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On Feit's definition of the Schur index

Joujuu (Zyozyu) OHMORI

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Abstract. We show that the definition of the Schur index which was given by W. Feit in the book [F1]: Characters of finite Groups, Benjamin, 1967, is well-defined.

Key words: Schur index, unipotent representation, motive, crystalline cohomology.

Introduction

In his arithmetic study of complex representations of finite groups ([Sch]), I. Schur introduced the notion of the index:

Definition 1 (see Curtis and Reiner [CR1], (70.4), p. 406; also see the books of Dornhoff, Huppert, Isaacs, etc, on the representation theory of finite groups) Let G be a finite group. Let K be a field of characteristic 0 and let K^* be an algebraically closed field-extension of K. Let $U: G \to \mathrm{GL}(d, K^*)$ be an absolutely irreducible matrix representation of G over K^* with character χ. Let $K(\chi) = K(\chi(g) \mid g \in G)$. Then the positive integer

$$m_K(U) = m_K(\chi) = \min[L : K(\chi)]$$

where the minimum is taken over all the subfields L of K^* such that U is realizable in L will be called the Schur index of U (or of χ) with respect to K.

On the other hand, the definition of the Schur indices given by W. Feit in his book [F1] is slightly different:

Definition 2 Let G be a finite group. Let χ be an absolutely irreducible character of G over some field K of characteristic 0. Let F be any field of characteristic 0. Then $F(\chi)$ is the field generated over F by the values of χ and $m_F(\chi)$ is defined to be the smallest positive integer such that $m_F(\chi)\chi$ is afforded by a matrix representation of G over $F(\chi)$. (See [F1], p. 10, lines 16-9; p. 13 lines 22-5; p. 61, lines 20-2.)

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In this paper, we shall prove the following theorem:

Theorem 1 *Let G be a finite group. Let χ be an absolutely irreducible character of G over some field K of characteristic* 0. *Let F be any field of characteristic* 0. *Let $F^{*^{(1)}}$, $F^{*^{(2)}}$ be any two (sufficiently large) field-extensions of K and let $\sigma_1 \colon K \to F^{*^{(1)}}$, $\sigma_2 \colon K \to F^{*^{(2)}}$ be any embeddings. Then $F(\sigma_1 \circ \chi)$ and $F(\sigma_2 \circ \chi)$ are isomorphic over F and we have $m_F(\sigma_1 \circ \chi) = m_F(\sigma_2 \circ \chi)$ (in the sense of Definition* 1).

Clearly, Theorem 1 justifies Feit's definition of $F(\chi)$ and $m_F(\chi)$.

Historically, Benard [B] proved the following theorem:

Theorem 2 (Benard [B], Theorem 1) *Let k be a subfield of a cyclotomic extension of $\mathbb{Q}$. Let A be a finite-dimensional central simple algebra over k which is similar to a simple direct summand of the group algebra $k[G]$ of a finite group G over k. Let p be any rational prime (possibly $p = \infty$). Let P_1, P_2 be any two primes of k lying above p. Let k_{P_1} (resp. k_{P_2}) be the completion of k at P_1 (resp. P_2). Then $k_{P_1} \otimes_k A$ and $k_{P_2} \otimes_k A$ have the same index.*

We shall show that the second assertion of Theorem 1 for $F = \mathbb{Q}_p$ is equivalent to Theorem 2. Thus, in particular, for any complex irreducible character χ of a finite group G and for any rational prime p, we can consider the p-local Schur index $m_{\mathbb{Q}_p}(\chi)$ of χ with respect to $\mathbb{Q}_p$.

Such a recongnition can be clearly seen in Feit's paper [F2] (see [F2], p. 278, lines 1-7). It seems that it can be also seen in the book of Curtis and Reiner [CR2] (see [CR2], p. 750, lines 13-5). But there seems to exist some confusions in the argument of [CR2], p. 750, lines21-. In fact, in this book, they are considering, for any complex irreducible character χ of G, $K(\chi)$ and $m_K(\chi)$ for any field K of characteristic 0. This would make the arguments about Theorem (74.26) of [CR2] meaningless.

The direct motivation for writing this paper arose in the study of the rationality-properties of the unipotent representations of finite gropus G^F of Lie type (as to G^F, see Theorem 3 below) ([L1], [O1, O2]; also cf. [L3], [Ge1, Ge2, Ge3]).

Let G^F be a finite group of Lie type where p is the defining characteristic of G^F. Then, by the definition (see Deligne and Lusztig [DL]), a unipotent representation ρ of G^F is an absolutely irreducible submodule of a certain $\mathbb{Q}_\ell[G^F]$-module $H_c^i(X(w), \mathbb{Q}_\ell)$ where ℓ is any fixed prime number different from p, and, via a fixed ispmorphism $\overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \mathbb{C}$ (where $\overline{\mathbb{Q}_\ell}$ is an algebraic closure of $\mathbb{Q}_\ell$), we consider ρ as a complex irreducible representation of G^F. Furthermore, by using a property of the Schur index, we conclude that $m_{\mathbb{Q}_\ell}(\rho)$ divides the multiplicity $\big(\rho, H_c^i(X(w), \mathbb{Q}_\ell)\big)_{G^F}$.

One of the purpose of this paper is to make such an argument clear.

In this connection, we can state the following theorem:

Theorem 3 (cf. Lusztig [L3]) *Let p be a prime number and let k be an algebraic closure of the prime field of characteristic p. Let G be a connected, reductive linear algebraic group over k, let $F\colon G \to G$ be a surjective endomorphism of G such that some power of F is the Frobenius endomorphism of G corresponding to a rational structure on G over a finite subfield of k and let G^F be the (finite) group of F-fixed points of G. Then, for any (complex) unipotent representation ρ of G^F, we have $m_{\mathbb{Q}_\ell}(\rho) = 1$ for any prime number $\ell \neq p$.*

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1. Benard's theorem

Let G be a finite group. For an absolutely irreducible character ζ of G over a field L of characteristic 0 and for a subfield k of L, we denote by $A(\zeta, k)$ the simple direct summand of $k[G]$ associated with ζ: if ζ is extended linearly to a character of $k[G]$, then $A(\zeta, k)$ can be characterized as the unique simple direct summand of $k[G]$ such that $\zeta\big(A(\zeta, k)\big) \neq \{0\}$. In this case, for each $\sigma \in \mathrm{Gal}\big(k(\zeta)/k\big)$, if we let

$$e(\zeta^\sigma) = \frac{\zeta^\sigma(1)}{|G|} \sum_{g\in G} \zeta^\sigma(g^{-1})g \quad (\in L[G]),$$

and, if we let

$$a(\zeta) = \sum_{\sigma \in \mathrm{Gal}\left(k(\zeta)/k\right)} e(\zeta^{\sigma}),$$

then we have $A(\zeta, k) = k[G]a(\zeta)$ (see Yamada [Y], Proposition 1.1).

Let K^* be an algebraically closed field of characteristic 0, let χ be an absolutely irreducible character of G over K^* and let $k = \mathbb{Q}(\chi)$. In this section we show that the second assertion of Theorem 1 for $k = \mathbb{Q}_p$ is equivalent to Theorem 2.

