_0 + Y_1$ be an arbitrary element of $V$, where $Y_0 \in \mathfrak{a} + \mathfrak{m}_2$; $Y_1 \in \mathfrak{m}_1$. Then we have $[Y_i^0, Y_0 + Y_1] = [Y_i^0, Y_0] + [Y_i^0, Y_1] \in \mathfrak{t}_0$. However, since $[Y_i^0, Y_0] \in \mathfrak{t}_1$ and $[Y_i^0, Y_1] \in \mathfrak{t}_0 + \mathfrak{t}_2$, we have $[Y_i^0, Y_0] = 0$. Therefore we have $Y_0 = 0$, because $\text{rank}(G/K) = 1$. This proves $V \subset \mathfrak{m}_i$. The other cases $V \cap \mathfrak{a} \neq 0$ and $V \cap \mathfrak{m}_2 \neq 0$ are similarly dealt with. □

We say that a pseudo-abelian subspace $V$ is categorical if it can be decomposed into a direct sum $V = V \cap \mathfrak{a} + V \cap \mathfrak{m}_1 + V \cap \mathfrak{m}_2$. By Lemma 8 we immediately have

Proposition 9 Let $V \subset \mathfrak{m}$ be a pseudo-abelian subspace of $\mathfrak{m}$. If $V$ is categorical and $V \neq 0$, then $V$ is contained in one of $\mathfrak{a}$, $\mathfrak{m}_1$ and $\mathfrak{m}_2$.

By this proposition, we can easily estimate $\dim V$ for a categorical pseudo-abelian subspace $V$ in $\mathfrak{m}$: $\dim V \leq 1$ if $V \subset \mathfrak{a}$; $\dim V \leq \dim \mathfrak{m}_2$ if $V \subset \mathfrak{m}_2$. In the case $V \subset \mathfrak{m}_1$ we proved in [7] $\dim V \leq n-1$ (see Theorem 3.2 of [7]). For completeness, we review this proof and show an additional property of $V \subset \mathfrak{m}_1$.

Let $E(\mathfrak{m}_1)$ denote the space of all linear endomorphisms of $\mathfrak{m}_1$. Let $X \in \mathfrak{t}_2$. We define an element $X^\dagger \in E(\mathfrak{m}_1)$ by

$X^\dagger(Y) = [X, Y], \quad Y \in \mathfrak{m}_1.$

(Note that $[\mathfrak{t}_2, \mathfrak{m}_1] \subset \mathfrak{m}_1$.) It is easy to see that $X^\dagger$ is skew-symmetric with respect to the inner product $(\cdot, \cdot)$. We denote by $\mathfrak{t}_2^\dagger$ the subspace of $E(\mathfrak{m}_1)$
consisting of all $X^\dagger$ ($X \in \mathfrak{t}_2$). Set $\mathfrak{F}^\dagger = R1_{m_1} + \mathfrak{t}_2^\dagger (\subset E(m_1))$, where $1_{m_1}$ denotes the identity mapping of $m_1$. We have proved in [7] (Theorem 3.5) the following

**Proposition 10**  Let $G/K = P^n(C)$ or $P^n(H)$. Then, $\mathfrak{F}^\dagger$ forms a subalgebra of $E(m_1)$, i.e., $\mathfrak{F}^\dagger$ is closed under addition and multiplication of $E(m_1)$. Further, in the case $G/K = P^n(C)$ ($n \geq 2$), $\mathfrak{F}^\dagger$ is isomorphic to the field $C$ of complex numbers and in the case $G/K = P^n(H)$ ($n \geq 2$), $\mathfrak{F}^\dagger$ is isomorphic to the field $H$ of quaternion numbers.

We now set $f = \dim_R \mathfrak{F}^\dagger$, i.e., $f = 2$ if $G/K = P^n(C)$; $f = 4$ if $G/K = P^n(H)$. By the definition we have $\dim m_2 = f - 1$, $\dim m_1 = (n - 1)f$ and $\dim G/K = \dim m = nf$. As seen in Proposition 10, $m_1$ can be regarded as a vector space over the field $\mathfrak{F}^\dagger$. For an element $Y_1 \in m_1$ we denote by $\mathfrak{F}^\dagger(Y_1)$ the subspace of $m_1$ spanned by $Y_1$ over $\mathfrak{F}^\dagger$. Then we easily have $\mathfrak{F}^\dagger(\mathfrak{F}^\dagger(Y_1)) = \mathfrak{F}^\dagger(Y_1)$ and $\dim_R \mathfrak{F}^\dagger(Y_1) = f$ if $Y_1 \neq 0$.

