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On solutions of $x'' = t^{\alpha\lambda-2}x^{1+\alpha}$ where $\alpha > 0$ and $\lambda = 0, -1$

Ichiro TSUKAMOTO

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Abstract. Here we investigate asymptotic behavior of positive solutions of an initial value problem of the second order nonlinear differential equation written in the title. This is done from obtaining analytical expressions of the solution valid in the neighborhoods of both end points of its domain.

Key words: asymptotic behavior, an initial value problem, a two dimensional autonomous system, the Briot-Bouquet differential equation.

1. Introduction

In this paper, we shall discuss asymptotic behavior of a solution of an initial value problem

$$x'' = t^{\alpha\lambda-2}x^{1+\alpha} \quad ('= d/dt) \tag{E}$$

$$x(T) = A, \quad x'(T) = B \tag{I}$$

where t and x are positive variables, α, T, and A are positive constants, and B is a real constant. We shall consider the cases $\lambda = 0$ and $\lambda = -1$ here.

The differential equation (E) has a form similar to that of the Thomas-Fermi differential equation in atomic physics and is expected to enter into mathematical physics (cf. [1]). Moreover (E) is an equation of motion in the potential field, so Euler's equation of a variational problem and an equation which positive radial solutions of an elliptic partial differential equation satisfy. That is, (E) is related to various another fields and worth solving.

Actually there are many papers in which (E) is considered (cf. [5], [6], [13], [16], and references of [4]). Taking [7] and [17] from references of [4] for example, these treat the solutions continuable to ∞. On the other hand in [5], [6], [13], and [16], the initial value problem (E) and (I) is treated. In [5] and [6], the case $T = 0$ and $\alpha\lambda > 1$ is considered and in [13] and [16],

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the cases $\lambda > 0$ and $\lambda < -1$ are respectively.

The method which we shall use in this paper is almost the same as in [5], [6], [13], and [16]. In these papers we first transform (E) into a first order rational differential equation, using a transformation

$$y = \psi(t)^{-\alpha} x^{\alpha}, \quad z = ty'$$

where

$$\psi(t) = \{\lambda(\lambda+1)\}^{1/\alpha} t^{-\lambda}$$

is a particular solution of (E). However this is no use in our cases $\lambda = 0$ and $\lambda = -1$ and another transformation will be adopted.

We shall discuss the case $\lambda = 0$ in Sections 2 through 5, and the case $\lambda = -1$ in Section 6.

2. The main conclusions of the case $\boldsymbol{\lambda = 0}$

In this section we suppose $\lambda = 0$. Then (E) becomes

$$x'' = t^{-2} x^{1+\alpha}. \tag{E$_0$}$$

If $x = x(t)$ denotes a solution of (E$_0$) satisfying an initial condition (I) and (ω_-, ω_+) the domain of $x(t)$, then we get the following:

Theorem 2.1 *There exists a number B_1 such that if $B = B_1$, then we have*

$$0 < \omega_- < \infty, \quad \omega_+ = \infty,$$

$$x(t) \sim \frac{1}{(\alpha \log t)^{1/\alpha}} \left\{ 1 + \sum_{m+n>0} x_{mn} \left(\frac{\log \log t}{\log t} \right)^m \left(\frac{1}{\log t} \right)^n \right\} \tag{2.1}$$

as $t \to \infty$, and

$$\begin{aligned} x(t) = &\left\{ \frac{2(\alpha+2)\omega_-^2}{\alpha^2} \right\}^{1/\alpha} (t - \omega_-)^{-2/\alpha} \\ &\times \left\{ 1 + \sum_{m>0} (t-\omega_-)^m q_m(\log(t-\omega_-)) \right\} \\ &\textit{in the case } 4/\alpha \in \boldsymbol{N} \end{aligned} \tag{2.2}$$

$$
\begin{aligned}
x(t) = & \left\{ \frac{2(\alpha+2)\omega_-^2}{\alpha^2} \right\}^{1/\alpha} (t-\omega_-)^{-2/\alpha} \\
& \times \left\{ 1 + \sum_{m+n>0} x_{mn}(t-\omega_-)^m (t-\omega_-)^{(2+4/\alpha)n} \right\} \\
& \qquad \textit{in the case } 4/\alpha \notin \boldsymbol{N}
\end{aligned} \tag{2.3}
$$

in the neighborhood of $t = \omega_-$. Here x_{mn} are constants and q_m are polynomials with degrees not greater than $[m\alpha/2(\alpha+2)]$, $[\ \ ]$ denoting Gaussian symbol.

