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On totally geodesic foliations perpendicular to Killing fields

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Abstract. We study the codimension-one totally geodesic foliation perpendicular to a non-singular Killing field of a Lorentzian manifold. We determine the structure of the totally geodesic foliation perpendicular to a non-singular Killing field of a two-dimensional Lorentzian torus containing at least two kinds of leaves.

Key words: Lorentzian manifolds, totally geodesic foliations, Killing fields.

1. Introduction

Totally geodesic foliations of Lorentzian manifolds are studied by several authors ([BMT], [CR], [M], [Y1], [Y2], [Y3], [Y4], [Z1], [Z2], [Z3]). In the paper [Y1], we constructed an example of a codimension-1 totally geodesic foliation of a Lorentzian 2-torus containing three kinds of leaves among spacelike, timelike, and lightlike ones, and proved that there exists no totally geodesic foliation of a lightlike complete Lorentzian 2-torus containing at least two kinds of leaves. To study totally geodesic foliations of Lorentzian manifolds, we introduced the concept of the STL-decompositions of the ambient Lorentzian manifolds by totally geodesic foliations. First we recall the definition of the STL-decompositions. Let (M, g) be a Lorentzian manifold and $\mathcal{F}$ a totally geodesic foliation of M. Denote the union of spacelike, timelike, or lightlike leaves of $\mathcal{F}$ by $\mathbf{S}$, $\mathbf{T}$, or $\mathbf{L}$, respectively. The decomposition

$$M = \mathbf{S} \sqcup \mathbf{T} \sqcup \mathbf{L}$$

is called the STL-decomposition of M by $\mathcal{F}$. We proved that the sets $\mathbf{S}$ and $\mathbf{T}$ are open in M, and $\mathbf{L}$ is closed in M ([Y1]). We want to know the structure of a connected component S of $\mathbf{S} \sqcup \mathbf{T}$ more precisely. For an open saturated set S of a codimension-1 foliation of a closed manifold M, we can consider the completion $\hat{S}$ of S with respect to the restriction of a Riemannian metric g on M. Note that $\hat{S}$ does not depend on the choice of

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g. When the manifold is a 2-torus, the completion $\hat{S}$ of an open connected saturated set $S \subsetneq T^2$ is diffeomorphic to either $S^1 \times [0, 1]$ or $\mathbf{R} \times [0, 1]$ by using an octopus decomposition of $\hat{S}$ (see [CC]). Thus our question is reduced to the following.

Question 1 *Let (T^2, g) be a Lorentzian 2-torus, $\mathcal{F}$ a totally geodesic foliation and $T^2 = \mathbf{S} \sqcup \mathbf{T} \sqcup \mathbf{L}$ the STL-decomposition of T^2 by $\mathcal{F}$. Is the completion $\hat{S}$ of a connected component $S \subsetneq T^2$ of $\mathbf{S} \sqcup \mathbf{T}$ bounded by compact leaves?*

To make our question easier, we restrict ourselves to the codimension-1 totally geodesic foliation perpendicular to a non-singular Killing field X, that is, the foliation defined by $\ker g(X, \cdot)$. We characterize the non-singular Killing field X of (T^2, g) having a closed orbit as follows.

Theorem 12 *Let (T^2, g) be a Lorentzian 2-torus, X a non-singular Killing field, and $\mathcal{F}$ the foliation defined by $\ker g(X, \cdot)$. Let $\mathcal{F}^{\perp}$ denote the foliation perpendicular to $\mathcal{F}$ with respect to g. Assume that $\mathcal{F}^{\perp}$ has a compact leaf. Then all the leaves of $\mathcal{F}^{\perp}$ are compact.*

As a corollary to this theorem, we classify the totally geodesic foliation which is perpendicular to a non-singular Killing field and contains more than one kind of leaves among spacelike, timelike, and lightlike ones as follows.

