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A new constructive version of Baire's theorem

Douglas BRIDGES, Hajime ISHIHARA and Luminiţa VÎŢĂ

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Abstract. A new constructive version of Baire's theorem is given and then applied to two problems in functional analysis. The second of these applications provides a new proof that compactly generated Banach spaces are finite dimensional.

Key words: Constructive, Baire, open mapping, compactly generated.

1. Introduction

Baire's (category) theorem states that the intersection of a sequence of dense open subsets of a complete metric space is dense in that space. The standard classical proof of this theorem passes over unchanged to the constructive setting (that is, the one in which classical logic is replaced by intuitionistic logic). However, various classically equivalent versions of Baire's theorem do not pass over unscathed; for a discussion of this, see Chapter 2 of [5]. In this note we present one alternative constructive version of Baire's theorem in the context of a Banach space, and then use this result, first to produce a new constructive version of the open mapping theorem (see [3, 4, 7, 13]); secondly to give conditions that ensure the existence of a certain type of projection in a Hilbert space; and finally to give a new proof that compactly generated Banach spaces are finite-dimensional (cf. [8]).

Our proofs, like all proofs that use only intuitionistic logic [18] (and an appropriate set-theoretic foundation such as that presented in [1]), embody algorithms for the computation/construction of the objects whose existence is asserted in the proposition being proved. These algorithms can be extracted and implemented (see [9, 10, 15]). Moreover, the constructive proof of the proposition is actually a proof that the embodied algorithm is correct—that is, meets its specifications.

What little background material in constructive analysis is needed for this paper can be gleaned from Chapters 2, 3, and 7 of [2]. However, one or two important differences between the constructive and the classical

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approaches are worth highlighting here. First, we observe that the classical least-upper-bound principle cannot be proved constructively. If S is a nonempty[1] subset of $\mathbf{R}$ that is bounded above, then in order to construct the least upper bound of S it is necessary and sufficient that for all real numbers a, b with $a < b$, either b is an upper bound of S or else there exists $x \in S$ with $x > a$. (Note that here, as always in the constructive setting, "either ... or" means that we can decide which of the alternatives holds.) In turn, we cannot guarantee that the distance

$$\rho(x, S) = \inf\{\rho(x, s) : s \in S\}$$

from a point of $\mathbf{R}$ to an arbitrary nonempty set S can be computed. If it can for all $x \in \mathbf{R}$, then S is said to be **located**. In accordance with our comments on the least-upper-bound principle, in order to prove that S is located, for each $x \in X$ we must compute an associated number d, show that $\rho(x, s) \geq d$ for each $s \in S$, and, given $\varepsilon > 0$, construct $s \in S$ such that $\rho(x, s) < d + \varepsilon$.

For any subset S of X we define the **metric complement** to be

$$X - S = \{x \in X : \exists \delta > 0 \, \forall s \in S \, (\rho(x, s) \geq \delta)\}.$$

We write $-S$, rather than $X - S$, for this metric complement when no confusion is likely.

Normally we denote the open and closed balls with center x and radius r in a metric space X by $B(x, r)$ and $\overline{B}(x, r)$, respectively; but sometimes the need for clarity dictates that we use the notations $B_X(x, r)$ and $\overline{B}_X(x, r)$ instead.

We begin with our new version of Baire's theorem.

Theorem 1 *Let X be a Banach space, and C a closed, convex, balanced and located subset of X such that $X = \bigcup_{n=1}^{\infty} nC$ and $\rho(0, -C)$ exists. Then C° is nonempty.*

Proof. For each positive integer n let

$$U_n = (X - nC) \cup \{x \in X : C^\circ \text{ is nonempty}\}.$$

[1]When we refer to a set S as "nonempty" we mean that it is **inhabited**, in the sense that there exists—we can construct—an element of S. This is constructively stronger than the impossibility of S being empty.

