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Rationality of certain cuspidal unipotent representations in crystalline cohomology groups

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Abstract. We complete the determination of the local Schur indices of each unipotent representation of the group $G(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of a simple algebraic group G defined over a finite field $\mathbb{F}_q$.

Key words: unipotent representations, Schur indices, crystalline cohomology groups.

Introduction

Let $\mathbb{F}_q$ be a finite field with q elements of characteristic p. Let G be a connected, reductive linear algebraic group, defined over $\mathbb{F}_q$, with Frobenius map F, and let G^F be the (finite) group of fixed points of G by F. Then the problem of determining of the local Schur indices of the (complex) irreducible unipotent representations ρ of G^F can be reduced to the case where G is a simple algebraic group of adjoint type and ρ is cuspidal ([DL, Propositon 7.10], [Ge II, Remark 2.6], [Lu II, p. 28], [Ge I, Propositions 5.5, 5.6]).

Suppose that G is a simple algebraic group of adjoint type and that ρ is cuspidal unipotent representation of G^F. Then, in almost all cases, the local Schur indices of ρ are determined by Lusztig [Lu V] and Geck [Ge I, II], more or less by a general method. However there are two remaining cases for which the above general method cannot be applied. They are the following:

(i) the characters $E_7[\pm\xi]$ in the group $G^F = E_7(q)$, where q is an even power of p such that $p \equiv 4 \pmod 4$;

(ii) the characters $E_8[\pm\sqrt{-1}\,]$ for $G^F = E_8(q)$ with $p = 5$.

(see [Ge II]; as to te notations of characters of G^F, we follow those in [Ca, p. 483, p. 488].)

The first case was dealt with by Geck [Ge III], by investigating certain generalized Gelfand-Graev representations. For this, he has to assume that p is large enough so that the result of Lusztig [Lu IV] on generalized

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Gelfand-Graev representations can be applied (note that it relies on the general theory of Lie algebra, which requires that p is not too small). Also it involves some explicit computations by using computer.

The idea for treating the second case was explained briefly in [Ge II] and was discussed in Ph. D. thesis of Hezard [He]. It also use the generalized Gelfand-Graev representations of $E_8(q)$. Since $p = 5$, the general theory of Lie algebra cannot be applied, and very delicate and precise computations are required.

In this paper, we with to propose a method of using crystalline cohomology groups in order to treat the above two cases. By realizing the above cuspidal unipotent representations ρ on crystalline cohomology groups, it is possible to determine the p-local Schur index $m_{\mathbb{Q}_p}(\rho)$ of ρ. For any prime $\ell \neq p$, the ℓ-local Schur index $m_{\mathbb{Q}_p}(\rho)$ of ρ can be determined by making use of the realization of ρ on the ℓ-adic cohomology groups due to Lusztig [Lu V]. Thus the Schur index $m_{\mathbb{Q}_p}(\rho)$ of ρ with respect to $\mathbb{Q}$ is determined by the Hasse principle. In particular, the argument works without restriction on p.

The method of making use of crystalline cohomology seems to be comparatively general since, by modifying our method, one can prove the following:

Theorem A *Let G be a simple algebraic group, defined over $\mathbb{F}_q$, with $\mathbb{F}_q$-rank r. Let ρ by any cuspidal unipotent representation of G^F with character χ_ρ and let $A(\rho, \mathbb{Q}_p)$ be the simple direct summand of the group algebra $\mathbb{Q}_p[G^F]$ associated with ρ. Let $\mathbb{Q}_p(\chi_\rho) = \mathbb{Q}_p(\chi_\rho(g_0),\, g_0 \in G^F)$. Then the Hasse invariant of the simple algebra $A(\rho, \mathbb{Q}_p)$ (central over $\mathbb{Q}_p(\chi_\rho)$) can be given by $-(r/2)[\mathbb{Q}_p(\chi_\rho) : \mathbb{Q}_p]$.*

Our result, with combining Lusztig's realization in the ℓ-adic cohomology, can be interpreted in terms of motives over finite fields (see Milne [Mi II]).

Let ρ be a cuspidal unipotent representation of G^F (G simple). Let w be a Weyl group element with minimal length $n = \ell(w)$ such that $(R^1(w), \rho)_{G^F} \neq 0$, where $R^1(w)$ is the Deligne-Lusztig virtual representation of G^F associated with w ([DL]). Let $\lambda q^{n\delta/2}$ be the eigenvalue of Frobenius on $H^n(\overline{X}(w), \mathbb{Q}_\ell)$ associated with ρ ([Lu II]; [DM, Théoremè 2.3, p. 48]), Here $\overline{X}(w)$ is the Hansen-Demazure-Deligne-Lusztig compactification of the Deligne-Lusztig variety $X(w)$ associated with w ([DL, (9.10)]),

$\overline{\mathbb{Q}_\ell}$ is an algebraic closure of the ℓ-adic field $\mathbb{Q}_\ell$ ($\ell \neq p$), δ is the minimal natural number such that F^δ acts trivially on the Weyl group of G, and λ is a certain root of unity (cf. [DM]). Let X be a simple motive with Weil q^δ-number $q^{n\delta/2}$ (uniquely determined up to isomorphisms; see [Mi II, p. 415]). Then

"Theorem B" *Assume that Tate conjecture over finite fields holds* (*see* [*Mi II*]). *Then in the Brauer group of* $\mathbb{Q}(\chi_\rho)$, *the class of the simple direct summand* $A(\rho, \mathbb{Q})$ *of* $\mathbb{Q}[G^F]$ *associated with* ρ *and the class of the endomorphism ring* $\mathrm{End}(X)$ *of* X *are the same.*

Theorem A also holds for cuspidal unipotent representations of the Suzuki and Ree groups ${}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$ except for the unique representation ρ of ${}^2F_4(q)$, such that $(R^1(w), \rho)_{G^F}$ is even for all w ([Lu III, p. 375]); for this representation, the formula in Theorem A does not hold; ρ has the property that $m_{\mathbb{Q}_\ell}(\rho) = 1$ for $\ell \neq p$ and $m_{\mathbb{R}}(\rho) = m_{\mathbb{Q}_p}(\rho) = 2$ (see [Ge I]).

Notation

p is a fixed prime number and k is an algebraic closure of the prime field of characteristic p. $q = p^{a'}$ is a power of p and $\mathbb{F}_q$ is the subfield of k with q elements. By a variety, we mean a separated reduced scheme of finite type over k and we identify it with the set of its k-rational points.

If ρ is an irreducible representation of a finite group H over an algebraically closed field C of characteristic 0, then χ_ρ denotes its character and for a field E of characteristic 0, $E(\rho) = E(\chi_\rho) = E(\chi_\rho(h),\, h \in H)$ and $m_E(\rho)$ or $m_E(\chi_\rho)$ denotes the Schur index of ρ with respect to E.

ℓ is any fixed prime number $\neq p$, and $\overline{\mathbb{Q}_\ell}$ is an algebraic closure of $\mathbb{Q}_\ell$. For a variety X, $H^i(X)$ (resp. $H^i_c(X)$) is the i-th étale cohomology group of X (resp. the i-th étale cohomology of X with compact supports) with coefficients in $\overline{\mathbb{Q}_\ell}$.