Corollary 15 *Let $(T^2; g)$ be a Lorentzian 2-torus, X a non-singular Killing field, and $\mathcal{F}$ the foliation defined by $\ker g(X, \cdot)$. Let $T^2 = \mathbf{S} \sqcup \mathbf{T} \sqcup \mathbf{L}$ denote the STL-decomposition of T^2 by $\mathcal{F}$. Assume that $\mathbf{L}$ is a proper subset of T^2. Let $\mathcal{U}$ denote the set of connected components S of $T^2 \setminus (\overline{\mathbf{S} \sqcup \mathbf{T}} \setminus \mathbf{S} \sqcup \mathbf{T})$. Then for each element $S \in \mathcal{U}$, the completion $\hat{S}$ of S is diffeomorphic to $S^1 \times [0, 1]$ and satisfies one of the following*:

(1) *$\mathcal{F}|_{\hat{S}}$ is diffeomorphic to the product foliation $\{S^1 \times \{*\}\}$ and S is contained in $\mathbf{L}$,*

(2) *$\mathcal{F}|_{\hat{S}}$ is diffeomorphic to a foliation of the $[0, 1]$-bundle $S^1 \times [0, 1]$ over S^1 constructed by a turbulization and S is a connected component of $\mathbf{S} \sqcup \mathbf{T}$.*

By using Corollary 15, we have a partial answer to Question 1 as follows.

Corollary 16 *Let $(T^2; g)$ be a Lorentzian 2-torus, X a non-singular Killing field, and $\mathcal{F}$ the foliation defined by $\ker g(X, \cdot)$. Let $T^2 = \mathbf{S} \sqcup \mathbf{T} \sqcup \mathbf{L}$ denote the STL-decomposition of T^2 by $\mathcal{F}$. Assume that a connected component S*

of $\mathbf{S} \sqcup \mathbf{T}$ *is a proper subset of* T^2. *Then* S *is bounded by compact leaves.*

Throughout this paper, we assume that manifolds, foliations and metrics under consideration are smooth, and do not assume the completeness of metrics.

2. Preliminaries

In this section, we recall definitions and gather fundamental results about totally geodesic foliations which will be used in the following sections (for generalities on Lorentzian manifolds, see [ON]).

We first recall an equation discriminating whether a foliation is totally geodesic or not.

Proposition 2 ([Y1]) *Let* (M, g) *be a pseudo-Riemannian manifold and* $\mathcal{F}$ *a codimension-*k *foliation of* M. *Then* $\mathcal{F}$ *is totally geodesic if and only if* $(\mathcal{L}_X g)(Y, Z) = 0$ *for all* $X \in \Gamma((T\mathcal{F})^{\perp})$ *and for all* $Y, Z \in \Gamma(T\mathcal{F})$, *where* $(T\mathcal{F})^{\perp}$ *is the distribution which consists of all vectors perpendicular to* $T\mathcal{F}$.

We have a corollary to this proposition as follows.

Corollary 3 *Let* (M, g) *be a Lorentzian manifold and* X *a non-singular Killing field of* g. *If the plane field* $\ker g(X, \cdot\,)$ *perpendicular to* X *is completely integrable, it defines the codimension-*1 *totally geodesic foliation perpendicular to* X.

Now we review the concept of the STL-decomposition.

Definition 4 Let (M, g) be a Lorentzian manifold and $\mathcal{F}$ a codimension-k totally geodesic foliation. Denote the union of all spacelike leaves, timelike ones, and lightlike ones of $\mathcal{F}$ by $\mathbf{S}$, $\mathbf{T}$, and $\mathbf{L}$, respectively. The decomposition $M = \mathbf{S} \sqcup \mathbf{T} \sqcup \mathbf{L}$ (disjoint union) is called the *STL-decomposition* of M by $\mathcal{F}$.

The STL-decomposition satisfies the following.

