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The local analytical triviality of a complex analytic singular foliation

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Abstract. A singular foliation on a complex manifold M is defined as an integrable coherent subsheaf E of tangent sheaf of M. We give a detailed proof of the fact that there exists a "leaf (integral submanifold)" of E passing through each point of M. The dimensions of the leaves are not constant on M in general, so the singular set $S(E)$, which is in fact an analytic subset of M, is given as the set of points where the dimension of the leaf of E is not maximal. Using this, we also prove that the structure of the foliation is locally analytically trivial along each of the leaves.

Key words: holomorphic foliations, structure of the singular set.

0. Introduction

There have been a number of fundamental works on foliations with singularities from various viewpoints, both in the complex and real cases (see, e.g., [C], [N], [Ss] and [St]). In the complex analytic case, a general theory in terms of coherent sheaves appeared in [BB]. This point of view not only unifies various definitions and clarifies the situation, but also in suitable in other contexts, e.g., in the studies of characteristic classes, residues, unfolding and deformation theories and $\mathcal{D}$-module theory of singular foliations. Combined with the stratification theory etc., this would give us a nice way of describing and understanding singular foliations.

The purpose of this paper is to give detailed proofs of some of the basic facts concerning the structure of the singular set, in the framework of theories of coherent sheaves and of stratification of analytic sets. We note that some arguments are done in parallel with those for logarithmic vector fields (cf. [Sa], [BR]). We should also note that some of the results explained

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here may be obtained by other means as well.

We first review and summarize basic definitions and facts about complex analytic singular foliations in Section 1. In this paper, we only consider "reduced" singular foliations (see Definition 1.4 below), as in [BB]. This apparently technical condition is in fact a natural one to impose when we wish to study "genuine" foliation singularities, since the singular set of a nonreduced foliation may contain the zero set of a function, which is too large. There is also a simple canonical process to obtain a reduced foliation from the given one. For a complex analytic singular foliation E, we have the fundamental "Tangency Lemma" (stated in Theorem 1.18 below), which says that each vector field defining the foliation is "tangential" to the singular set $S(E)$. Using this lemma, we prove the existence of a leaf passing through each point of M (even on $S(E)$) in Section 2 (Corollary 2.10). For the proof we use the method of Whitney stratification of the singular set and in fact we prove the existence of leaves "compatible" with the natural Whitney stratification (Theorem 2.9).

In Section 3, we mainly explain and prove the local analytical triviality of E along each leaf (Theorem 3.1). This is done using Theorem 2.9 and we reprove the "normal form theorem" of [C] on the way, in the case of reduced foliations. We also give some applications and examples. In a situation as in Example 3.32, Theorem 3.1 does not give us much information about the structure of a singular foliation near the singular set, since the dimension of the leaf containing each singular point is zero. In such a case, it is more appropriate to look into the problem of local topological triviality, which is treated in [Y].

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1. Complex analytic singular foliations

First we recall some generalities about complex analytic singular foliations on complex manifolds. For further details, see [B], [BB] and [Sw].

Let M be a (connected) complex manifold of dimension n, and let $\mathcal{O}_M$, Θ_M and Ω_M denote, respectively, the sheaf of holomorphic functions on M, the tangent sheaf and the cotangent sheaf of M.

Let E be a coherent subsheaf of Θ_M. Note that, in this case, E is coherent if and only if E is locally finitely generated, since Θ_M is locally free. We set

$$S(E) = \{p \in M \mid (\Theta_M/E)_p \text{ is not } \mathcal{O}_{M,p}\text{-free}\},$$

and call it the *singular set* of E. For each point p of $S(E)$, we also say that p is a *singular point* of E. If we restrict E to a sufficiently small coordinate neighborhood U with coordinates $(z_1, z_2, \ldots, z_n)$, we can express E on U explicitly as follows:

$$E_p = \sum_{i=1}^{m} \mathcal{O}_{M,p} v_i, \qquad v_i = \sum_{j=1}^{n} f_{ij}(z) \frac{\partial}{\partial z_j}, \quad 1 \le i \le m, \tag{1.1}$$

where f_{ij} are holomorphic functions defined on U, and m is a non-negative integer. Then the singular set $S(E)$ is given on U by

$$S(E) = \{p \in U \mid \operatorname{rank}\bigl(f_{ij}(p)\bigr) \text{ is not maximal}\}.$$

