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The Weierstrass representation for pluriminimal submanifolds

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Abstract. In this paper a Weierstrass representation formula for pluriminimal submanifolds of the Euclidean space is proposed. Holomorphic 1.0 forms on Kähler varieties are used to globalize the local data. As an application we construct immersions of $\mathbb{C}^2$ in $\mathbb{R}^6$ generalizing the example given by Furuhata. We also show that any affine algebraic variety admits a pluriminimal immersion into some Euclidean space.

Key words: Pluriminimal varieties, Weierstrass representation.

Introduction

The classical methods to describe minimal submanifolds of riemannian manifolds are complex analysis for 2-dimensional domains, and the study of the minimal equation for hypersurfaces. In the intermediate cases these tools do not give a satisfactory description of the picture. It is then natural to restrict the class of minimal submanifolds. For example, when M is a complex manifold of dimension m, (X, g) is a riemannian manifold, following Eschenburg and Tribuzy ([7]) we set:

Definition An immersion $f: M \to X$ is called *pluriminimal* if the restriction to any smooth complex curve in M is a minimal immersion into X.

We remark that if $m = 1$ pluriminimal is equivalent to minimal.

The first problem is to show that this class of submanifolds contains interesting examples.

In this paper we study the case when (X, g) is the Euclidean space. We propose an analogue of the Weierstrass representation for pluriminimal

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maps. As for minimal surfaces, this formula allows to construct many examples, either by explicit calculations, or by using techniques of complex geometry to establish existence results. We give an application of both ways by constructing infinite families of pluriminimal immersions in particular of $\mathbb{C}^2$ into $\mathbb{R}^6$, generalizing the fundamental example found by Furuhata ([8]), where implicitely the representation formula was given.

We also solve a general existence problem in arbitrary dimension, once again drawing analogy with the case of minimal surfaces. In fact we prove that all affine algebraic manifolds (i.e. compact projective minus an ample divisor) admit a pluriminimal nonholomorphic immersion into some Euclidean space. The corresponding result for minimal surface in 3-space has been proved in [13], where it is shown that any compact Riemann surface minus any set of point can be minimally immersed into $\mathbb{R}^3$.

Using the Weierstrass representation we also give a simple proof of the fact that pluriminimal immersions induce a Kähler metric on the domain. Thus these submanifolds can be seen as isometric pluriharmonic immersions of Kähler manifolds, which have been extensively studied by many authors, in particular we refer to the work of Dajczer, Gromoll and Rodriguez ([2], [3], [5], [6]). We underline that this relation holds only for submanifolds of Euclidean spaces. This suggests that pluriminimal immersions have a variational characterization, which greatly enhances interest in their study, and which has been successfully used by many authors to solve rigidity questions for Kähler manifolds, see e.g. Siu ([14]).

Because of these considerations, it seems natural to ask to which extent the analogy with the two dimensional case carries over. In particular, we point out the problem of the extension of Osserman's Theorem ([12]), which states that, if the minimal surface has finite total curvature, the holomorphic 1-forms which appear in the Weierstrass formula extend to meromorphic data on a compact Riemann surface. We believe it would be very interesting to find the geometric hypothesis which allow to compactify the pluriminimal submanifold in such a way that the Weierstrass formula extend to meromorphic data on the compactification.

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1. The Weierstrass formula

Let M be a complex manifold of dimension m, we will denote by J its complex structure. Let $f\colon M \to \mathbb{R}^n$ be a smooth map. We can write:

$$f(Q) = \int_P^Q df + f(P) = \operatorname{Re} \int_P^Q (\omega_1, \ldots, \omega_n) + f(P),$$

where the ω_i are (1, 0)-forms on M, that is they are smooth sections of the complex cotangent bundle Ω_M^1 of M. In local coordinates:

$$\omega_i(z_1, \ldots, z_m) = \sum_{j=1}^{m} \omega_{ij}(z_1, \ldots, z_m) dz_j.$$

The *conformality tensor* will be the section of $\operatorname{Sym}^2 \Omega_M^1$ defined by:

$$\Omega = \sum_{i=1}^{n} \omega_i \otimes \omega_i.$$

The following result characterizes the pluriminimal immersions.