Proposition 5 ([Y1]) *The sets* $\mathbf{S}$ *and* $\mathbf{T}$ *are open in* M, *and* $\mathbf{L}$ *is closed in* M.

We review the concept of an element of isometric holonomy.

Definition 6 Let (M, g) be a Lorentzian manifold, $\mathcal{F}$ a codimension-k totally geodesic foliation, and $\mathcal{H}$ a distribution of M. A piecewise smooth curve $\sigma\colon [0,\, t_0] \to M$ is called an *$\mathcal{H}$-curve* if its tangent vectors lie in $\mathcal{H}$. An *element of isometric holonomy* along the $\mathcal{H}$-curve σ is a family of maps $\{\psi_t\colon V_{\sigma(0)} \to V_{\sigma(t)}\}_{t\in[0,\, t_0]}$ which satisfies the following:

(1) The set $V_{\sigma(t)}$ is a plaque of the leaf of $\mathcal{F}$ containing the point $\sigma(t)$ for each $t \in [0,\, t_0]$.
(2) The map ψ_t is an isometry from $(V_{\sigma(0)},\, g|_{V_{\sigma(0)}})$ to $(V_{\sigma(t)},\, g|_{V_{\sigma(t)}})$ for each $t \in [0,\, t_0]$.
(3) The curve $\psi_t(x)$ with parameter $t \in [0,\, t_0]$ is an $\mathcal{H}$-curve for each $x \in V_{\sigma(0)}$ and $\psi_t(\sigma(0)) = \sigma(t)$.
(4) The map ψ_0 is the identity map of $V_{\sigma(0)}$.

Finally, we review a result about an element of isometric holonomy.

Proposition 7 ([Y1]) *Let $(M,\, g)$ be a Lorentzian manifold, $\mathcal{F}$ a codimension-k totally geodesic foliation of M, and $\mathcal{H}$ the distribution perpendicular to $T\mathcal{F}$ with respect to g. If an $\mathcal{H}$-curve $\sigma\colon [0,\, t_0] \to M$ intersects only spacelike or timelike leaves, then there exists an element of isometric holonomy along σ.*

3. Totally geodesic foliations perpendicular to Killing fields

In this section, we consider the plane field E perpendicular to a non-singular Killing field X of a Lorentzian manifold. If E is completely integrable, it defines the codimension-1 totally geodesic foliation perpendicular to the Killing field X.

First we prove the following.

Proposition 8 *Let $(M,\, g)$ be a Lorentzian manifold and X a non-singular Killing field. Let E be the plane field defined by $\ker g(X,\, \cdot\,)$. Then the flow generated by X preserves E, that is, $[X,\, Y] \in \Gamma(E)$ for all $Y \in \Gamma(E)$.*

Proof. Let $Y \in \Gamma(E)$. We have

$$\begin{aligned} 0 &= (\mathcal{L}_X g)(X,\, Y) = X(g(X,\, Y)) - g([X, X], Y) - g(X, [X,\, Y]) \\ &= -g(X,\, [X,\, Y]). \end{aligned}$$

Therefore $[X,\, Y] \in \Gamma(E)$. This proves Proposition 8. □

We have a corollary to this proposition as follows.

Corollary 9 *Let (M, g) be a Lorentzian manifold and X a non-singular Killing field. Assume that the plane field $\ker g(X, \cdot)$ is completely integrable and denote by $\mathcal{F}$ the codimension-1 foliation defined by $\ker g(X, \cdot)$. Then the flow generated by X preserves $\mathcal{F}$.*

Now we establish an elementary proposition about the Lorentzian vector space.