1.

Let n be a positive integer, and let $\Lambda_n = \mathbb{Z}/\ell^n\mathbb{Z}$. Let X, Y be varieties and let $f\colon X \to Y$ be a proper morphism. Then there exists a spectral sequence

$$(R^i_c \pi_*)(R^j f_*)F \Longrightarrow R^{i+j}_c(\pi f)_* F$$

where $\pi\colon Y \to \mathrm{Spec}(k)$ is the structural morphism of Y (see [Mi I, Theorem 3.2(c), p. 228]) (note that $R^i_c f_* = R^j f_*$ since f is proper) and F is any torsion (étale) sheaf of Λ_n-modules on X. Thus one of the edge homomorphisms of this spectral sequence gives Λ_n-homomorphisms ([CE, p. 329, Case B])

$$(*) \qquad H^i_c(Y, f_*F) \longrightarrow H^i_c(X, F) \qquad (i \geqq 0).$$

(Note that $(R^i_c\pi_*)(R^0 f_*)F = (R^i_c\pi_*)f_*F = H^i_c(Y, f_*F)$ and $R^i_c(\pi f)_*F = H^i_c(X, F)$.)

Let F' be a torsion sheaf on Y of Λ_n-modules, and let $F = f^*F'$. Then, by composing the homomorphism $H^i_c(Y, F') \to H^i_c(Y, f_*f^*F')$ induced by the natural morphism $F' \to f_*f^*F'$ with the homomorphism $(*)$, we get a Λ_n-homomorphism $H^i_c(Y, F') \to H^i_c(X, f^*F')$. By letting $F' = \Lambda_n$, and by using the canonical isomorphism $f^*\Lambda_n \xrightarrow{\sim} \Lambda_n$, we get a Λ_n-homomorphism

$$(**) \qquad f^*_n\colon\ H^i_c(Y, \Lambda_n) \longrightarrow H^i_c(X, \Lambda_n) \qquad (i \geqq 0).$$

We note that if Z is a variety and $g\colon Y \to Z$ is a proper morphism, then we have

$$(***) \qquad (gf)^*_n = f^*_n g^*_n.$$

Assume that X, Y are proper over $\mathrm{Spec}(k)$. Then, by the functoriality ([Sri. p. 41]), we get a natural Λ_n-homomorphism $H^i(Y, F') \to H^i(X, f^*F')$, which, as we can check, coincides with the above homomorphism $H^i(Y, F') \to H^i(Y, f_*f^*F') \to H^i(X, f^*F')$.

Returning to the general case with $f\colon X \to Y$ proper, let $\psi_n\colon H^i_c(Y, \Lambda_{n+1}) \to H^i_c(Y, \Lambda_n)$, $\phi_n\colon H^i_c(X, \Lambda_{n+1}) \to H^i_c(X, \Lambda_n)$ be homomorphisms which are induced by the natural morphism $\Lambda_{n+1} \to \Lambda_n$. Then we have $\phi_n f^*_{n+1} = f^*_n \psi_n$. Hence, by taking projective limits, we get a $\mathbb{Z}_\ell$-homomorphism

$$\varprojlim_n f^*_n\colon\ H^i_c(Y, \mathbb{Z}_\ell) = \varprojlim_n H^i_c(Y, \Lambda_n) \longrightarrow H^i_c(X, \mathbb{Z}_\ell).$$

By tensoring with $\mathbb{Q}_\ell$, we get $\mathbb{Q}_\ell$-linear maps

$$H^i_c(Y, \mathbb{Q}_\ell) \longrightarrow H^i_c(X, \mathbb{Q}_\ell) \qquad (i \geqq 0),$$

hence we get $\overline{\mathbb{Q}_\ell}$-linear maps

$$f^*\colon\ H^i_c(Y) \longrightarrow H^i_c(X) \qquad (i \geqq 0).$$

We note that, if $g\colon Y \to Z$ is proper, then

$$(gf)^* = f^*g^*.$$

Now let G be a connected, reductive linear algebraic group over k, defined over $\mathbb{F}_q$, with Frobenius map F. Let X_G be the projective variety of all Borel subgroups of G. Let $F\colon X_G \to X_G$ be the map defined by $B \to F(B)$, which is the Frobenius map corresponding to the natural $\mathbb{F}_q$-rational structure of X_G. G acts on X_G by the conjugations: $B \to gBg^{-1}$, $g \in G$, $B \in X_G$.

For the sake of later use, let me allow to explain this action of G on X_G. Let $k[G]$ be the k-algebra of regular functions on G. Then G acts on it by $(h \cdot g)(x) = h(xg^{-1})$, $g \in G$, $h \in k[G]$, $x \in G$. Then there is a finite-dimensional, G-stable subspace V of $k[G]$ and a line L through 0 ($\subset V$) such that $B^* = \{g \in G \,|\, g(L) = L\}$, where B^* is a previously fixed F-stable Borel subgroup of G. Let $\mathbb{P}(V)$ be the projective space associated with V and let $[L]$ be the class of L in $\mathbb{P}(V)$. Note that V and L can be chosen so that they are defined over $\mathbb{F}_q$. The homogeneous space G/B^* is defined to be the orbit $G \cdot [L]$ in $\mathbb{P}(V)$. Since G/B^* is projective, it is complete, hence $G/B^* = G \cdot [L]$ is closed in $\mathbb{P}(V)$. Let $\rho\colon G \to \mathrm{GL}(V)$ be the representation which is determined by the G-module V. Then, for each $g \in G$, $\rho(g)$ is an automorphism of the affine space V, which hence induces a k-algebra automorphism $\theta(g)$ of $k[V]$. With respect to a basis of the $\mathbb{F}_q$-structure V_0 of V, $k[V]$ can be viewed naturally as a polynomial ring $k[T_0, \ldots, T_d]$ over k ($d+1 = \dim_{\mathbb{F}_q}(V_0)$). Then, for $g \in G$, $\theta(g)$ is a homogeneous automorphism of $k[T_0, \ldots, T_d]$ of degree 0, so that it induces a ring automorphism $\overline{\theta}(g)$ of $k[G/B^*] = k[G \cdot [L]]$. Then it is well known that, for $g \in G$, the automorphism $\overline{\theta}(g)$ induces an automorphism of G/B^*, which coincides with the mapping $hB^* \to ghB^*$, $hB^* \in G/B^*$. Since X_G is isomorphic to G/B^* naturally, the adjoint action of $g \in G$ on X_G is induced by a k-algebra automorphism of a k-algebra A such that $X_G = \mathrm{Proj}(A)$.

