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Strongly almost (w, λ)-summable sequences defined by Orlicz functions

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Abstract. The purpose of this paper is to introduce the spaces of sequences that are strongly almost (w, λ)-summable with respect to an Orlicz function. We give some relations related to these sequence spaces. It is also shown if a sequence is strongly (w, λ)-summable with respect to an Orlicz function, then it is $\bar{S}_\lambda$-statistically convergent.

Key words: (V, λ)-summability, statistical convergence, Orlicz function.

1. Introduction

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\| = \sup |x_k|$. A sequence $x \in \ell_\infty$ is said to be almost convergent if all its Banach limits [14] coincide and the set of all almost convergent sequences is denoted by $\hat{c}$. Lorentz [14] proved that $x \in \hat{c}$ if and only if $\lim_n (1/n) \sum_{k=1}^n x_{k+m}$ exists uniformly in m.

Several authors including Lorentz [14], Duran [4] and King [10] have studied almost convergent sequences. Maddox [15], [17] has defined x to be strongly almost convergent to a number L if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - L| = 0, \quad \text{uniformly in } m.$$

By $[\hat{c}]$ we denote the space of all strongly almost convergent sequences. It is easy to see that $c \subset [\hat{c}] \subset \hat{c} \subset \ell_\infty$.

The space of strongly almost convergent sequences was generalized by Nanda [18]. Let $p = (p_k)$ be a sequence of strictly positive real numbers.

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Nanda [18] defined

$$[\hat{c},\, p] = \left\{ x = (x_k)\colon\ \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m} - L|^{p_k} = 0,\ \text{uniformly in } m \right\}$$

$$[\hat{c},\, p]_0 = \left\{ x = (x_k)\colon\ \lim_n \frac{1}{n} \sum_{k=1}^{n} |x_{k+m}|^{p_k} = 0,\ \text{uniformly in } m \right\}$$

$$[\hat{c},\, p]_\infty = \left\{ x = (x_k)\colon\ \sup_{n,\, m} \frac{1}{n} \sum_{k=1}^{n} |x_{k+m}|^{p_k} < \infty \right\}.$$

Later Das and Sahoo [3] defined the sequence spaces

$$(w) = \left\{ x\colon\ \begin{array}{r} \displaystyle\frac{1}{n+1} \sum_{k=0}^{n} (t_{km}(x) - L) \to 0 \text{ as } n \to \infty, \\ \text{uniformly in } m, \text{ for some } L \end{array} \right\}$$

and

$$[w] = \left\{ x\colon\ \begin{array}{r} \displaystyle\frac{1}{n+1} \sum_{k=0}^{n} |t_{km}(x) - L| \to 0 \text{ as } n \to \infty, \\ \text{uniformly in } m, \text{ for some } L \end{array} \right\}$$

where $t_{km}(x) = (x_m + \cdots + x_{m+k})/(k+1)$.

The idea of statistical convergence was introduced by Fast [8] and studied by various authors ([2], [9], [11], [22], [23]).

A sequence $x = (x_k)$ is said to be $\bar{S}$-statistically convergent to L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k \le n\colon |t_{km}(x) - L| \ge \varepsilon\}| = 0, \quad \text{uniformly in } m.$$

In this case we write $\bar{S} - \lim x = L$ or $x_k \to L(\bar{S})$ and $\bar{S}$ denotes the set of all $\bar{S}$-statistically convergent sequences [5].

Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ x \in w\colon \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \quad \text{for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = \inf \left\{ \rho > 0 \colon \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. Lindenstrauss and Tzafriri proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p for some $p \geq 1$. For $M(t) = t^p$, $1 \leq p < \infty$, the space ℓ_M coincides with the classical sequence space ℓ_p.

Recently, Parashar and Chaudhary [20] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence space ℓ_M and strongly summable sequence spaces $[C, 1, p]$, $[C, 1, p]_0$ and $[C, 1, p]_\infty$. It may be noted here that the spaces of strongly summable sequences were discussed by Maddox [15].