We first show that Theorem 2 implies the second assertion of Theorem 1. In fact, let p be any rational prime (possibly $p = \infty$), and let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ at p. Thus, if $p = \infty$, then $\mathbb{Q}_p = \mathbb{R}$, and if p is finite, then $\mathbb{Q}_p$ is the quotient field of proj-lim $\mathbb{Z}/p^n\mathbb{Z}$. Let $\overline{\mathbb{Q}_p}$, $\overline{\mathbb{Q}_p}'$ be any two algebraic closure of $\mathbb{Q}_p$, and let $\sigma\colon k \to \overline{\mathbb{Q}_p}$, $\sigma'\colon k \to \overline{\mathbb{Q}_p}'$ be any embeddings of k. Let $F = \mathbb{Q}_p \cdot \sigma(k) = \mathbb{Q}_p(\sigma \circ \chi)$ and $F' = \mathbb{Q}_p \cdot \sigma'(k) = \mathbb{Q}_p(\sigma' \circ \chi)$. Then F and F' are finite algebraic extensions of $\mathbb{Q}_p$, so that they are local fields in the sense of Weil [W], p. 20, lines 31-2. We see that $\sigma(k)$ (resp. $\sigma'(k)$) is dense in F (resp. F'). In fact, let F_0 be the topological closure of $\sigma(k)$ if F. Then, since $\sigma(k)$ contains $\mathbb{Q}$, the topological closure $\mathbb{Q}_p$ of $\mathbb{Q}$ in F is containd in F_0. Thus, since F_0 contains both of $\sigma(k)$ and $\mathbb{Q}_p$, we must have $F_0 = F$. So the embedding $\sigma\colon k \to F$ (resp. $\sigma'\colon k \to F'$) defines a place v (resp. v') of k (see [W], Definition 2, pp. 43-4). The place v (resp. v') determines a prime P (resp. P') of k. Here a prime of k means an equivalent class of valuations on k. (See [W], p. 44, lines 6-14.) The completion k_P (resp. $k_{P'}$) of k at P (resp. P') is nothing but the completion k_v (resp. $k_{v'}$) of k at v (resp. v'). Therefore σ (resp. σ') can be extended uniquely to a topological-field isomorphism σ_v (resp. $\sigma'_{v'}$) from $k_v = k_P$ (resp. $k_{v'} = k_{P'}$) onto F (resp. F'). Let $\overline{k_v}$ (resp. $k_{v'}$) be an algebraic closure of k_v (resp. $k_{v'}$). Then σ_v (resp. $\sigma'_{v'}$) can be extended to an isomorphism $\overline{\sigma_v}$ (resp. $\overline{\sigma'_{v'}}$) from $\overline{k_v}$ (resp. $\overline{k_{v'}}$) onto $\overline{\mathbb{Q}_p}$ (resp. $\overline{\mathbb{Q}_p}'$)

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in K^* (see Theorem 1). Then, since $k \subset \overline{\mathbb{Q}}$, there is an embedding τ (resp. τ') over k of $\overline{\mathbb{Q}}$ into $\overline{k_v}$ (resp. $\overline{k_{v'}}$). Let $U\colon G \to \mathrm{GL}(d, \overline{\mathbb{Q}})$ be a matrix representation of G over $\overline{\mathbb{Q}}$ with character χ. Let $U^{\tau}\colon G \to \mathrm{GL}(d, \overline{k_v})$ (resp. $U^{\tau'}\colon G \to \mathrm{GL}(d, \overline{k_{v'}})$) be the matrix representation of G over $\overline{k_v}$ (resp. $\overline{k_{v'}}$) defined by $U^{\tau}(g) = [\tau(r_{ij}(g))]$ (resp. $U^{\tau'}(g) = [\tau'(r_{ij}(g))]$) if $U(g) = [r_{ij}(g)]$ for $g \in G$. Then U^{τ} and $U^{\tau'}$ have the character χ. Thus one can consider $m_{k_v}(\chi)$ and $m_{k_{v'}}(\chi)$ (see Definition 1).

Let $A = A(\chi, k)$. Let $\phi\colon k_v \otimes_k k[G] \to k_v[G]$ be the canonical isomorphism over k_v. We have $A = k[G]e(\chi)$ and A contains $1 \cdot e(\chi) = e(\chi)$. Let $A_v = k_v \otimes_k A$, which we consider as a subalgebra of $k_v \otimes_k k[G]$. Then $\phi(1 \otimes e(\chi)) = 1 \cdot e(\chi) \in k_v[G]e(\chi) = A(\chi, k_v)$, so $\phi(A_v) = A(\chi, k_v)$. Thus A_v and $A(\chi, k_v)$ are isomorphic over k_v, so that $m_{k_v}(\chi)$ is equal to the index of A_v. Similarly, $m_{k_{v'}}(\chi)$ is equal to the index of $A_{v'} = k_{v'} \otimes_k A$. Therefore, by Theorem 2, we have $m_{k_v}(\chi) = m_{k_{v'}}(\chi)$.

Now, if $V\colon G \to \mathrm{GL}(d, \overline{k_v})$ is a matrix representation of G over $\overline{k_v}$ with character χ, then $V^{\overline{\sigma_v}}\colon G \to \mathrm{GL}(d, \overline{\mathbb{Q}_p})$ is a matrix representation of G over $\overline{k_v}$ with character $\sigma \circ \chi$. And $\overline{\sigma_v}(k_v(\chi)) = \overline{\sigma_v}(k_v) = \sigma_v(k_v) = F = F(\sigma \circ \chi)$. If L is any subfield of $\overline{k_v}$ such that V is realizable in L, then $M = \overline{\sigma_v}(L)$ is a subfield of $\overline{\mathbb{Q}_p}$ such that $V^{\overline{\sigma_v}}$ is realizable in M, and $[L : k_v] = [M : F]$. Conversely, if M is any subfield of $\overline{\mathbb{Q}_p}$ such that $V^{\overline{\sigma_v}}$ is realizable in M, then $L = \overline{\sigma_v}^{-1}(M)$ is a subfield of $\overline{k_v}$ such that V is realizable in L, and $[M : F] = [L : k_v]$. Therefore, by Definition 1, we see that $m_{k_v}(\chi) = m_F(\sigma \circ \chi)$. Similarly, we have $m_{k_{v'}}(\chi) = m_{F'}(\sigma' \circ \chi)$. Thus we have:

$$\begin{aligned} m_{\mathbb{Q}_p}(\sigma \circ \chi) &= m_{\mathbb{Q}_p}(\sigma \circ \chi) = m_F(\sigma \circ \chi) = m_{k_v}(\chi) = m_{k_{v'}}(\chi) \\ &= m_{F'}(\sigma' \circ \chi) = m_{\mathbb{Q}_p(\sigma' \circ \chi)}(\sigma' \circ \chi) = m_{\mathbb{Q}_p}(\sigma' \circ \chi). \end{aligned}$$

This proves the second assertion of Theorem 1. □

Remark By the above result, we see that the value $m_{\mathbb{Q}_p}(\sigma \circ \chi)$ is independent of the choice of $\overline{\mathbb{Q}_p}$ and an embedding $\sigma\colon k \to \overline{\mathbb{Q}_p}$ (cf. Curtis and Reiner [CR2], p. 750, lines 13-5).