Now, we let G act on $X_G \times X_G$ diagonally. Then $W_G = G\backslash(X_G \times X_G)$ has a natural group structure, which is called the Weyl group of G ([DL, 1.2], [Lu I, (1.2)]). For $w \in W_G$, let $X(w) = \{B \in X_G \,|\, (B, F(B)) \in w\}$. Then $X(w)$ is a locally closed smooth subvariety of X_G, purely of dimension $\ell(w)$, where $\ell(\ )$ is the length function on W_G ([DL, 1.4]). For $g_0 \in G^F$, $X(w)$ is g_0-stable, so, for each $i \geqq 0$, we have an automorphism g_0^* of $H_c^i(X(w))$. We consider $H_c^i(X(w))$ as G^F-modules by $(g_0^{-1})^*$, $g_0 \in G^F$. For $i \geqq 0$, $H_c^i(X(w))$ is a $\overline{\mathbb{Q}_\ell}[G^F]$-module with $\mathbb{Q}_\ell$-structure $H_c^i(X(w), \mathbb{Q}_\ell)$.

Let δ be the minimal positive integer such that F^δ acts trivially on W_G. Then, for $w \in W_G$, $X(w)$ is F^δ-stable. Let $w \in W_G$. Then the morphism $F^\delta\colon X(w) \to X(w)$ is finite, hence proper, so, for $i \geqq 0$, F^δ induces a $\overline{\mathbb{Q}_\ell}$-linear map $(F^\delta)^*\colon H_c^i(X(w)) \to H_c^i(X(w))$.

Let M be a simple $\overline{\mathbb{Q}_\ell}[G^F]$-module. Then we say that M has depth t if there is an F-stable subset I of the set S of simple reflections in W_G with $|I_F| = r - t$, where I_F is the set of orbits of F on I and r is the semisimple $\mathbb{F}_q$-rank of G, such that, for an F-stable parabolic subgroup P_I of G corresponding to I, with unipotent radical U_I, $M^{U_I^F}$ contains a non-zero cuspidal L^F-module, where $L = P_I/U_I$ and $M^{U_I^F}$ is the subspace of M consisting of elements of M fixed by U_I^F (see [Lu I, §4]).

Let M be any (finitely generated) $\overline{\mathbb{Q}_\ell}[G^F]$-module For an integer $t \geqq 0$, let $M^{(t)}$ be the subspace of M defined as the sum of all simple $\overline{\mathbb{Q}_\ell}[G^F]$-submodules of M of depth t. Then we have $M = \bigoplus_{t \geqq 0} M^{(t)}$. $M^{(0)}$ is the cuspidal part of M.

Now assume that G is a simple algebraic group of type (E_7). Let $s_1, \dots, s_7$ be the simple reflections in W_G. Put $c = s_1 \cdots s_7$ and $f = (s_1, \dots, s_7)$. Let $X_f = X(c)$. Then X_f is a smooth affine irreducible subvariety of dimension 7 ([Lu I, (2.8), (4.8)]). We have $H_c^i(X_f)^{(0)} = 0$ for $i \neq 7$ and $H_c^7(X_f)^{(0)} = H_c^7(X_f)_{\sqrt{-q^7}} \oplus H_c^7(X_f)_{-\sqrt{-q^7}}$, where $H_c^7(X_f)_{\sqrt{-q^7}}$ (resp. $H_c^7(X_f)_{-\sqrt{-q^7}}$) is the subspace of $H_c^7(X_f)$ on which F^* acts by multiplication by $\sqrt{-q^7}$ (resp. $-\sqrt{-q^7}$) ([Lu I, (6.1), (7.1), (7.3), (7.4)(c)]). They afford two non-isomorphic cuspidal unipotent representations of G^F over $\overline{\mathbb{Q}_\ell}$ (see [Lu III, pp. 364–5], Cater [Ca, pp. 482–3]). And they are all the cuspidal unipotent representations of G^F.

Let ρ be a (complex) cuspidal unipotent representation of G^F. Then $\mathbb{Q}(\chi_\rho) = \mathbb{Q}(\sqrt{-q^7})$ (cf. [Ge I, p. 21]). Since χ_ρ is not real, we have $m_{\mathbb{Q}_\infty}(\rho) = 1$. Let $\tau\colon \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}_\ell}$ be an isomorphism. Then, since $(H_c^7(X_f), \rho^\tau)_{G^F} = 1$ and $H_c^7(X_f)$ is defined over $\mathbb{Q}_\ell$, by a property of the Schur index, we have $m_{\mathbb{Q}_\ell}(\rho) = 1$. Since ℓ is any prime number $\neq p$, by Hasse's sum formula, we must have $m_{\mathbb{Q}_p}(\rho) = 1$ (hence $m_{\mathbb{Q}}(\rho) = 1$) if the number of the places of $\mathbb{Q}(\chi_\rho)$ lying above p is equal to one, and this is the case unless q is an even power of p such that $p \equiv 1 \pmod 4$.

To treat the remaining case we use crystalline cohomology. To do so we need some analysis of Lusztig's results in [Lu I].

Let $X_f^{\cdot} = \{(B_0, B_1, \ldots, B_7) \in X_G^8 \mid (B_{i-1}, B_i) \in s_i \cup e$ for $1 \leqq i \leqq 7$, and $F(B_0) = B_7\}$. Then $X_f^{\cdot}$ is a smooth projective variety and X_f can be identified with the open dense subvariety $\{(B_0, B_1, \ldots, B_7) \in X_f^{\cdot} \mid B_0 \neq B_1 \neq \cdots \neq B_7\}$ of $X_f^{\cdot}$ (by Bruhat lemma) ([DL, 9.10]). Let $F\colon X_G^8 \to X_G^8\colon (B_0, B_1, \ldots, B_7) \to (F(B_0), F(B_1), \ldots, F(B_7))$. Then $X_f^{\cdot}$ is F-stable and G^F acts on it diagonally. The inclusion $X_f \hookrightarrow X_f^{\cdot}$ is F-G^F-equivariant. Then this inclusion map induces an isomorphism $H_c^i(X_f)^{(0)} \xrightarrow{\sim} H_c^i(X_f^{\cdot})^{(0)}$ $(i \geqq 0)$ ([Lu I, (4.3.1)]).

Lemma 1 ([Lu I, §4]) *We have* $H^7(X_f^{\cdot}) = H^7(X_f^{\cdot})^{(0)}$.

In fact, let me allow to use the notations of [Lu I, §4] freely.

The exact sequence (4.2.3) of [Lu I, §4] for $a = 7$ and $i = 7$ can be read:

$$\cdots \longrightarrow H_c^7(X_f)^{(t)} \xrightarrow{\alpha^{(t)}} H_c^7(X_f^{\cdot})^{(t)} \xrightarrow{\beta^{(t)}} H_c^8(D_6)^{(t)} \longrightarrow \cdots .$$

We see from the table on page 146 of [Lu I] that the absolute value of each eigenvalue of F^* on $H_c^7(X_f)$ other than $\pm\sqrt{-q^7}$ is an integral power of q. On the other hand, we know from Deligne's theorem on the eigenvalues of Frobenius [De] that the absolute value of any eigenvalue of F^* on $H^7(X_f^{\cdot})$ is $q^{7/2}$. Since the actions of F and G^F commute, we see that $\alpha^{(t)} = 0$ for $t \geqq 1$.

