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Affine differential geometry of the unit normal vector fields of hypersurfaces in the real space forms

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Abstract. In this paper, for a hypersurface in the real space form of constant curvature, we prove that the unit normal vector field is an affine imbedding into a certain sphere bundle with canonical metric. Moreover, we study the relations between a hypersurface and its unit normal vector field as an affine imbedding. In particular, several hypersurfaces are characterized by affine geometric conditions which are independent of the choice of the transversal bundle.

Key words: section of sphere bundle, canonical metric, metrically minimal affine immersion, metrically totally umbilic affine immersion.

1. Introduction

Let M (resp. $\tilde{M}$) be a Riemannian manifold with a metric g (resp. $\tilde{g}$) and $f\colon (M,g) \to (\tilde{M},\tilde{g})$ an isometric immersion. We set $E := f^{\#}(T\tilde{M})$, where $f^{\#}(T\tilde{M})$ is the pull back bundle of the tangent bundle $T\tilde{M}$ by f. The (unit) sphere bundle of E is denoted by UE. We take the canonical metric G on E relative to the pull back connection and metric of $\tilde{M}$, and also denote induced metric on UE by G. In submanifold geometry, it is important and interesting to study maps from submanifolds to the suitable spaces, and the relations between the maps and submanifolds have been studied by many researchers. See [4], [13] and [14], for example. The main purpose of this paper is to study the relations between the section of UE and f from the view point of affine differential geometry. The inclusion map from the normal bundle $T^{\perp}M$ to E is denoted by ι. Let $Q^{n+1}(c)$ be the real space form of constant curvature c and dimension $n+1$. We denote the Levi-Civita connection on M (resp. UE) by ∇ (resp. $\bar{\nabla}^G$). Let $\Gamma(V)$ be the space of all sections of a vector bundle V. One of our main theorems in this paper is

Theorem *Let M be an orientable hypersurface in $Q^{n+1}(c)$ immersed by f and ξ the unit normal vector field. Then $\iota \circ \xi\colon (M,\nabla) \to (UE,\bar{\nabla}^G)$ is an*

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affine imbedding with the transversal bundle $N = (\iota \circ \xi)^{\#}(\ker(p|_{UE})_*) \cong TM$*, where* $p\colon E \to M$ *is the bundle projection. The affine fundamental form* α*, the affine shape operator* A *and the transversal connection* ∇^N *of* $\iota \circ \xi$ *are given by*

$$\alpha(X,Y) = -((\nabla_X S)(Y))^t, \quad A_{X^t}Y = 0, \quad \nabla^N_X Y^t = (\nabla_X Y)^t$$

for all X, $Y \in \Gamma(TM)$*, where* S *is the shape operator of* f *relative to* ξ *and* X^t *stands for the tangential lift of* $X \in \Gamma(TM)$*.*

In this case, affine differential geometry naturally appears in Riemannian one. Hence, the relations between a hypersurface in the real space form and its unit normal vector field as an affine imbedding are of interest. Several hypersurfaces in $Q^{n+1}(c)$ are characterized by affine geometric conditions for $\iota \circ \xi$, which are independent of the choice of the transversal bundle. For example, we prove that f has the parallel second fundamental form if and only if $\iota \circ \xi$ is totally geodesic.

In Section 2, we recall affine immersions with transversal bundles. The fundamental lemmas for sections of sphere bundles are obtained in Section 3. Finally, in Section 4, we study the unit normal vector fields of hypersurfaces in the real space forms as affine imbeddings.

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2. Affine immersions with transversal bundles

In this section, we recall the definition of the affine immersion with transversal bundle. Let (M, ∇) and $(\hat{M}, \hat{\nabla})$ be smooth manifolds with torsion free affine connections and $F\colon M \to \hat{M}$ an immersion. The pull back bundle of $T\hat{M}$ by F is denoted by $F^{\#}T\hat{M}$. An immersion $F\colon M \to \hat{M}$ is called an immersion with a transversal bundle N if $F^{\#}T\hat{M} = TM \oplus N$ holds. Let π_{TM} and π_N be the projections from $F^{\#}T\hat{M}$ onto TM and N, respectively. We say that $F\colon (M, \nabla) \to (\hat{M}, \hat{\nabla})$ is an affine immersion with a transversal bundle N if F is an immersion with a transversal bundle N and $\pi_{TM}((F^{\#}\hat{\nabla})_X Y) = \nabla_X Y$ for all X, $Y \in \Gamma(TM)$, where $F^{\#}\hat{\nabla}$ is the pull back connection of $\hat{\nabla}$ by F. Set $\alpha(X,Y) := \pi_N((F^{\#}\hat{\nabla})_X Y)$, $A_\nu X := -\pi_{TM}((F^{\#}\hat{\nabla})_X \nu)$ and $\nabla^N_X \nu := \pi_N((F^{\#}\hat{\nabla})_X \nu)$ for X, $Y \in \Gamma(TM)$ and $\nu \in \Gamma(N)$. Then α, A and ∇^N are called the affine fundamental form, the affine shape operator and the transversal connection, respectively. We see that