We next show that the second assertion of Theorem 1 implies Theorem 2. Let p be any rational prime, and let P, Q be any two primes of $k = \mathbb{Q}(\chi)$ lying above p. Let $\mathbb{Q}_p^{(P)}$ (resp. $\mathbb{Q}_p^{(Q)}$) be the topological closure of $\mathbb{Q}$ in k_P (resp. k_Q). Let v_P (resp. v_Q) be a valuation of k_P (resp. k_Q) whose restriction to k belongs to P (resp. Q). Then the restriction of v_P (resp. v_Q) to $\mathbb{Q}$ is equivalent to a p-adic valuation v_p of $\mathbb{Q}$. For $x \in \mathbb{Q}_p$, taking a Cauchy sequence (a_n) in $\mathbb{Q}$ such that $x = v_p\text{-}\lim a_n$, we set, by noting that (a_n) is also a Cauchy sequence in k with respect to v_P (resp. v_Q), $\sigma_P(x) = v_P\text{-}\lim a_n$ in k_P (resp. $\sigma_Q(x) = v_Q\text{-}\lim a_n$ in k_Q), which, as one can easily check, is independent of the choice of (a_n). Then σ_P (resp. σ_Q) is a topological-field isomorphism from Q_p onto $\mathbb{Q}_p^{(P)}$ (resp. $\mathbb{Q}_p^{(Q)}$). Thus there is an embedding τ_P (resp. τ_Q) of $k_P = \mathbb{Q}_p^{(P)}(\chi)$ (resp. $k_Q = \mathbb{Q}_p^{(Q)}(\chi)$) into $\overline{\mathbb{Q}_p}$ such that $\tau_P|\mathbb{Q}_p^{(P)} = \sigma_P^{-1}$ (resp. $\tau_Q|\mathbb{Q}_p^{(Q)} = \sigma_Q^{-1}$). We have $\tau_P(k_P) = \mathbb{Q}_p(\tau_P \circ \chi)$ and $\tau_Q(k_Q) = \mathbb{Q}_p(\tau_Q \circ \chi)$. By Theorem 1, we have $m_{\mathbb{Q}_p}(\tau_p \circ \chi) = m_{\mathbb{Q}_p}(\tau_Q \circ \chi)$. Let $A = A(\chi, k)$. Then $m_{k_P}(\chi)$ (resp. $m_{k_Q}(\chi)$) is equal to the index m_P (resp. m_Q) of $A(\chi, k_P) = k_P \otimes_k A$ (resp. $A(\chi, k_Q) = k_Q \otimes_k A$). τ_P (resp. τ_Q) can be extended to an isomorphism $\overline{\tau_P}$ (resp. $\overline{\tau_Q}$) from an algebraic closure $\overline{k_P}$ of k_P (resp. $\overline{k_Q}$ of k_Q) onto $\overline{\mathbb{Q}_p}$, and $\overline{\tau_P}(k_P) = \mathbb{Q}_p(\tau_P \circ \chi)$ (resp. $\overline{\tau_Q}(k_Q) = \mathbb{Q}_p(\tau_Q \circ \chi)$). Therefore we have

$$m_{k_P}(\chi) = m_{\mathbb{Q}_p(\tau_P \circ \chi)}(\tau_P \circ \chi) = m_{\mathbb{Q}_p}(\tau_P \circ \chi)$$

and

$$m_{k_Q}(\chi) = m_{\mathbb{Q}_p(\tau_Q \circ \chi)}(\tau_Q \circ \chi) = m_{\mathbb{Q}_p}(\tau_Q \circ \chi).$$

Thus:

$$m_P = m_{k_P}(\chi) = m_{\mathbb{Q}_p}(\tau_P \circ \chi) = m_{\mathbb{Q}_p}(\tau_Q \circ \chi) = m_{k_Q}(\chi) = m_Q.$$

This proves Theorem 2. □

2. The Brauer-Witt Theorem

Let k be a field of characteristic 0 and $\overline{k}$ an algebraic closure of k. Let ζ be a primitive n-th root of unity in $\overline{k}$ where n is some positive integer, and let Γ be the Galois group $\mathrm{Gal}(k(\zeta)/k)$ of $k(\zeta)$ over k. Let $W(k(\zeta))$ be the group of roots of unity in $k(\zeta)$. Let $\beta\colon \Gamma \times \Gamma \to W(k(\zeta))$ be a factor set 2-cocycle of Γ with values in $W(k(\zeta))$. Let $(\beta, K(\zeta)/k)$ be the crossed product algebra over k associated with β:

$$(\beta, k(\zeta)/k) = \sum_{\sigma \in \Gamma} k(\zeta) u_\sigma \quad \text{(direct sum)},$$

$$u_\sigma u_\tau = \beta(\sigma, \tau) u_{\sigma\tau} \quad (\sigma, \tau \in \Gamma) \quad u_\sigma x = x^\sigma u_\sigma \quad (\sigma \in \Gamma,\ x \in k(\zeta))$$

Such an algebra over k is called a cyclotomic algebra over k (see Yamada [Y]).

Let $\mathrm{Br}(L)$ denote the Brauer group of a field L. If A is a finite-dimensional, central simple algebra over a field L, then $[A]$ denotes the class of A in the Brauer group $\mathrm{Br}(L)$.

Let L be a field of characteristic 0. Then we say that a finite group H is L-elementary with respect to a prime p if the following two conditions are satisfied:

(i) H can be expressed as a semidirect product CP where C is a cyclic normal subgroup of H whose order is relatively prime to p and P is a p-group.

(ii) Let c be a generator of the cyclic group C, of order m, and let ζ_m be a primitive m-th root of unity in some algebraic closure of L. If c^i and c^j are conjugate in H, then there exists an automorphism σ in $\mathrm{Gal}\big(L(\zeta_m)/L\big)$ such that $\sigma(\zeta_m^i) = \zeta_m^j$.

We quote from [Y], p. 31, the following:

The Brauer-Witt Theorem. *Let k be a field of characteristic 0 and $\overline{k}$ an algebraic closure of k. Let G be a finite group of exponent n and let χ be an absolutely irreducible character of G over $\overline{k}$ such that $k(\chi) = k$. Let ζ_n be a primitive n-th root of unity in $\overline{k}$. Let p be a prime number.*

(I) *Let L be the subfield of $k(\zeta_n)$ over k such that $[k(\zeta_n) : L]$ is a power of p and $[L : k] \not\equiv 0 \pmod{p}$. Then there is subgroup H of G which is L-elementary with respect to p and an absolutely irreducible character θ of H over $\overline{k}$ such that $L(\theta) = L$ and that $(\chi|H, \theta)_H \neq 0$ and the following statement* (II) *holds.*

(II) *There is a normal subgroup N of H and a linear character ψ of N over $\overline{k}$ such that* (i) *$\theta = \psi^H$, the character of H induced by ψ,* (ii) *for each $h \in H$, there exists $\tau(h) \in \mathrm{Gal}\big(L(\psi)/L\big)$ such that $\psi^h = \psi^{\tau(h)}$ ($\psi^h(x) = \psi(hxh^{-1})$, $x \in N$), and, by the mapping $h \to \tau(h)$, $H/N \cong \mathrm{Gal}\big(L(\psi)/L\big)$,* (iii) *$A(\theta, L)$ is isomorphic over L to the cyclotomic algebra $\big(\beta, L(\psi)/L\big)$ over L, where, if T is a set of complete system of representatives of N in H ($1 \in T$) with $hh' = x(h, h')h''$ for $h, h', h'' \in T$, $x(h, h') \in N$, then $\beta\big(\tau(h), \tau(h')\big) = \psi\big(x(h, h')\big)$.*

(III) *If the notation is as in* (II), *then we have $[A(\chi, L)] = [A(\theta, L)] = [(\beta, L(\psi)/L)]$ in $\mathrm{Br}(L)$ and the p-part of $m_k(\chi)$ is equal to $m_L(\theta)$.*

Now let K be a field of characteristic 0 and $\overline{K}$ an algebraic closure of K. Let G be a finite group of exponent n and let χ be an absolutely irreducible character of G over $\overline{K}$. Let α be any automorphism of $\mathbb{Q}(\chi)$ ($\subset \overline{K}$).

Proposition 1 *We have $m_K(\chi) = m_K(\chi^\alpha)$.*

Remark Since χ and χ^α are algebraically conjugate over $\mathbb{Q}$, we have $m_{\mathbb{Q}}(\chi) = m_{\mathbb{Q}}(\chi^\alpha)$. But, over K, χ and χ^α are not necessarily algebraically conjugate, so it is not clear whether $m_K(\chi) = m_K(\chi^\alpha)$ or not.

The following proof of Proposition 1, which we shall use the Brauer-Witt theorem, was inspired by the argument in the proof of Theorem 1 of [B].

Set $k = K(\chi)$ and let $\overline{k} = \overline{K}$. Let ζ_n be primitive n-th root of unity in $\overline{k}$. Let the notation be as in the Brauer-Witt theorem. Let p be any prime number. Then, by that theorem, we see that the p-part of $m_k(\chi) = m_K(\chi)$ is equal to the index of $\big(\beta,\, L(\psi)/L\big)$.