Next we show that $\beta^{(t)} = 0$ for all $t \geqq 0$, which would imply the desired assertion.

Assume that $t \geqq 2$. Then, by the statement on page 122, lines 7–8, of [Lu I], we see that $H^7(D_6)^{(t)}$ is isomorphic as G^F-modules to a quotient of $\bigoplus_{|I| \leqq 6} H_c^7(X_f^{\cdot}(I))$. Moreover, by a standard argument from linear algebra by using the exact sequences in lines 5, 6 on page 122 in [Lu I], that the set of eigenvalues of F^* on $H^7(D_6)^{(t)}$ is contained in the set of eigenvalues of F^* on $\bigoplus_{|I| \leqq 6} H_c^7(X_f^{\cdot}(I))$. By (4.2.1) of [Lu I, p. 119], by the Künneth formula, and by the table on page 146 of [Lu I, p. 119], we see that the absolute value of each eigenvalue of F^* on $\bigoplus_{|I| \leqq 6} H_c^7(X_f^{\cdot}(I))$, hence on $H^7(D_6)^{(t)}$, is an integral power of q. Thus $\beta^{(t)} = 0$ for $t \geqq 2$. by the formula on page 121, line 13, of [Lu I], we see that $H^7(D_6)^{(1)} = 0$. And, by the formula on page 120, line 15, of [Lu I], we have $H^7(D_6)^{(0)} = 0$. Thus $\beta^{(t)} = 0$ for all $t \geqq 0$.

2.

Let $W(\mathbb{F}_q)$ be the ring of Witt vectors of $\mathbb{F}_q$ and let K be its quotient field. Let σ be the Frobenius automorphism of $W(\mathbb{F}_q)$ (induced by the automorphism $x \to x^p$ of $\mathbb{F}_q$); we also denote by σ its extension to K. For a proper smooth scheme X_0 over $\mathbb{F}_q$, let $H^i(X_0/W(\mathbb{F}_q))$ be the i-th crystalline cohomology group of X_0 over $W(\mathbb{F}_q)$ (see Berthelot [Ber, p. 525]; also see Illusite [Ill, 1.2, p. 44]), and let $H^i_{\mathrm{crys}}(X_0) = H^i(X_0/W(\mathbb{F}_q)) \otimes_{W(\mathbb{F}_q)} K$.

Let n be a positive integer, and let $W_n = W(\mathbb{F}_q)/p^n W(\mathbb{F}_q)$ ($W_1 = \mathbb{F}_q$). Let $g\colon X_0 \to Y_0$ be a morphism of proper smooth schemes X_0, Y_0 over $\mathbb{F}_q$, and suppose that the diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{\quad g \quad} & Y_0 \\ \downarrow & & \downarrow \\ S_n = \mathrm{Spec}(W_n) & \xrightarrow{h} & S_n \end{array}$$

commutes. Here, $X_0 \to S_n$ is the composition: $X_0 \to \mathrm{Spec}(\mathbb{F}_q) = \mathrm{Spec}(W_1) \to S_n$ ($Y_n \to S_n$ is defined similarly) and h is a PD morphism (see [Ber, p. 30] or [BO, p. 3.1]). Then we have a morphism of topoi:

$$g_{\mathrm{cris}} = (g^*_{\mathrm{cris}}, g_{\mathrm{cris}*})\colon (X_0/S_n)_{\mathrm{cris}} \longrightarrow (Y_0/S_n)_{\mathrm{cris}}$$

(see [Ber, Théorème 2.2.3, p. 197] or [BO, p. 5.3, p. 5.16]). Here $(X_0/S_n)_{\mathrm{cris}}$ is the topos of sheaves on the site $\mathrm{Cris}(X_0/S_n)$ (see [Ber, p. 180] or [BO, p. 5.3]) ($(Y_0/S_n)_{\mathrm{cris}}$ is defined similarly). Let O_{X_0/S_n} (resp. O_{Y_0/S_n}) be the "structural sheaf" of X_0 over S_n (resp. Y_0 over S_n) ([Ber, p. 183] or [BO, p. 5.4]; also cf. [Ill, p. 44]). Then, by the functoriality (or by the spectral sequence in [BO, p. 5.16]), there is a natural map $H^i(\mathrm{Cris}(Y_0/S_n), O_{Y_0/S_n}) \to H^i(\mathrm{Cris}(X_0/S_n), g^*_{\mathrm{cris}} O_{X_0/S_n})$ for each i, where $H^i(\mathrm{Cris}(Y_0/S_n), O_{Y_0/S_n})$ is the i-th cohomology group of the site $\mathrm{Cris}(Y_0/S_n)$ with coefficients in O_{Y_0/S_n} ([Ber, p. 180, p. 184]) ($H^i(\mathrm{Cris}(X_0/S_n), g^*_{\mathrm{cris}} O_{X_0/S_n})$ is defined similarly). By composing this map with the natural map $H^i(\mathrm{Cris}(X_0/S_n), g^*_{\mathrm{cris}} O_{Y_0/S_n}) \to H^i(\mathrm{Cris}(X_0/S_n), O_{X_0/S_n})$ induced by the natural morphism $g^*_{\mathrm{cris}} O_{Y_0/S_n} \to O_{X_0/S_n}$ (see [Ber, (2.2.4), p. 199]), we get a map

$$g_n^*\colon\ H^i(\mathrm{Cris}(Y_0/S_n), O_{Y_0/S_n}) \longrightarrow H^i(\mathrm{Cris}(X_0/S_n), O_{X_0/S_n}).$$

If

$$\begin{array}{ccc} Y_0 & \xrightarrow{g'} & Z_0 \\ \downarrow & & \downarrow \\ S_n & \xrightarrow{h'} & S_n \end{array}$$

is another commutative diagram, where Z_0 is a proper smooth variety over $\mathbb{F}_q$ and h' is a PD morphism, we have

$$(g'g)_n^* = (g_n^*)(g_n'^*)$$

(cf. [Ber, Proposition 2.2.6, p. 200]).

We also have natural maps $p_n : H^i(\mathrm{Cris}(X_0/S_{n+1}), O_{X_0/S_{n+1}}) \to H^i(\mathrm{Cris}(X_0/S_n),\ O_{X_0/S_n})$, $q_n : H^i(\mathrm{Cris}(Y_0/S_{n+1}),\ O_{Y_0/S_{n+1}}) \to H^i(\mathrm{Cris}(Y_0/S_n), O_{Y_0/S_n})$, and we have $p_n g_{n+1}^* = g_n^* q_n$. Therefore, by taking projective limits, we get a map

$$g^* := \varprojlim_n g_n^*\colon\ H^i(Y_0/W(\mathbb{F}_q)) \longrightarrow H^i(X_0/W(\mathbb{F}_q)) \qquad (i \geqq 0).$$

We have $(g'g)^* = g^* g'^*$.