Then U_n is open in X. To prove that U_n is also dense in X, consider $y \in X$ and $\varepsilon > 0$. Note that $\rho(y, nC)$ exists and equals $n\rho\big((1/n)y, C\big)$. Either $\rho(y, nC) > 0$ or $\rho(y, nC) < \varepsilon$. In the first case, $y \in X - nC$. In the second, choose $z \in nC$ such that $\|y - z\| < \varepsilon$. Noting that

$$\rho(0, -2nC) = 2n\rho(0, -C)$$

exists, we have either $\rho(0, -2nC) < 2\varepsilon$ or $\rho(0, -2nC) > \varepsilon$. In the former case, choose $z' \in -2nC$ such that $\|z'\| < 2\varepsilon$. For each $w \in nC$ we have $-w \in nC$ (since C is balanced), so

$$z - w \in nC + nC = 2nC,$$

by the convexity of C. Hence

$$\|(z - z') - w\| = \|z' - (z - w)\| \geq \rho(z', 2nC) > 0.$$

Thus $z - z' \in -nC$. Since also

$$\|y - (z - z')\| \leq \|y - z\| + \|z'\| < 3\varepsilon,$$

we see that $\rho(y, U_n) < 3\varepsilon$. Finally, in the case $\rho(0, -2nC) > \varepsilon$, we have $\overline{B}(0, \varepsilon) \subset \overline{2nC} = 2nC$; whence $(2nC)^\circ$, and therefore C°, is nonempty. In this case, $U_n = X$. This completes the proof that U_n is dense in X.

Since X is complete, it follows from the standard version of Baire's theorem that $\bigcap_{n=1}^{\infty} U_n$ is dense in X and therefore, in particular, contains a point ξ. Choose n such that $\xi \in nC$. Since also $\xi \in U_n$, we must have $\xi \in \{x \in X : C^\circ \text{ is nonempty}\}$. Hence C° is indeed nonempty. $\square$

2. An open mapping theorem

Certain classical Banach spaces may not carry a well-defined norm in the constructive setting. To get round this, Johns [14] introduced a notion that we call a **quasinorm** on a linear space X: a family $(\|\cdot\|_i)_{i \in I}$ of seminorms on X such that for each $x \in X$ the set $\{\|x\|_i : i \in I\}$ is bounded in $\mathbf{R}$. An element x of X is then said to be **normable** if

$$\|x\| = \sup\{\|x\|_i : i \in I\}$$

exists. Given a quasinormed space $(X, (\|\cdot\|_i)_{i \in I})$, we define topological and uniform notions in natural ways. For details, we refer to [2] (Chapter 7, Section 5).

For example, on the space $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H the natural quasinorm is defined by taking I to be the (closed) unit ball $\overline{B}(0,1)$ of H and

$$\|T\|_x = \|Tx\| \quad (\|x\| \leq 1,\ T \in \mathcal{B}(H)).$$

An element of $\mathcal{B}(H)$ is normable relative to this quasinorm if and only if its operator norm, in the usual sense, exists. Moreover, $\mathcal{B}(H)$ is complete relative to its quasinorm, and so any quasinorm-closed subset of $\mathcal{B}(H)$ is quasinorm-complete.

In order to obtain a version of the open mapping theorem from Theorem 1, we extend Lemma 4.4 of [7], and improve its proof, to cover mappings defined on a quasinormed space; see also Lemma 3 of [11].

Lemma 2 *Let $(X, (\|\cdot\|_i)_{i\in I})$ be a complete quasinormed space, Y a normed space, and $u\colon X \to Y$ a bounded linear mapping of X onto a dense subspace of Y such that $u(B_X(0,r))$ is located in Y. Let r be a positive number, and y an element of $B_Y(0,r)$. Then there exists $x \in \overline{B}_X(0,2)$ such that if $y \neq u(x)$, then $\rho\big(y', u(B_X(0,1))\big) > 0$ for some $y' \in B_{u(X)}(0,r)$.*

Proof. Let $y \in Y$ and $\|y\| < r$. Define an increasing binary sequence $(\lambda_n)_{n=0}^{\infty}$ and a sequence $(x_n)_{n=0}^{\infty}$ of elements of $B_X(0,2)$ such that for each $n \geq 1$,

- ▷ if $\lambda_n = 0$, then

$$\rho\left(2^{n-1}y - \sum_{i=1}^{n-1} 2^{n-1-i}u(x_i), u(B_X(0,1))\right) < r/2$$

and

$$\left\|2^n y - \sum_{i=1}^{n} 2^{n-i}u(x_i)\right\| < r;$$

- ▷ if $\lambda_n = 1 - \lambda_{n-1}$, then

$$\rho\left(2^{n-1}y - \sum_{i=1}^{n-1} 2^{n-1-i}u(x_i), u(B_X(0,1))\right) > 0$$

and $x_i = 0$ for all $i \geq n$.