Now, notice that $f(t) \sim g(t)$ as $t \to \infty$ means

$$
\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.
$$

Moreover we conclude the following:

Theorem 2.2 *There exists a number B_2 $(> B_1)$ such that if $B = B_2$, then we get*

$$
\omega_- = 0, \quad 0 < \omega_+ < \infty,
$$

$$
x(t) = \Gamma t \left(1 + \sum_{n=1}^{\infty} x_n t^{\alpha n} \right) \tag{2.4}
$$

in the neighborhood of $t = 0$, and

$$
\begin{aligned}
x(t) = & \left\{ \frac{2(\alpha+2)\omega_+^2}{\alpha^2} \right\}^{1/\alpha} (\omega_+ - t)^{-2/\alpha} \\
& \times \left\{ 1 + \sum_{m>0} (\omega_+ - t)^m q_m(\log(\omega_+ - t)) \right\} \\
& \qquad \textit{in the case } 4/\alpha \in \boldsymbol{N}
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
x(t) = & \left\{ \frac{2(\alpha+2)\omega_+^2}{\alpha^2} \right\}^{1/\alpha} (\omega_+ - t)^{-2/\alpha} \\
& \times \left\{ 1 + \sum_{m+n>0} x_{mn}(\omega_+ - t)^m (\omega_+ - t)^{(2+4/\alpha)n} \right\} \\
& \qquad \textit{in the case } 4/\alpha \notin \boldsymbol{N}
\end{aligned} \tag{2.6}
$$

in the neighborhood of $t = \omega_+$. Here Γ and x_n are constants.

If $B \neq B_1$, B_2, then we obtain the following:

Theorem 2.3

(i) *If* $B < B_1$*, then we get*

$$x(t) = \Gamma(\omega_+ - t)\left\{1 + \sum_{m+n>0} x_{mn}(\omega_+ - t)^m(\omega_+ - t)^{\alpha n}\right\} \tag{2.7}$$

in the neighborhood of $t = \omega_+$*, and* (2.2) *or* (2.3) *in the neighborhood of* $t = \omega_-$*.*

(ii) *If* $B_1 < B < B_2$*, then we get* (2.2) *or* (2.3) *in the neighborhood of* $t = \omega_-$*, and* (2.5) *or* (2.6) *in the neighborhood of* $t = \omega_+$*.*

(iii) *If* $B > B_2$*, then we get*

$$x(t) = \Gamma(t - \omega_-)\left\{1 + \sum_{m+n>0} x_{mn}(t - \omega_-)^m(t - \omega_-)^{\alpha n}\right\} \tag{2.8}$$

in the neighborhood of $t = \omega_-$*, and* (2.5) *or* (2.6) *in the neighborhood of* $t = \omega_+$*.*

(iv) *In* (i), (ii), *and* (iii), *we have*

$$0 < \omega_- < \omega_+ < \infty.$$

For proving these theorems, we transform (E_0). For this we put

$$y = x^\alpha, \quad z = ty' \tag{2.9}$$

in (E_0) and get a first order rational differential equation

$$\frac{dz}{dy} = \frac{(\alpha - 1)z^2 + \alpha yz + \alpha^2 y^3}{\alpha yz}. \tag{2.10}$$

Using a parameter s, we rewrite this as a two dimensional autonomous system

$$\begin{aligned} \frac{dy}{ds} &= \alpha yz \\ \frac{dz}{ds} &= (\alpha - 1)z^2 + \alpha yz + \alpha^2 y^3. \end{aligned} \tag{2.11}$$

If $\alpha \neq 1$, then $(0, 0)$ is the only singular point of (2.11) and if $\alpha = 1$, then every point of the z axis is the singular point. Notice that we always get $y > 0$ from $x > 0$ and an orbit of (2.11) is a solution of (2.10).

The proof will be carried out in Section 5, since it is necessary to obtain lemmas of Sections 3 and 4.

3. The investigation of (2.10) in the neighborhood of $y = 0$

First, let us consider (2.10) in the neighborhood of $y = 0$. For this, we put

$$w = y^{-2}z$$

and get

$$\frac{dw}{dy} = \frac{\alpha^2 + \alpha w - (\alpha+1)yw^2}{\alpha y^2 w}. \tag{3.1}$$

If $y = 0$, then the numerator of the righthand side of (3.1) vanishes if and only if

$$w = -\alpha.$$

Let γ be an accumulation point of a solution $w = w(y)$ of (3.1) as $y \to 0$. Then we conclude the following:

Lemma 3.1 *γ is the limit and $\gamma = -\alpha, \pm\infty$. If $\gamma = -\alpha$, then we get the solution of* (3.1) *represented as*

$$w = -\alpha + \sum_{n=1}^{N-1} a_n y^n + O(y^N) \qquad \text{as } y \to 0 \tag{3.2}$$

where N is a positive integer and a_n are constants.

Proof. There exists a sequence $\{y_n\}$ ($y_n > 0$) such that $y_n \to 0$ and $w(y_n) \to \gamma$ as $n \to \infty$. If $\gamma \neq -\alpha, \pm\infty$, then dw/dy does not vanish at $y = y_n$ from (3.1) and hence we get the inverse function $y = y(w)$ such that if $w_n = w(y_n)$, then $w_n \to \gamma$ and $y(w_n) \to 0$ as $n \to \infty$. Therefore from Painlevé's theorem (cf. Theorem 3.2.1 of [2]), $y = y(w)$ is a solution of

$$\frac{dy}{dw} = \frac{\alpha y^2 w}{\alpha^2 + \alpha w - (\alpha+1)yw^2}$$

satisfying an initial condition $y(\gamma) = 0$. However from the uniqueness of the solution, such a solution is only $y \equiv 0$. Namely we get a contradiction $y_n = 0$. Hence $\gamma = -\alpha, \pm\infty$ and γ is the limit.