An Orlicz function is a function $M\colon [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If the convexity of an Orlicz function M is replaced by

$$M(x + y) \leq M(x) + M(y)$$

then this function is called modulus function, defined and discussed by Ruckle [21] and Maddox [16].

An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$, $u \geq 0$.

It is easy to see that always $K > 2$. The Δ_2-condition is equivalent to the satisfaction of inequality $M(\ell u) \leq K\ell M(u)$ for all values of u and for $\ell > 1$.

Subsequently Orlicz sequence spaces have been studied by Nuray and Gülcü [19], Esi and Et [6], Esi [7] and Bhardwaj and Singh [1].

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ and $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$.

The generalized de la Vallée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ is said to be (V, λ)-summable to a number L [12]

if $t_n(x) \to L$ as $n \to \infty$. If $\lambda_n = n$, then (V, λ)-summability is reduced to $(C, 1)$-summability. The set of sequences $x = (x_k)$ which are strongly almost (V, λ)-summable was defined by Savaş [23] such as

$$[\hat{V}, \lambda] = \left\{ x = (x_k) \colon \begin{array}{r} \text{for some } L,\ \lim_n \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in I_n} |x_{k+m} - L| = 0, \\ \text{uniformly in } m \end{array} \right\}.$$

Definition 1 [23] A sequence $x = (x_k)$ is said to be almost λ-statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n \colon |x_{k+m} - L| \geq \varepsilon \right\} \right| = 0, \quad \text{uniformly in } m.$$

In this case we write $\hat{s}_\lambda - \lim x = L$ or $x_k \to L(\hat{s}_\lambda)$ and

$$\hat{s}_\lambda = \{x \colon \text{ for some } L,\ \hat{s}_\lambda - \lim x = L\}.$$

2. $\bar{S}_\lambda$-Statistical convergence

This paper extends the statistical convergence to the $\bar{S}_\lambda$-statistical convergence and finds its relations between $[w, \lambda]$ and $\bar{S}_\lambda$.

Before giving some inclusion relations we will give two new definitions.

Definition 2 A sequence $x = (x_k)$ is said to be $\bar{S}_\lambda$-statistical convergent to L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n \colon |t_{km}(x) - L| \geq \varepsilon \right\} \right| = 0, \quad \text{uniformly in } m.$$

In this case we write $\bar{S}_\lambda - \lim x = L$ or $x_k \to L(\bar{S}_\lambda)$ and

$$\bar{S}_\lambda = \{x \colon \text{ for some } L,\ \bar{S}_\lambda - \lim x = L\}.$$

Definition 3 The sequence $x = (x_k)$ is said to be strongly (w, λ)-summable to$L$, if

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} |t_{km}(x) - L| = 0, \quad \text{uniformly in } m.$$

In this case we write

$$[w, \lambda] = \left\{ x = (x_k) \colon \begin{array}{r} \text{for some } L, \ \lim_n \dfrac{1}{\lambda_n} \displaystyle\sum_{k \in I_n} |t_{km}(x) - L| = 0, \\ \text{uniformly in } m \end{array} \right\}$$

for the set of sequences $x = (x_k)$ which are strongly (w, λ)-summable to L, i.e. $x_k \to L[w, \lambda]$.

It can be shown that if $x_k \to L[w]$, then $x_k \to L(\bar{S})$.

Theorem 2.1 *Let $\lambda = (\lambda_n)$ be the same as above, then*

i) $x_k \to L[w, \lambda] \Rightarrow x_k \to L(\bar{S}_\lambda)$.

ii) *If $x \in \ell_\infty$ and $x_k \to L(\bar{S}_\lambda)$, then $x_k \to L[w, \lambda]$, especially if $\lambda_n = n$, $x_k \to L(w)$ provided $x = (x_k)$ is not eventually constant.*

iii) $\bar{S}_\lambda \cap \ell_\infty = [w, \lambda] \cap \ell_\infty$.