If $\rho\big(y, u(B_X(0,1))\big) > 0$, then, since $B_{u(X)}(0,r)$ is dense in $B_Y(0,r)$, we can find $y' \in B_{u(X)}(0,r)$ such that $\rho\big(y', u(B_X(0,1))\big) > 0$; to complete the proof, we can then take x to be any element of $B_X(0,2)$ such that $y \neq Tx$. So we may assume that $\rho\big(y, u(B_X(0,1))\big) < r/2$. We then choose $x_1 \in B_X(0,2)$ such that $\big\|y - (1/2)x_1\big\| < r/2$, so that $\|2y - x_1\| < r$, and we set $\lambda_1 = 0$. Now suppose that we have found λ_{n-1} and x_{n-1} with the applicable properties. If $\lambda_{n-1} = 1$, we set $\lambda_n = 1$ and $x_n = 0$. If $\lambda_{n-1} = 0$, we consider the two cases,

$$\rho\left(2^{n-1}y - \sum_{i=1}^{n-1} 2^{n-1-i}u(x_i), u(B_X(0,1))\right) > 0$$

and

$$\rho\left(2^{n-1}y - \sum_{i=1}^{n-1} 2^{n-1-i}u(x_i), u(B_X(0,1))\right) < \frac{r}{2}.$$

In the first case we set $\lambda_i = 1$ and $x_i = 0$ for all $i \geq n$. In the second case we choose $x_n \in B_X(0,2)$ such that

$$\left\|2^{n-1}y - \sum_{i=1}^{n-1} 2^{n-1-i}u(x_i) - u\Big(\frac{1}{2}x_n\Big)\right\| < \frac{r}{2}$$

and set $\lambda_n = 0$. Then

$$\left\|2^n y - \sum_{i=1}^{n} 2^{n-i}u(x_i)\right\| < r.$$

This completes the induction.

Since X is complete, the series $\sum_{i=1}^{\infty} 2^{-i}x_i$ converges to a sum $x \in \overline{B}_X(0,2)$. Suppose that $y \neq u(x)$. Then there exists N such that

$$\left\|y - \sum_{i=1}^{N} 2^{-i}u(x_i)\right\| > 2^{-N}r$$

and therefore

$$\left\|2^N y - \sum_{i=1}^{N} 2^{N-i}u(x_i)\right\| > r.$$

We must therefore have $\lambda_N = 1$; so there exists $n \leqslant N$ such that $\lambda_n =$

$1-\lambda_{n-1}$. Setting

$$z = 2^{n-1}y - \sum_{1i=1}^{n-1} 2^{n-1-i}u(x_i),$$

we then have $\rho(z, B_X(0,1)) > 0$ (as $\lambda_n = 1$) and $\|z\| < r$ (as $\lambda_{n-1} = 0$). Since $B_{u(X)}(0,r)$ is dense in $B_Y(0,r)$, we can find $y' \in B_{u(X)}(0,r)$ such that $\|y' - z\| < \rho\big(z, u(B_X(0,1))\big)$ and $\rho\big(y', u(B_X(0,1))\big) > 0$. □

We now have a new constructive version of the open mapping theorem, which should be compared with those found in [7, 3, 4] and in Chapter 2 of [5]. This new version is not constructively equivalent to those other versions, and will enable us to prove the locatedness of certain subspaces associated with a closed linear subset of $\mathcal{B}(H)$ when H is a Hilbert space (Theorem 5 below).