If $\gamma = -\alpha$, then we put $\theta = w + \alpha$ and get from (3.1)

$$y^2 \frac{d\theta}{dy} = -\frac{1}{\alpha}\theta + (\alpha+1)y + \cdots \tag{3.3}$$

where $\cdots$ denotes terms whose degrees are greater than the previous terms. Therefore from Hukuhara's theorem stated in p.66 of [3], there exists a solution of (3.3) uniquely such that

$$\theta \to 0 \qquad \text{as } y \to 0$$

and this is represented as

$$\theta = \sum_{n=1}^{N-1} a_n y^n + O(y^N) \qquad \text{as } y \to 0 \tag{3.4}$$

where $\sum_{n=1}^{N-1} a_n y^n$ is a partial sum of the formal solution of (3.3). Moreover (3.4) implies (3.2). $\square$

From (3.2) we get a solution of (2.10) represented as

$$z = -\alpha y^2 \left\{ 1 + \sum_{n=1}^{N-1} b_n y^n + O(y^N) \right\} \qquad \text{as } y \to 0 \tag{3.5}$$

where $b_n = -a_n/\alpha$. Since existence of (3.4) is unique, so is that of (3.2) and (3.5). Thus we denote (3.5) as $z = z_1(y)$.

Lemma 3.2 *From $z = z_1(y)$, we get a solution of* (E_0) *represented as* (2.1) *as $t \to \infty$.*

Proof. Applying (2.9) to $z = z_1(y)$, we have

$$ty' = -\alpha y^2 (1 + o(1))$$

and

$$y = \frac{1 + o(1)}{\alpha \log t} \tag{3.6}$$

as $y \to 0$. Hence $y \to 0$ is equivalent to $t \to \infty$. Moreover from (2.9) and $z = z_1(y)$ we get

$$\frac{y'}{y^2 \left\{ 1 + \sum_{n=1}^{N-1} b_n y^n + O(y^N) \right\}} = -\frac{\alpha}{t}$$

and integrating both sides,

$$\frac{1}{y}\left\{1 + c_1 y \log y + \sum_{n=2}^{N-1} c_n y^n + O(y^N)\right\} = \alpha \log t + D$$

where c_n $(n = 1, 2, \ldots, N-1)$ and D are constants. Therefore we have

$$y = \frac{1}{\alpha \log t}\left(1 + c_1 y \log y + \sum_{n=2}^{N-1} c_n y^n\right)(1+o(1)) \qquad \text{as } t \to \infty.$$

Substituting (3.6) into this, we obtain

$$y = \frac{1}{\alpha \log t}\left\{1 + c_1 \frac{1}{\alpha \log t} \log \frac{1}{\alpha \log t} + \sum_{n=2}^{N-1} c_n \left(\frac{1}{\alpha \log t}\right)^n\right\}(1+o(1))$$

and (2.1), since $x = y^{1/\alpha}$ from (2.9). □

If $\gamma = \pm\infty$, then we put $w = 1/\theta$ and get

$$\theta \to 0 \qquad \text{as } y \to 0,$$
$$y^2 \frac{d\theta}{dy} = -\theta^2 + \frac{\alpha+1}{\alpha} y\theta - \alpha\theta^3.$$

Moreover putting $u = y^{-1}\theta$, we have

$$y\frac{du}{dy} = \frac{1}{\alpha}u - u^2 - \alpha y u^3. \tag{3.7}$$

Let δ be an accumulation point of a solution $u = u(y)$ of (3.7) as $y \to 0$. Then we obtain the following:

Lemma 3.3 *Suppose $\gamma = \pm\infty$. Then δ is the limit and $\delta = 0, 1/\alpha$. If $\delta = 0$, then from $u(y)$ we get a solution of (2.10) represented in the neighborhood of $y = 0$ as*

$$z = C^{-1} y^{1-1/\alpha}\left\{1 + \sum_{m+n>0} \tilde{a}_{mn} y^{m+n/\alpha}\right\} \tag{3.8}$$

and if $\delta = 1/\alpha$, then

$$z = \alpha y\left(1 + \sum_{n=1}^{\infty} \tilde{a}_n y^n\right). \tag{3.9}$$

Here C, $\tilde{a}_{mn}$, and $\tilde{a}_n$ are constants.

Proof. If $\delta \neq 0,\ 1/\alpha,\ \pm\infty$, then from (3.7) we get a contradiction $y \equiv 0$, since the righthand side of (3.7) does not vanish. Thus $\delta = 0,\ 1/\alpha,\ \pm\infty$ and δ is the limit.

If $\delta = 0$, then since

$$u = y^{-1}\theta = \frac{1}{yw} = \frac{y}{z},$$

the same transformation as in Section 5 of [6] is used. Therefore applying the discussion done there to (3.7) we get

$$u = \sum_{m+n>0} a_{mn} y^m (Cy^{1/\alpha})^n$$

and (3.8) from this. Here $a_{01} = 1$, $a_{m0} = 0$, and a_{mn} are constants.