$$(F^{\#}\hat{\nabla})_X Y = \nabla_X Y + \alpha(X,Y) \quad \text{and} \quad (F^{\#}\hat{\nabla})_X \nu = -A_\nu X + \nabla^N_X \nu$$

for X, $Y \in \Gamma(TM)$ and $\nu \in \Gamma(N)$. We refer to [10] and [11] for affine immersions. Set

$$(\nabla'_X \alpha)(Y,Z) := \nabla^N_X \alpha(Y,Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$$

for X, Y, $Z \in \Gamma(TM)$. An affine immersion F with a transversal bundle N is called *totally geodesic* if $\alpha = 0$ on M and *parallel* if $\nabla'\alpha = 0$ on M. We set

$$T(X,Y,Z) = (\nabla'_X \alpha)(Y,Z) + (\nabla'_Y \alpha)(Z,X) + (\nabla'_Z \alpha)(X,Y)$$

for any tangent vectors X, Y, Z on M. An affine immersion F with a transversal bundle N is said to be *cyclic parallel* if $T = 0$. For the cyclic parallel condition, we have

Lemma 2.1 *Let $F\colon (M,\nabla) \to (\hat{M},\hat{\nabla})$ be an affine immersion with a transversal bundle N. The following three statements are mutually equivalent*:
(1) *F is cyclic parallel,*
(2) $(\nabla'_X \alpha)(X,X) = 0$ *for all $X \in TM$,*
(3) *For any geodesic $\gamma\colon [0,t] \to M$, it holds that $P^N_\gamma(\alpha(\gamma'(0),\gamma'(0))) = \alpha(\gamma'(t),\gamma'(t))$, where $P^N_\gamma : N_{\gamma(0)} \to N_{\gamma(t)}$ is the parallel translation along γ with respect to ∇^N.*

Proof. It is easy to prove that (1) $\Longleftrightarrow$ (2) by the polarization. Next, assume that (2) holds and take any geodesic $\gamma\colon [0,t] \to M$. We obtain

$$0 = (\nabla'_{\gamma'}\alpha)(\gamma',\gamma') = \nabla^N_{\gamma'}\alpha(\gamma',\gamma').$$

Hence we can see that (3) holds. Conversely, for any point $x \in M$ and any tangent vector X at x, take a geodesic γ such that $\gamma(0) = x$ and $\gamma'(0) = X$. Then we have

$$(\nabla'_X \alpha)(X,X) = \nabla^N_X \alpha(\gamma',\gamma') = \frac{d}{dt}(P^N_\gamma)^{-1}\alpha(\gamma',\gamma')\bigg|_{t=0} = 0. \qquad \square$$

From (3) in Lemma 2.1, if F is a cyclic parallel isometric immersion, then for each geodesic γ with arc length parameter on M, the first Frenet curvature of $F\circ\gamma$ is constant along the curve. Moreover, from the view point of affine differential geometry, the cyclic parallel condition is independent of the choice of the transversal bundle such that the induced connection coincides (see bellow). In Section 4, we consider a weaker condition than the cyclic parallel condition for unit normal vector fields of hypersurfaces in the Euclidean spaces. We prepare the following notations to prove the independence of the choice of the transversal bundle for conditions studying in this paper. Let $\bar{N}$ be another transversal bundle. The corresponding objects associated with $\bar{N}$ are denoted by the symbol with "$-$". Let $\iota_{TM}\colon TM \to F^{\#}T\hat{M}$ (resp. $\iota_N\colon N \to F^{\#}T\hat{M}$) be the inclusion from TM (resp. N) to $F^{\#}T\hat{M}$ and $\pi_{TM}\colon F^{\#}T\hat{M} \to TM$ (resp. $\pi_N\colon F^{\#}T\hat{M} \to N$) the projection from $F^{\#}T\hat{M}$ to TM (resp. N) with respect to the decomposition $F^{\#}T\hat{M} = TM \oplus N$. Set

$$\begin{aligned}
&P := \bar{\pi}_{\bar{N}}\iota_N\colon N \to \bar{N},\\
&Q := \bar{\pi}_{TM}\iota_N\colon N \to TM,\\
&\hat{P} := \pi_N\bar{\iota}_{\bar{N}}\colon \bar{N} \to N,\\
&\hat{Q} := \pi_{TM}\bar{\iota}_{\bar{N}}\colon \bar{N} \to TM.
\end{aligned}$$

Since the following lemma is proved in [1] and [9], we give a short proof here.

Lemma 2.2 *The following equations hold*:

$$\begin{aligned}
&\bar{\nabla}_X Y = \nabla_X Y + Q(\alpha(X,Y)),\\
&\bar{\alpha}(X,Y) = P(\alpha(X,Y)),\\
&\bar{A}_\nu X = -\nabla_X\hat{Q}(\nu) + A_{\hat{P}(\nu)}X - Q(\alpha(X,\hat{Q}(\nu))) - Q(\nabla^N_X\hat{P}(\nu)),\\
&\nabla^{\bar{N}}_X\nu = P(\alpha(X,\hat{Q}(\nu))) + P(\nabla^N_X\hat{P}(\nu))
\end{aligned}$$

for X, $Y \in \Gamma(TM)$ *and* $\nu \in \Gamma(\bar{N})$.