Theorem 3 *Let $(X, (\|\cdot\|_i)_{i\in I})$ be a complete quasinormed space, Y a Banach space, and $u\colon X \to Y$ a bounded linear mapping of X onto Y such that $u(B_X(0,1))$ is located and $\rho\big(0, -u(B_X(0,1))\big)$ exists. Then u is an open mapping.*

Proof. Since

$$Y = \bigcup\nolimits_{n=1}^{\infty} n\overline{u(B_X(0,1))}$$

we can apply Theorem 1 to show that there exist $y \in Y$ and $r > 0$ such that

$$B_Y(y,r) \subset \overline{u(B_X(0,1))}.$$

A standard argument ([16], proof of Theorem 1.8.4) now shows that

$$B_Y(0,r) \subset \overline{u(B_X(0,1))}. \tag{1}$$

Consider any $y \in Y$ with $\|y\| < r$. Choose $x \in \overline{B}_X(0,2)$ as in the conclusion of Lemma 2. If $y \neq u(x)$, then there exists $y' \in B_Y(0,r)$ such that $\rho\big(y', u(B_X(0,1))\big) > 0$, which contradicts (1); hence $y = u(x)$. Thus $B_Y(0,r) \subset u(\overline{B}_X(0,2))$, from which it follows that u is an open mapping. □

Let H be a Hilbert space, $\mathcal{B}(H)$ the space of all bounded operators on H, and $\mathcal{B}_1(H)$ the unit ball of $\mathcal{B}(H)$. Given a subset $\mathcal{A}$ of $\mathcal{B}(H)$, we denote its unit ball $\mathcal{A} \cap \mathcal{B}_1(H)$ by $\mathcal{A}_1$.

As well as the uniform topology—the one associated with the family $(\|\cdot\|_x)_{\|x\|\leq 1}$ of quasinorms introduced earlier—there are at least two other important topologies on $\mathcal{B}(H)$:

- the **strong-operator topology**, which is the weakest with respect to which the mapping $T \rightsquigarrow Tx$ is continuous for each $x \in H$;
- the **weak-operator topology**, which is the weakest with respect to which the mapping $T \rightsquigarrow \langle Tx, y\rangle$ is continuous for all x, $y \in H$.

Note that $\mathcal{B}_1(H)$ is totally bounded with respect to the weak-operator topology; but that, in contrast to the classical situation, when H is infinite-dimensional, it cannot be proved constructively that $\mathcal{B}_1(H)$ is weak-operator complete [6].

Projections on the closure of subspaces of the form $\mathcal{A}x$, with $\mathcal{A}$ a subalgebra of $\mathcal{B}(H)$, play an important part in the classical theory of operator algebras. Constructively, it seems hard to find conditions that ensure the locatedness of $\mathcal{A}x$, a condition that is necessary and sufficient for the existence of the associated projection. Spitters [17] (Proposition 9.8.4) shows that when $\mathcal{A}$ is an abelian von Neumann algebra, then $\mathcal{A}x$ is located for all x in a dense subset of H. We use the foregoing open mapping theorem to give conditions on a linear subset $\mathcal{R}$ of $\mathcal{B}(H)$, rather than a subalgebra, and on the element x of H, that ensure the locatedness of $\mathcal{R}x$. First we need

Proposition 4 *Let H be a Hilbert space, and $\mathcal{R}$ a linear subset of $\mathcal{B}(H)$ that has weak-operator totally bounded unit ball $\mathcal{R}_1$. Let x be an element of H such that the linear mapping $u\colon R \rightsquigarrow Rx$ of the quasinormed subspace $\mathcal{R}$ of $\mathcal{B}(H)$ onto $\mathcal{R}x$ is open. Then $\mathcal{R}x$ is located in H.*