Theorem 1.1 *Let $\omega_1, \ldots, \omega_n$ be (1, 0) smooth forms of M, such that* $\operatorname{Re} \omega_i$ *is exact for every i. Then*

$$f(Q) = \operatorname{Re} \int_P^Q (\omega_1, \ldots, \omega_n) + const, \tag{1}$$

defines a pluriminimal immersion if and only if:
a) *the ω_i are closed holomorphic;*
b) *the conformality tensor vanishes:*

$$\sum_{i=1}^{n} \omega_i \otimes \omega_i = 0; \tag{2}$$

c) *the (complex) jacobian matrix (ω_{ik}) has maximal rank at every point.*

Proof. The classical Weierstrass representation formula for minimal surfaces implies that if the properties *a*), *b*) and *c*) hold, the map f defined in (1) is a pluriminimal immersion.

Conversely, let us first prove that each one of the ω_i is holomorphic: indeed, we know that $\omega_i|_C$ is holomorphic on each holomorphic curve C (see [12]). Chosen $P \in M$, we can find local coordinates $z_1, \ldots, z_m$ such that $z_j(P) = 0$ for every j. Since $\omega_i = \sum_{j=1}^{m} \omega_{ij}(z_1, \ldots, z_m)dz_j$, we can write

$$\bar{\partial}\omega_i = \sum_{j,k=1}^{m} \frac{\partial \omega_{ij}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_j.$$

Restricting this form to each line $z = a\xi$, $a = (a_j) \in \mathbb{C}^n$, $\xi \in \mathbb{C}$, we get

$$\sum_{j,k=1}^{m} \frac{\partial \omega_{ij}}{\partial \bar{z}_k} a_j \bar{a}_k d\bar{t} \wedge dt = 0,$$

where t is a complex coordinate of the curve. This clearly implies $\partial\omega_{ij}/\partial\bar{z}_k = 0$ for any j and k.

Let us now prove that ω_i has to be closed: we know that $\operatorname{Re}\omega_i$ is closed and ω_i is holomorphic. Then $0 = (\partial + \bar{\partial})(\omega_i + \bar{\omega}_i) = \partial\omega_i + \bar{\partial}\bar{\omega}_i$. Since $\partial\omega_i$ is of type (2, 0) and $\bar{\partial}\bar{\omega}_i$ is of type (0, 2) we get $\partial\omega_i = 0$ and $\bar{\partial}\bar{\omega}_i = 0$ which immediately imply $d\omega_i = 0$.

The conformality condition b) follows directly from the fact that given any vector in complexified tangent space $v \in T_{\mathbb{C}}M$, there exists a complex curve with v as tangent vector. On this curve f has to be minimal, which, by the classical Weierstrass representation formula, implies $\Omega(v, v) = 0$.

Condition c) follows by contradiction. Indeed, if $v \in \ker D_{\mathbb{C}}(f)$, where D stands for the jacobian, we can take a complex curve C in M tangent to v. By restricting f to this Riemann surface we get $\omega_i(v) = 0$ for any i, and then $f|_C$ is not an immersion. □

The geometrical meaning of condition c) is given in the following:

Remark 1.2 Let $W = \operatorname{span}_{\mathbb{C}}\{\omega_1, \ldots, \omega_n\}$ be the space generated by the ω_i, and consider the natural map $\lambda\colon \bigwedge^m W \to H^0(M, \Omega_M^m)$, where Ω_M^m is the canonical bundle of M. Then, the immersion property c) holds if and only if the linear system $|\lambda(\bigwedge^m W)|$ is base point free. We note that the associate map $g\colon M \to |\lambda(\bigwedge^m W)|$ is the composition of the (complex) Gauss map with the Plücker embedding. This explains why it is more difficult for $m > 1$ to see the appearance of the Gauss map in the Weierstrass formula.

We also note that $\dim W = \dim \operatorname{span}_{\mathbb{C}}\{\omega_1, \ldots, \omega_n\}$ is an invariant of the pluriminimal immersion.