Proof. Since $\iota_{TM}\pi_{TM} + \iota_N\pi_N = \mathrm{id}_{F^{\#}T\hat{M}}$, we have

$$\begin{aligned}
\bar{\nabla}_X Y &= \bar{\pi}_{TM}((F^{\#}\hat{\nabla})_X Y) = \bar{\pi}_{TM}(\iota_{TM}\pi_{TM} + \iota_N\pi_N)((F^{\#}\hat{\nabla})_X Y)\\
&= \nabla_X Y + Q(\alpha(X,Y))
\end{aligned}$$

for X, $Y \in \Gamma(TM)$. Other equations are also obtained by a similar calculation. $\square$

Proposition 2.3 *With the notations as above, if* $\nabla = \bar{\nabla}$, *we have*

$$P((\nabla'_X\alpha)(Y,Z)) = (\bar{\nabla}'_X\bar{\alpha})(Y,Z)$$

for all tangent vectors X, Y, Z on M.

Proof. By Lemma 2.2 and $\nabla = \bar{\nabla}$, we have $P(\nabla^N_X \alpha(Y,Z)) = \nabla^{\bar{N}}_X \bar{\alpha}(Y,Z)$ for X, Y, $Z \in \Gamma(TM)$. Therefore we have the conclusion. □

In the case where F is an affine immersion with a transversal bundle N from a Riemannian manifold (M, g), we set

$$H^F := \frac{1}{n}\sum_{i=1}^{n} \alpha(e_i, e_i),$$

where $n = \dim M$ and $e_1, \ldots, e_n$ is an orthonormal frame of (M, g). An affine immersion $F\colon (M, \nabla) \to (\hat{M}, \hat{\nabla})$ is said to be a *metrically minimal* (resp. *metrically totally umbilic*) affine immersion if $H^F = 0$ (resp. $\alpha(X, Y) = g(X, Y)H^F$ for all X, $Y \in TM$). From Lemma 2.2, all conditions $\alpha = 0$, $H^F = 0$ and $\alpha(X, Y) = g(X, Y)H^F$ for all X, $Y \in TM$ are independent of the choice of the transversal bundle.

3. Sphere bundles with canonical metrics and sections

In this section, we study the sphere bundles with canonical metrics. Let E be a Riemannian vector bundle over a Riemannian manifold (M, g) with a fiber metric g^E and a metric connection ∇^E. We denote the connection map with respect to ∇^E by K^E. The canonical metric G on E is defined by

$$G(\zeta, \zeta) = g(p_*(\zeta), p_*(\zeta)) + g^E(K^E(\zeta), K^E(\zeta)),$$

where $\zeta \in TE$ and $p\colon E \to M$ is the bundle projection. Note that p is a Riemannian submersion (see [2] and [12]). Let $\mathcal{V}$ (resp. $\mathcal{H}$) be the projection from TE onto $\ker p_*$ (resp. $\ker K^E$). The curvature form of ∇^E is denoted by R^E. We define $\hat{R}^E_{\xi,\eta}$ for ξ, $\eta \in \Gamma(E)$ by

$$g(\hat{R}^E_{\xi,\eta}X, Y) = g^E(R^E_{X,Y}\xi, \eta),$$

where X, $Y \in TM$. Let ∇^G (resp. ∇) be the Levi-Civita connection of G (resp. g) on E (resp. M). We set

$$H^{\nabla^E}(X, Y)\xi := -\nabla^E_X \nabla^E_Y \xi + \nabla^E_{\nabla_X Y}\xi$$

for X, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(E)$. For $\xi \in \Gamma(E)$, its vertical lift is denoted by ξ^v, and X^h stands for the horizontal lift of $X \in \Gamma(TM)$. We note that

$K^E(\xi^v) = \xi$ for $\xi \in \Gamma(E)$. The following equations hold at $u \in E$ (see [3]):

$$\begin{aligned}
&\nabla^G{}_{\xi^v}\zeta^v = 0,\\
&\nabla^G{}_{\xi^v}Y^h = \frac{1}{2}(\hat{R}^E_{u,\xi}Y)^h,\\
&\nabla^G{}_{X^h}\xi^v = \frac{1}{2}(\hat{R}^E_{u,\xi}X)^h + (\nabla^E_X\xi)^v,\\
&\nabla^G{}_{X^h}Y^h = (\nabla_X Y)^h - \frac{1}{2}(R^E_{X,Y}u)^v.
\end{aligned}$$

Set $UE := \{u \in E \,|\, g^E(u,u) = 1\}$. The restriction of G to UE is also denoted by G. A unit normal vector field η on UE in E is the vertical vector such that $\eta_u = u^v$ for $u \in UE$. For $\xi \in E$, we define the tangential lift ξ^t of ξ to $u \in UE$ by $\xi^t = \xi^v - g^E(\xi,u)\eta_u$. The tangential lift of a section $\xi \in \Gamma(E)$ is the vertical vector field ξ^t on UE whose value at $u \in UE$ is the tangential lift of $\xi(p(u))$. Let $\mathcal{A}$ be the shape operator of UE in E with respect to η.