Proof. By Theorem 4 of [12], $\mathcal{R}_1 x$ is located in H; whence

$$\mathcal{R}_n x = \{Rx : R \in n\mathcal{R}_1\}$$

is located in H for each positive integer n. Compute $\delta > 0$ such that $B_{\mathcal{R}x}(0,\delta) \subset u(\overline{B}_{\mathcal{R}}(0,1))$. Given $y \in H$, choose a positive integer N such that $N\delta > 2\|y\|$. Suppose that $\|y - Rx\| < \rho(y, \mathcal{R}_N x)$ for some $R \in \mathcal{R}$. Then $\rho(Rx, \mathcal{R}_N x) > 0$, so $N^{-1}Rx \notin u(\overline{B}_{\mathcal{R}}(0,1))$ and therefore $\|N^{-1}Rx\| \geq \delta$. Hence

$$\|y - Rx\| \geq \|Rx\| - \|y\| \geq N\delta - \|y\| > \|y\|$$

and therefore $\|y - Rx\| > \rho(y, \mathcal{R}_N x)$, a contradiction. We conclude that

$\|y - Rx\| \geq \rho(y, \mathcal{R}_N x)$ for all $R \in \mathcal{R}$; whence $\rho(y, \mathcal{R}x)$ exists and equals $\rho(y, \mathcal{R}_N x)$. $\square$

Theorem 5 *Let H be a Hilbert space, and $\mathcal{R}$ a closed linear subset of the quasinormed space $\mathcal{B}(H)$ that has weak-operator totally bounded unit ball $\mathcal{R}_1$. Let x be an element of H such that $\rho(0, \mathcal{R}x - \mathcal{R}_1 x)$ exists and $\mathcal{R}x$ is a closed subspace of H. Then $\mathcal{R}x$ is located in H.*

Proof. By Theorem 4 of [12], $\mathcal{R}_1 x$ is located in H. Applying Theorem 3 with $Y = \mathcal{R}x$, we see that $R \rightsquigarrow Rx$ is an open mapping of $\mathcal{R}$ onto $\mathcal{R}x$. The desired conclusion now follows from Proposition 4. $\square$

3. Compactly generated Banach spaces

In this section we show how our new version of Baire's theorem can be used to give another proof of the theorem, originally proved in [8], that compactly generated Banach spaces are finite dimensional.

Note that for a subset C of a normed space we define

- the **logical complement**,

$$\neg C = \{x \in X : x \notin C\},$$

- and the **complement**,

$$\sim C = \{x \in X : \forall y \in C \ (\|x - y\| > 0)\}.$$

If these two sets are provably equal, then Markov's Principle (a to-us-unacceptable form of unbounded search) holds: for each binary sequence (a_n)

$$\neg \forall n \ (a_n = 0) \Rightarrow \exists n \ (a_n = 1).$$

Our discussion hinges on some interesting geometrical properties of convex subsets of a Banach space.

Proposition 6 *If C is a convex generating set for a Banach space X, then $\sim C$ is dense in $\neg C$.*

Proof. Let $x \in -C$, let $\varepsilon > 0$, and choose $\delta > 0$ such that $\delta \|x\| < \varepsilon$. Then

$$x' = (1 + \delta)x \notin (1 + \delta)C.$$

Let $y \in C$, and construct an increasing binary sequence (λ_n) such that

$$\lambda_n = 0 \Rightarrow \|x' - y\| < 1/(n+1)^2$$
$$\lambda_n = 1 \Rightarrow x' \neq y.$$

Define a sequence (z_n) in X as follows: if $\lambda_n = 0$, set $z_n \equiv 0$; if $\lambda_n = 1 - \lambda_{n-1}$, set $z_k = n(x' - y)$ for all $k \geq n$. Then (z_n) is a Cauchy sequence and so converges to a limit $z \in X$. Choosing a positive integer N so that $z \in N\delta C$, consider any integer $n \geq N$. If $\lambda_n = 1 - \lambda_{n-1}$, then $z = n(x' - y)$; whence

$$x' = y + n^{-1}z \in y + \frac{N\delta}{n}C \subset C + \delta C = (1+\delta)C,$$

a contradiction. Hence $\lambda_n = \lambda_{n-1}$ for all $n \geq N$. It follows that if $\lambda_n = 0$ for all $n < N$, then $\lambda_n = 0$ for all n, and therefore $x' = y \in C \subset (1+\delta)C$. This contradiction ensures that $\lambda_n = 1$ for some $n < N$; whence $x' \neq y$. So $x' \in {\sim}C$ and $\|x' - x\| = \delta\|x\| < \varepsilon$. $\square$