Moreover if $\delta = 1/\alpha$, then we put $v = u - 1/\alpha$ and get

$$v \to 0 \qquad \text{as } y \to 0 \tag{3.10}$$

$$y\frac{dv}{dy} = -\frac{1}{\alpha}v - v^2 - \alpha y\left(v + \frac{1}{\alpha}\right)^3. \tag{3.11}$$

This is a Briot-Bouquet differential equation and since $-1/\alpha < 0$ it follows from Lemma 2.5 of [12] that there exists a solution $v = v(y)$ of (3.11) uniquely such that (3.10) holds. Moreover $v(y)$ is holomorphic in the neighborhood of $y = 0$ and hence we denote this as

$$v = \sum_{n=1}^{\infty} a_n y^n.$$

Returning the variables, we get (3.9).

Finally if $\delta = \pm\infty$, then we put $u = 1/v$ and have

$$v \to 0 \qquad \text{as } y \to 0 \tag{3.12}$$

$$\frac{dv}{dy} = \frac{\alpha^2 y + \alpha v - v^2}{\alpha y v}. \tag{3.13}$$

On the other hand, we get

$$y^{-1}v = w \to \pm\infty \qquad \text{as } y \to 0$$

since $\gamma = \pm\infty$. Hence from (3.13) we have

$$\frac{dv}{dy} = \frac{\alpha^2 y v^{-1} + \alpha - v}{\alpha y}, \ \text{ namely } \frac{dv}{dy} = \frac{1 + o(1)}{y} \qquad \text{as } y \to 0.$$

Integrating both sides from y to y_* ($y < y_*$), we get

$$v_* - v = \left(\log \frac{y_*}{y}\right)(1+o(1)) \qquad \text{as } y \to 0 \text{ and } y_* \to 0$$

where $v = v_*$ if $y = y_*$. If y_* is fixed and $y \to 0$, then we obtain a contradiction

$$v_* = \infty.$$

Therefore $\delta \neq \pm\infty$, which completes the proof. □

Here, notice that from (3.8) we get

$$z \to 0 \text{ if } \alpha > 1, \quad z \to C^{-1} \neq 0 \text{ if } \alpha = 1, \quad z \to \pm\infty \text{ if } 0 < \alpha < 1,$$
as $y \to 0$.

Moreover let $z = z_2(y)$ denote a solution of (2.10) represented as (3.9) in the neighborhood of $y = 0$, since the existence of (3.9) is unique from the uniqueness of the solution $v = v(y)$ of (3.11). Next, use a transformation (2.9) as in the proof of Lemma 3.3 to (3.8) and (3.9). Then we conclude the following:

Lemma 3.4 *From* (3.8) *we have a solution of* (E_0) *represented as* (2.7) *if* $z < 0$, *and* (2.8) *if* $z > 0$.

Proof. If $z < 0$, then from (2.9) we get $y' < 0$ and t tends to the right end point of the domain of the solution as $y \to +0$. Noticing this, it suffices to follow the discussion of Section 5 of [6] again. If $z > 0$, then the same discussion follows. □

Lemma 3.5 *We have a solution of* (E_0) *represented as* (2.4) *from* (3.9).

Proof. From (2.9) and (3.9) we get

$$ty' = \alpha y \left(1 + \sum_{n=1}^{\infty} \tilde{a}_n y^n\right)$$

and solving this differential equation

$$y = (\Gamma t)^{\alpha} \left\{1 + \sum_{n=1}^{\infty} \tilde{b}_n (\Gamma t)^{\alpha n}\right\}$$

where $\tilde{b}_n$ are constants. Hence from (2.9) we have (2.4). □

4. The investigation of (2.10) in the neighborhood of $y = \infty$

If we define (y, z) from a solution $x = x(t)$ of (E_0) through (2.9), then z is a solution of (2.10) and (y, z), (2.11). Let (ω_-, ω_+) be a domain of $x(t)$. Then we conclude the following:

Lemma 4.1 *As $t \to \omega_\pm$, (y, z) does not converge to a finite nonsingular point.*

This is Lemma 2 of [13] or Lemma 3.3 of [16] and so we omit the proof.

Lemma 4.2 *An arbitrary solution $z = z(y)$ of (2.10) exists for $y_* < y < \infty$ where y_* is a nonnegative constant. If $y_* \neq 0$, then $z(y)$ is bounded as $y \to y_*$. Moreover $z(y) \to \pm\infty$ as $y \to \infty$.*

Proof. Suppose that there exists a sequence $\{y_n\}$ such that $y_n \to c$ where $y_* \leq c < \infty$ and $z(y_n) \to \pm\infty$ as $n \to \infty$. Then if we put $z = 1/\zeta$ in (2.10), we get

$$\frac{d\zeta}{dy} = -\frac{(\alpha - 1 + \alpha y\zeta + \alpha^2 y^3 \zeta^2)\zeta}{\alpha y} \tag{4.1}$$

and a contradiction $\zeta = 1/z(y) \equiv 0$ from Painleve's theorem and the uniqueness of the solution (cf. Proof of Lemma 3.3). Hence $z(y)$ is defined for $y_* < y < \infty$. Moreover if $y_* \neq 0$, then $z(y)$ is bounded as $y \to y_*$.