Definition 1.3 A plurinimimal map $f\colon M \to \mathbb{R}^n$ is full if $\dim W = n$.

Remark 1.4 The notion of *full* is here opposite to *holomorphic*. In fact for holomorphic immersion $f\colon M \to \mathbb{C}^k \equiv \mathbb{R}^{2k}$, s.t. $f(M)$ is not contained in any affine proper subspace, it holds $\dim W = k$, (see 2.2 for further explanations).

The condition $b)$ can be used to give simple proofs of two well-known results:

Proposition 1.5 *The riemannian metric induced by a pluriminimal immersion is Kähler.*

Proof. Let us consider local coordinates $z_k = x_k + iy_k$ on M. We can write

$$\omega_j = \sum_{k=1}^{m} \alpha_{jk} dx_k - \beta_{jk} dy_k + i\left(\sum_{l=1}^{m} \alpha_{jl} dy_l + \beta_{jl} dx_l\right),$$

$j = 1, \ldots, n$, and in matrix form

$$\underline{\omega} = A\underline{x} - B\underline{y} + i(A\underline{y} + B\underline{x}). \tag{3}$$

Therefore, having set g the pull-back of the Euclidean metric, we can write the matrix associated to g as

$$\begin{pmatrix} A & -B \end{pmatrix} \begin{pmatrix} A^t \\ -B^t \end{pmatrix} = \begin{pmatrix} AA^t & -AB^t \\ -BA^t & BB^t \end{pmatrix}.$$

The vanishing of the tensor $\Omega = \sum_{i=1}^{n} \omega_i \otimes \omega_i = 0$ gives, by equation (3), the following system of equations

$$-AB^t = -BA^t = 0, \quad AA^t = BB^t.$$

Therefore the matrix associated to g is hermitian and its associated form can be written, as in the classical case of minimal surfaces, as $\sum_{r=1}^{n} \omega_r \wedge \bar{\omega}_r$ which is clearly positive definite and of type (1, 1). It is also closed since each ω_r is closed. □

The above proposition is crucial to link our definition and more standard notions in the theory of higher dimensional submanifolds of Euclidean spaces. In particular let us observe that pluriminimal immersions are part of a broader class of submanifolds studied in general by many authors (e.g. [2], [3], [5], [6]).

Proposition 1.6 *The second fundamental form B of a pluriminimal immersion satisfies*

$$B(X,\, JY) = B(JX,\, Y),$$

i.e. a pluriminimal immersion is circular.

Proof. Since the induced metric is Kähler, at every point of M we can choose an orthonormal basis for the tangent space of the form $\{e_1, \ldots, e_m, Je_1, \ldots, Je_m\}$. Since the map restricted to every holomorphic direction has to be minimal, $B(e_j, e_j)+B(Je_j, Je_j) = 0$ for any $j = 1, \ldots, m$. Therefore

$$\begin{aligned} &B(e_j + e_k,\, e_j + e_k) + B\big(J(e_j + e_k),\, J(e_j + e_k)\big) \\ &= 2\big(B(e_j,\, e_k) + B(Je_j,\, Je_k)\big) = 0 \end{aligned}$$

Thus, $(B) = -(J^tBJ)$, which implies directly the conclusion. □

2. Constructions of pluriminimal immersions

We look for holomorphic functions of two complex variables x and y, whose differentials satisfy the quadratic relation

$$dP_1 \otimes dP_2 + dP_3 \otimes dP_4 + dP_5 \otimes dP_6 = 0. \tag{4}$$

By a diagonalization process on the above tensor, in such a way that the condition $b)$ of the Theorem 1.1 is satisfied, we can write the map f defined in (1) as:

$$(x,\, y) \mapsto \begin{pmatrix} \mathrm{Re}(P_1 + P_2) \\ \mathrm{Im}(P_2 - P_1) \\ \mathrm{Re}(P_3 + P_4) \\ \mathrm{Im}(P_4 - P_3) \\ \mathrm{Re}(P_5 + P_6) \\ \mathrm{Im}(P_6 - P_5) \end{pmatrix} \tag{5}$$