**Lemma 11**  Let $Y_1, Y'_1 \in m_1$. Then $[Y_1, Y'_1] \in \mathfrak{t}_0$ if and only if $(\mathfrak{t}_2(Y_1), Y'_1) = 0$. Accordingly, a subspace $V \subset m_1$ is pseudo-abelian if and only if $(\mathfrak{t}_2(V), V) = 0$.

**Proof.** Since $[Y_1, Y'_1] \in \mathfrak{t}_0 + \mathfrak{t}_2$, $[Y_1, Y'_1] \in \mathfrak{t}_0$ holds if and only if $([Y_1, Y'_1], \mathfrak{t}_2) = 0$. Clearly, the last equality is equivalent to $(\mathfrak{t}_2(Y_1), Y'_1) = 0$.

Utilizing the above lemma, we can show the following

**Proposition 12**  Let $V$ be a pseudo-abelian subspace of $m$. Assume that $V \subset m_1$. Then:

1. $\dim \mathfrak{F}^\dagger(V) = f \dim V$. Accordingly, $\dim V \leq n - 1$.
2. Let $\xi \in V$ ($\xi \neq 0$). Then there is a subspace $U$ of $m_1$ satisfying $U \supset V$, $[\xi, U] \subset \mathfrak{t}_0$ and $\dim U = (n - 2)f + 1$.

**Proof.** Let $\{Y_1^1, \ldots, Y_s^1\}$ ($s = \dim V$) be an orthonormal basis of $V$. Let $i, j$ be integers such that $1 \leq i \neq j \leq s$. Then, since $(\mathfrak{t}_2(Y_i^1), Y_j^1) = (Y_i^1, \mathfrak{t}_2(Y_j^1)) = 0$ (see Lemma 11) and since $(\mathfrak{t}_2^2) \subset \mathfrak{F}^\dagger$, we have

$$(\mathfrak{F}^\dagger(Y_i^1), \mathfrak{F}^\dagger(Y_j^1)) = (R Y_i^1 + \mathfrak{t}_2(Y_i^1), R Y_j^1 + \mathfrak{t}_2(Y_j^1))$$

$$\subset (Y_i^1, (\mathfrak{t}_2^2(Y_j^1))) = 0.$$

This proves $\mathfrak{F}^\dagger(V) = \bigoplus_{1 \leq i \leq s} \mathfrak{F}^\dagger(Y_i^1)$ (orthogonal direct sum) and hence
A lower bound for the class number of $P^n(C)$ and $P^n(H)$

Next we prove (2). Since $V$ is pseudo-abelian and $\xi \in V$, we have $(t^*(\xi), V) = 0$. Let $U$ be the orthogonal complement of $t^*(\xi)$ in $m_1$. Then $U$ satisfies $U \supset V$ and $[\xi, U] \subset f_0$ (see Lemma 11). Moreover, since $\dim t^*(\xi) = f - 1$ and $\dim m_1 = (n - 1)f$, we immediately obtain the equality $\dim U = (n - 2)f + 1$.

Finally, we refer to non-categorical pseudo-abelian subspaces. Let $V$ be a pseudo-abelian subspace of $m$. Assume that $V$ is not categorical, i.e., $V$ cannot be represented by a direct sum of subspaces $V \cap a$, $V \cap m_1$ and $V \cap m_2$. Then it is clear that $V \not\subseteq a$, $V \not\subseteq m_1$ and $V \not\subseteq m_2$. In view of Lemma 8, we know that $V \cap a = V \cap m_1 = V \cap m_2 = 0$. Apparently, this condition is sufficient for a pseudo-abelian subspace $V$ to be non-categorical. Hence we have

**Proposition 13** Let $V$ be a pseudo-abelian subspace of $m$ such that $V \neq 0$.

1. $V$ is non-categorical if and only if $V \cap a = V \cap m_1 = V \cap m_2 = 0$.
2. If $V$ is non-categorical, then $\dim V \leq 2$.

For the proof of (2), see Proposition 5.2 (1) of [7].