Let $\tilde{\alpha}$ be an extension of α to an automorphism of $\mathbb{Q}(\zeta_n)$. Then, in the statement (I) of the Brauer-Witt theorem, we have $L(\theta^{\tilde{\alpha}}) = L(\theta) = L$ (since $\mathbb{Q}(\theta^{\tilde{\alpha}}) = \mathbb{Q}(\theta)$), and $(\chi^{\tilde{\alpha}}|H,\, \theta^{\tilde{\alpha}})_H = (\chi|H,\, \theta)_H^{\tilde{\alpha}} = (\chi|H,\, \theta)_H \not\equiv 0 \pmod p$. In the statement (II) of that theorem, we clearly have $\theta^{\tilde{\alpha}} = (\psi^H)^{\tilde{\alpha}} = (\psi^{\tilde{\alpha}})^H$. Let $h \in H$, and suppose that $\psi^h = \psi^{\tau(h)}$, $\tau(h) \in \mathrm{Gal}\big(L(\psi)/L\big)$. Let $\tilde{\tau}$ be an extension to $\mathbb{Q}(\zeta_n)$ of the restriction of $\tau(h)$ to $\mathbb{Q}(\psi)$. Then, since $\mathrm{Gal}\big(\mathbb{Q}(\zeta_n)/\mathbb{Q}\big)$ is an abelian group, we have $\tilde{\tau}\tilde{\alpha} = \tilde{\alpha}\tilde{\tau}$. From this we easily see that $(\psi^{\tilde{\alpha}})^h = (\psi^{\tilde{\alpha}})^{\tau(h)}$. Hence, by Proposition 3.5 of [Y], we have $A(\theta^{\tilde{\alpha}},\, L) = \big(\beta^{\tilde{\alpha}},\, L(\psi^{\tilde{\alpha}})/L\big) = \big(\beta^{\tilde{\alpha}},\, L(\psi)/L\big)$. Thus, by the statement (III) of the Brauer-Witt theorem, we see that the p-part of $m_k(\chi^{\alpha}) = m_K(\chi^{\alpha})$ is equal to the index of $\big(\beta^{\tilde{\alpha}},\, L(\psi^{\tilde{\alpha}})/L\big) = \big(\beta^{\tilde{\alpha}},\, L(\psi)/L\big)$.

Let ζ' be a root of unity in $\mathbb{Q}(\zeta_n)$ such that whose order n' is maximum. Then we have $\mathbb{Q}(\zeta_n) = \mathbb{Q}(\zeta')$ and $\mathrm{Gal}\big(\mathbb{Q}(\zeta')/\mathbb{Q}\big) \cong (\mathbb{Z}/n'\mathbb{Z})^{\times}$. So there exist positive integers a, b, which are relatively prime to n', such that $(\zeta')^{\tilde{\alpha}} = (\zeta')^a$ and $(\zeta')^{\tilde{\alpha}^{-1}} = (\zeta')^b$. Therefore, since the values of ψ are roots of unity in $\mathbb{Q}(\zeta')$, we have $\psi^{\tilde{\alpha}} = \psi^a$ and $\psi = (\psi^{\tilde{\alpha}})^b$, so that we have $\beta^{\tilde{\alpha}} = \beta^a$ and $\beta = (\beta^{\tilde{\alpha}})^b$. Thus

$$\big[\big(\beta^{\tilde{\alpha}},\, L(\psi)/L\big)\big] = \big[\big(\beta^{a},\, L(\psi)/L\big)\big] = \big[\big(\beta,\, L(\psi)/L\big)\big]^a$$

and

$$\big[\big(\beta,\, L(\psi)/L\big)\big] = \big[\big((\beta^{\tilde{\alpha}})^b,\, L(\psi)/L\big)\big] = \big[\big(\beta^{\tilde{\alpha}},\, L(\psi)/L\big)\big]^b.$$

Put $B = \big(\beta,\, L(\psi)/L\big)$ and $B' = \big(\beta^{\tilde{\alpha}},\, L(\psi)/L\big)$, and call m (resp. m') the index of B (resp. B'). Let E ($\supset L$) be any splitting field of B of minimal degree over L; we have $[E : L] = m$. Then E is also a splitting field of $B \otimes_L B \otimes_L \cdots \otimes_L B$ (a times) $\sim B'$ (similar). Hence m' divides m. Conversely, if E' is any splitting field of B, then it is also a splitting field of B, so m divides m'. Thus we must have $m = m'$. This shows that the p-part of $m_k(\chi)$ is equal to the p-part of $m_k(\chi^{\alpha})$. Since p is any prime number, we conclude that $m_k(\chi) = m_k(\chi^{\alpha})$.

This completes the proof of Proposition 1. □

Remark Deligne informed me (personal communication) another way for looking at Benard's theorem and the second assertion of Theorem 1. In our point of view, he gives an alternating proof of Proposition 1. Furthermore (in our point of view) he proves the following (cf. [BS]):

Theorem (Deligne) *Let χ be an absolutely irreducible character of a finite group of exponent n. For an integer a coprime to n, let α be the automorphism of $\mathbb{Q}(\zeta_n)$ corresponding to a via the natural isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^\times$, where ζ_n is a primitive n-th root of unity in an algebraically closed extention K of $k = \mathbb{Q}(\chi)$. Then we have $[A(\chi^\alpha, F)] = a[A(\chi, F)]$ in $\mathrm{Br}(F)$ for any subfield F of K containing k.*

3. Proof of Theorem 1

Let G be a finite group. Let K be a field of characteristic 0. Let $U\colon G \to \mathrm{GL}(d, K)$ be an absolutely irreducible matrix representation of G over K with character χ. Let F be any field of characteristic 0, and, for $i = 1, 2$, let $\sigma_i\colon K \to F^{*(i)}$ be any embedding of K into an algebraically closed extension $F^{*(i)}$ of F (we assume that, for $i = 1, 2$, $F^{*(i)}$ is sufficiently large so that such an embedding exists). For $i = 1, 2$, let $U_i = U^{\sigma_i}$ and $\chi_i = \sigma_i \circ \chi$. Then it is clear that, for $i = 1, 2$, $U_i\colon G \to \mathrm{GL}(d, F^{*(i)})$ is an absolutely irreducible matrix representation of G over $F^{*(i)}$ with character χ_i. For $i = 1, 2$, set $F_i = F(\chi_i) = F\big(\sigma_i(\chi(g)) \mid g \in G\big)$.

Proposition 2 *There is an isomorphism ρ over F from F_1 onto F_2 such that $\rho \circ \chi_1 = \chi_2^\alpha$ for some automorphism α of $\mathbb{Q}(\chi_2)$.*

For $i = 1, 2$, we consider σ_i as an isomorphism from K onto $\sigma_i(K)$. Then $\sigma_2 \circ \sigma_1^{-1}$ is an isomorphism from $\sigma_1(K)$ onto $\sigma_2(K)$ so that, by enlarging $F^{*(2)}$ if necessary, we can extend $\sigma_2 \circ \sigma_1^{-1}$ to an embedding τ of $F^{*(1)}$ into $F^{*(2)}$. We then have $\tau \circ \sigma_1 = \sigma_2$. Let $\overline{\mathbb{Q}}^{(2)}$ be the algebraic closure of $\mathbb{Q}$ in $F^{*(2)}$. Then $\mathbb{Q}(\chi_2) \subset \overline{\mathbb{Q}}^{(2)}$. Since $\tau\big(\mathbb{Q}(\chi_1)\big) = \mathbb{Q}(\chi_2)$, $\tau|\big(\mathbb{Q}(\chi_1) \cap F\big)$ can be considered as an embedding of $\mathbb{Q}(\chi_1) \cap F$ into $\overline{\mathbb{Q}}^{(2)}$. Since $\mathbb{Q}(\chi_1) \cap F \subset F \subset F^{*(2)}$ and $\mathbb{Q}(\chi_1) \cap F$ is algebraic over $\mathbb{Q}$, $\mathbb{Q}(\chi_1) \cap F$ is contained in $\overline{\mathbb{Q}}^{(2)}$. Therefore, since $\mathbb{Q}(\chi_1) \cap F$ is a normal extension of $\mathbb{Q}$, we must have $\tau\big(\mathbb{Q}(\chi_1) \cap F\big) = \big(\tau|(\mathbb{Q}(\chi_1) \cap F)\big)\big(\mathbb{Q}(\chi_1) \cap F\big) = \mathbb{Q}(\chi_1) \cap F$, hence $\tau\big(\mathbb{Q}(\chi_1) \cap$