Proof. i) Let $\varepsilon > 0$ and $x_k \to L[w, \lambda]$. Since

$$\begin{aligned} \sum_{k \in I_n} |t_{km}(x) - L| &\geq \sum_{\substack{k \in I_n \\ |t_{km}(x) - L| \geq \varepsilon}} |t_{km}(x) - L| \\ &\geq \varepsilon \left| \left\{ k \in I_n \colon |t_{km}(x) - L| \geq \varepsilon \right\} \right|. \end{aligned}$$

Therefore $x_k \to L[w, \lambda] \Rightarrow x_k \to L(\bar{S}_\lambda)$.

ii) Suppose that $x_k \to L(\bar{S}_\lambda)$ and $x \in \ell_\infty$, say that $|t_{km}(x) - L| \leq K$ for all k and m. Given $\varepsilon > 0$, we get

$$\begin{aligned} \lambda_n^{-1} \sum_{k \in I_n} |t_{km}(x) - L| &= \lambda_n^{-1} \sum_{\substack{k \in I_n \\ |t_{km}(x) - L| \geq \varepsilon}} |t_{km}(x) - L| \\ &\quad + \lambda_n^{-1} \sum_{\substack{k \in I_n \\ |t_{km}(x) - L| < \varepsilon}} |t_{km}(x) - L| \\ &\leq \frac{K}{\lambda_n} \left| \left\{ k \in I_n \colon |t_{km}(x) - L| \geq \varepsilon \right\} \right| + \varepsilon \end{aligned}$$

which implies that $x_k \to L[w, \lambda]$.

Further, we have

$$\begin{aligned} \frac{1}{n} \left| \sum_{k=1}^{n} (t_{km}(x) - L) \right| &= \frac{1}{n} \left| \sum_{k=1}^{n-\lambda_n} (t_{km}(x) - L) + \sum_{k \in I_n} (t_{km}(x) - L) \right| \\ &\leq \frac{1}{\lambda_n} \sum_{k=1}^{n-\lambda_n} |t_{km}(x) - L| + \frac{1}{\lambda_n} \sum_{k \in I_n} |t_{km}(x) - L| \end{aligned}$$

$$\leq \frac{2}{\lambda_n} \sum_{k \in I_n} |t_{km}(x) - L|.$$

Hence $x_k \to L(w)$, since $x_k \to L[w, \lambda]$.

iii) This immediately follows from (i) and (ii). □

Theorem 2.2 *If* $\lim_n \inf(\lambda_n/n) > 0$, *then* $\bar{S} \subset \bar{S}_\lambda$.

Proof. Given $\varepsilon > 0$ we have

$$\{k \leq n \colon |t_{km}(x) - L| \geq \varepsilon\} \supset \{k \in I_n \colon |t_{km}(x) - L| \geq \varepsilon\}.$$

Therefore

$$\begin{aligned} \frac{1}{n}\left|\{k \leq n \colon |t_{km}(x) - L| \geq \varepsilon\}\right| &\geq \frac{1}{n}\left|\{k \in I_n \colon |t_{km}(x) - L| \geq \varepsilon\}\right| \\ &\geq \frac{\lambda_n}{n}\frac{1}{\lambda_n}\left|\{k \in I_n \colon |t_{km}(x) - L| \geq \varepsilon\}\right|. \end{aligned}$$

Taking the limit as $n \to \infty$ we get $x_k \to L(\bar{S}) \Rightarrow x_k \to L(\bar{S}_\lambda)$. □

3. Some new sequence spaces defined by an Orlicz function

In this section, we introduce and examine some properties of some new sequence spaces defined by using an Orlicz function. It is also shown that if a sequence is strongly (w, λ)-summable with respect to an Orlicz function then it is $\bar{S}_\lambda$-statistically convergent.