Proposition 10 *Let $n = 2, 3$. Let $(\mathbf{R}^{n-1,1}, \langle \cdot, \cdot \rangle)$ be the standard Lorentzian vector space and $(e_1, \ldots, e_n)$ the standard orthonormal basis with $\langle e_n, e_n \rangle = -1$. Let $W \subset \mathbf{R}^{n-1,1}$ be an $(n-1)$-dimensional lightlike subspace and $f \in \mathrm{SO}_0(n-1, 1)$. Let $w_1 \in W$ be a non-zero lightlike vector. Assume that $f(W) = W$. Let λ_1 denote the eigenvalue of the eigenvector w_1. When $n = 3$ and $\lambda_1 = 1$, assume furthermore that there exists a spacelike eigenvector $w_0 \in W$. Then there exists a 1-dimensional lightlike subspace V of $\mathbf{R}^{n-1,1}$ such that*

$$\mathbf{R}^{n-1,1} = V \oplus W$$

is an f-invariant splitting.

Moreover $\lambda_1\lambda_2 = 1$, where λ_2 denotes the eigenvalue corresponding to non-zero lightlike vectors $v_2 \in V$.

Proof. Assume that $n = 2$. By taking the other lightlike subspace V, we have a splitting $\mathbf{R}^{1,1} = V \oplus W$. The map f preserves an orientation and a time orientation of $\mathbf{R}^{1,1}$. Hence $\mathbf{R}^{1,1} = V \oplus W$ is an f-invariant splitting and $\det f = \lambda_1\lambda_2 = 1$.

Assume that $n = 3$. Since W is f-invariant, we have $\det(t\mathrm{I}_2 - f|_W) = (t-\lambda_1)(t-\lambda_0)$ for some λ_0. Hence $\det(t\mathrm{I}_3 - f) = (t-\lambda_2)(t-\lambda_1)(t-\lambda_0)$ for some λ_2 and $\lambda_1\lambda_2\lambda_0 = 1$. We have that $\lambda_1 > 0$ by the assumption that λ_1 is the eigenvalue of the lightlike eigenvector w_1 and f preserves an orientation and a time-orientation of $\mathbf{R}^{2,1}$.

If $\lambda_1 = 1$, there exists a spacelike eigenvector $w_0 \in W$ by the additional assumption for this case. We have $\langle w_0, w_0 \rangle = (\lambda_0)^2 \langle w_0, w_0 \rangle$. We have that $\lambda_0 = 1$ because f preserves an orientation of W and $\dim W = 2$. There exists an f-invariant splitting $\mathbf{R}^{2,1} = \langle w_0 \rangle \oplus \langle w_0 \rangle^{\perp}$, where $\langle w_0 \rangle^{\perp}$ denote the orthogonal complement of $\langle w_0 \rangle$ with respect to the metric $\langle \cdot, \cdot \rangle$. The subspace $\langle w_0 \rangle^{\perp}$ is a 2-dimensional timelike subspace. We have that the lightlike eigenvector w_1 satisfies that $w_1 \in \langle w_0 \rangle^{\perp}$ since $\langle w_1, w_0 \rangle = 0$. Note that $\langle w_1\rangle^\perp = W$. Since $\langle w_0\rangle^\perp$ is an f-invariant 2-dimensional timelike subspace, there exists the other 1-dimensional lightlike subspace $V \subset \langle w_0\rangle^\perp$ such that the splitting $\mathbf{R}^{2,1} = V \oplus W$ is invariant by f. (In this case, f is equal to the identity.)

If $\lambda_1 \neq 1$ and $\lambda_1 \neq \lambda_0$, an eigenvector $w_0 \in W$ having the eigenvalue λ_0 must be spacelike. Hence $\lambda_0 = 1$ and we can prove the proposition in the same way as above.