Lemma 3.1 *For any vertical vector field U and any horizontal vector field X tangent to UE, we have $\mathcal{A}(U) = -U$ and $\mathcal{A}(X) = 0$.*

Proof. We may assume that U is the tangential lift ξ^t of $\xi \in \Gamma(E)$. Fix $u \in UE$ and take a vertical curve $\bar{u}\colon I \to UE$ defined on an open interval I containing 0 such that $u = \bar{u}(0)$ and $\bar{u}'(0) = (\xi^t)_u$. We have

$$\mathcal{A}(\xi^t)_u = -(\nabla^G_{\xi^t}\eta)_u = -\frac{d}{dt}\bar{u}(t)^v\Big|_{t=0} = -(\xi^t)_u.$$

Similarly, we obtain $\mathcal{A}(X) = 0$ taking a horizontal curve. □

Let $\bar{\nabla}^G$ be the Levi-Civita connection of UE relative to G. Using Lemma 3.1, we have the following lemma.

Lemma 3.2 *For ξ, $\zeta \in \Gamma(E)$ and X, $Y \in \Gamma(TM)$, at $u \in UE$, we have*

$$\begin{aligned}
&\bar{\nabla}^G{}_{\xi^t}\zeta^t = -g^E(\zeta(p(u)),u)\xi^t,\\
&\bar{\nabla}^G_{\xi^t}Y^h = \frac{1}{2}(\hat{R}^E_{u,\xi}Y)^h,\\
&\bar{\nabla}^G_{X^h}\xi^t = (\nabla^E_X\xi)^t + \frac{1}{2}(\hat{R}^E_{u,\xi}X)^h,\\
&\bar{\nabla}^G_{X^h}Y^h = (\nabla_X Y)^h - \frac{1}{2}(R^E_{X,Y}u)^t.
\end{aligned}$$

Proof. We only prove the first equation, since $\mathcal{A}(X) = 0$ for any horizontal vector X. Take a vertical curve $\bar{u}\colon I \to UE$ defined on an open interval I containing 0 such that $u = \bar{u}(0)$ and $\bar{u}'(0) = (\xi^t)_u$. Since $\zeta^t_u = \zeta^v_u - g^E(\zeta(p(u)), u)u^v$, we have

$$\begin{aligned}
(\bar{\nabla}^G{}_{\xi^t}\zeta^t)_u &= \nabla^G{}_{u'(0)}\zeta^t + G_u(\xi^t, \zeta^t)\eta_u \\
&= \frac{d}{dt}\big(\zeta^v_{\bar{u}(t)} - g^E(\zeta(p(\bar{u}(t))), \bar{u}(t))\bar{u}(t)^v\big)\Big|_{t=0} + G_u(\xi^t, \zeta^t)\eta_u \\
&= -g^E\left(\zeta(p(u)), \frac{d}{dt}\bar{u}(t)\Big|_{t=0}\right)u^v \\
&\quad - g^E(\zeta(p(u)), u)\,\frac{d}{dt}\bar{u}(t)^v\Big|_{t=0} + G_u(\xi^t, \zeta^t)\eta_u \\
&= -g^E(\zeta(p(u)), u)\xi^t_u,
\end{aligned}$$

where we used Lemma 3.1. $\square$

Let R (resp. $\bar{R}^G$) be the curvature tensor of ∇ (resp. $\bar{\nabla}^G$). By Lemma 3.2, the curvature tensor $\bar{R}^G$ is given as follows.

Lemma 3.3 *At $u \in UE$, we have*

$$\begin{aligned}
\bar{R}^G_{\xi_1{}^t, \xi_2{}^t}\xi_3{}^t &= g^E(\xi_2, \xi_3)\xi_1^t - g^E(\xi_1, \xi_3)\xi_2^t, \\
\bar{R}^G_{\xi_1{}^t, \xi_2{}^t}Z^h &= \left(\hat{R}^E_{\xi_1, \xi_2}Z + \frac{1}{4}\hat{R}^E_{u,\xi_1}\hat{R}^E_{u,\xi_2}Z - \frac{1}{4}\hat{R}^E_{u,\xi_2}\hat{R}^E_{u,\xi_1}Z\right)^h, \\
\bar{R}^G_{X^h, \xi_2{}^t}\xi_3{}^t &= -\left(\frac{1}{2}\hat{R}^E_{\xi_2,\xi_3}X + \frac{1}{4}\hat{R}^E_{u,\xi_2}\hat{R}^E_{u,\xi_3}X\right)^h, \\
\bar{R}^G_{X^h, \xi_2{}^t}Z^h &= \frac{1}{2}((\nabla_X\hat{R}^E)_{u,\xi_2}Z)^h + \left(\frac{1}{2}R^E_{X,Z}\xi_2 + \frac{1}{4}R^E{}_{\hat{R}^E_{u,\xi_2}Z, X}u\right)^t, \\
\bar{R}^G_{X^h, Y^h}\xi_3{}^t &= \frac{1}{2}((\nabla_X\hat{R}^E)_{u,\xi_3}Y) - (\nabla_Y\hat{R}^E)_{u,\xi_3}X))^h \\
&\quad + \left(R^E_{X,Y}\xi_3 + \frac{1}{4}R^E_{\hat{R}^E_{u,\xi_3}Y, X}u - \frac{1}{4}R^E_{\hat{R}^E_{u,\xi_3}X, Y}u\right)^t, \\
\bar{R}^G_{X^h, Y^h}Z^h &= \left(R_{X,Y}Z + \frac{1}{4}\hat{R}^E_{u, R^E_{Z,Y}u}X + \frac{1}{4}\hat{R}^E_{u, R^E_{X,Z}u}Y\right. \\
&\quad \left. + \frac{1}{2}\hat{R}^E_{u, R^E_{X,Y}u}Z\right)^h + \frac{1}{2}((\nabla^E_Z R^E)_{X,Y}u)^t
\end{aligned}$$