$F) \subset \mathbb{Q}(\chi_2) \cap F$. Similarly, we have $(\tau|\mathbb{Q}(\chi_1))^{-1}(\mathbb{Q}(\chi_2) \cap F) \subset \mathbb{Q}(\chi_1) \cap F$. Therefore we see easily that $\tau(\mathbb{Q}(\chi_1) \cap F) = \mathbb{Q}(\chi_2) \cap F$, hence we have $\mathbb{Q}(\chi_1) \cap F = \mathbb{Q}(\chi_2) \cap F$. Put $\tau_0 = \tau|(\mathbb{Q}(\chi_1) \cap F)$, which we consider as an automorphism of $\mathbb{Q}(\chi_2) \cap F$. Let α be an extension of τ_0^{-1} to $\mathbb{Q}(\chi_2)$ into $\overline{\mathbb{Q}}^{(2)}$. Then, since $\mathbb{Q}(\chi_2)$ is a normal extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}^{(2)}$, we must have $\alpha(\mathbb{Q}(\chi_2)) = \mathbb{Q}(\chi_2)$, so that α can be considered as an automorphism of $\mathbb{Q}(\chi_2)$. Put $\tau_1 = \tau|\mathbb{Q}(\chi_1)$, which we consider as an isomorphism from $\mathbb{Q}(\chi_1)$ onto $\mathbb{Q}(\chi_2)$, and put $\upsilon = \alpha \circ \tau_1$. Then υ is an isomorphism from $\mathbb{Q}(\chi_1)$ onto $\mathbb{Q}(\chi_2)$ over $\mathbb{Q}(\chi_1) \cap F = \mathbb{Q}(\chi_2) \cap F$. Let $\mathbb{Q}(\chi_1) = (\mathbb{Q}(\chi_1) \cap F)(\theta_1)$, and put $\theta_2 = \upsilon(\theta_1)$. Then we have $\mathbb{Q}(\chi_2) = (\mathbb{Q}(\chi_2) \cap F)(\theta_2)$, and, for $i = 1, 2$, we have $F_i = F \cdot \mathbb{Q}(\chi_i) = F \cdot ((\mathbb{Q}(\chi_i) \cap F)(\theta_i)) = F(\theta_i)$ Moreover, we have

$$\begin{aligned}
[F(\theta_1) : F] &= [F(\theta_1) : F] \\
&= [\mathbb{Q}(\chi_1) : \mathbb{Q}(\chi_1) \cap F] \\
&= [\mathbb{Q}(\chi_2) : \mathbb{Q}(\chi_2) \cap F] \\
&= [F(\chi_2) : F] \\
&= [F(\theta_2) : F].
\end{aligned}$$

Call s this common value. Then, for $i = 1, 2$, $\{1, \theta_i, \theta_i^2, \ldots, \theta_i^{s-1}\}$ is a basis of the vector space $F(\theta_i)$ over F. We note that, for $0 \le u, v \le s-1$, if $\theta_1^u \cdot \theta_1^v = \sum_{w=0}^{s-1} b_{uvw}\theta_1^w$ with $b_{uvw} \in \mathbb{Q}(\chi_1) \cap F = \mathbb{Q}(\chi_2) \cap F$, then $\theta_2^u \cdot \theta_2^v = \upsilon(\theta_1^u \cdot \theta_1^v) = \sum_{w=0}^{s-1} b_{uvw}\theta_2^w$. Now for an element $x = a_0 + a_1\theta_1 + a_2\theta_1^2 + \cdots + a_{s-1}\theta_1^{s-1}$ of $F(\theta_1)$ with $a_0, a_1, \ldots, a_{s-1} \in F$, we set $\rho(x) = a_0 + a_1\theta_2 + a_2\theta_2^2 + \cdots + a_{s-1}\theta_2^{s-1}$, an element of $F(\theta_2)$. Then it is easy to see that ρ is a field-isomorphism from $F(\theta_1) = F(\chi_1)$ onto $F(\theta_2) = F(\chi_2)$ over F. We note that $\rho \circ \chi_1 = \upsilon \circ \chi_1 = \alpha \circ (\tau|\mathbb{Q}(\chi_1)) \circ \sigma_1 \circ \chi = \alpha \circ (\sigma_2|\mathbb{Q}(\chi)) \circ \chi = \alpha \circ \chi_2$, and α is an automorphism of $\mathbb{Q}(\chi_2)$.

This completes the proof of Proposition 2. □

Proof of Theorem 1. Let the notation be as above. For $i = 1, 2$, let $\overline{F}^{(i)}$ be the algebraic closure of F in $F^{*(i)}$. Let ρ and α be as in Proposition 2. Then ρ can be extended to an isomorphism $\overline{\rho}$ from $\overline{F}^{(1)}$ onto $\overline{F}^{(2)}$. There is matrix representation $V_1 \colon G \to \mathrm{GL}(d, \overline{F}^{(1)})$ with character χ_1. Then $V_2 = V_1^{\overline{\rho}} \colon G \to \mathrm{GL}(d, \overline{F}^{(2)})$ is a matrix representation of G with character χ_2^{α}. The mapping $W_1 \to W_2 = W_1^{\overline{\rho}}$ define a bijection from the set of all matrix representations W_1 of G over $\overline{F}^{(1)}$ with character χ_1 onto the set of all matrix representations W_2 of G over $\overline{F}^{(2)}$ with character χ_2^{α}. Therefore, in view of Definition 1, by Propositions 1, 2, we have

$$\begin{aligned} m_F(\sigma_1 \circ \chi) = & m_F(\chi_1) = m_{F_1}(\chi_1) \\ & = m_{F_2}(\chi_2^\alpha) = m_F(\chi_2^\alpha) = m_F(\chi_2) = m_F(\sigma_2 \circ \chi). \end{aligned}$$

This completes the proof of Theorem 1. □

Let the notation be as above: χ is an abosolutely irreducible character of a finite group G over a field K of characteristic 0 and F is any field of characteristic 0. Then we define $F(\chi)$ (resp. $m_F(\chi)$) to be $F(\sigma \circ \chi)$ (resp. $m_F(\sigma \circ \chi)$) where σ is some embedding of K into some algebraically closed extension F^* of F. These notations can be justified by Theorem 1. Therefore the definition of the Schur index which was given by Feit in [F1] (Definition 2) is well-defined.

Proposition 3 *Let G be a finite group and let χ be an absolutely irreducible character of G over some field K of characteristic* 0. *Let F be any field of characteristic* 0 *and let σ be any embedding of K into an algebraically closed extension F^* of F. Then $m_F(\chi) = m_F(\sigma \circ \chi)$ divides the inner product $(\xi, \sigma \circ \chi)_H$ for any actual character ξ of G that is realizable in F.*

In fact, let $\psi_1, \ldots, \psi_k$ be all the irreducible characters of G over F (the F-irreducible characters). Then we have $\xi = m_1\psi_1 + \cdots + m_k\psi_k$ where $m_1, \ldots, m_k$ are some non-negative integers. For each i, $1 \le i \le k$, there is an absolutely irreducible character η_i of G over F^* such that $\psi_i = m_F(\eta_i)(\eta_{i1} + \eta_{i2} + \cdots + \eta_{is_i})$ where $\eta_{i1}, \eta_{i2}, \ldots, \eta_{is_i}$ are the algebraically conjugate characters of η_i over F (see Schur [Sch], Theorem 3, pp. 170-1). We have $\sigma \circ \chi = \eta_{ij}$ for some i, j. Therefore $(\xi, \sigma \circ \chi)_G = m_i m_F(\eta_i)$. We have $m_F(\eta_i) = m_F(\eta_{ij}) = m_F(\sigma \circ \chi) = m_F(\chi)$. Therefore $m_F(\chi)$ divides $(\xi, \sigma \circ \chi)_G$.