If $\lambda_1 \neq 1$ and $\lambda_1 = \lambda_0$, we have $\lambda_2 = (\lambda_1)^{-2}$ by $\lambda_1\lambda_2\lambda_0 = 1$. The assumption $\lambda_1 \neq 1$ means that $\lambda_2 \neq 1$ and $\lambda_2 \neq \lambda_1$. Take a 1-dimensional subspace V which satisfies $f(v_2) = \lambda_2 v_2$ for all $v_2 \in V$. Since $\lambda_2 \neq 1$ and $\lambda_2 > 0$, the subspace V must be lightlike. Hence f has a timelike invariant subspace $V \oplus \langle w_1\rangle$. The orthogonal complement of $V \oplus \langle w_1\rangle$ is a 1-dimensional spacelike subspace invariant by f. Therefore $\lambda_0 = 1$, which is a contradiction. Hence this case does not happen. $\square$

Remark 11 When $n = 3$ and $\lambda_1 = 1$, we can not remove the assumption of the existence of a spacelike eigenvector $w_0 \in W$. The reason is as follows. Let $f \in \mathrm{SO}_0(2, 1)$ be the matrix given by

$$\begin{pmatrix} 1 & a & -a \\ -a & (2-a^2)/2 & a^2/2 \\ -a & -a^2/2 & (a^2+2)/2 \end{pmatrix},$$

where $a \neq 0$. It has an eigenvector $e_2 + e_3$ and the characteristic equation of f equals $(t-1)^3$. We have $(f - \mathrm{I}_3)^2 \neq 0$. Hence the dimension of the eigenspace corresponding to the eigenvalue 1 is 1.

4. In the case of 2-tori

In this section, we consider totally geodesic foliations perpendicular to non-singular Killing fields of Lorentzian 2-tori. Let (T^2, g) be a Lorentzian 2-torus and X a non-singular Killing field of g. In this case, $\ker g(X, \cdot\,)$ is completely integrable because $\dim(\ker g(X, \cdot\,)) = 1$. So the foliation $\mathcal{F}$ defined by $\ker g(X, \cdot\,)$ is totally geodesic by Corollary 3. However the foliation $\mathcal{F}^\perp$ perpendicular to $\mathcal{F}$ with respect to g is not always totally geodesic. Note that the foliation $\mathcal{F}^\perp$ is equal to the orbit foliation defined by X.

In the case of the existence of a closed orbit of a non-singular Killing field, we have the following.

Theorem 12 *Let (T^2, g) be a Lorentzian 2-torus, X a non-singular Killing field, and $\mathcal{F}$ the foliation defined by $\ker g(X, \cdot)$. Let $\mathcal{F}^{\perp}$ denote the foliation perpendicular to $\mathcal{F}$ with respect to g. Assume that $\mathcal{F}^{\perp}$ has a compact leaf. Then all the leaves of $\mathcal{F}^{\perp}$ are compact.*

Proof. First, notice that $\mathcal{F}$ is totally geodesic and the leaves of $\mathcal{F}^{\perp}$ are the orbits of X. Let φ_t be the flow generated by X. Denote by C the union of all compact leaves of $\mathcal{F}^{\perp}$. The set C is compact ([CC]). Since T^2 is connected, it is sufficient to prove that C is open.

Case 1: The case where $\mathcal{F}^{\perp}$ has a lightlike compact leaf L.

Fix $x \in L$. We can assume that $\varphi_1(x) = x$ by considering cX, $c \in \mathbf{R}_{>0}$, if necessary. Take the φ_{1*}-invariant splitting $T_xT^2 = T_xL \oplus V$ so that $V \subset T_xT^2$ is the other lightlike subspace. Fix non-zero lightlike vectors $X_x \in T_xL$ and $v_2 \in V$. By $\varphi_{1*}(X_x) = X_x$ and Proposition 10, eigenvalues of X_x and v_2 are 1. Hence we have

$$\varphi_{1*}|_{T_xT^2} = \mathrm{id}_{T_xT^2}.$$

Fix $v \in V \subset T_xT^2$ such that $\exp_x v$ is defined. We have that $\varphi_1(\exp_x v) = \exp_x(\varphi_{1*}v) = \exp_x v$. Hence the point $\exp_x v$ is a fixed point of φ_1. This means that leaves of $\mathcal{F}^{\perp}$ near a lightlike compact $\mathcal{F}^{\perp}$-leaf L are compact.