for X, Y, $Z \in TM$ *and* ξ_1, ξ_2, $\xi_3 \in E$ *such that* $g^E(u,\xi_i) = 0$ $(i = 1, 2, 3)$.

We also write ζ^t for the pull back section of ζ^t by $\xi \in \Gamma(UE)$, for simplicity. With respect to the Levi-Civita connections ∇ and $\bar{\nabla}^G$, we obtain

Lemma 3.4 *A section* $\xi \in \Gamma(UE)$ *is an imbedding with the transversal bundle* $\xi^{\#}(\ker(p|_{UE})_*)$. *At* $u = \xi(x)$ $(x \in M)$, *we have*

$$\begin{aligned}
\pi_{TM}((\xi^{\#}\bar{\nabla}^G)_X Y) &= \nabla_X Y + \frac{1}{2}\hat{R}^E_{u,\nabla^E_Y \xi} X + \frac{1}{2}\hat{R}^E_{u,\nabla^E_X \xi} Y, \\
\pi_N((\xi^{\#}\bar{\nabla}^G)_X Y) &= -\frac{1}{2}\Big(\nabla^E_{\hat{R}^E_{u,\nabla^E_Y \xi} X}\xi\Big)^t - \frac{1}{2}\Big(\nabla^E_{\hat{R}^E_{u,\nabla^E_X \xi} Y}\xi\Big)^t \\
&\quad - \Big(H^{\nabla^E}(X,Y)\xi\Big)^t - \frac{1}{2}(R^E_{X,Y}u)^t, \\
\pi_{TM}((\xi^{\#}\bar{\nabla}^G)_X \zeta^t) &= \frac{1}{2}\hat{R}^E_{u,\zeta} X, \\
\pi_N((\xi^{\#}\bar{\nabla}^G)_X \zeta^t) &= (\nabla^E_X \zeta)^t - \frac{1}{2}\Big(\nabla^E_{\hat{R}^E_{u,\zeta} X}\xi\Big)^t - g^E(\zeta(x),u)(\nabla^E_X \xi)^t
\end{aligned}$$

for all X, $Y \in \Gamma(TM)$ *and* $\zeta \in \Gamma(E)$, *where* $N = \xi^{\#}(\ker(p|_{UE})_*)$.

Proof. From Lemma 3.2 and $X^h = \xi_*(X) - (\nabla^E_X \xi)^t$ on $\xi(M)$ for $X \in TM$, it is easy to obtain the conclusion. For example, we have

$$\begin{aligned}
(\xi^{\#}\bar{\nabla}^G)_X \zeta^t &= (\nabla^E_X \zeta)^t + \frac{1}{2}(\hat{R}^E_{u,\zeta} X)^h - g^E(\zeta,u)(\nabla^E_X \xi)^t \\
&= (\nabla^E_X \zeta)^t + \frac{1}{2}\xi_*(\hat{R}^E_{u,\zeta} X) - \frac{1}{2}\Big(\nabla^E_{\hat{R}^E_{u,\zeta} X}\xi\Big)^t \\
&\quad - g^E(\zeta(x),u)(\nabla^E_X \xi)^t.
\end{aligned}$$

□

Using Lemma 3.4, the following proposition can be obtained immediately.

Proposition 3.5 *Let* ξ *be a section of* UE. *If* $\hat{R}^E_{\xi,\nabla^E_Y \xi} X = 0$ *for all* X, $Y \in TM$, *then* $\xi\colon (M,\nabla) \to (UE, \bar{\nabla}^G)$ *is an affine imbedding with the transversal bundle* $N = \xi^{\#}(\ker(p|_{UE})_*)$. *At* $u = \xi(x)$ $(x \in M)$, *the affine fundamental form* α, *the affine shape operator* A *and the transversal connection* ∇^N *of* ξ *are given by*

$$\alpha(X,Y) = -\Big(H^{\nabla^E}(X,Y)\xi\Big)^t - \frac{1}{2}(R^E_{X,Y}u)^t,$$

$$A_{\zeta^t}X = -\frac{1}{2}\hat{R}^E_{u,\zeta}X,$$
$$\nabla^N_X\zeta^t = (\nabla^E_X\zeta)^t - \frac{1}{2}\Big(\nabla^E_{\hat{R}^E_{u,\zeta}X}\xi\Big)^t - g^E(\zeta(x),u)(\nabla^E_X\xi)^t$$

for all X, $Y \in \Gamma(TM)$ *and* $\zeta \in \Gamma(E)$.