Lemma 7 *Let G be a compact generating set for a nontrivial Banach space X. Then there exists a convex, balanced, compact generating set C for X such that $\rho(0, -C)$ exists.*

Proof. We may assume that G contains 0 and is both balanced and convex. Given $x \in -G$, choose $r \geq 0$ and $g \in G$ such that $x = rg$; if $r \geq 2$, then $(2/r)x \in 2G - G$ and $\left\|(2/r)x\right\| = (2/r)\|x\| \leq \|x\|$. It follows that $\rho(0, -G) = \rho(0, 2G - G)$.

Since $2G$ is compact and the mapping $x \rightsquigarrow \rho(x, G)$ is uniformly continuous on X, there exists $\delta > 0$ such that both the sets

$$C = \{x \in 2G : \rho(x, G) \leq \delta\},$$
$$B = \{x \in 2G : \rho(x, G) \geq \delta\}$$

are either compact or empty. Note that C is also convex and balanced. Since X is nontrivial, we may assume that δ is so small that both C and B are compact. We show that $-C$ is dense in B. To this end, consider any $x \in B$ and any $\varepsilon > 0$. Choose $t > 1$ so that $(t-1)\|x\| < \varepsilon/2$, and suppose that $tx \in C$. Then, since C is balanced, $x \in C$; whence $x \in C \cap B$ and therefore $\rho(x, G) = \delta$. But then for each $g \in G$ we have

$$\|tx - g\| = t\|x - t^{-1}g\| \geq t\rho(x, G) = t\delta > \delta,$$

which is absurd as $tx \in C$. We conclude that $tx \notin C$. It follows from Proposition 6 that there exists $y \in {\sim}C$ such that $\|tx - y\| < \varepsilon/2$ and therefore $\|x - y\| < \varepsilon$. Applying Bishop's Lemma ([2], page 92, Lemma (3.8)), we see that $y \in -C$. This completes the proof that $-C$ is dense in B.

Since the norm function is uniformly continuous on the compact set B, it now follows that

$$\rho(0, -C) = \inf\{\|x\| : x \in -C\} = \inf\{\|x\| : x \in B\}$$

exists. $\square$

Theorem 8 *A compactly generated Banach space is finite-dimensional.*

Proof. Let X be a compactly generated Banach space. We first suppose that X contains a nonzero vector. By Lemma 7, X has a balanced, convex, compact generating set C such that $\rho(0, -C)$ exists. Applying Theorem 1, we see that C° is nonempty; whence C contains a nontrivial ball. But every ball in a normed space is located, so the ball in question is totally bounded, and therefore X is finite-dimensional.

It remains to remove the restriction that X be nontrivial. To do this, we work in the Banach space $X \oplus \mathbf{F}$ (where $\mathbf{F}$ is the groundfield of X) with the norm

$$\|(x, \lambda)\| = \|x\| + |\lambda|.$$

This space is generated by the complete, totally bounded, and therefore compact set $G \oplus \{1\}$; so, by the foregoing, $X \oplus \mathbf{F}$ is finite-dimensional. It follows that X, being isomorphic to the quotient space $(X \oplus \mathbf{F})/\mathbf{F}$, is finite-dimensional. $\square$

Theorem 8 is not without interesting applications. For example, an immediate consequence of it is that if the range of a compact linear mapping is finite-dimensional, then the mapping has finite rank.

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D. Bridges
Department of Mathematics & Statistics
University of Canterbury
Private Bag 4800, Chistchurch, New Zealand
E-mail: d.bridges@math.canterbury.ac.nz

H. Ishihara
School of Information Science
Japan Advanced Institute of Science and Technology, Hokuriku
Ishikawa 923-1292, Japan
E-mail: ishihara@jaist.ac.jp

L. Vîţă
Department of Mathematics & Statistics
University of Canterbury
Private Bag 4800, Chistchurch, New Zealand