4. Applications of Benard's theorem

Let p be a fixed prime number and let k be an algebraic closure of the prime field of characteristic p. If q is a power of p, then $\mathbb{F}_q$ denotes the subfield of k with q elements.

Let G, F and G^F be as in Theorem 3 (cf. Lusztig [L1], (1.4), and Carter [C], 1.17). Let X be the (projective) variety of all Borel subgroups of G. Then F acts on X naturally. G acts on X by the conjugation: $g \cdot B =$

gBg^{-1}, $g \in G$, $B \in X$. We let G act on $X \times X$ diagonally. Then $W = G \setminus (X \times X)$ has a natural group structure, which we call the Weyl group of G (see Deligne and Lusztig [DL], 1.2, and Lustig [L1], (1.2)). For $w \in W$, let $X(w) = \{B \in X \mid (B, F(B)) \in w\}$. Then, for $w \in W$, $X(w)$ is a locally closed smooth subvariety of X, purely of dimension $\ell(w)$, where $\ell(\ ) \colon W \to \mathbb{Z}_{>0}$ is the length function ([DL], 1.3). For $w \in W$, let $\overline{X}(w)$ be the closure of $X(w)$ in X.

Let ℓ be any fixed prime number $\neq p$. Let $w \in W$. Let $R^1(w)$ be the virtual G^F-module over $\mathbb{Q}_\ell$

$$\sum_{i=0}^{2\ell(w)} (-1)^i H_c^i(X(w), \mathbb{Q}_\ell)$$

(an element of the Grothendieck group of the category of the finitely generated $\mathbb{Q}_\ell[G^F]$-modules). The character R_w of $R^1(w)$ has rational integral values and is independent of ℓ ([DL], Proposition 3.3). Thus there is a virtual representation of G^F over $\mathbb{C}$, uniquely determined up to isomorphisms, with character R_w. We say that a complex irreducible representation ρ of G^F, with character χ_ρ, is unipotent if $(R_w, \chi_\rho)_{G^F} \neq 0$ for some $w \in W$ ([DL], Definition 7.8). Recall that ℓ is any fixed prime number $\neq p$. Let $\sigma \colon \mathbb{C} \to \overline{\mathbb{Q}_\ell}$ be an isomorphism as abstract fields, where $\overline{\mathbb{Q}_\ell}$ is an algebraic closure of $\mathbb{Q}_\ell$. Assume that G is an almost simple algebraic group, defined over $\mathbb{F}_q$ for some power q of p, and that F is the corresponding Frobenius endomorphism of G. For $w \in W$, let $\mathbb{H}^j(\overline{X}(w), \mathbb{Q}_\ell)$ be the j-th ℓ-adic intersection cohomology group of $\overline{X}(w)$. This is a G^F-module over $\mathbb{Q}_\ell$.

Proposition 4 (Lusztig [L4], Lemma 1.2, 1.13) *Assume that G is of adjoint type. Then for any (complex) unipotent representation ρ of G^F, there is some element x of W such that $\left(\mathbb{H}^j(\overline{X}(x), \mathbb{Q}_\ell), \rho^\sigma\right)_{G^F} = 1$ for some j.*

Corollary (Lusztig [L4]) *For any (complex) unipotent representation ρ of G^F, we have $m_{\mathbb{Q}_\ell}(\rho) = 1$ for any prime number $\ell \neq p$.*

In fact, when G is of adjoint type the corollary follows from Propositions 4, 3. Suppose that G is not necessarily of adjoint type, and let G^{ad} be the adjoint group of G. Let $\pi \colon G \to G^{\mathrm{ad}}$ be the natural morphism. Let ρ^{ad} be any unipotent representation of $(G^{\mathrm{ad}})^F$. Then $\rho = \rho^{\mathrm{ad}} \circ \pi$ is also a unipotent representation of G^F (see [DL], Proposition 7.10). Then, by the argument of Geck in [Ge2], Remark 2.6, we have $m_{\mathbb{Q}_\ell}(\rho) = m_{\mathbb{Q}_\ell}(\rho^{\mathrm{ad}}) = 1$ for any prime number $\ell \neq p$.

Remark Assume that G is an almost simple algebraic group, defined and split over $\mathbb{F}_q$ for some power q of p, and that F is the corresponding Frobenius endomorphism of G. Let ρ be a (complex) unipotent representation of G^F. Then, according to [Ge1, Ge2, Ge3], we have $m_{\mathbb{Q}}(\rho) = 1$ except if G is of type E_7 and ρ is a cuspidal unipotent representation of G^F, or G is of type E_8 and ρ is a component of the representation induced by a cuspidal unipotent representation of a parabolic of type E_7. Assume that we are in this exceptional case. Then, if q is non-square, or if q is an even power of $p \equiv 3 \pmod 4$, we still have $m_{\mathbb{Q}}(\rho) = 1$. Assume that q is an even power of $p \equiv 1 \pmod 4$ and that p is sufficiently large so that the results of Lusztig in [L3] can be applicable. Then we have $m_{\mathbb{Q}_p}(\rho) = 2$ and $m_{\mathbb{Q}_\infty}(\rho) = 1$ (note that $\mathbb{Q}(\chi_\rho) = \mathbb{Q}(iq^{7/2})$). I wish to propose here a motivic explanation for this fact (under the assumption that Tate conjecture holds). The basic reference is Milne's lecture [Mi].

Assume that G is of type E_7 and ρ is a cuspidal unipotent representation of G^F. Let c be a Coxeter element in W. Then we see from [L1] that ρ^σ is contained in $H_c^7(X(c), \mathbb{Q}_\ell)$ with multiplicity one (see [L1], p. 146). Facts on the space $\bigoplus_{i=7}^{14} H_c^i(X(c), \mathbb{Q}_\ell)$ semisimply ($X(c)$ is affine); for any eigenvalue λ of F on this space, let M_λ be the corresponding eigen-subspace of it. Then $\rho^\sigma = M_\lambda$ where $\lambda = iq^{7/2}$ or $-iq^{7/2}$ (see [loc. cit.]).

Let $s_1, \ldots, s_7$ be the simple reflections in W. We may assume that $c = s_1s_2s_3s_4s_5s_6s_7$ (for a suitable numbering of $s_1, \ldots, s_7$). Let $f = (s_1, \ldots, s_7)$, and let $X_f = X(c)$ (cf. [L1], (1.7), (4.2)). As in [L1], (4.2), let $\bar{X}_f$ be the set of all sequences $(B_0, B_1, \ldots, B_7)$ of Borel subgroups of G such that $(B_{i-1}, B_i) \in s_i$ or $B_{i-1} = B_i$ $(1 \leq i \leq 7)$ and $F(B_0) = B_7$. Then $\bar{X}_f$ is a smooth projective subvariety of X^8 and we can regard X_f as a open dense subvariety of $\bar{X}_f$. By the arguments in [L1], pp. 119-120, and the information in [L1], p. 146, we can observe that ρ^σ appears with multiplicity one in the space $H^7(\bar{X}_f, \mathbb{Q}_\ell)$. In fact, the eigenvalues of F on it are $\pm iq^{7/2}$ and the two eigen-subspaces afford the two non-isomorphic cuspidal unipotent representations of G^F.