In the case where $E = TM$, vector fields with unit length have been studied from the view point of geometric variational problems. We refer to [5], [6], [7], [15], [16] and [17] for example. An isometric immersion $F\colon M \to \hat{M}$ is an affine immersion with the transversal bundle $T^{\perp}M$ with respect to Levi-Civita connections of M and $\hat{M}$, where $T^{\perp}M$ is the normal bundle. Proposition 3.5 gives an example of non isometric affine immersions between Riemannian manifolds relative to Levi-Civita connections.

4. Affine differential geometry of unit normal vector fields

Let $f\colon (M,g) \to (Q^{n+1}(c),\tilde{g})$ be an isometric immersion from an orientable n-dimensional Riemannian manifold M into the real space form $Q^{n+1}(c)$ of constant curvature c and dimension $n+1$. The inclusion map from the normal bundle $T^{\perp}M$ to $f^{\#}(TQ^{n+1}(c))$ is denoted by ι. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections of M and $Q^{n+1}(c)$, respectively. The pull back connection of $\tilde{\nabla}$ by f is denoted by $f^{\#}\tilde{\nabla}$ and $f^{\#}\tilde{R}$ stands for the curvature form of $f^{\#}\tilde{\nabla}$. Let S be the shape operator relative to the unit normal vector field ξ. The mean curvature function H is defined by $H = (1/n)\operatorname{tr} S$. We set $E := f^{\#}(TQ^{n+1}(c))$ and give the canonical metric G on E with respect to $f^{\#}\tilde{\nabla}$. Let $\bar{\nabla}^G$ be the Levi-Civita connection of UE. From Proposition 3.5, we have

Theorem 4.1 *Let M be an orientable hypersurface in $Q^{n+1}(c)$ immersed by f and ξ the unit normal vector field. Then $\iota\circ\xi\colon (M,\nabla) \to (UE,\bar{\nabla}^G)$ is an affine imbedding with the transversal bundle $N = (\iota\circ\xi)^{\#}(\ker(p|_{UE})_*) \cong TM$. The affine fundamental form α, the affine shape operator A and the transversal connection ∇^N of $\iota\circ\xi$ are given by*

$$\alpha(X,Y) = -((\nabla_X S)(Y))^t, \quad A_{X^t}Y = 0, \quad \nabla^N_X Y^t = (\nabla_X Y)^t$$

for all X, $Y \in \Gamma(TM)$.

Proof. Since M is a hypersurface, each fiber $(\iota\circ\xi)^{\#}(\ker(p|_{UE})_*)_x$ can be identified with T_xM at $x \in M$ by the map $T_xM \ni X \mapsto X^t_{(\iota\circ\xi)(x)} \in$

$(\iota\circ\xi)^{\#}(\ker(p|_{UE})_*)_x$. From Proposition 3.5, we obtain

$$\begin{aligned}
&\alpha(X,Y) = -((\nabla_X S)(Y))^t - \frac{1}{2}((f^{\#}\tilde{R})_{X,Y}u)^t = -((\nabla_X S)(Y))^t,\\
&A_{X^t}Y = -\frac{1}{2}(f^{\#}\tilde{R})^{\hat{}}_{u,X}Y = 0,\\
&\nabla^N_X Y^t = (\nabla_X Y)^t + \tilde{g}(Y,u)(-SX)^t = (\nabla_X Y)^t
\end{aligned}$$

for all X, $Y \in \Gamma(TM)$, since $(f^{\#}\tilde{R})^{\hat{}}_{u,X}Y = 0$ and $(f^{\#}\tilde{R})_{X,Y}u = 0$ at $u = (\iota\circ\xi)(x)$ $(x \in M)$. □

Corollary 4.2 *Let M be an orientable hypersurface in $Q^{n+1}(c)$ immersed by f and ξ the unit normal vector field. Then we have*
(1) *f has the parallel second fundamental form if and only if $\iota\circ\xi$ is totally geodesic,*
(2) *In the case where $\dim M \geq 2$, f has the parallel second fundamental form if and only if $\iota\circ\xi$ is metrically totally umbilic,*
(3) *f has a constant mean curvature if and only if $\iota\circ\xi$ is metrically affine minimal,*
(4) *For X, Y, $Z \in \Gamma(TM)$, the equation $(R_{X,Y}Z)^t = R^N_{X,Y}Z^t$ holds, where R is the curvature tensor of M and R^N is the curvature form of ∇^N. In particular, M is flat if and only if the transversal connection of $\iota\circ\xi$ is flat.*