We define the *rank* (we sometimes call it *dimension*) of E to be the rank of the locally free sheaf $E|_{M-S(E)}$, and denote it by $\operatorname{rank} E$. If we use the notation in (1.1), we can rewrite it as

$$\operatorname{rank} E = \max_{p \in M} \operatorname{rank}\bigl(f_{ij}(p)\bigr).$$

Remark 1.2 The set $S(E)$ is the set of points where the quotient sheaf Θ_M/E fails to be free. Thus it is possible that E is locally free on M, with $S(E) \neq \emptyset$.

Next we give the definition of a singular foliation on M in terms of vector fields. Later, we will introduce it again from another viewpoint.

Definition 1.3 A (complex analytic) singular foliation on M is a coherent subsheaf E of Θ_M which is *integrable* in the sense that

$$[E_p, E_p] \subset E_p, \quad \text{for} \quad p \in M - S(E),$$

where $[\ ,\ ]$ denotes the Lie bracket.

It is clear that a singular foliation E induces a non-singular foliation on $M - S(E)$, whose dimension is equal to $\operatorname{rank} E$.

Definition 1.4 A singular foliation E is said to be *reduced* if it is "full" in Θ_M, i.e.,

$$v \in \Gamma(U, \Theta_M),\ v|_{U-S(E)} \in \Gamma(U - S(E), E) \Longrightarrow v \in \Gamma(U, E)$$

holds for every open set U in M.

Note that a reduced foliation E is *involutive* in the sense that

$$[E_p, E_p] \subset E_p, \quad \text{for} \quad p \in M. \tag{1.5}$$

Remark 1.6 If a singular foliation E is locally free, then (cf. [Sw])

$$E \text{ is reduced} \Longleftrightarrow \operatorname{codim} S(E) \geq 2.$$

In the following (Proposition 3.25), we prove the implication "$\Longrightarrow$" without assuming that E is locally free. In general, the implication "$\Longleftarrow$" is false as the following example shows:

Let $M = \mathbb{C}^2 = \{(z_1, z_2)\}$ and let E be generated by $v_1 = z_1(\partial/\partial z_1)$ and $v_2 = z_2(\partial/\partial z_1)$. Then $[v_1, v_2] = -v_2$ and E defines a one-dimensional foliation on $\mathbb{C}^2$. We have $\operatorname{codim} S(E) = 2$, as $S(E) = \{0\}$. However, E is not reduced, since the vector field $\partial/\partial z_1$ is in E away from $S(E)$, but not over $S(E)$.

Next we represent singular foliations in terms of holomorphic 1-forms. It is not so difficult to rewrite it from the viewpoint of its "dual", however there are several points which require a little care.

Definition 1.7 Let F be a coherent subsheaf of Ω_M. Then we set

$$S(F) = \{p \in M \mid (\Omega_M/F)_p \text{ is not } \mathcal{O}_{M,p}\text{-free}\},$$

and call it the *singular set* of F. A point in $S(F)$ is called a *singular point* of F.

Definition 1.8 A (complex analytic) singular foliation on M is a coherent subsheaf F of Ω_M which is *integrable* in the sense that

$$dF_p \subset \Omega_p \wedge F_p, \quad \text{for} \quad p \in M - S(F).$$

Moreover, the *rank* of F is defined to be the rank of the locally free sheaf $F|_{M-S(F)}$, and denote it by $\operatorname{rank} F$.

A singular foliation F induces a non-singular foliation on $M - S(F)$, whose codimension is equal to $\operatorname{rank} F$.