5. Proof of Theorem 1

Let $G/K = P^n(C)$ ($n \geq 2$) or $P^n(H)$ ($n \geq 2$). In the following we assume that the Gauss equation in codimension $r$ admits a solution $\Psi \in S^2 m^* \otimes R^r$. We first prove

**Lemma 14** Let $X \in m (X \neq 0)$ and let $k$ be an element of $K$ satisfying $\text{Ad}(k)\mu \in RX$. Then $\text{Ad}(k^{-1})\text{Ker}(\Psi_X)$ is a pseudo-abelian subspace of $m$.

**Proof.** By Lemma 3 we have $[[\text{Ker}(\Psi_X), \text{Ker}(\Psi_X)], X] = 0$. Applying $\text{Ad}(k^{-1})$ to this equality, we have $[[\text{Ad}(k^{-1})\text{Ker}(\Psi_X), \text{Ad}(k^{-1})\text{Ker}(\Psi_X)], \mu] = 0$. This proves that $\text{Ad}(k^{-1})\text{Ker}(\Psi_X)$ is a pseudo-abelian subspace of $m$. \hfill $\square$

Let $X \in m (X \neq 0)$. If $\text{Ker}(\Psi_X) = 0$, then we say $X$ is of type $P_{inj}$. Now assume $\text{Ker}(\Psi_X) \neq 0$. Let $k \in K$ be an element satisfying $\text{Ad}(k)\mu \in RX$. As is shown in Lemma 14, $\text{Ad}(k^{-1})\text{Ker}(\Psi_X)$ is a pseudo-abelian subspace of $m$. If $\text{Ad}(k^{-1})\text{Ker}(\Psi_X)$ is categorical and is contained in $m_i$ ($i = 0, 1, 2$), then we say $X$ is of type $P_i$ ($i = 0, 1, 2$). We also say $X$ is of
type $P_{\text{non}}$ if $\text{Ad}(k^{-1})\text{Ker}(\Psi_X)$ is non-categorical, i.e., $\text{Ad}(k^{-1})\text{Ker}(\Psi_X) \cap m_i = 0 \ (i = 0, 1, 2)$.

The following lemma asserts that the type of $X$ does not depend on the choice of $k \in K$ satisfying $\text{Ad}(k)\mu \in RX$.

**Lemma 15** Let $X \in m \ (X \neq 0)$. Let $i = 0, 1$ or 2 and let $k_j \ (j = 1, 2)$ be elements of $K$ satisfying $\text{Ad}(k_j)\mu \in RX$. Then:

1. $\text{Ad}(k_1^{-1})\text{Ker}(\Psi_X) \subseteq m_i$ if and only if $\text{Ad}(k_2^{-1})\text{Ker}(\Psi_X) \subseteq m_i$.
2. $\text{Ad}(k_1^{-1})\text{Ker}(\Psi_X) \cap m_i = 0$ if and only if $\text{Ad}(k_2^{-1})\text{Ker}(\Psi_X) \cap m_i = 0$.

**Proof.** Set $k' = k_1^{-1}k_2 \in K$. By the assumption we have $\text{Ad}(k')\mu = \pm \mu$. Therefore it is easily seen that $\text{Ad}(k')m_i = m_i$ for any $i = 0, 1, 2$. Since $\text{Ad}(k')\text{Ad}(k_2^{-1}) = \text{Ad}(k_1^{-1})$, the lemma follows immediately. \qed

Let us denote by $p_i \ (i = 0, 1, 2, \text{non, inj})$ the subset of $m$ consisting of all elements of type $P_i$. Then it is clear that

$$m \setminus \{0\} = p_0 \cup p_1 \cup p_2 \cup p_{\text{non}} \cup p_{\text{inj}} \quad \text{(disjoint union).} \quad (5.1)$$

**Proposition 16** Let $X, Y \in m \ (X \neq 0, Y \neq 0)$. Assume that $\Psi(X, Y) = 0$. Then $X \in p_i$ if and only if $Y \in p_i \ (i = 0, 1, 2, \text{non})$.

**Proof.** We note that under the assumption $\Psi(X, Y) = 0$ we have $X \notin p_{\text{inj}}$ and $Y \notin p_{\text{inj}}$, because $Y \in \text{Ker}(\Psi_X)$ and $X \in \text{Ker}(\Psi_Y)$.