Next, suppose that there exists a sequence $\{y_n\}$ such that $y_n \to \infty$ and $\{z(y_n)\}$ is bounded as $n \to \infty$. Then if we put $y = 1/\eta$ in (2.10), we have

$$\frac{dz}{d\eta} = -\frac{(\alpha - 1)\eta^3 z^2 + \alpha\eta^2 z + \alpha^2}{\alpha\eta^4 z} \tag{4.2}$$

and a contradiction $\eta \equiv 0$, using Painleve's theorem and the uniqueness of the solution again. Hence $z(y) \to \pm\infty$ as $y \to \infty$. $\square$

Now for considering (2.10) in the neighborhood of $y = \infty$, we put

$$z = 1/\zeta \tag{4.3}$$

in (4.2) and obtain

$$\eta^4 \frac{d\zeta}{d\eta} = \frac{\alpha - 1}{\alpha}\eta^3\zeta + \eta^2\zeta^2 + \alpha\zeta^3.$$

Moreover we put

$$\theta = \eta^{-3/2}\zeta, \quad \xi = \eta^{1/2} \tag{4.4}$$

and get

$$\xi\frac{d\theta}{d\xi} = -\frac{\alpha+2}{\alpha}\theta + 2\xi\theta^2 + 2\alpha\theta^3. \tag{4.5}$$

If $\xi = 0$, then the righthand side of (4.5) vanishes if and only if

$$\theta = 0, \ \pm\rho$$

where

$$\rho = \frac{1}{\alpha}\sqrt{\frac{\alpha+2}{2}}.$$

Let γ be an accumulation point of a solution $\theta = \theta(\xi)$ of (4.5) as $\xi \to 0$. Then we get the following:

Lemma 4.3 *γ is the limit and $\gamma = \pm\rho$.*

Proof. If $\gamma \neq 0, \pm\rho, \pm\infty$, then using Painleve's theorem and the uniqueness of the solution we have $\xi \equiv 0$ from (4.5). This implies a contradiction $y \equiv 0$. Hence we obtain $\gamma = 0, \pm\rho, \pm\infty$ and so γ is the limit.

If $\gamma = 0$, then from Lemma 2.5 of [12] we obtain $\theta \equiv 0$, since $-(\alpha+2)/\alpha < 0$ and θ divides the righthand side of (4.5). Returning the variables, this implies a contradiction $z \equiv \infty$. If $\gamma = \pm\infty$, then putting $\theta = 1/u$ in (4.5) we get

$$u \to 0 \qquad \text{as } \xi \to 0$$

$$\frac{d\xi}{du} = \frac{\alpha\xi u}{(\alpha+2)u^2 - 2\alpha\xi u - 2\alpha^2}$$

which implies a contradiction $\xi \equiv 0$ again. Therefore we conclude $\gamma = \pm\rho$. $\square$

Lemma 4.4 *If $\gamma = \rho$, then from $\theta = \theta(\xi)$ we get a solution $z = z(y)$ of* (2.10) *represented as*

$$z^{-1} = \xi^3\left[\gamma + \sum_{m+n>0} u_{mn}\xi^m\left\{\xi^{2+4/\alpha}(b\log\xi + C)\right\}^n\right] \tag{4.6}$$

in the neighborhood of $\xi = 0$. *Here* u_{mn}, b, *and* C *are constants and* $b = 0$ *unless* $4/\alpha$ *is a positive integer. Moreover from* $z = z(y)$ *we get a solution* $x = x(t)$ *of* (E_0) *represented as* (2.5) *or* (2.6) *in the neighborhood of* $t = \omega_+$. *Here* ω_+ *is a constant with* $t < \omega_+ < \infty$.

Proof. Put $u = \theta - \rho$. Then we have

$$\xi \frac{du}{d\xi} = \frac{\alpha + 2}{\alpha^2} \xi + \left(2 + \frac{4}{\alpha}\right) u + \cdots . \tag{4.7}$$

This has the form similar to (15) of [6] and hence it suffices to follow the discussion of Section 3 of [6]. □

Similarly we conclude the following:

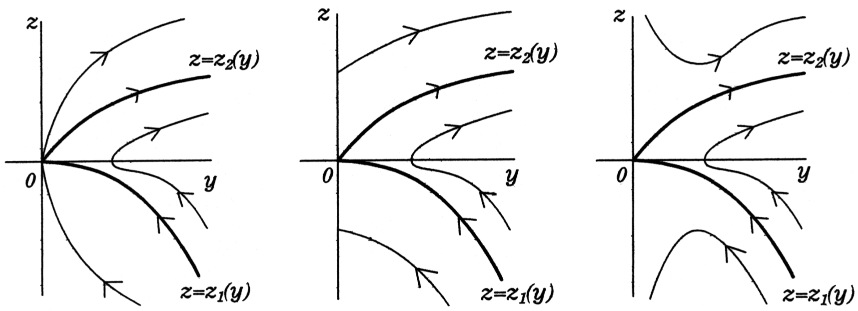
Lemma 4.5 *If* $\gamma = -\rho$, *then from* $\theta = \theta(\xi)$ *we obtain a solution* $z = z(y)$ *of* (2.10) *represented as* (4.6) *in the neighborhhood of* $\xi = 0$. *Furthermore from* $z = z(y)$ *we obtain a solution* $x = x(t)$ *of* (E_0) *represented as* (2.2) *or* (2.3) *in the neighborhood of* $t = \omega_-$. *Here* ω_- *is a constant with* $0 < \omega_- < \infty$.