Definition 4 Let M be an Orlicz function and $p = (p_k)$ be any sequence of strictly positive real numbers. We define the following sequence sets.

$$[w, \lambda, M, p] = \left\{ x = (x_k) \colon \begin{array}{r} \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|t_{km}(x) - L|}{\rho} \right) \right]^{p_k} = 0 \\ \text{uniformly in } m, \\ \text{for some } L \text{ and for some } \rho > 0 \end{array} \right\},$$

$$[w, \lambda, M, p]_0 = \left\{ x = (x_k) \colon \begin{array}{r} \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|t_{km}(x)|}{\rho} \right) \right]^{p_k} = 0 \\ \text{uniformly in } m, \text{ for some } \rho > 0 \end{array} \right\},$$

$$[w, \lambda, M, p]_\infty = \left\{ x = (x_k) \colon \begin{array}{r} \sup_{n,m} \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|t_{km}(x)|}{\rho} \right) \right]^{p_k} < \infty \\ \text{for some } \rho > 0 \end{array} \right\}.$$

We denote $[w, \lambda, M, p]$, $[w, \lambda, M, p]_0$ and $[w, \lambda, M, p]_\infty$ by $[w, \lambda, M]$, $[w, \lambda, M]_0$ and $[w, \lambda, M]_\infty$, respectively, when $p_k = 1$ for all k. If $x \in [w, \lambda, M]$ then we say that x is strongly (w, λ)-summable with respect to the Orlicz function M.

If $M(x) = x$ and $\lambda_n = n$, then $[w, \lambda, M, p] = [w(p)]$ for $\rho = 1$ (for the definition of $[w(p)]$ see [3]).

If $M(x) = x$, $\lambda_n = n$ then $[w, \lambda, M] = [w]$ for $\rho = 1$ (for the definition of $[w]$ see [3]).

Theorem 3.1 *For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers $[w, \lambda, M, p]$, $[w, \lambda, M, p]_0$ and $[w, \lambda, M, p]_\infty$ are linear spaces over the complex numbers field $\mathbb{C}$.*

Proof. We shall only prove for $[w, \lambda, M, p]_0$. The others can be treated similarly. Let $x, y \in [w, \lambda, M, p]_0$ and $\alpha, \beta \in \mathbb{C}$, $\alpha\beta \neq 0$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|t_{km}(x)|}{\rho_1} \right) \right]^{p_k} = 0$$

and

$$\lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|t_{km}(y)|}{\rho_2} \right) \right]^{p_k} = 0, \quad \text{unifomly in } m.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing and convex

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|t_{km}(\alpha x + \beta y)|}{\rho_3} \right) \right]^{p_k} \\
\leq &\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|t_{km}(\alpha x)|}{\rho_3} + \frac{|t_{km}(\beta y)|}{\rho_3} \right) \right]^{p_k} \\
= &\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|\alpha t_{km}(x)|}{\rho_3} + \frac{|\beta t_{km}(y)|}{\rho_3} \right) \right]^{p_k} \\
\leq &\frac{1}{\lambda_n} \sum_{k \in I_n} \frac{1}{2^{p_k}} \left[M\left(\frac{|t_{km}(x)|}{\rho_1} \right) + M\left(\frac{|t_{km}(y)|}{\rho_2} \right) \right]^{p_k} \\
\leq &\frac{1}{\lambda_n} \sum_{k \in I_n} \left[M\left(\frac{|t_{km}(x)|}{\rho_1} \right) + M\left(\frac{|t_{km}(y)|}{\rho_2} \right) \right]^{p_k}
\end{aligned}$$

$$\leq C\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|t_{km}(x)|}{\rho_1}\right)\right]^{p_k}+C\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|t_{km}(y)|}{\rho_2}\right)\right]^{p_k}\to 0$$

as $n\to\infty$ uniformly in m, where $C=\max(1,\,2^{H-1})$, $H=\sup_k p_k$. Hence $\alpha x+\beta y\in[w,\,\lambda,\,M,\,p]_0$, this proves that $[w,\,\lambda,\,M,\,p]_0$ is a linear space. □