Let X_0 be a projective smooth scheme over $\mathbb{F}_q$. Let $F_{\mathrm{abs}}\colon X_0 \to X_0$ be the absolute Frobenius endomorphism of X_0: F_{abs} is the identity map on the underlying space of X_0 and, for each section h in the structural sheaf O_{X_0} of X_0, we have $F_{\mathrm{abs}}(h) = h^p$. Then we have a commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{F_{\mathrm{abs}}} & X_0 \\ \downarrow & & \downarrow \\ S_n & \xrightarrow[h_n]{} & S_n, \end{array}$$

where h_n is the PD morphism induced by $\sigma\colon W_n \to W_n$. Then we have a σ-linear endomorphism $(F_{\mathrm{abs}})^*$ of $H^i(X_0/W(\mathbb{F}_q))$ for each i. Hence we get a σ-linear endomorphism $\phi = (F_{\mathrm{abs}})^* \otimes \sigma$ of $H^i_{\mathrm{crys}}(X_0)$ for each i. This makes each $(H^i_{\mathrm{crys}}(X_0), \phi)$ an isocrystal over K, i.e. a finite-dimensional vector space over K with σ-linear bijection.

Let X_0 be as above. Recall that $q = p^{a'}$. Then $F_0 = (F_{\mathrm{abs}})^{a'}$ is the Frobenius endomorphism of X_0; if $X = X_0 \times_{\mathbb{F}_q} k$, then $F = F_0 \times 1$ is the Frobenius endomorphism of X which corresponds to the $\mathbb{F}_q$-rational structure X_0 on X.

Let $X = \mathrm{Proj}(A)$, $Y = \mathrm{Proj}(B)$ be two projective varieties defined over $\mathbb{F}_q$, and let g and h be automorphisms of X and Y respectively, defined over $\mathbb{F}_q$; Assume that g (resp. h) is the restriction to X (resp. Y) of an automorphism of an ambient projective space, with the standard $\mathbb{F}_q$-rational structure, defined over $\mathbb{F}_q$. Then we see that $g \times h$ is the automorphism of $X \times X$ which is induced by a k-algebra automorphism of $A \otimes_k B$.

Now let the assumptions and the notations be as in §1. Recall that $X^{\cdot}_f$ is an F-stable closed subvariety of X^8_G. Suppose that $X^{\cdot}_f = \mathrm{Proj}(A)$. Then, for each $g_0 \in G^F$, the automorphism $g_0 \colon X^{\cdot}_f \to X^{\cdot}_f$ is induced by a k-algebra automorphism $\theta(g_0)$ of A which is homogeneous of degree 0. Let

$$A_0 = \{x \in A \mid F(x) = x^q\}.$$

Then $X^{\cdot}_{f,0} = \mathrm{Proj}(A_0)$ is the $\mathbb{F}_q$-rational structure on $X^{\cdot}_f$ determined by $F \colon X^{\cdot}_f \to X^{\cdot}_f$. Let $g_0 \in G^F$. Then, since $\theta(g_0) \colon A \to A$ is a ring automorphism, for $x \in A_0$, we have

$$F(\theta(g_0)(x)) = \theta(g_0)(F(x)) = \theta(g_0)(x^q) = \theta(g_0)(x)^q,$$

so $\theta(g_0)(x) \in A_0$. So $\theta(g_0)$ induces a ring automorphism of A_0, hence induces an endomorphism g_0 of $X^{\cdot}_{f,0}$. Thus we get an endomorphism $(g_0)^*$ of $H^i(X^{\cdot}_{f,0}/W(\mathbb{F}_q))$ for $i \geqq 0$. It is clear that $(h_0 g_0)^* = (g_0)^*(h_0)^*$ ($h_0 \in G^F$). Thus each $H^i(X^{\cdot}_{f,0}/W(\mathbb{F}_q))$ is a G^F-module by the actions $(g_0^{-1})^*$, $g_0 \in G^F$.

Let $g_0 \in G^F$. Then the graph of $g_0 \colon X^{\cdot}_{f,0} \to X^{\cdot}_{f,0}$ defines a cycle in $X^{\cdot}_{f,0} \times_{S_n} X^{\cdot}_{f,0}$ of codimension 7 ($n \geqq 1$), hence, by the Künneth formula and the Poincarè duality theorem for crystalline cohomology, its class in $H^{14}((X^{\cdot}_{f,0} \times_{S_n} X^{\cdot}_{f,0})/S_n, O_{(X^{\cdot}_{f,0} \times_{S_n} X^{\cdot}_{f,0})/S_n})$, hence in $H^{14}_{\mathrm{crys}}(X^{\cdot}_{f,0} \times_{\mathbb{F}_q} X^{\cdot}_{f,0})$ determines a linear endomorphism of $H^1_{\mathrm{crys}}(X^{\cdot}_{f,0})$ for each i, which is just $(g_0)^* \otimes 1$ (cf. Kleimann [Kl, 3, pp. 11–2] and Berthelot [Ber, Chap. VII, §3, Lemma 3.1.4, p. 575]). Similar statements also hold for étale cohomology (cf. [Mi I, Chap. VI, §12, Lemma 12.1]). Thus by Theorem 2 of Katz and Messing [KM], we see that, for each i, the characteristic polynomial of $(g_0)^* \otimes 1$ on $H^i_{\mathrm{crys}}(X^{\cdot}_{f,0})$ coincides with the characteristic polynomial of $(g_0)^*$ on $H^i(X^{\cdot}_f)$ (they have coefficients in $\mathbb{Z}$). In particular, we have

$$\mathrm{Tr}((g_0)^* \otimes 1, H^i_{\mathrm{crys}}(X^{\cdot}_{f,0})) = \mathrm{Tr}((g_0)^*, H^i(X^{\cdot}_f)) \qquad (i \geqq 0). \tag{1}$$

(This argument was inspired by Lusztig [Lu I, p. 121, line 24].) We also see that, by Theorem 1 of [KM], the eigenvalues of $(F_0)^* \otimes 1$ on $H^7_{\text{crys}}(X^{\cdot}_{f,0})$ coinside the eigenvalue of F^* on $H^7(X^{\cdot}_f)$.

Lemma 2 *Let $g_0 \in G^F$. Then $F_{\text{abs}} g_0 = g_0 F_{\text{abs}}$ on the scheme $X^{\cdot}_{f,0}$. Thus $\phi((g_0)^* \otimes 1) = ((g_0)^* \otimes 1)\phi$ on $H^i_{\text{crys}}(X^{\cdot}_{f,0})$ $(i \geqq 0)$ (recall that $\phi = (F_{\text{abs}})^* \otimes \sigma$).*

In fact, on the underlying space of $X^{\cdot}_{f,0}$, F_{abs} is the identity. For $x \in A_0$, we have $\theta(g_0)(F_{\text{abs}}(x)) = \theta(g_0)(x^p) = \theta(g_0)(x)^p = F_{\text{abs}}(\theta(g_0)(x))$. The last assertion is clear.