Definition 1.9 A singular foliation F is said to be *reduced* if it is "full" in Ω_M, i.e.,

$$\omega \in \Gamma(U, \Omega_M),\ \omega|_{U-S(F)} \in \Gamma(U - S(F), F) \Longrightarrow \omega \in \Gamma(U, F)$$

holds for every open set U in M.

Remark 1.10 If we impose the condition $dF_p \subset \Omega_p \wedge F_p$ for all p in M, it is too strong. It is not satisfied even by a reduced foliation as the following example shows:

Let $M = \mathbb{C}^2 = \{(z_1, z_2)\}$ and let F be generated by $\omega = z_2 dz_1 - z_1 dz_2$. Then F defines a reduced singular foliation of codimension one on $\mathbb{C}^2$. However, the integrability condition is satisfied only away from its singular set, which is $\{0\}$.

In the following we describe the relation between the two Definitions 1.3 and 1.8.

For singular foliations $E \subset \Theta_M$ and $F \subset \Omega_M$, we consider their "annihilators";

$$E^a = \{\omega \in \Omega_M \mid \langle v, \omega \rangle = 0 \quad \text{for all } v \in E\},$$
$$F^a = \{v \in \Theta_M \mid \langle v, \omega \rangle = 0 \quad \text{for all } \omega \in F\},$$

where $\langle\ ,\ \rangle$ denotes the natural pairing between a vector field and a 1-form. Then it is not difficult to see that E^a ($\subset \Omega_M$) and F^a ($\subset \Theta_M$) define reduced singular foliations on M.

Remark 1.11 Note that $S(E^a) \subset S(E)$ and $S(F^a) \subset S(F)$ hold.

Definition 1.12 We call $(E^a)^a$ (resp. $(F^a)^a$) the reduction of E (resp. F).

In the notation above, a singular foliation $E \subset \Theta_M$ (resp. $F \subset \Omega_M$) is reduced if and only if $(E^a)^a = E$ (resp. $(F^a)^a = F$). In this way we can make any singular foliation reduced by taking its reduction. Moreover, if we consider only reduced foliations, then the two definitions of singular foliation stated above are equivalent and the singular sets in terms of vector fields and of 1-forms are the same.

In the sequel, we only consider reduced singular foliations, which will be usually expressed in terms of vector fields.

Next, let us summarize the basic properties of the singular set of a singular foliation. Hereafter, we assume E ($\subset \Theta_M$) to be a reeuced singular foliation on a complex manifold M and set $r = \operatorname{rank} E$.

Definition 1.13 For each point p in M, we set

$$E(p) = \{v(p) \mid v \in E_p\},$$

where $v(p)$ denotes the evaluation of the vector field germ v at p. Note that $E(p)$ is a sub-vector space of the tangent space T_pM.

Definition 1.14 For an integer k with $0 \leq k \leq r$, we set

$$L^{(k)} = \{p \in M \mid \dim_{\mathbb{C}} E(p) = k\},$$
$$S^{(k)} = \{p \in M \mid \dim_{\mathbb{C}} E(p) = k\},$$

and set $L^{(-1)} = S^{(-1)} = \emptyset$ for convenience. Clearly we have

$$L^{(k)} = S^{(k)} - S^{(k-1)}, \quad S^{(k)} = \bigcup_{i=0}^{k} L^{(i)}$$

for $k = 0, 1, 2, \ldots, r$.

Proposition 1.15 *$S^{(k)}$ is an analytic set and $L^{(k)}$ is a locally analytic set for every integer k with $0 \leq k \leq r$.*

Proof. If we use the notation in (1.1), $S^{(k)}$ is locally expressed on a small open set U in M as follows:

$$S^{(k)} \cap U = \big\{z \in U \mid \operatorname{rank}\big(f_{ij}(z)\big) \leq k\big\}.$$

All f_{ij} are holomorphic on U, so $S^{(k)}$ is analytic. And besides, we come to the conclusion that $L^{(k)}$ ($= S^{(k)} - S^{(k-1)}$) is locally analytic because $S^{(k)}$ is analytic and $S^{(k-1)}$ is closed in M. □