Let $\bar{X}_{f,0}$ be the $\mathbb{F}_q$-structure of $\bar{X}_f$ determined by F, and let $Y = h(\bar{X}_{f,0})$ be the motive of $\bar{X}_{f,0}$ in the category $\mathrm{Mot}(\mathbb{F}_q)$ (see [Mi]). Let $\mathbb{Z}$ be the simple component of Y such that $[\pi_{\mathbb{Z}}] = [iq^{7/2}]$ (cf. [Mi], Proposition 2.6). Let $E = \mathrm{End}(\mathbb{Z})$. Then E is a division algebra whose centre is $K =$

$\mathbb{Q}(iq^{7/2}) = \mathbb{Q}(\chi_\rho)$ ([Mi], Proposition 2.4). By using Theorem 2.16 of [Mi], we can calculate the Hasse invariants of E:
(i) If q is non-square, or q is an even power of $p \equiv 3 \pmod 4$, then $E \sim K$.
(ii) Assume that q is an even power of $p \equiv 1 \pmod 4$. Then, if v is a finite place of K lying above p, we have $\mathrm{inv}_v(E) = 1/2$, and $\mathrm{inv}_v(E) = 0$ otherwise.
This seems to correspond to the result of Geck in [Ge3].

The rationality of the other cuspidal unipotent representations associated with Coxeter elements have similar motivic explanations.

At any rate, this observation suggest that we can expect that crystalline cohomology groups might play some roles in the study of the rationality-properties of (cuspidal) unipotent representations. In this connection, when $G^F = E_7(\mathbb{F}_q)$ where q is an even power of $p \equiv 1 \pmod 4$ we can prove that F acts on $H^7_{\mathrm{crys}}(X^{\cdot}_{f,0}/W(\mathbb{F}_q)) \otimes_{W(\mathbb{F}_q)} K(\mathbb{F}_q)$ semisimply and the two eigen-subspaces of it (with eigenvalues $\pm iq^{7/2}$) afford two non-isomorphic cuspidal unipotent representations of G^F (where $W(\mathbb{F}_q)$ is the Witt ring of $\mathbb{F}_q$ and $K(\mathbb{F}_q)$ is its quotient field).

Assumt that G is a simple algebraic group of adjoint type, of type B_2 (resp. G_2), and that F is an exceptional isogeny such that F^2 is the Frobenius endomorphism of G coresponding to a rational structure on G over the finite subfield of k with $q^2 = 2^{2n+1}$ (resp. $q^2 = 3^{2n+1}$) elements. Then G^F is the Suzuki group ${}^2B_2(q)$ (resp. the Ree group ${}^2G_2(q)$ of type G_2). We see from the table in Lusztig [L2], pp. 373-4, that, for any unipotent representation ρ of G^F, there is some $w \in W$ such that $(R^1(w), \rho^\sigma)_{G^F} = \pm 1$. Therefore, by Proposition 3, we see that, for any (complex) unipotent representation ρ of G^F, we have $m_{\mathbb{Q}_\ell}(\rho) = 1$ for any prime number $\ell \neq p$. (We note that R. Gow [Go] has proved that any complex irreducible representation of ${}^2B_2(q)$ or ${}^2G_2(q)$ has the Schur index 1 over $\mathbb{Q}$.)

Assume that G is a simple adjoint algebraic group of type F_4 and that F is an exceptional isogeney such that F^2 is the Frobenius endomorphism of G corresponding to a rational structure on G over the finite subfield of k with $q^2 = 2^{2n+1}$ elements. Then G^F is the Ree group ${}^2F_4(q)$ of type F_4. The isomorphism classes of the unipotent representations of G^F were classified by Lusztig (in [L2]) and their character values were calculated by Malle in [Ma]. According to the notation of [Ma], the unipotent characters are $\chi_1, \chi_2, \ldots, \chi_{21}$. We find from [L2] that, for each i, $1 \leq i \leq 20$, there is some $w \in W$ such that $(R_w, \chi_i)_{G^F} = \pm 1$, so that, for such i, we have

$m_{\mathbb{Q}_\ell}(\chi_i) = 1$ for any prime number $\ell \neq p = 2$. (We have $m_{\mathbb{Q}}(\chi_i) = 1$ for $1 \leq i \leq 20$ (see Geck [Ge1]).) For $i = 21$, we find from [L2] that $(R_w, \chi_{21})_{G^F}$ is even for each $w \in W$. But we still have $m_{\mathbb{Q}_\ell}(\chi_{21}) = 1$ for any prime number $\ell \neq p$.

In fact, let H be a finite group and let ℓ be a prime number which divides the order of H. Then we say that an element x of H is ℓ-singular if ℓ divides the order of x. For a field L, let $R_L(H)$ denote the Grothendieck group of the category of finitely generated $L[H]$-modules, and, for a commutative ring R with 1, $P_R(H)$ denotes the Grothendieck group of the category of finitely generated projective $R[H]$-modules. Let K be a finite extension of $\mathbb{Q}_\ell$ and let A be the integer ring of K. Let $e\colon P_A(H) \to R_K(H)$ be the additive homomorphism which is induced by the corerspondence $M \to K \otimes_A M$.

Lemma 1 (Swan and Serre; see Serre [Se], Theorem 37) *Let K' be a finite extension of K. Then, for an element x of $R_{K'}(h)$, x belongs to the image of $e\colon P_A(H) \to R_K(H)$ if and only if the character of x has the values in K and vanishes at all ℓ-singular elements of H.*

Let $H = G^F = {}^2F_4(q)$, and let ℓ be any prime number. Let $\chi = \chi_{21}$. Then, if ℓ deoes not divide the order of H, then it is well known that $m_{\mathbb{Q}_\ell}(\chi) = 1$. Suppose that ℓ divides the order of H and that $\ell \nmid 2(q^2 + 1)(q^4 + 1)$ (cf. $3|q^2 + 1$). Let $K = \mathbb{Q}_\ell$. Let $\sigma\colon \mathbb{C} \to \overline{\mathbb{Q}_\ell}$ be an isomorphism where $\overline{\mathbb{Q}_\ell}$ is an algebraic closure of $\mathbb{Q}_\ell$. Let ρ be a (complex) unipotent representation of H with character χ. Then ρ^σ is an absolutely irreducible representation of H over $\overline{\mathbb{Q}_\ell}$ with character χ (cf. $\mathbb{Q}(\chi) = \mathbb{Q}$). Let n be the expoent of H. Then it is well known that ρ^σ is realizable in $K' = \mathbb{Q}(\zeta_n)$ where ζ_n is a primitive n-th root of unity in $\overline{\mathbb{Q}_\ell}$ (Brauer; see, e.g., [Se], Theorem 24). Thus ρ^σ defines an element x of $R_{K'}(H)$. The character χ_x of x is χ. χ_x takes the values in $\mathbb{Q}$ ($\subset K$) and we see from [Ma] that it vanishes at all ℓ-singular elements of H. Therefore, by Lemma 1, we see that x belongs to $R_{\mathbb{Q}_\ell}(H)$. By Proposition 33 of [Se], we see that ρ^σ is realizable in $\mathbb{Q}_\ell$. Therefore, by Proposition 3, we have $m_{\mathbb{Q}_\ell}(\rho) = 1$.