Let $W = \mathrm{span}_{\mathbb{C}}[dP_1, \ldots, dP_6]$; if W satisfies the condition $c)$, we can choose, in local coordinates, $P_3 = x$ and $P_5 = y$. Moreover, we set $P_1(x, y) = xy$. Then the equation (4) translates into the system:

$$\begin{cases} y(P_2)_x + (P_4)_x = 0 \\ x(P_2)_y + (P_6)_y = 0 \\ y(P_2)_y + x(P_2)_x + (P_4)_y + (P_6)_x = 0 \end{cases} \tag{6}$$

A simple calculation shows that the solutions of (6) are of the following form:

$$P_2(x,\, y) = \frac{g'(x) + f'(y)}{2}$$
$$P_4(x,\, y) = f(y) - y\frac{g'(x) + f'(y)}{2}$$
$$P_6(x,\, y) = g(x) - x\frac{g'(x) + f'(y)}{2}$$

where f and g are arbitrary holomorphic functions of one complex variable.

We observe that by taking f, g entire functions we obtain pluriminimal immersions of $\mathbb{C}^2$ in $\mathbb{R}^6$. Moreover when the quadratic relation satisfied by the dP_i is of rank greater than zero, the pluriminimal immersions constructed cannot be holomorphic w.r.t. any complex structure of R^6 compatible with the standard euclidean metric. This will be shown in general in Remark 2.2.

The only example of this sort known to us is due to Furuhata ([8]), and belongs to our class for $f = x^3$ and $g = 0$, up to a real constant. We remark that the Furuhata example is (see 1.3) *full*. Nevertheless by direct computation it is possible to show that Furuhata's example is not an embedding, i.e. the map is not injective. We believe this should be true for all such maps.

Conjecture Any complete pluriminimal full immersion from $\mathbb{C}^2$ to $\mathbb{R}^6$ is not an embedding.

It is clear that with a similar procedure, choosing meromorphic or algebraic functions, we could construct families of pluriminimal immersions of more complicated domains. For other examples, see also [4].

We underline that the Weierstrass representation theorem also allows to prove general existence theorems, without explicitly finding the holomorphic differentials, in total analogy with the theory of minimal surfaces in $\mathbb{R}^3$.

We now prove that every affine algebraic manifold admits many pluriminimal embeddings into Euclidean spaces. Let us then start with a smooth projective manifold, X, of complex dimension m, and let H be a hyperplane section. Set $M = X \setminus H$. Let $H^0(X,\, \mathcal{O}_X(nH))$ (and $H^0(X,\, \Omega^1_X((n+1)H))$) be the vector space of meromorphic functions (and respectively ((1, 0)-forms) on X, holomorphic on M and having poles on H of order at most n. By Hirzebruch-Riemann-Roch' s Theorem (see, for example, [10] and [11,

p. 432]) and vanishing Theorem B (see [9, p. 159]) we know that, for large n:

$$\dim H^0(X, \mathcal{O}_X(nH)) = \frac{H^m}{m!} n^m + P(n),$$

where P is a polynomial of degree at most $m-1$ in the variable n. In order to construct holomorphic (1, 0)-forms on the manifold $M = X \setminus H$, we consider the image V of the exterior differential $d\colon H^0(X, \mathcal{O}_X(nH)) \to H^0(X, \Omega^1_X((n+1)H))$.

By restriction, the cup product map

$$\mu_n\colon \operatorname{Sym}^2 H^0(X, \Omega^1_X((n+1)H)) \to H^0(X, \operatorname{Sym}^2 \Omega^1_X(2(n+1)H))$$

defines an application

$$\mu'_n\colon \operatorname{Sym}^2 V \to H^0(X, \operatorname{Sym}^2 \Omega^1_X(2(n+1)H)).$$

Any element in the kernel of μ'_n represents a quadratic relation satisfied by holomorphic (1, 0)-forms, and therefore we can diagonalize the tensor in order to satisfy the condition *b*) in the Theorem 1.1.