Theorem 3.2 *For any Orlicz function M and a bounded sequence $p=(p_k)$ of strictly positive real numbers, $[w,\,\lambda,\,M,\,p]_0$ is a paranormed space (not necessarily totally paranormed) with*

$$g(x)=\inf_{\substack{\rho>0\\ n\geq 1}}\left\{\rho^{p_n/H}\colon\ \sup_k M\left(\frac{|t_{km}(x)|}{\rho}\right)\leq 1,\quad \text{uniformly in } m\right\}$$

where $H=\sup_k p_k$.

Proof. Clearly $g(x)=g(-x)$. Since $M\,(0)=0$, we get $\inf\{\rho^{p_n/H}\}=0$ for $x=0$. Now let (x_k), $(y_k)\in[w,\,\lambda,\,M,\,p]_0$ and let us choose $\rho_1>0$ and $\rho_2>0$ such that

$$\sup_k M\left(\frac{|t_{km}(x)|}{\rho_1}\right)\leq 1,\quad \text{uniformly in } m$$

and

$$\sup_k M\left(\frac{|t_{km}(y)|}{\rho_2}\right)\leq 1,\quad \text{uniformly in } m.$$

Let $\rho=\rho_1+\rho_2$. Then we get

$$\begin{aligned}\sup_k M\left(\frac{|t_{km}(x+y)|}{\rho}\right)&\leq\sup_k M\left(\frac{|t_{km}(x)|}{\rho_1+\rho_2}+\frac{|t_{km}(y)|}{\rho_1+\rho_2}\right)\\&\leq\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_k M\left(\frac{|t_{km}(x)|}{\rho_1+\rho_2}\right)\\&\quad+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_k M\left(\frac{|t_{km}(y)|}{\rho_1+\rho_2}\right)\\&\leq 1,\quad \text{uniformly in } m.\end{aligned}$$

Therefore $g(x+y)\leq g(x)+g(y)$.

For the continuity of scalar multiplication let $r\neq 0$ be any complex number. By the definition

$$g(rx) = \inf\left\{\rho^{p_n/H}\colon \sup_k M\left(\frac{|t_{km}(rx)|}{\rho}\right) \le 1, \quad \text{uniformly in } m\right\}$$

$$= \inf\left\{(|r|s)^{p_n/H}\colon \sup_k M\left(\frac{|t_{km}(x)|}{s}\right) \le 1, \quad \text{uniformly in } m\right\}$$

where $s = \rho/|r|$. Since $|r|^{p_n} \le \max(1, |r|^H)$, we have

$$g(rx) \le \max(1, |r|^H) \inf\left\{s^{p_n/H}\colon \sup_k M\left(\frac{|t_{km}(x)|}{s}\right) \le 1, \quad \text{uniformly in } m\right\}$$

$$= \max(1, |r|^H) g(x)$$

and therefore $g(rx)$ converges to zero when $g(x)$ converges to zero in $[w, \lambda, M, p]_0$.

Now let x be a fixed element in $[w, \lambda, M, p]_0$. Then there exists $\rho > 0$ such that

$$g(x) = \inf\left\{\rho^{p_n/H}\colon \sup_k M\left(\frac{|t_{km}(x)|}{\rho}\right) \le 1, \quad \text{uniformly in } m\right\}.$$

Now

$$g(rx) = \inf\left\{\rho^{p_n/H}\colon \sup_k M\left(\frac{|rt_{km}(x)|}{\rho}\right) \le 1, \quad \rho > 0 \quad \text{uniformly in } m\right\} \to 0$$

as $r \to 0$. This completes the proof. □

Lemma 3.3 [1] *Let M be an Orlicz function which satisfies Δ_2-condition and let $0 < \delta < 1$. Then for each $x \ge \delta$ we have $M(x) < Kx\delta^{-1}M(2)$ for some constant $K > 0$.*

Theorem 3.4 *For any Orlicz function M which satisfies Δ_2-condition, we have $[w, \lambda] \subset [w, \lambda, M]$.*

Proof. Let $x \in [w, \lambda]$ so that

$$A_n \equiv \frac{1}{\lambda_n} \sum_{k \in I_n} |t_{km}(x) - L| \to 0$$

as $n \to \infty$, for some L, uniformly in m.