By the proposition stated above, we get the *natural filtration* which consists of analytic sets:

$$\begin{array}{ccccccc} S^{(r)} & \supset & S^{(r-1)} & \supset S^{(r-2)} \supset \cdots\cdots \supset S^{(1)} \supset S^{(0)} \supset & S^{(-1)}. \\ \| & & \| & & \| \\ M & & S(E) & & \emptyset \end{array} \tag{1.16}$$

Now we recall the following result, which will be the basis of our subsequent arguments. The proof is originally due to T. Suwa and is given in

[Y]. It is done in a similar way as for logarithmic vector fields (cf. [BR], [Sa]).

Proposition 1.17 *Let p be a point in M and v a germ in E_p. Let $\{\varphi_t = \exp tv\}$ be the local 1-parameter group of transformations induced by v. Then, for all t sufficiently close to 0, we have*

$$(\varphi_t)_* E_p = E_{\varphi_t(p)},$$

where $(\varphi_t)_*$ denotes the differential of φ_t.

It is not difficult to see that Proposition 1.17 implies the following (cf. [Y]):

Theorem 1.18 (Tangency Lemma) *Let k be an integer with $0 \leq k \leq r$ and p a point in $S^{(k)}$. Then we have*

$$E(p) \subset C_p S^{(k)},$$

where $C_p S^{(k)}$ denotes the tangent cone of $S^{(k)}$ at p.

Remark 1.19 Theorem 1.18 was proved by P. Baum under the hypotheses that E is reduced, $k = r - 1$ and p is a non-singular point of $S^{(k)}$ ($= S^{(r-1)} = S(E)$) (see [B]). The general case may also be obtained as a consequence of a theorem of D. Cerveau ([C] Théorème 1.1), which gives explicitly "normal forms" of vector fields generating the foliation locally. Note that, for this we only need the involutiveness (1.5) for E and E need not be reduced. This normal form theorem also holds in the C^∞, real analytic and formal cases.

For related results in the case of real singular foliations, we refer to [N], [Ss] and [St], see also Remark 2.11 below.

2. Existence of the integral submanifolds

Let E be a singular foliation of rank r on M. In the preceding section we recalled that E induces a non-singular foliation on $M - S(E)$, so if a point $p \in M$ does not belong to $S(E)$, it is clear that there exists an integral submanifold (of dimension r) passing through p. The main purpose in this section is to prove that there also exist integral submanifolds on the singular set $S(E)$, whose dimensions are lower than r.

Since the singular set $S(E)$ is not a smooth submanifold of M in general, we have to take a stratification of $S(E)$. However we must be careful in the choice of the stratification, because if we take a stratification too much fine, then the space $E(p)$ is not always contained in the tangent space of the stratum at p. We adopt here the famous method of *natural Whitney stratification* which is due to H. Whitney. We introduce just the essence below. (For details, see [W].)

Let A be an analytic set. We denote by $\mathrm{Sing}(A)$ the singular set of A and denote by $\mathrm{Reg}(A)$ the set of regular (i.e., non-singular) points of A. Moreover we set

$$\Sigma(A) = \mathrm{Sing}(A) \cup \{p \in \mathrm{Reg}(A) \mid \dim_p A < \dim A\},$$

where $\dim_p A$ denotes the dimension of A at each point $p \in \mathrm{Reg}(A)$. For two manifolds X and Y, we define a subset $B(X, Y)$ of X by

$$B(X, Y) = \{p \in X \mid Y \text{ is not Whitney regular over } X \text{ at } p\}.$$

Also for two analytic sets A and A' we set

$$W(A, A') = \Sigma(A) \cup B\big(A - \Sigma(A), A' - \Sigma(A')\big). \tag{2.1}$$

$W(A, A')$ is a analytic subset of A whose dimension is lower than $\dim A$.