Case 2: The case where $\mathcal{F}^{\perp}$ has a compact leaf L' which is not lightlike.

Let L' be a compact $\mathcal{F}^{\perp}$-leaf which is not lightlike and $\sigma\colon [0, 1] \to L'$ be a simple loop in L'. The curve σ intersects only spacelike or timelike $\mathcal{F}$-leaves. So we can consider an element of isometric holonomy along σ

$$\{\psi_t\colon V_{\sigma(0)} \to V_{\sigma(t)}\}_{t\in[0,1]}$$

by Proposition 7, where $V_{\sigma(t)}$ is an $\mathcal{F}$-plaque containing $\sigma(t)$ and ψ_t is an isometry from $(V_{\sigma(0)}, g|_{V_{\sigma(0)}})$ to $(V_{\sigma(t)}, g|_{V_{\sigma(t)}})$. Since ψ_1 is an isometry from $V_{\sigma(0)}$ to $\psi_1(V_{\sigma(0)})$, $\psi_1(\sigma(0)) = \sigma(1) = \sigma(0)$, and $\dim V_{\sigma(t)} = 1$, we have that the map ψ_1 is the identity. Hence all the $\mathcal{F}^{\perp}$-leaves near L' are compact.

From Case 1 and 2, we have that C is open. Since T^2 is connected, we have that $C = T^2$. This proves the proposition. □

We have a corollary to this theorem, which characterizes the lightlike totally geodesic foliation perpendicular to a non-singular lightlike Killing field.

Corollary 13 *Let (T^2, g) be a Lorentzian 2-torus, X a non-singular Killing field, and $\mathcal{F}$ the foliation defined by $\ker g(X, \cdot)$. Suppose that X is lightlike. Then $\mathcal{F}$ is lightlike and satisfies one of the following*:

(1) *All the leaves of $\mathcal{F}$ are compact,*

(2) *$\mathcal{F}$ contains no compact leaves.*

Proof. By the assumption that X is lightlike, we have that $g(X, X) = 0$. So $\mathcal{F}$ is equal to the orbit foliation defined by X. Hence $\mathcal{F}$ is lightlike and equal to the foliation $\mathcal{F}^{\perp}$ perpendicular to $\mathcal{F}$ with respect to g. So we can apply Theorem 12. If $\mathcal{F} = \mathcal{F}^{\perp}$ has a compact lightlike leaf, then all the leaves of $\mathcal{F}$ are compact by Theorem 12. □

Remark 14 By Corollary 13, the lightlike totally geodesic foliation of the torus of Clifton-Pohl ([CR]), which has two Reeb components, can not be perpendicular to a non-singular lightlike Killing field even if we change the metric.

Now we consider the totally geodesic foliation which is perpendicular to a non-singular Killing field and contains more than one kind of leaves among spacelike, timelike, and lightlike ones. We have the following.

Corollary 15 *Let (T^2, g) be a Lorentzian 2-torus, X a non-singular Killing field, and $\mathcal{F}$ the foliation defined by $\ker g(X, \cdot)$. Let $T^2 = \mathbf{S} \sqcup \mathbf{T} \sqcup \mathbf{L}$ denote the STL-decomposition of T^2 by $\mathcal{F}$. Assume that $\mathbf{L}$ is a proper subset of T^2. Let $\mathcal{U}$ denote the set of connected components S of $T^2 \setminus (\overline{\mathbf{S} \sqcup \mathbf{T}} \setminus \mathbf{S} \sqcup \mathbf{T})$. Then for each element $S \in \mathcal{U}$, the completion $\hat{S}$ of S is diffeomorphic to $S^1 \times [0, 1]$ and satisfies one of the following*:

(1) *$\mathcal{F}|_{\hat{S}}$ is diffeomorphic to the product foliation $\{S^1 \times \{*\}\}$ and S is contained in $\mathbf{L}$,*