Assume that q is an even power of p such that $p \equiv 1 \pmod 4$. Then we have $\sqrt{-q^7} \in \mathbb{Q}_p$. The eigenvalues of $(F_0)^* \otimes 1 = \phi^{a'}$ $(q = p^{a'})$ on $H^7_{\text{crys}}(X^{\cdot}_{f,0})$ are $\pm\sqrt{-q^7}$. Let M_+ (resp. M_-) be the generalized $\sqrt{-q^7}$-eigenspace (resp. $-\sqrt{-q^7}$-eigenspace) of $H^7_{\text{crys}}(X^{\cdot}_{f,0})$. Then, since the action of G^F and $(F_0)^* \otimes 1$ on $H^7_{\text{crys}}(X^{\cdot}_{f,0})$ commute, we see that M_+ and M_- are G^F-submodules of $H^7_{\text{crys}}(X^{\cdot}_{f,0})$. Hence, by (1), we see that they are absolutely irreducible G^F-modules over K and $H^7_{\text{crys}}(X^{\cdot}_{f,0}) = M_+ \oplus M_-$; moreover we see that the actions of $(F_0)^* \otimes 1$ on M_+ and M_- are semisimple.

By Lemma 2, we see that $\phi(M_+)$ is a G^F-module. For $g_0 \in G^F$, we have

$$\mathrm{Tr}((g_0)^* \otimes 1, \phi(M_+)) = \sigma(\mathrm{Tr}((g_0)^* \otimes 1, M_+)) = \mathrm{Tr}((g_0)^* \otimes 1, M_+)$$

since $\mathbb{Q}_p(\chi_\rho) = \mathbb{Q}_p(\sqrt{-q^7}\,) = \mathbb{Q}_p$ (ρ is the representation afforded by M_+) (Geck [Ge I, §5]). So $\phi(M_+)$ is isomorphic to M_+ as G^F-modules. Since M_+ and M_- are not isomorphic, we must have $\phi(M_+) = M_+$. Similarly, we must have $\phi(M_-) = M_-$. Therefore (M_+, ϕ) and (M_-, ϕ) are semisimple isocrystals over K (cf. Milne [Mi II, Proposition 2.10, p. 417]).

Let M be M_+ or M_-, and let

$$U\colon\ K[G^F] \longrightarrow \mathrm{End}_K(M)$$

be the corresponding representation of $K[G^F]$. Since U is absolutely irreducible, we have $U(K[G^F]) = \mathrm{End}_K(M) \cong M_d(K)$, where $M_d(K)$ is the K-algebra of all $d \times d$-matrices over K with $d = \dim_K M$. Let $B = U(\mathbb{Q}_p[G^F])$. Then there is a division algebra D, central over $\mathbb{Q}_p = \mathbb{Q}_p(\sqrt{-q^7}\,)$, such that $B \cong M_n(D)$, where if m denotes the index of D, then $d = nm$ (cf. Curtis and Reiner [CR, p. 468]). We note that $m =$

$m_{\mathbb{Q}_p}(U)$. We have $B \otimes_{\mathbb{Q}_p} K \simeq M_d(K) \simeq \mathrm{End}_K(M)$ ([loc. cit.]).

Let

$$\mathrm{End}(M, \phi) = \{h \in \mathrm{End}_K(M) \mid \phi h = h\phi\}.$$

This is a $\mathbb{Q}_p$-form of the centralizer $Z_{\mathrm{End}_K(M)}(\pi_M)$ of $\pi_M = \phi^{a'}$ in $\mathrm{End}_K(M)$ (see Kottwitz [Ko, p. 410]; also see Milne [Mi II, p. 417]). By Lemma 2, we see that B is contained in $\mathrm{End}(M, \phi)$. But,, as $B \otimes_{\mathbb{Q}_p} K = \mathrm{End}_K(M)$, we must have $B = \mathrm{End}(M, \phi)$. Therefore, as (M, ϕ) is semisimple, there is a simple subisocrystal (X, ϕ) of (M, ϕ) such that $\mathrm{End}(X, \phi) \cong D$. By Lemma 11.3 of [Ko] (also see [Mi II, Proposition 2.14]), we see that the Hasse invariant of D is $1/2$. Therefore $m_{\mathbb{Q}_p}(U) = 2$.

We note that $G^F = E_7(q)$ has just two isomorphism classes of cuspidal unipotent representations.

The following theorem is due to Geck [Ge III] except for (ii) where he had to assume that p is large enough. Our argument can remove this assumption.

Theorem 1 (cf. Geck [Ge III]) *Let G be a simple algebraic group of type (E_7), defined over $\mathbb{F}_q$, with Frobenius map F. Let ρ be a (complex) cuspidal unipotent representation of G^F with character χ_ρ. Then the value field $\mathbb{Q}(\chi_\rho)$ of χ_ρ is $\mathbb{Q}(\sqrt{-q^7}\,)$.* (i) *If $p = 2$, or q is an odd power of p, or q is an even power of p such that $p \equiv 3 \pmod 4$, then $m_{\mathbb{Q}}(\rho) = 1$.* (ii) *Assume that q is an even power of p such that $p \equiv 1 \pmod 4$, Then we have $m_{\mathbb{Q}_\infty}(\rho) = m_{\mathbb{Q}_\ell}(\rho) = 1$ for any prime number $\ell \neq p$ and $m_{\mathbb{Q}_p}(\rho) = 2$. Thus $m_{\mathbb{Q}}(\rho) = 2$.*

By Propositions 5.5, 5.6 of [Ge I], we see that a unipotent representations of $E_8(q)$ with character $E_7[\xi]$, 1, $E_7[-\xi]$, 1, $E_7[\xi], \varepsilon$ or $E_7[-\xi], \varepsilon$ has the same rationality.

Remark Let $h(X^{\cdot}_{f,0})$ be the motive over $\mathbb{F}_q$ corresponding to $X^{\cdot}_{f,0}$ (see Milne [Mi II]), and let Z be the simple submotive of $h(X^{\cdot}_{f,0})$ such that $[\pi_Z] = [\sqrt{-q^7}\,]$ (cf. [Mi II, Proposition 2.6]). Then we see from Theorem 2.16 of [Mi II] that the distribution of the Hasse invariants of the division algebra $\mathrm{End}(Z)$ coincides with the results of Theorem 1.

3.

Let G be a simple algebraic group, defined over $\mathbb{F}_q$, with Frobenius map F. Let δ be the minimal natural number such that F^δ acts trivially on W_G.

Let $\underline{s} = (s_1, \dots, s_n)$ be a sequence of simple reflections in W_G, and let

$$X_{\underline{s}} = X(s_1, \dots, s_n) = \{(B_0, B_1, \dots, B_n) \in X_G^{n+1} \mid (B_{i-1}, B_i) \in s_i \\ \text{for } 1 \leq i \leq n \text{ and } F(B_0) = B_n\}.$$

Then $X_{\underline{s}}$ is a locally closed subvariety of X_G^{n+1} on which G^F acts diagonally. We can prove that, for each i, each irreducible component of the G^F-module $H_c^i(X_{\underline{s}})$ is unipotent (We use [Lu II, p. 25–6] and [DL, Theorem 6.2]).