First consider the case $X \in p_i \ (i = 0, 1, 2)$. Let $k \in K$ be an element such that $\text{Ad}(k)\mu \in RX$. Then we have $\text{Ad}(k^{-1})Y \in m_i$, because $\text{Ad}(k^{-1})Y \in \text{Ad}(k^{-1})\text{Ker}(\Psi_X) \subseteq m_i$. Take an element $k' \in K$ satisfying $\text{Ad}(k')\mu \in R\text{Ad}(k^{-1})Y$ and set $k'' = kk'$ (see Proposition 7). Then we have $\text{Ad}(k'')\mu = \text{Ad}(k)\text{Ad}(k')\mu \in \text{Ad}(k)R\text{Ad}(k^{-1})Y = RY$ and $\text{Ad}(k''^{-1})X = \text{Ad}(k')^{-1}\text{Ad}(k^{-1})X \in R\text{Ad}(k')^{-1}\mu = r\text{Ad}(k^{-1})Y \subseteq m_i$. Since $X \in \text{Ker}(\Psi_Y)$, it follows that $\text{Ad}(k''^{-1})\text{Ker}(\Psi_Y) \cap m_i \neq 0$. Hence $\text{Ad}(k''^{-1})\text{Ker}(\Psi_Y)$ is categorical (see Proposition 13) and $\text{Ad}(k''^{-1})\text{Ker}(\Psi_Y) \subseteq m_i$ (see Proposition 9). This means $Y \in p_i$. The converse can be proved in the same manner.

By these arguments we know that $X \in p_{\text{non}}$ if and only if $Y \in p_{\text{non}}$. \qed

**Lemma 17** Let $G/K = P^n(C) \ (n \geq 2)$ or $P^n(H) \ (n \geq 2)$. Then:

1. $p_0 = \emptyset$.
2. Let $X \in m \ (X \neq 0)$. Then:
Therefore we have
\[ \dim \ker(\varPsi_X) \leq \begin{cases} 
 n - 1, & \text{if } X \in p_1; \\
 f - 1, & \text{if } X \in p_2; \\
 2, & \text{if } X \in p_{\text{non}}.
\end{cases} \tag{5.2} \]

Proof. Suppose that \( p_0 \neq \emptyset \). Let \( X \in p_0 \) and let \( k \in K \) be an element such that \( \text{Ad}(k)\mu \in RX \). Then we have \( \text{Ad}(k^{-1})\ker(\varPsi_X) \subset a = R\mu \). Hence we have \( \ker(\varPsi_X) = R\text{Ad}(k)\mu = RX \), i.e., \( \varPsi(X, X) = 0 \). Let \( Y \in m \) such that \( Y \notin RX \). By (3.1) we have
\[
(\langle [X, Y], X \rangle, Y) = \langle \varPsi(X, X), \varPsi(Y, Y) \rangle - \langle \varPsi(X, Y), \varPsi(Y, X) \rangle
= -\langle \varPsi_X(Y), \varPsi_X(Y) \rangle.
\]

Since \( G/K \) is of positive curvature, the left side of the above equality is \( \geq 0 \). Therefore we have \( \varPsi_X(Y) = 0 \), which contradicts \( Y \notin RX \). Thus we have, \( p_0 = \emptyset \).

The assertion (2) follows from Propositions 12, Proposition 13, \( \dim m_2 = f - 1 \) and the discussions in the previous section. \( \square \)

**Proposition 18** \ Let \( G/K = P^n(C) \ (n \geq 2) \) or \( P^n(H) \ (n \geq 2) \). Then:

1. \( p_{\text{inj}} = \emptyset \) if \( r \leq nf - 1 \);
2. \( p_1 = \emptyset \) if \( r \leq 2(n - 1)(f - 1) \);
3. \( p_2 = \emptyset \) if \( r \leq (n - 1)f \);
4. \( p_{\text{non}} = \emptyset \) if \( r \leq nf - 3 \).

Proof. We first note that \( \dim \ker(\varPsi_X) \geq \dim G/K - r = nf - r \) holds for any \( X \in m \). By this fact we can easily prove (1), (3) and (4). In fact, if \( r \leq nf - 1 \), then it is clear that \( \ker(\varPsi_X) \neq 0 \) for any \( X \in m \). Hence \( X \notin p_{\text{inj}} \), which implies \( p_{\text{inj}} = \emptyset \). Similarly, if \( r \leq (n - 1)f \) (resp. \( r \leq nf - 3 \)), then \( \dim \ker(\varPsi_X) \geq f \) (resp. \( \dim \ker(\varPsi_X) \geq 3 \)) holds for any \( X \in m \) and hence \( p_2 = \emptyset \) (resp. \( p_{\text{non}} = \emptyset \)) (see Lemma 17).