Proof. The statements (1) and (4) can be obtained by Theorem 4.1 immediately. From Theorem 4.1, we have

$$H^{\iota\circ\xi} = -(\operatorname{grad} H)^t. \tag{4.1}$$

Hence we see the statement (3). Finally, to prove (2), we assume that $\iota\circ\xi$ is metrically totally umbilic, that is, $\alpha(X,Y) = g(X,Y)H^{\iota\circ\xi}$ holds for all X, $Y \in TM$. By Theorem 4.1 and (4.1), we have

$$g((\nabla_X S)(Y), Z) = g(X,Y)g(\operatorname{grad} H, Z) \tag{4.2}$$

for all X, Y, $Z \in TM$. Since the left hand side of (4.2) is symmetric, it holds that

$$g(X,Y)g(\operatorname{grad} H, Z) = g(X,Z)g(\operatorname{grad} H, Y)$$

for all X, Y, $Z \in TM$. Therefore, by $\dim M \geq 2$, we have $H^{\iota\circ\xi} = -(\operatorname{grad} H)^t = 0$. Hence we see that f has the parallel second fundamental form. Because the converse is trivial, we obtain the statement (2). □

We note that all conditions $\alpha = 0$, $H^{\iota\circ\xi} = 0$ and $\alpha(X,Y) = g(X,Y)H^{\iota\circ\xi}$ for all X, $Y \in TM$ are independent of the choice of the transversal bundle. In [8], we proved that $f\colon M \to Q^{n+1}(c)$ is a constant mean curvature hypersurface if and only if $\iota\circ\xi$ is a harmonic section.

In general, the covariant derivative $\nabla'\alpha$ is not symmetric for $\iota\circ\xi$. We set

$$T(X,Y,Z) = (\nabla'_X\alpha)(Y,Z) + (\nabla'_Y\alpha)(Z,X) + (\nabla'_Z\alpha)(X,Y),$$
$$\hat{T}(X) = \sum_{i=1}^{n} T(e_i,e_i,X)$$

for all X, Y, $Z \in TM$, where $e_1,\dots,e_n$ is an orthonormal frame of M. Hereafter, we consider hypersurfaces in the Euclidean spaces with $\hat{T} = 0$. Since

$$\mathcal{V}((\iota\circ\xi)_*(X)) = -(SX)^t$$

for $X \in TM$, the horizontal lift X^h of $X \in TM$ can be decomposed into

$$X^h = (\iota\circ\xi)_*(X) + (SX)^t \tag{4.3}$$

on $(\iota\circ\xi)(M)$.

Lemma 4.3 *Let $F\colon (M,\nabla)\to(\hat{M},\hat{\nabla})$ be an affine immersion with a transversal bundle N. We have*

$$T(X,Y,Z) = 3(\nabla'_Z\alpha)(X,Y) + \pi_N((F^{\#}R')_{X,Z}Y) + \pi_N((F^{\#}R')_{Y,Z}X)$$

for any tangent vectors X, Y, Z on M, where $F^{\#}R'$ is the curvature form of $F^{\#}\hat{\nabla}$ and α is the affine fundamental form of F.

Proof. By the Codazzi equation of F,

$$\pi_N((F^{\#}R')_{X,Y}Z) = (\nabla'_X\alpha)(Y,Z) - (\nabla'_Y\alpha)(X,Z)$$

for any tangent vectors X, Y, Z on M, we have the conclusion. $\square$

We define $\nabla^2 S$ by

$$(\nabla^2_{X,Y}S)(Z) = \nabla_X((\nabla_Y S)(Z)) - (\nabla_{\nabla_X Y}S)(Z) - (\nabla_Y S)(\nabla_X Z)$$

for X, Y, $Z \in \Gamma(TM)$.

Lemma 4.4 *Let M be an orientable hypersurface in $\mathbf{R}^{n+1}$ immersed by f and ξ the unit normal vector field. Then we have*

$$\begin{aligned} T(X,Y,Z) = \bigl\{&-3(\nabla^2_{Z,X}S)(Y) + g(SZ,Y)S^2X - g(SX,Y)S^2Z \\ &- g(SZ,SY)SX + g(SZ,X)S^2Y - g(SY,X)S^2Z \\ &- g(SZ,SX)SY + 2g(SY,SX)SZ\bigr\}^t, \end{aligned} \tag{4.4}$$

$$\hat{T}(X) = \bigl\{-3n\nabla_X \operatorname{grad} H - 2nHS^2X + 2\|S\|^2SX\bigr\}^t, \tag{4.5}$$

$$\operatorname{tr}\hat{T} = 3n\triangle H \tag{4.6}$$

for any tangent vectors X, Y, Z on M.

Proof. By virtue of Theorem 4.1, it holds that

$$(\nabla'_Z\alpha)(X,Y) = -((\nabla^2_{Z,X}S)(Y))^t$$

for all X, Y, $Z \in TM$. Set $N = (\iota\circ\xi)^{\#}(\ker(p|_{UE})_*)$. By Lemma 3.3 and (4.3), we have

$$\begin{aligned} \pi_N((\xi^{\#}\bar{R}^G)_{X,Z}Y) &= \pi_N\bigl(\bar{R}^G_{X^h-(SX)^t,Z^h-(SZ)^t}(Y^h - (SY)^t)\bigr) \\ &= \bigl\{g(SZ,Y)S^2X - g(SX,Y)S^2Z \\ &\qquad - g(SZ,SY)SX + g(SX,SY)SZ\bigr\}^t \end{aligned}$$

for any tangent vectors X, Y, Z. Then we can obtain (4.4) combined with Lemma 4.3. For (4.5), take an orthonormal frame $e_1,\dots,e_n$ on M. Since

$$\sum_{i=1}^n(\nabla^2_{X,e_i}S)(e_i) = n(\nabla_X \operatorname{grad} H),$$

we have the desired conclusion by calculating the trace of (4.4). From (4.5), it is easy to obtain the last equation. □