(2) *$\mathcal{F}|_{\hat{S}}$ is diffeomorphic to a foliation of the $[0, 1]$-bundle $S^1 \times [0, 1]$ over S^1 constructed by a turbulization and S is a connected component of $\mathbf{S} \sqcup \mathbf{T}$.*

Proof. Let $\mathcal{F}^{\perp}$ denote the foliation perpendicular to $\mathcal{F}$. First we prove that all the leaves of $\mathcal{F}^{\perp}$ are compact. By the assumption that $\mathbf{L}$ is a proper subset of T^2 and Proposition 5, the set $\mathbf{L}$ is a non-empty closed saturated set. Hence $\mathbf{L}$ contains a minimal set $\mathcal{M}$ (see [CC]). Since $\mathcal{F}$ is smooth, $\mathcal{M} \subset \mathbf{L}$ must be a single closed leaf. So there exists a lightlike compact $\mathcal{F}$-leaf. Note that an $\mathcal{F}$-leaf L is lightlike if and only if an $\mathcal{F}$-leaf L is also an $\mathcal{F}^{\perp}$-leaf. Hence all the leaves of $\mathcal{F}^{\perp}$ are compact by Theorem 12.

Let $S \in \mathcal{U}$ and denote by $\hat{S}$ the completion of S. We prove that $\hat{S}$ is diffeomorphic to $S^1 \times [0, 1]$. Note that S is an open connected $\mathcal{F}$-saturated set. The set S is contained in $\mathbf{S} \sqcup \mathbf{T}$ or $\mathbf{L}$ by the definition of $\mathcal{U}$. In both cases, we have that $\overline{S} \setminus S \subset \mathbf{L}$ by Proposition 5. So $\mathcal{F}^{\perp}|_{\overline{S}}$ is tangent to $\overline{S} \setminus S$. Hence S is also an open connected $\mathcal{F}^{\perp}$-saturated set. Since all the leaves of $\mathcal{F}^{\perp}$ are compact, an open connected $\mathcal{F}^{\perp}$-saturated set S is diffeomorphic to $S^1 \times (0, 1)$. Therefore $\hat{S}$ is diffeomorphic to $S^1 \times [0, 1]$.

Now we determine the structure of $\mathcal{F}|_{\hat{S}}$.

Case 1: The case where S is contained in $\mathbf{L}$.

Since all the leaves of $\mathcal{F}|_S$ are lightlike, we have that $\mathcal{F}|_S = \mathcal{F}^{\perp}|_S$. So we have $\mathcal{F}|_{\hat{S}} = \mathcal{F}^{\perp}|_{\hat{S}}$. Since $(S, \mathcal{F}^{\perp}|_S)$ is diffeomorphic to $(S^1 \times (0, 1), \{S^1 \times \{*\}\})$, the couple $(\hat{S}, \mathcal{F}|_{\hat{S}})$ is diffeomorphic to $(S^1 \times [0, 1], \{S^1 \times \{*\}\})$.

Case 2: The case where S is contained in $\mathbf{S} \sqcup \mathbf{T}$.

The couple $(\hat{S}, \mathcal{F}^{\perp}|_{\hat{S}})$ is diffeomorphic to $(S^1 \times [0, 1], \{S^1 \times \{*\}\})$. On $S \cong S^1 \times (0, 1)$, the foliation $\mathcal{F}$ is transverse to $\mathcal{F}^{\perp}|_S \cong \{S^1 \times \{*\}\}$. Since $\overline{S} \setminus S \subset \mathbf{L}$, the foliation $\mathcal{F}$ is tangent to $\partial\hat{S} \cong \partial(S^1 \times [0, 1])$. Hence $(\hat{S}, \mathcal{F}|_{\hat{S}})$ is a slope component or a Reeb component (see Fig. 1). Therefore $\mathcal{F}|_{\hat{S}}$ is diffeomorphic to a foliation of the $[0, 1]$-bundle $S^1 \times [0, 1]$ constructed by a turbulization (for the definition of a turbulization, see [CC]). $\square$

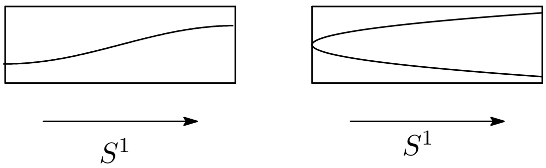

Fig. 1. a slope component and a Reeb component

Now we give a partial answer to Question 1 as follows.