Let $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \varepsilon$ for $0 \le t \le \delta$.

We can write

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n} M(|t_{km}(x)-L|) &= \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ |t_{km}(x)-L|<\delta}} M(|t_{km}(x)-L|)\\ &\quad+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ |t_{km}(x)-L|\geq\delta}} M(|t_{km}(x)-L|)\\ &< \lambda_n^{-1}(\lambda_n\varepsilon)+K\delta^{-1}M(2)A_n,\end{aligned}$$

by Lemma 3.3 and letting $n\to\infty$, it follows that $x\in[w,\,\lambda,\,M]$ for $\rho=1$. □

Theorem 3.5 *For any Orlicz function M, $[w,\,\lambda,\,M]\subset\bar{S}_\lambda$.*

Proof. Let $x\in[w,\,\lambda,\,M]$ and $\varepsilon>0$ be given. Then

$$\begin{aligned}&\frac{1}{\lambda_n}\sum_{k\in I_n}\left[M\left(\frac{|t_{km}(x)-L|}{\rho}\right)\right]\\ &\qquad= \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ |t_{km}(x)-L|\geq\varepsilon}} M\left(\frac{|t_{km}(x)-L|}{\rho}\right)\\ &\qquad\quad+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ |t_{km}(x)-L|<\varepsilon}} M\left(\frac{|t_{km}(x)-L|}{\rho}\right)\\ &\qquad\geq \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ |t_{km}(x)-L|\geq\varepsilon}} M\left(\frac{|t_{km}(x)-L|}{\rho}\right)\\ &\qquad\geq \lambda_n^{-1}M\left(\frac{\varepsilon}{\rho}\right)\left|\left\{k\in I_n\colon |t_{km}(x)-L|\geq\varepsilon\right\}\right|.\end{aligned}$$

Hence $x\in\bar{S}_\lambda$. □

Theorem 3.6 *If M is a bounded function which does not satisfy only the condition $M(x)\to\infty$ as $x\to\infty$ in the definition of Orlicz function, then $\bar{S}_\lambda\subset[w,\,\lambda,\,M]$.*

Proof. Suppose that $M(y)\leq K$ for some positive constant K and all $y\geq 0$. Let $\varepsilon>0$ and choose $\delta>0$ such that $M(t)<\varepsilon$ for $0\leq t\leq\delta$. Then we have

$$
\begin{aligned}
&\frac{1}{\lambda_n}\sum_{k\in I_n} M(|t_{km}(x)-L|)\\
=&\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ |t_{km}(x)-L|\geq\delta}} M(|t_{km}(x)-L|)+\frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ |t_{km}(x)-L|<\delta}} M(|t_{km}(x)-L|)\\
\leq&\frac{K}{\lambda_n}\left|\left\{k\in I_n\colon |t_{km}(x)-L|\geq\delta\right\}\right|+\frac{M(\delta)}{\lambda_n}\lambda_n\\
=&\frac{K}{\lambda_n}\left|\left\{k\in I_n\colon |t_{km}(x)-L|\geq\delta\right\}\right|+M(\delta)\\
<&\frac{K}{\lambda_n}\left|\left\{k\in I_n\colon |t_{km}(x)-L|\geq\delta\right\}\right|+\varepsilon
\end{aligned}
$$

Hence $x\in[w,\,\lambda,\,M]$. □

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