Let ρ be a unipotent representation of G^F. Then we have $(R^1(w), \rho)_{G^F} \neq 0$ for some $w \in W_G$. We note that $R^1(w) = \sum_i (-1)^i H_c^i(X(w))$. Let $w = s_1 \cdots s_n$ be a reduced expression for w ($n = \ell(w)$). Then $X(w)$ is isomorphic to $X_{\underline{s}}$ with $\underline{s} = (s_1, \dots, s_n)$. Therefore there is an integer i such that $(H_s^i(X_{\underline{s}}), \rho)_{G^F} \neq 0$.

Let $\underline{s} = (s_1, \dots, s_n)$ be a minimal sequence such that $(H_c^i(X_{\underline{s}}), \rho)_{G^F} \neq 0$ for some i. Then we see that $\ell(s_1 \cdots s_n) = n$ and $X_{\underline{s}} \xrightarrow{\sim} X(w)$ with $w = s_1 \cdots s_n$ (cf. [Lu II, pp. 25–6]). In the following, we fix one of such $\underline{s}$.

We have $(H_c^i(X_{\underline{s}}), \rho)_{G^F} = 0$ for $i \neq n$ (Haastert [Ha, Korollar 4.4 (1)]). Therefore w is an element of W_G with minimal length such that $(R^1(w), \rho)_{G^F} \neq 0$. If ρ is cuspidal, then $(R^1(w), \rho)_{G^F} = (-1)^r$, where r is the $\mathbb{F}_q$-rank of G (Lusztig [Lu V]).

Let

$$\overline{X}_{\underline{s}} = \overline{X}(s_1, \dots, s_n) = \{(B_0, B_1, \dots, B_n) \in X_G^{n+1} \mid (B_{i-1}, B_i) \in s_i \cup e \\ \text{for } 1 \leq i \leq n \text{ and } F(B_0) = B_n\}.$$

Then $\overline{X}_{\underline{s}}$ is a smooth closed subvariety of X_G^{n+1} ([DL, 9.10]) and $X_{\underline{s}}$ is an open dense subvariety of $\overline{X}_{\underline{s}}$. By the minimality of $\underline{s}$, we see that the inclusion $X_{\underline{s}} \hookrightarrow \overline{X}_{\underline{s}}$ induces an isomorphism as G^F-modules from ρ-isotropic part $H_c^n(X_{\underline{s}})_\rho$ of $H_c^n(X_{\underline{s}})$ onto the ρ-isotropic part $H^n(\overline{X}_{\underline{s}})_\rho$ of $H^n(\overline{X}_{\underline{s}})$ ([Lu II, p. 26]).

Let $X^{\cdot} = \overline{X}_{\underline{s}}$. Let m be any multiple of δ, and let $N_{\underline{s}}^m(g_0) = \left| {X^{\cdot}}^{\, g_0 F^m} \right|$ ($g_0 \in G^F$). Then we have (Digne and Michel [DM, pp. 60–61]):

$$N_{\underline{s}}^m(g_0) = \sum_{i=0}^{2n} (-1)^i \operatorname{Tr}((g_0 F^m)^*, H^i(X^{\cdot}))$$

$$= \sum_{i=0}^{2n} (-1)^i \operatorname{Tr}((F^\delta)^{*m/\delta}(g_0)^*, H^i(X^\cdot))$$
$$= \sum_{i=0}^{2n} (-1)^i q^{mi/2} \sum_{\rho' \in U} (H^i(X^\cdot), \rho')_{G^F} \omega_{\rho'}^{m/\delta} \chi_{\rho'}(g_0). \qquad (2)$$

(Note that one can prove that any irreducible component of $H^i(X^\cdot)$ is unipotent (cf. [Lu II, p. 26].).) Here U is the set of isomorphism classes of the unipotent representations of G^F and, for $\rho' \in U$, $\omega_{\rho'}$ is a root if unity such that $\omega_{\rho'} q^{i\delta/2}$ is the eigenvalue of $(F^\delta)^*$ on $H^i(X^\cdot)$ associated with ρ' ([Lu II]).

Suppose that ρ is cuspidal. Then $X^\cdot$ is irreducible (Lusztig [Lu II, pp. 26–27]). Let $W(\mathbb{F}_{q^\delta})$ be the ring of Witt vectors over $\mathbb{F}_{q^\delta}$, let K be its quotient field and let $\overline{K}$ be an algebraic closure of K. Let $X_0^\cdot$ be the $\mathbb{F}_{q^\delta}$-rational structure on $X^\cdot$ determined by the Frobenius $F^\delta \colon X^\cdot \to X^\cdot$. Let $F_0 \colon X_0^\cdot \to X_0^\cdot$ be the Frobenius endomorphism of $X_0^\cdot$ ($F_0 = (F_{\mathrm{abs}})^{a'\delta}$, $q = p^{a'}$). Then, by Theorem 2 of [KM], we have

$$\operatorname{Tr}((g_0 F_0^m)^*, H^i_{\mathrm{crys}}(X_0^\cdot)) = \operatorname{Tr}((g_0 F^m)^*, H^i(X^\cdot)) \qquad (i \geqq 0). \qquad (3)$$

Let α be an eigenvalue of $(F_0)^* \otimes 1$ on $H^i_{\mathrm{crys}}(X_0^\cdot) \otimes_K \overline{K}$ and let $H^i_{\mathrm{crys}}(X_0^\cdot)_\alpha$ be the generalized α-eigensubspace of $H^i_{\mathrm{crys}}(X_0^\cdot) \otimes_K \overline{K}$. $H^i_{\mathrm{crys}}(X_0^\cdot)_\alpha$ is a $\overline{K}[G^F]$-submodule of $H^i_{\mathrm{crys}}(X_0^\cdot) \otimes_K \overline{K}$. In views of (2), (3), together with Grothendieck's trace formula for the étale cohomology, we see, by using the linearly independence of the irreducible characters of G^F and the linearly independence of the functions $m/\delta \to \omega_{\rho'}^{m/\delta}$, that if ρ' is contained in $H^i_{\mathrm{crys}}(X_0^\cdot)_\alpha$, then α is of the form $\omega_{\rho'} q^{i\delta/2}$.

Assume that G is of type (E_8) and that ρ is a cuspidal unipotent representation of G^F such that $\chi_\rho = E_8[i]$ or $E_8[-i]$. Then $\mathbb{Q}(\chi_\rho) = \mathbb{Q}(i)$ ([Ge I, §5]) and $n = \ell(w) = 10$ ([Lu V]). Therefore, by Hasse's sum formula, we get $m_{\mathbb{Q}_p}(\rho) = 1$ if $p = 2$ or $p \equiv 3 \pmod 4$.

Assume that $p \equiv 1 \pmod 4$. Then we have $\mathbb{Q}_p(\chi_\rho) = \mathbb{Q}_p(i) = \mathbb{Q}_p$, and we see that, by taking $M = H^{10}_{\mathrm{crys}}(X_{f,0}^\cdot)_\rho$, (M, ϕ) is an isocrystal over K. Thus, by considering the representation

$$R\colon\ K[G^F] \longrightarrow \operatorname{End}_K(M),$$

the argument goes as §2 (note that we see that (M, ϕ) is a semisimple isocrystal). Thus we have $m_{\mathbb{Q}_p}(\rho) = 1$, hence $m_{\mathbb{Q}}(\rho) = 1$.