Next, suppose that $\ell | q^2 + 1$ and $\ell \neq 3$. Set $\chi' = 2\chi_{13} + \chi$. Let ρ_{13} be a (complex) unipotent representation of H with character χ_{13}. Let $\sigma\colon \mathbb{C} \to \overline{\mathbb{Q}_\ell}$ be an isomorphism where $\overline{\mathbb{Q}_\ell}$ is an algebraic closure of $\mathbb{Q}_\ell$. Then $\rho_{13}^\sigma + \rho_{13}^\sigma + \rho^\sigma$ is realizable in $K' = \mathbb{Q}_\ell(\zeta_n)$, where ζ_n is a primitive n-th root of unity in $\overline{\mathbb{Q}_\ell}$, and defines an element x' of $R_{K'}(H)$ with character χ'. We see from [Ma] that χ' has the values in $\mathbb{Q}$ and vanishes at all ℓ-singular elements of H. Therefore, by Lemma 1, we see that x' lies in $R_{\mathbb{Q}_\ell}(H)$, hence, by Proposition 33 of [Se], we see that $\rho_{13}^\sigma + \rho_{13}^\sigma + \rho^\sigma$ is realizable in $\mathbb{Q}_\ell$. Thus, since $(\rho_{13}^\sigma + \rho_{13}^\sigma + \rho^\sigma,\ \rho^\sigma)_H = 1$, by Proposition 3, we have $m_{\mathbb{Q}_\ell}(\rho) = 1$.

Thirdly, let $\chi'' = \chi - \chi_{15} - \chi_{16}$ and suppose that $\ell | q^2 - \sqrt{2}q + 1$. Let ρ_{15} (resp. ρ_{16}) be a (complex) unipotent representation of H with character χ_{15} (resp. χ_{16}). Let $\sigma\colon \mathbb{C} \to \overline{\mathbb{Q}_\ell}$ be an isomorphism where $\overline{\mathbb{Q}_\ell}$ is an algebraic closure of $\mathbb{Q}_\ell$. Let $K' = \mathbb{Q}_\ell(\zeta_n)$ where ζ_n is a primitive n-th root of unity in $\overline{\mathbb{Q}_\ell}$. Let $[\rho^\sigma]$ (resp. $[\rho_{15}^\sigma]$, $[\rho_{16}^\sigma]$) be the element of $R_{K'}(H)$ which is determined by ρ^σ (resp. ρ_{15}^σ, ρ_{16}^σ). Let $x'' = [\rho^\sigma] - [\rho_{15}^\sigma] - [\rho_{16}^\sigma]$ ($\in R_{K'}(H)$). Then x'' has the character χ''. We see from [Ma] that χ'' has the values in $\mathbb{Q}$ and vanishes at all ℓ-singular elements of H.

Thus, by Lemma 1, we see that x'' belongs to the image of the homomorphism $e\colon P_{\mathbb{Z}_\ell}(H) \to R_{\mathbb{Q}_\ell}(H)$. So there exist finitely many finitely generated projective $\mathbb{Z}_\ell[H]$-modules $P_1, \ldots, P_t$ and signatures $\varepsilon_1, \ldots, \varepsilon_t = \pm 1$ such that $e(\varepsilon_1[P_1] + \cdots + \varepsilon_t[P_t]) = x''$, where, for $1 \le i \le t$, $[P_i]$ is the element of $P_{\mathbb{Z}_\ell}(H)$ determined by P_i. For $1 \le i \le t$, let $V_i = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} P_i$, which is ralizable in $\mathbb{Q}_\ell$. By Proposition 3, we see that $m_{\mathbb{Q}_\ell}(\rho)$ divides $(\rho^\sigma,\ V_i)_H$ for each i. Hence $m_{\mathbb{Q}_\ell}(\rho)$ divides $\sum_{i=1}^t \varepsilon_i(\rho^\sigma,\ V_i)_H = (\chi,\ \chi'')_H = 1$.

Similarly, by considering $\chi''' = \chi - \chi_{17} - \chi_{18}$ and $\chi'''' = \chi + \chi_{20} + \chi_{19} + 2\chi_{14} + \chi_{13} + \chi_{12}$ (there characters are rational integral valued), we see that, for any prime number ℓ such that $\ell | q^2 + \sqrt{2}q + 1$ (resp. $\ell = 3$), we have $m_{\mathbb{Q}_\ell}(\rho) = 1$.

Thus we get

Proposition 5 *For any unipotent representation ρ of ${}^2B_2(q)$ (resp. ${}^2G_2(q)$, ${}^2F_4(q)$), we have $m_{\mathbb{Q}_\ell}(\rho) = 1$ for each prime number $\ell \neq 2$ (resp. $\neq 3$, $\neq 2$)*

Let us give a sketch of the proof of Theorem 3.

Let p, k, G and F be as in Theorem 3. Let G^{ad} be the adjoint group of G, let $\pi\colon G \to G^{\mathrm{ad}}$ be the natural morphism and let $\tilde{\pi}\colon \tilde{G} \to G$ be the simply-connected covering of the derived group of G. Let $U(G)$ (resp. $U(G^{\mathrm{ad}})$, $U(\tilde{G})$) be the set of isomorphism classes of the unipotent representations of G^F (resp. $(G^{\mathrm{ad}})^F$, $\tilde{G}^F$). Then the mapping $\rho^{\mathrm{ad}} \to \rho^{\mathrm{ad}} \circ \pi$ (resp. $\rho \to \rho \circ \tilde{\pi}$) defines a bijection from $U(G^{\mathrm{ad}})$ onto $U(G)$ (resp. $U(G)$ onto $U(\tilde{G})$) ([DL], Proposition 7.10). Then, for $\rho^{\mathrm{ad}} \in U(G^{\mathrm{ad}})$, we have $\mathbb{Q}(\chi_{\rho^{\mathrm{ad}}}) = \mathbb{Q}(\chi_{\rho^{\mathrm{ad}} \circ \pi}) = \mathbb{Q}(\chi_{\rho^{\mathrm{ad}} \circ \pi \circ \tilde{\pi}})$ (cf. Geck [Ge2], Remark 2.6). Thus it clearly suffices to prove the theorem for G^{ad}. Then, by a standard reduction argument, we are reduced to the case where G is a simple algebraic group of adjoint type. In this case the assertion is clear from the corollary of Proposition 4 and Proposition 5.

Remark Assume that $H = {}^2F_4(q)$ and $\chi = \chi_{21}$. Then, in [Ge1], Geck has proved that $m_R(\chi) = 2$ by showing that

$$\nu(\chi) = \frac{1}{|H|} \sum_{x \in H} \chi(x^2) = -1$$

(cf. [Se], Proposition 39). His calculation is very interesting. It will be convenient to note that the same result also follows from the following datum (cf. Shinoda [Shi]):

Class of x	1	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9
Class of x^2	1	1	1	u_1	u_1	u_1	u_1	u_2	u_2	u_2

u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	u_{16}	u_{17}	u_{18}	t_1	t_1u_1
u_5	u_5	u_5	u_8	u_7	u_{11}	u_{12}	u_{12}	u_{11}	t_1	t_1

t_1u_3	t_1u_4	t_2	t_2u_2	t_3	t_4	t_4u_2	t_4u_7	t_4u_9	t_4u_9'
t_1u_1	t_1u_1	t_2	t_2	t_3	t_4	t_4	t_4u_2	t_4u_2	t_4u_2

t_5	t_5u_2	t_6	t_7	t_7u_1	t_7u_3	t_7u_4	t_8	t_9	t_9u_1
t_5	t_5	t_6	t_7	t_7	t_7u_1	t_7u_1	t_8	t_9	t_9

t_9u_3	t_9u_4	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}	t_{16}	t_{17}
t_9u_1	t_9u_1	t_{10}	t_{11}	t_{12}	t_{13}	t_{14}	t_{15}	t_{16}	t_{17}

Let R be a set of the class representatives of H. For $x \in R$, let $\mathbb{Z}_H(x)$ be the centralizer of x in H. Then we have $\nu(\chi) = \sum_{x \in R} \chi(x^2)/|\mathbb{Z}_H(x)|$. The left hand side of this equation is a rational integer and the right hand side is a rational function on q (see [Ma]). Thus, by letting $q \to \infty$, we get $\nu(\chi) = -1$.

By the same method, we can show that $\nu(\eta) = 1$ for any other rational-valued unipotent character η of H.

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Mathematical Department
Hokkaido University of Education
Iwamizawa Campus
2-34-1 Midorigaoka Iwamizawa
068-8642 Hokkaido, Japan