Proof. If we put $u = \theta + \rho$, then we get (4.7) and it suffices to follow the same discussion. □

Finally, notice that a solution (y, z) of (2.11) satisfies

$$\frac{dy}{ds} = 0, \quad \frac{dz}{ds} = \alpha^2 y^3 > 0$$

on the y axis. Moreover, notice that from Lemmas 3.1 and 3.3 solutions



The case $\alpha > 1$ The case $\alpha = 1$ The case $0 < \alpha < 1$

Fig. 1.

of (2.10) continuable to $y = 0$ are $z = z_1(y)$, $z = z_2(y)$, and in addition (3.8) in the case $\alpha > 1$ which are continuable to $y = \infty$ from Lemma 4.2. Then the phase portrait of (2.11) is denoted as Figure 1.

5. Proof of theorems

Let $x = x(t)$ be a solution of (E_0) satisfying an initial condition (I) given in Section 1. Then from $x = x(t)$ and (2.9) we get a function y of t and a solution $z = z(y)$ of (2.10) satisfying an initial condition

$$z(y_0) = z_0 \tag{5.1}$$

where

$$y_0 = A^\alpha, \quad z_0 = \alpha T A^{\alpha-1} B. \tag{5.2}$$

Indeed we have

$$z = ty' = \alpha t x(t)^{\alpha-1} x'(t)$$

from (2.9). Moreover $z = z(y)$ is an orbit of a solution $(y, z) = (y(s), z(s))$ of (2.11) passing a point (y_0, z_0). Conversely from $z = z(y)$ or $(y, z) = (y(s), z(s))$ we get a solution $x = x(t)$ of an initial value problem (E_0) and (I).

Now, fix T and A in (I) arbitrarily. Then y_0 is fixed and z_0 is an increasing function of B. Therefore as B moves the locus of (y_0, z_0) is a line parallel to the z axis. Let ℓ denote such a line.

Proof of Theorem 2.1. Take (y_0, z_0) to be an intersection of ℓ and $z = z_1(y)$, and suppose that if $B = B_1$, then (5.2) is satisfied. Then if $B = B_1$, $z = z_1(y)$ is the solution of (E_0) satisfying (5.1). Moreover it follows from Lemma 3.2 that we get a solution $x = x(t)$ of (E_0) and (I) represented as (2.1) as $t \to \infty$. Here, recall that (ω_-, ω_+) denotes the domain of $x(t)$. Then as $t \to \omega_-$ we have y $(= x(t)^\alpha) \to \infty$ from Lemma 4.1 and Figure 1, and if γ is the accumulation point of θ obtained from $z = z_1(y)$ and (4.4), then Lemma 4.3 implies $\gamma = -\rho$ since $z_1(y) < 0$. Hence from Lemma 4.5, $x(t)$ just obtained is defined for (ω_-, ∞) where $0 < \omega_- < \infty$ and represented as (2.2) or (2.3) in the neighborhood of $t = \omega_-$. This completes the proof. □

Proof of Theorem 2.2. Now, take (y_0, z_0) to be an intersection of ℓ and

$z = z_2(y)$, and let B_2 denote B satisfying (5.2). Then using Lemmas 3.5 and 4.4 instead of Lemmas 3.2 and 4.5 respectively, it suffices to follow the same discussion as of the proof of Theorem 2.1. $\square$

Proof of Theorem 2.3. If $B < B_1$, then (y_0, z_0) lies below the orbit $z = z_1(y)$ and the orbit $z = z(y)$ passing (y_0, z_0) does. If follows from (2.9), Lemma 4.1, and Figure 1 that y is the function of t satisfying

$$y \to 0 \quad \text{as } t \to \omega_+, \qquad y \to \infty \quad \text{as } t \to \omega_-.$$

Therefore Lemmas 3.4 and 4.5 imply (i).

If $B_1 < B < B_2$, then (y_0, z_0) and $z = z(y)$ passing (y_0, z_0) lie between $z = z_1(y)$ and $z = z_2(y)$. Therefore we get $y \to \pm\infty$ as $t \to \omega_\pm$ respectively from Lemma 4.1 and Figure 1. Hence we conclude from Lemmas 4.4 and 4.5 that $x(t)$ is represented as (2.5) or (2.6) in the neighborhood of $t = \omega_+$ and (2.2) or (2.3), $t = \omega_-$. Here $0 < \omega_- < \omega_+ < \infty$. This proves(ii).

Finally if $B > B_2$, then the same discussion as of (i) implies (iii). Now the proof is complete. $\square$

6. On the case $\lambda = -1$

Now let us consider (E) in the case $\lambda = -1$. That is, we treat

$$x'' = t^{-\alpha-2} x^{1+\alpha} \tag{E$_{-1}$}$$

under the initial condition (I). Let $x = x(t)$ be a solution of an initial value problem (E$_{-1}$) and (I), and (ω_-, ω_+) a domain of $x(t)$ also here. Asymptotic behavior of $x(t)$ is as follows:

Theorem 6.1 *There exists a number B_1 such that if $B = B_1$, then we get*

$$\omega_- = 0, \quad 0 < \omega_+ < \infty,$$
$$x(t) \sim \frac{t}{(\alpha \log(1/t))^{1/\alpha}} \left\{ 1 + \sum_{m+n>0} x_{mn} \left(\frac{\log\log(1/t)}{\log(1/t)} \right)^m \left(\frac{1}{\log(1/t)} \right)^n \right\} \tag{6.1}$$

as $t \to +0$, and (2.5) *or* (2.6) *in the neighborhood of $t = \omega_+$.*

Theorem 6.2 *There exists a number B_2 $(< B_1)$ such that if $B = B_2$, then we get*

$$0 < \omega_- < \infty, \quad \omega_+ = \infty,$$

$$x(t) = \Gamma\left(1 + \sum_{n=1}^{\infty} x_n t^{-\alpha n}\right) \tag{6.2}$$

in the neighborhood of $t = \infty$, and (2.2) *or* (2.3) *in the neighborhood of $t = \omega_-$. Here Γ is a constant.*

In the case $B \neq B_1$, B_2, we conclude the following:

Theorem 6.3
(i) *If $B > B_1$, then the conclusion of* (iii) *of Theorem* 2.3 *follows.*
(ii) *If $B_2 < B < B_1$, then the conclusion of* (ii) *of Theorem* 2.3 *follows.*
(iii) *If $B < B_2$, then the conclusion of* (i) *of Theorem* 2.3 *follows.*

Proof of these theorems and the proof of Theorems 2.1, 2.2, and 2.3 are almost the same. So we state only the outline of the proof.

Outline of the proof. First we use a transformation

$$x = ty^{1/\alpha} \quad (\text{namely } y = t^{-\alpha}x^{\alpha}), \quad z = ty' \tag{6.3}$$

and transform (E_{-1}) into a first order rational differential equation

$$\frac{dz}{dy} = \frac{(\alpha-1)z^2 - \alpha yz + \alpha^2 y^3}{\alpha yz}. \tag{6.4}$$

Moreover using a parameter s we rewrite this as a two dimensional autonomous system

$$\begin{aligned} \frac{dy}{ds} &= \alpha yz \\ \frac{dz}{ds} &= (\alpha-1)z^2 - \alpha yz + \alpha^2 y^3. \end{aligned} \tag{6.5}$$

The singular points of (6.5) are the same as of (2.11), namely $(0,0)$ in the case $\alpha \neq 1$ and all points of the z axis in the case $\alpha = 1$.

Next we consider (6.4) in the neighborhood of $y = 0$. For this we put

$$w = y^{-2}z$$

in (6.4) and get

$$y^2\frac{dw}{dy} = \frac{\alpha^2 - \alpha w - (\alpha+1)yw^2}{\alpha w}. \tag{6.6}$$

The numerator of the righthand side vanishes in the case $y = 0$, if and only if $w = \alpha$. Let γ be an accumulation point of a solution $w = w(y)$ of (6.6) as $y \to 0$. Then applying the proof of Lemma 3.1 to our case, we conclude that γ is the limit and equal to α, $\pm\infty$.

If $\gamma = \alpha$, then we put

$$\theta = w - \alpha$$

and get

$$\theta \to 0 \qquad \text{as } y \to 0 \tag{6.7}$$

$$y^2 \frac{d\theta}{dy} = -(\alpha + 1)y - \frac{1}{\alpha}\theta + \cdots . \tag{6.8}$$

If follows from Hukuhara's theorem used for (3.3) that there exists uniquely a solution of (6.8) satisfying (6.7) and this is represented as

$$\theta = \sum_{n=1}^{N-1} a_n y^n + O(y^N) \qquad \text{as } y \to 0 \tag{6.9}$$

where $\sum_{n=1}^{N-1} a_n y^n$ is a partial sum of the formal solution of (6.8). Returning the variables, we get a solution of (6.4) represented as

$$z = \alpha y^2 \left\{ 1 + \sum_{n=1}^{N-1} b_n y^n + O(y^N) \right\} \qquad \text{as } y \to 0 \tag{6.10}$$

where $b_n = a_n/\alpha$. Since the existence of (6.9) is unique, so is that of (6.10). Thus we denote (6.10) as $z = z_1(y)$. Applying the discussion done in the proof of Lemma 3.2 to $z_1(y)$, we have (6.1).

Moreover if $\gamma = \pm\infty$, then we put

$$w = \frac{1}{\theta}$$

and obtain

$$\theta \to 0 \qquad \text{as } y \to 0 \tag{6.11}$$

$$y^2 \frac{d\theta}{dy} = \frac{\alpha + 1}{\alpha} y\theta + \theta^2 - \alpha\theta^3. \tag{6.12}$$

Furthermore if $\theta = yu$, then we get

$$y \frac{du}{dy} = \frac{1}{\alpha} u + u^2 - \alpha y u^3. \tag{6.13}$$

The righthand side of (6.13) vanishes in the case $y = 0$ if and only if $u = 0, -1/\alpha$.