Using the notation stated above, for an analytic subset A we define a family of analytic subsets $\{\Pi^i A\}_{i=0,1,2,\ldots}$ (inductively) as follows:

$$\begin{cases} \Pi^0 A = A \\ \Pi^1 A = \Sigma(A) \\ \text{For each integer } i \text{ with } i \geq 2, \\ \Pi^i A = \overline{\bigcup_{j=0}^{i-2} W(\Pi^{i-1}A, \Pi^j A - \Pi^{j+1} A)}, \end{cases} \tag{2.2}$$

where the closure is taken in A. Note that $\Pi^i A = \emptyset$ for sufficiently large i. Thus we have a sequence of analytic subsets of A:

$$\begin{array}{ccccc} \Pi^0 A & \supset & \Pi^1 A & \supset \Pi^2 A \supset \cdots\cdots \supset \Pi^l A \supset \Pi^{l+1} A \supset \cdots\cdots \supset \emptyset. \\ \| & & \| & \\ A & & \Sigma(A) & \end{array} \tag{2.3}$$

Then we set $\mathcal{A} = \{\Pi^i A - \Pi^{i+1} A \,(\neq \emptyset) \mid i = 0, 1, 2, \ldots\}$. By the construction of $\Pi^i A$, $\mathcal{A}$ is a Whitney stratification of A. This stratification is called the natural Whitney stratification of A. Note that each stratum of $\mathcal{A}$ is not always connected, but if X and Y are connected components of a stratum $\Pi^i A - \Pi^{i+1} A$ then $\dim X = \dim Y$.

Now let us prepare a lemma which plays an important role in the proof of the existence of integral submanifolds.

Lemma 2.4 *Let E ($\subset \Theta_M$) be a singular foliation on a complex manifold M and S be an analytic subset of M. Suppose that $E(p) \subset C_p S$ holds for every point $p \in S$ ($C_p S$ denotes the tangent cone of S at p, same notation as in §1). Let $\mathcal{S}$ be the natural Whiteley stratification of S. Then we have $E(p) \subset T_p X$ for every point $p \in S$ where X ($\in \mathcal{S}$) is the stratum containing p.*

Proof. It is sufficient to show that

$$E(p) \subset C_p \Pi^i S \quad (\text{for } \forall p \in \Pi^i S) \tag{2.5}$$

holds for every non-negative integer i. Suppose we have already showed (2.5). For any point $p \in S$, take the stratum $X \in \mathcal{S}$ passing through p. By the definition of the natural Whitney stratification, X can be expressed as

$$X = \Pi^i S - \Pi^{i+1} S$$

for some integer i. p belongs to $\Pi^i S - \Pi^{i+1} S$ and the singular points of $\Pi^i S$ are contained in $\Pi^{i+1} S$, hence p is a non-singular point of $\Pi^i S$. Then, by (2.5), we have

$$E(p) \subset C_p \Pi^i S = T_p \Pi^i S = T_p X,$$

which completes the proof. □

In the following let us show (2.5) by the induction on i under the assumption of Lemma (2.4). In the case of $i = 0$, (2.5) is nothing but the assumption of Lemma (2.4). Suppose (2.5) holds for every integer i with $0 \leq i \leq l$ and take a point $p \in \Pi^{i+1} S$ arbitrarily. Our purpose is to show that

$$E(p) \subset C_p \Pi^{l+1} S.$$

In order to do this, it is enough to prove that

$$v(p) \in C_p \Pi^{l+1} S \tag{2.6}$$

holds for any vector field germ v in the stalk of E at p. If $v(p) = 0$ then (2.6) is clearly fulfilled, so let us consider the case of $v(p) \neq 0$. Take a coordinate neighborhood U of p with coordinates $(z_1, z_2, \dots, z_n)$ on U such that $p = (0, 0, \dots, 0)$. Since $v(p) \neq 0$, we may assume that the expression of v using the local coordinates $(z_1, z_2, \dots, z_n)$ is given by

$$v = \frac{\partial}{\partial z_1}. \tag{2.7}$$

Next, for each point $q \in U$ we set

$$L(q) = \{q' \in U \mid z_k(q') = z_k(q) \text{ for } (k = 2, 3, \dots, n)\}.$$

It may be assumed that U has been chosen such that all $L(q)$ are connected. Note that $L(q)$ is the integral curve of $v = \partial/\partial z_1$ passing through q. Furthermore, we set

$$D = \{z \in U \mid z_1 = 0\},$$

and let $\pi \colon U \longrightarrow D$ be the natural projection from U onto D (i.e. $\pi(z_1, z_2, \dots, z_n) = (0, z_2, \dots, z_n)$).