Note that we consider the trace of $\hat{T}$ under the identification $N \cong TM$. For a hypersurface in the Euclidean space, we see that $\triangle H = 0$ if and only if $\operatorname{tr}\hat{T} = 0$ from (4.6). In the case where M is compact, we characterize a hypersurface with $\hat{T} = 0$ using the previous lemma.

Theorem 4.5 *Let M be a compact connected orientable hypersurface in $\mathbf{R}^{n+1}$ and ξ the unit normal vector field. If $\hat{T} = 0$ for $\iota\circ\xi\colon M \to UE$, then M is congruent to the standard sphere.*

Proof. From (4.6), H is constant on M. Hence, from (4.5), we have

$$-nHS^2X + \|S\|^2SX = 0 \tag{4.7}$$

for any tangent vector X on M. If $H = 0$, then we obtain $\|S\|^2SX = 0$. Therefore M is totally geodesic. So $H \neq 0$. Let $\lambda_1(p), \ldots, \lambda_n(p)$ be principal curvatures at $p \in M$. From (4.7), we have

$$0 = \lambda_1(p) = \cdots = \lambda_{k(p)}(p) < \lambda_{k(p)+1}(p) = \cdots = \lambda_n(p) = \frac{\|S\|_p^2}{nH} := \lambda(p)$$

at each point p, where $k(p) = \dim\ker S_p$. On M, the function λ is smooth, and hence $k = n(\lambda - H)/\lambda$ is also smooth. Therefore k is constant on M. Moreover, by $nH = k\lambda$, λ is constant. Then M is an isoparametric hypersurface. Since M is compact, we have the desired conclusion. □

We note that the condition $\hat{T} = 0$ is independent of the choice of the transversal bundle such that the induced connection coincides from Proposition 2.3. Similarly we obtain

Theorem 4.6 *Let M be an orientable hypersurface in $\mathbf{R}^{n+1}$ and ξ the unit normal vector field. If $\hat{T} = 0$ for $\iota \circ \xi \colon M \to UE$ and $\iota \circ \xi$ is metrically affine minimal, then M is locally congruent to the standard sphere, the hyperplane or the product of a k-dimensional sphere and an $(n-k)$-dimensional affine space.*

Proof. Since $\iota \circ \xi$ is metrically affine minimal, M is a constant mean curvature hypersurface. Using the similar way to the proof of Theorem 4.5, we see that M is an isoparametric hypersurface. □

Finally, we consider the two dimensional case.

Theorem 4.7 *Let M be an orientable surface in $\mathbf{R}^3$ with the Gaussian curvature $\mathcal{K} \neq 0$ on M and ξ the unit normal vector field. If $\hat{T} = 0$ for $\iota \circ \xi \colon M \to UE$ and the Hessian $\mathcal{H}^H$ of the mean curvature function H is semidefinite on M, then M is a part of the standard sphere.*

Proof. We may assume that $\mathcal{H}^H$ is positive semidefinite. By (4.5), we see that

$$3\mathcal{H}^H(X, X) = -2Hg(SX, SX) + \|S\|^2g(SX, X) \geq 0 \tag{4.8}$$

for all $X \in TM$. Let λ and μ be the principal curvatures of M. We have

$\mathcal{K}(\mu-\lambda) \geq 0$ and $\mathcal{K}(\lambda-\mu) \geq 0$, substituting the principal curvature vectors into (4.8). Therefore it holds that $\lambda = \mu$ from $\mathcal{K} \neq 0$. $\square$

In the case where M is a flat surface, we have

Theorem 4.8 *Let M be an orientable surface in $\mathbf{R}^3$ with the Gaussian curvature $\mathcal{K}$ and ξ the unit normal vector field. If $\iota \circ \xi \colon M \to UE$ is cyclic parallel and $\mathcal{K} = 0$, then we have $\nabla^2 S = 0$.*

Proof. By $\mathcal{K} = 0$, one of the principal curvatures of M vanishes. Then, from (4.4), if one of X, Y, Z is the principal curvature vector corresponding to zero principal curvature, then it holds that $(\nabla^2_{Z,X} S)(Y) = 0$. Moreover, if X, Y, Z are the principal curvature vectors for non zero one, we also have $(\nabla^2_{Z,X} S)(Y) = 0$ by (4.4). $\square$

Remark Let M be an orientable hypersurface in $Q^{n+1}(c)$ immersed by f and ξ the unit normal vector field. If $\iota \circ \xi$ is cyclic parallel, then $G(\alpha(\gamma', \gamma'), \alpha(\gamma', \gamma'))$ is constant along a geodesic γ on M.

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