So if δ denotes an accumulation point of a solution of (6.13) as $y \to 0$, then applying the proof of Lemma 3.3 we first conclude that δ is the limit and equal to $0, -1/\alpha, \pm\infty$. Moreover if $\delta = 0$, then from a solution of (6.13) tending to 0 as $y \to 0$ we get (3.8). If $\delta = -1/\alpha$, then we put $v = u + 1/\alpha$ and have

$$v \to 0 \qquad \text{as } y \to 0 \tag{6.14}$$

$$y\frac{dv}{dy} = -\frac{1}{\alpha}v + v^2 - \alpha y\left(v - \frac{1}{\alpha}\right)^3. \tag{6.15}$$

Applying the discussion done for (3.11) here, we get the unique solution of (6.4) represented as

$$z = -\alpha y\left(1 + \sum_{n=1}^{\infty}\tilde{a}_n y^n\right) \tag{6.16}$$

in the neighborhood of $y = 0$ where $\tilde{a}_n$ are constants. Let $z = z_2(y)$ denote this. If $\delta = \pm\infty$, then we put $u = 1/v$ and get

$$v \to 0 \qquad \text{as } y \to 0$$

$$\frac{dv}{dy} = \frac{\alpha^2 y - \alpha v - v^2}{\alpha y v}.$$

Here we apply the discussion done for (3.12) and (3.13) and conclude a contradiction. Hence we have $\delta \neq \pm\infty$ and $\delta = 0, -1/\alpha$.

Notice here that Lemma 3.4 holds for (E_{-1}), namely from (3.8) we get a solution of (E_{-1}) represented as (2.7) if $z < 0$, and (2.8) if $z > 0$. Moreover from (6.16) (namely $z_2(y)$) we have (6.2). This can be shown as in the proof of Lemma 3.5.

Now, let us consider (6.4) in the neighborhood of $y = \infty$. Then we put $z = 1/\zeta$ and obtain

$$\eta^4\frac{d\zeta}{d\eta} = \frac{(\alpha - 1)\eta^3\zeta - \alpha\eta^2\zeta^2 + \alpha^2\zeta^3}{\alpha}.$$

Moreover we put

$$\theta = \eta^{-3/2}\zeta, \quad \xi = \eta^{1/2}$$

and get

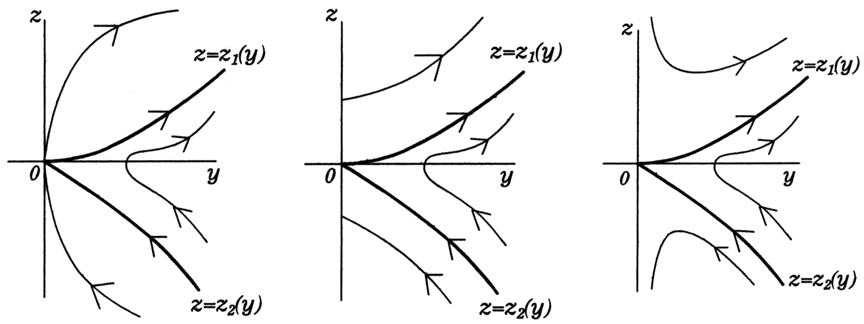
$$\xi \frac{d\theta}{d\xi} = -\frac{\alpha+2}{\alpha}\theta - 2\xi\theta^2 + 2\alpha\theta^3. \tag{6.17}$$

In the case $\xi = 0$, the righthand side of (6.17) vanishes if and only if

$$\theta = 0,\ \pm\rho \quad \left(\rho = \frac{1}{\alpha}\sqrt{\frac{\alpha+2}{2}}\right).$$

Here, let γ be an accumulation point of a solution $\theta = \theta(\xi)$ of (6.17). Then discussing as in the proof of Lemma 4.3, we conclude that γ is the limit and equal to $\pm\rho$. Next, follow the discussion of the proof of Lemmas 4.4 and 4.5. Then from $\theta(\xi)$ we get a solution of (6.4) with the representation (4.6). Moreover from (4.6) we have a solution of (E_{-1}) represented as (2.5) or (2.6) if $\gamma = \rho$, and (2.2) or (2.3) if $\gamma = -\rho$.

From the above discussion, solutions of (6.4) continuable to $y = 0$ are $z = z_1(y)$, $z = z_2(y)$, and in addition (3.8) if $\alpha > 1$. Moreover those continuable to $y = \infty$ are (2.2) or (2.3) and (2.5) or (2.6). Therefore noticing that Lemma 4.2 is valid for (E_{-1}) we obtain the phase portrait of (6.5) in Figure 2.



The case $\alpha > 1$ The case $\alpha = 1$ The case $0 < \alpha < 1$

Fig. 2.

Finally, let us take the initial condition (I) into account. Then from a solution $x = x(t)$ of the initial value problem (E_{-1}) and (I) and from (6.3) we get a solution $z = z(y)$ of (6.4) satisfying an initial condition

$$z(y_0) = z_0$$

where

$$y_0 = T^{-\alpha}A^{\alpha}, \quad z_0 = \alpha y_0\left(\frac{TB}{A} - 1\right).$$

Fix T and A arbitrarily also here. Then z_0 is an increasing function of B and the locus of (y_0, z_0) is a line parallel to the z axis. Moreover, notice that Lemma 4.1 is valid for (E_{-1}). Then the discussion of Section 5 can be applied here. This finishes the proof of Theorems 6.1, 6.2, and 6.3. □

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3-10-38 Higashi-kamagaya
Kamagaya-shi, Chiba 273-0104, Japan
E-mail: itkmt@k9.dion.ne.jp