In the following theorem, the case where $p = 5$ was discussed in [Ge II] and [He] in an individual way, as explained in Introduction. Out method gives a uniform and conceptual proof in the case $p \equiv 1 \pmod 4$.

Theorem 2 (cf. Geck [Ge I, II] and Hezard [He]) *The cuspidal unipotent characters $E_8[\pm i]$ of $E_8(q)$ have the Schur index* 1 *over* $\mathbb{Q}$.

The same argument can be applied to any unipotent cuspidal representation ρ with $\mathbb{Q}_p(\chi_\rho) = \mathbb{Q}_p$ for any G. Therefore it remains the case where G is of type (E_8) and ρ is such that $\chi_\rho = E_8[\zeta^j]$ $(1 \le j \le 4)$, $p \equiv 4 \pmod 5$. But, in this case, we can argue as follows.

Let $\chi = \chi_\rho = E_8[\zeta^j]$, and let χ' be the algebraically conjugate character of χ over $\mathbb{Q}_p$, i.e. $\chi' = E_8[\zeta^{4j}]$. Since the character of the $K[G^F]$-module $H^n_{\mathrm{crys}}(X_0^\cdot)$ takes values in $\mathbb{Z}$. we must have $(H^n_{\mathrm{crys}}(X_0^\cdot), \rho')_{G^F} = (H^n_{\mathrm{crys}}(X_0^\cdot), \rho)_{G^F} = 1$, where ρ' is a representation of G^F with character χ'. Therefore, by the property of the Schur index, we have $m_K(\rho) = m_K(\rho') = 1$, so that, by a threorem of Schur, we see that $\rho \oplus \rho'$ is a representation of G^F which is realizable in K. Hence there is a unique submodule M of $H^n_{\mathrm{crys}}(X_0^\cdot)$ with character $\chi + \chi'$. We must have $\phi(M) = M$, since $\phi(M)$ is a G^F-submodule of $H^n_{\mathrm{crys}}(X_0^\cdot)$ with character $\sigma(\chi + \chi') = \chi + \chi'$. Thus (M, ϕ) is an isocrystal over K.

Let us consider the representation

$$R\colon\ K[G^F] \longrightarrow \mathrm{End}_K(M).$$

Let $A(\chi, \mathbb{Q}_p)$ be the simple component of $\mathbb{Q}_p[G^F]$ $(\subset K[G^F])$ associated with χ. Then we see that $R(\mathbb{Q}_p[G^F]) = R(A(\chi, \mathbb{Q}_p))$ (cf. T. Yamada [Ya, Proposition 1.1, pp. 4–5]). Then since $A(\chi, \mathbb{Q}_p)$ is a central simple algebra over $\mathbb{Q}_p(\chi) = \mathbb{Q}_p(\zeta)$ and R is a ring homomorphism, we see that $B = R(\mathbb{Q}_p[G^F])$ is a simple algebra, isomorphic to $A(\chi, \mathbb{Q}_p)$. By Lemma 2 for $X_0^\cdot$, we must have $B \subset \mathrm{End}(M, \phi)$.

We have $M \otimes_K \overline{K} = M_\rho \oplus M_{\rho'}$, where M_ρ (resp. $M_{\rho'}$) is the ρ-isotropic part (resp. ρ'-isotropic part) of $M \otimes_K \overline{K}$. Let $\pi_M = \phi^{a'} = (F_0)^* \otimes 1$ $(q = p^{a'})$ on M. The eigenvalues of $(F_0)^* \otimes 1$ on $M_\rho \subset (M \otimes_K \overline{K})_{\zeta^j q^{n/2}}$ (resp. $M_{\rho'} \subset (M \otimes_K \overline{K})_{\zeta^{4j} q^{n/2}}$) are of the form $\zeta^j q^{n/2}$ (resp. $\zeta^{4j} q^{n/2}$). Since the actions of $(F_0)^* \otimes 1$ and G^F commute, by Schur's lemma, we must have $(F_0)^* \otimes 1 = \zeta^j q^{n/2}$ (resp. $= \zeta^{4j} q^{n/2}$) on M_ρ (resp. $M_{\rho'}$). Therefore the endomorphism π_M of M is semisimple, hence (M, ϕ) is a semisimple isocrystal over K (see Milne [Mi II, Proposition 2.10, p. 417]). Therefore $\mathrm{End}(M, \phi)$ is a $\mathbb{Q}_p$-form on the centralizer $C = Z_E(\pi_M)$ of π_M in $E = \mathrm{End}_K(M)$ ([Ko, p. 410]). We have $C \otimes_K \overline{K} \subset Z_{E\otimes_K\overline{K}}(\pi_M) \cong M_d(\overline{K}) \oplus M_d(\overline{K})$, where $d = \chi(1) = \chi'(1)$, and it is well known that $B \otimes_K \overline{K} \simeq A(\chi, \mathbb{Q}_p) \otimes_K \overline{K} = M_d(\overline{K}) \oplus M_d(\overline{K})$. Therefore we must have $B = \mathrm{End}(M, \phi)$. Therefore, as B is simple and (M, ϕ) is semisimple, there is a simple subisocrystal (X, ϕ) of (M, ϕ) such that $B = \mathrm{End}(M, \phi) \simeq M_t(D)$ with $D = \mathrm{End}(X, \phi)$ for some positive integer t. By Lemma 11.3 of [Ko], we see that the Hasse invariant of D can be given by $-(\mathrm{ord}_p(\pi_X)/\mathrm{ord}_p(q))[\mathbb{Q}_p(\pi_X) : \mathbb{Q}_p]$, where ord_p is the valuation of $\mathbb{Q}_p$ and its extension to the field $\mathbb{Q}_p[\pi_X]$ and $\pi_X = \phi^{a'}$ on X. But $\mathbb{Q}_p[\pi_X] \cong \mathbb{Q}_p(\chi) \cong \mathbb{Q}_p(\zeta)$ and $\mathrm{ord}_p(\pi_X) = a'n/2$, $\mathrm{ord}_p(q) = a'$, hence

$$\mathrm{inv}(A(\chi, \mathbb{Q}_p)) \equiv -\frac{n}{2}[\mathbb{Q}_p(\chi) : \mathbb{Q}_p] \equiv -\frac{r}{2}[\mathbb{Q}_p(\chi) : \mathbb{Q}_p] \equiv 0 \pmod 1$$

(note that $(-1)^n = (-1)^r$). Thus $m_{\mathbb{Q}_p}(\rho) = 1$ and $m_{\mathbb{Q}}(\rho) = 1$.

Remark The last argument works in general case (G is simple, ρ is cuspidal, and q, p arbitrary). Therefore we can prove Theorem A in the introduction.

"Theorem B" follows from this proof of Theorem A and Theorem 2.16 of Milne [Mi II].

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