Corollary 16 *Let* (T^2, g) *be a Lorentzian* 2*-torus,* X *a non-singular Killing field, and* $\mathcal{F}$ *the foliation defined by* $\ker g(X, \cdot)$*. Let* $T^2 = \mathbf{S} \sqcup \mathbf{T} \sqcup \mathbf{L}$ *denote the STL-decomposition of* T^2 *by* $\mathcal{F}$*. Assume that a connected component* S *of* $\mathbf{S} \sqcup \mathbf{T}$ *is a proper subset of* T^2*. Then* S *is bounded by compact leaves.*

Proof. Since S is a proper subset of T^2, the set $\overline{S} \setminus S$ is a non-empty closed saturated set and S is a connected component of $T^2 \setminus (\overline{\mathbf{S} \sqcup \mathbf{T}} \setminus \mathbf{S} \sqcup \mathbf{T})$. We have that $\overline{S} \setminus S \subset \mathbf{L}$ by Proposition 5. So $\mathbf{L}$ is also a proper subset of T^2.

Hence we can apply Corollary 15. Since S has no lightlike leaves, the Case (2) in Corollary 15 happens. $\square$

References

[BMT] Boubel C., Mounoud P. and Tarquini C., *Foliations admitting a transverse connection; applications in dimension three.* preprint.

[CC] Candel A. and Conlon L., *Foliations* I. Graduate Studies in Mathematics **23**, AMS (2000).

[CR] Carrière Y. and Rozoy L., *Complétude des Métriques Lorentziennes de* $\mathbb{T}^2$ *et Difféomorphismes du Cercle.* Bol. Soc. Brasil. Mat. **25** (1994), 223–235.

[M] Mounoud P., *Complétude et flots nul-géodésibles en géométrie lorentzienne.* to appear in Bulletin de la SMF.

[ON] O'Neill B., *Semi-Riemannian geometry.* Academic press, 1983.

[Y1] Yokumoto K., *Mutual exclusiveness among spacelike, timelike, and lightlike leaves in totally geodesic foliations of lightlike complete Lorentzian two-dimensional tori.* Hokkaido Math. J. **31** (2002), 643–663.

[Y2] Yokumoto K., *Examples of Lorentzian geodesible foliations of closed three-manifolds having Heegaard splittings of genus one.* to appear in Tôhoku Math. J.

[Y3] Yokumoto K., *Totally geodesic foliations of Lorentzian manifolds.* abstract of a talk at the conference "Geometry and Foliations 2003", Ryukoku University, Kyoto, Japan; see `http://gf2003.ms.u-tokyo.ac.jp/`

[Y4] Yokumoto K., *Some remarks on totally geodesic foliations of Lorentzian manifolds.* Ph. D. Thesis, Hokkaido University, 2002.

[Z1] Zeghib A., *Isometry groups and geodesic foliations of Lorentz manifolds. Part* I: *Foundations of Lorentz dynamics.* GAFA **9** (1999), 775–822.

[Z2] Zeghib A., *Isometry groups and geodesic foliations of Lorentz manifolds. Part* II: *Geometry of analytic Lorentz manifolds with large isometry groups.* GAFA **9** (1999), 823–854.

[Z3] Zeghib A., *Geodesic foliations in Lorentz* 3*-manifolds.* Comment. Math. Helv. **74** (1999), 1–21.

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