We now show that $\ker(\mu'_n)$ is not trivial for n sufficiently large. Indeed, applying to the vector bundle $\operatorname{Sym}^2 \Omega^1_X(2(n+1)H)$ Hirzebruch-Riemann-Roch's Theorem and the vanishing Theorem B quoted above, for large n we get:

$$\begin{aligned}&\dim H^0(X, \operatorname{Sym}^2 \Omega^1_X((n+1)H)) \\ &= \left(\frac{m(m+1)}{2}\right) \frac{H^m}{m!}(2n+2)^m + Q(n),\end{aligned}$$

Q being a polynomial of degree at most $m-1$. In fact the leading term in the formula depends only on the rank of the vector bundles, i.e. $m(m+1)/2$ and on the maximal intersection of the first Chern class: $((2n+2)H)^m$. Since $\dim \operatorname{Sym}^2 V$ grows as n^{2m}, the map μ'_n has nontrivial kernel for n large enough.

At this point we can construct a pluriminimal map by associating to a nontrivial element γ of $\ker \mu'_n$ a set of independent exact (1, 0)-forms $dF_1, \ldots, dF_k$, where k is the rank of γ, satisfying $\sum_{j=1}^k dF_j \otimes dF_j = 0$. Then, the map $\phi\colon X \setminus H \to \mathbb{R}^k$ defined by

$$\phi(p) = \operatorname{Re}(F_1, \ldots, F_k) + \text{const}$$

is a pluriminimal map.

It is immediate to check that for n large enough one can find F_i s.t. ϕ is an embedding. In fact if $\phi\colon M \to \mathbb{R}^n$ and $\psi\colon M \to \mathbb{R}^k$ are pluriminimal map then $(\phi, \psi)\colon M \to \mathbb{R}^{k+n}$ is a pluriminimal map. If $\psi\colon M \to \mathbb{R}^{2s} = \mathbb{C}^s$ is an holomorphic embedding into an affine space and ψ is any pluriminimal map, we get new pluriminimal embeddings. Because of the open nature of the embedding condition we remark that a small deformation of (ϕ, ψ) is still an embedding. Let $c(n)$ be the dimension of the variety which parameterises the pluriminimal maps, that is the *number moduli*; the previous estimates prove the following:

Theorem 2.1 *Let X an affine algebraic variety of dimension m; then we can find an integer $k(m)$ such that for every $n \geq k(m)$ there exist pluriminimal embeddings*

$$\phi\colon X \to \mathbb{R}^n$$

such that

1. *the Gauss map g defined in Remark* 1.2 *is algebraic;*
2. *the moduli number c_n satisfies*

$$c_n \geq O(n^{2m})_{n\to\infty}.$$

Remark 2.2 The fact that the $\ker \mu'_n$ contains nontrivial elements implies the existence of a non holomorphic map ϕ w.r.t. any complex structure compatible with the standard euclidean metric. The proof is the same as in the case $m = 1$ (e.g. see [1, Lemma 2.2]). We may assume $k = 2n$. Suppose now that ϕ is $\tilde{J}$-holomorphic, where $\tilde{J}$ is a complex structure on $\mathbb{R}^{2n}$, but the rank of the quadratic relation of the dF_i is not zero, i.e. $\gamma = \sum_i dF_i \otimes dF_i \in \ker \mu'_n$, $\gamma \neq 0$. Remark that $\gamma \neq 0$ when ϕ is full (see 1.3). We have $\tilde{J} = CJ_0C^{-1}$ for some orthogonal matrix C, where J_0 is the standard complex structure on $\mathbb{R}^{2n}$. Denoting by J the complex structure on M, the holomorphicity condition means that $d\phi(Jv) = CJ_0C^{-1}d\phi(v)$, and therefore $C^{-1}\phi$ is J_0-holomorphic. It is clear that any J_0-holomorphic map is of the form $\operatorname{Re}\int_p^q(\omega_1, \ldots, \omega_n, -i\omega_1, \ldots, -i\omega_n)$, and therefore the holomorphic differentials involved satisfy a quadratic relation of rank zero. If this was true the same would be true for ϕ, i.e. $\gamma = 0$, contradicting our hypothesis. Hence, the map ϕ cannot be holomorphic as claimed.

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