Next we prove (2). Suppose that \( p_1 \neq \emptyset \). Let \( X \in p_1 \). Take \( k \in K \) such that \( \text{Ad}(k)\mu \in RX \) and set \( V = \text{Ad}(k^{-1})\ker(\varPsi_X) \). Then \( V \) is a pseudo-abelian subspace such that \( V \subset m_1 \). Consequently, by Lemma 17 we have \( \dim V \leq n - 1 \).

Now let us take a non-zero element \( \xi \in V \) and a subspace \( U \subset m_1 \) satisfying \( U \supset V \), \( [\xi, U] \subset \xi_0 \) and \( \dim U = (n - 2)f + 1 \) (see Proposition 12 (2)). Put \( Y = \text{Ad}(k)\xi \in \ker(\varPsi_X) \) and \( U = \text{Ad}(k)U \subset m \). Then we have \( \varPsi(X, Y) = 0 \) and \( U \supset \ker(\varPsi_X) \). Moreover, we have \( [U, Y], X = 0 \), because \( [U, Y], X = \text{Ad}(k)[[U, \xi], \mu] = 0 \). Therefore, by Theorem 6 we
have the following inequality:

$$r \geq nf + (n-2)f + 1 - \dim \ker(\Psi_X) - \dim \ker(\Psi_Y).$$

Since $X$ and $Y \in p_1$ (see Proposition 16), it follows that $\dim \ker(\Psi_X) \leq n-1$ and $\dim \ker(\Psi_Y) \leq n-1$ (see Lemma 17). Consequently, we have $r \geq 2(n-1)(f-1) + 1$, which proves (2).

We are now in a position to prove Theorem 1. If there is a solution $\Psi$ of the Gauss equation in codimension $r$, then at least one of the sets $p_{\text{inj}}, p_0, p_1, p_2$ and $p_{\text{non}}$ is not empty (see (5.1)). Therefore, in view of Lemma 17 (1) and Proposition 18, we have

$$r \geq 1 + \min\{nf - 1, 2(n-1)(f-1), (n-1)f, nf - 3\}.$$ 

Accordingly, we have $r \geq 2n - 2$ if $G/K = P^n(C)$ and $r \geq 4n - 3$ if $G/K = P^n(H)$. Hence, $\text{Crank}(P^n(C)) \geq 2n - 2$ and $\text{Crank}(P^n(H)) \geq 4n - 3$. This, together with Lemma 2, shows Theorem 1. \hfill $\square$

**Remark 1** The proof of Theorem 1 stated above is effective in the case $n = 2$. We thereby have $\text{Crank}(P^2(C)) \geq 2$ and $\text{Crank}(P^2(H)) \geq 5$. However, for the spaces $P^2(C)$ and $P^2(H)$, we have already known the best results: $\text{Crank}(P^2(C)) = 3$ (see [1]) and $\text{class}(P^2(H)) = \text{Crank}(P^2(H)) = 6$ (see [8]).

As for the class number of $P^2(C)$ we have $\text{class}(P^2(C)) = 3$ or 4 (see Lemma 2 and Introduction). It is still an open question whether $\text{class}(P^2(C)) = 3$ or not (cf. [20]).

**Remark 2** Consider the following two cases:

1. $G/K = P^n(C)$ ($n \geq 3$) and $r = 2n - 2$;
2. $G/K = P^n(H)$ ($n \geq 3$) and $r = 4n - 3$.

If there is a solution $\Psi$ of the Gauss equation in codimension $r$, then it is shown by Lemma 17 (1) and Proposition 18 that $\Psi$ must satisfy the following condition:

Case (1) $p_0 = p_1 = p_2 = p_{\text{inj}} = \emptyset$, i.e., $m \setminus \{0\} = p_{\text{non}}$;
Case (2) $p_0 = p_1 = p_{\text{non}} = p_{\text{inj}} = \emptyset$, i.e., $m \setminus \{0\} = p_2$.

We conjecture that in both cases (1) and (2) there are no such solutions $\Psi$.

In other words:

$$\text{Crank}(P^n(C)) \geq 2n - 1 \quad (n \geq 3);$$
$$\text{Crank}(P^n(H)) \geq 4n - 2 \quad (n \geq 3).$$
If this is true, then we obtain an improvement of Theorem 1:

\[
\begin{align*}
\text{class}(P^n(C)) & \geq 2n - 1 \quad (n \geq 3); \\
\text{class}(P^n(H)) & \geq 4n - 2 \quad (n \geq 3).
\end{align*}
\]

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