Our purpose was to prove (2.6) under the inductive assumptions. However, in fact, it suffices to show the following claim:

$$L(q) \cap \Pi^i S \neq \emptyset \Rightarrow L(q) \subset \Pi^i S \quad \text{holds for } i = 0, 1, \dots, l. \tag{2.8}$$

If we assume that (2.8) is true, then for any point $y \in \Pi^i S$ we have $y \in L(y) \cap \Pi^i S$, so (2.8) assures $L(y) \subset \Pi^i S$. This implies that the structures of $\Pi^i S \cap U$ are trivial along the z_1-axis. To be more precise, there exist analytic subsets A^i of D such that $\pi^{-1}(A^i) = \Pi^i S \cap U$ (in fact A^i coincide with $\Pi^i S \cap D$). On the other hand, the way of construction of $\Pi^i S$ given in (2.2) tells us that the local structure of $\Pi^{l+1} S$ is determined using only the local structures of $\Pi^i S$ for $i = 0, 1, \dots, l$. Therefore the structure of $\Pi^i S \cap U$ is also trivial along the z_1-axis, i.e., there exists an analytic subset A^{l+1} of D such that $\pi^{-1}(A^{l+1}) = \Pi^{l+1} S \cap U$. Taking (2.7) into consideration, we obtain (2.6), thus the induction is completed.

From the preceding argument, all we have to do is to show (2.8) under the inductive assumptions. We set

$$L^+ = L(q) \cap \Pi^i S, \quad L^- = L(q) - \Pi^i S.$$

Note that $L(q)$ is the disjoint union L^+ and L^-. The inductive assumption implies that the vector field v is logarithmic for $(\Pi^i S, q)$, so the flow generated by v preserves $(\Pi^i S, q)$ (see, for example, [BR] §1). This fact tells us that L^+ is open in $L(q)$. On the other hand, L^- is also open in $L(q)$ since $\Pi^i S$ is a closed set of M. Then either L^+ or L^- must be empty by the connectedness of $L(q)$. In other words if L^+ is not empty then L^- is empty, and this is clearly equivalent to (2.8). □

Let E ($\subset \Theta_M$) be a reduced singular foliation of dimension r on a complex manifold M and $\mathcal{S}^{(k)}$ the natural Whitney stratification of $S^{(k)}$, $k = 0, \ldots, r$. Note that, for a stratum $X \in \mathcal{S}^{(k)}$, $X - S^{(k-1)}$ is a complex manifold, since $S^{(k-1)}$ is a closed set in M.

As a consequence of Theorem 1.18 and Lemma 2.4, we have

Theorem 2.9 *In the above situation, for each $X \in \mathcal{S}^{(k)}$ and $p \in X$, we have $E(p) \subset T_pX$. Thus E induces a non-singular foliation of dimension k on $X - S^{(k-1)}$.*

Corollary 2.10 (Existence of Integral Submanifolds) *There exist integral submanifolds (whose dimensions are lower than r) also on $S(E)$. To be more precise, there is a family $\mathcal{L}$ of submanifolds of M such that $M = \bigcup_{L \in \mathcal{L}} L$ is a disjoint union and that, for any $L \in \mathcal{L}$ and $p \in L$, we have $E(p) = T_pL$.*

Each member L in $\mathcal{L}$ is called a leaf of E.

Remark 2.11 For the above results to hold, we only need the involutiveness (1.5) for E and E need not be reduced (cf. Remark 1.19). In the real case, consider the following properties:
(1) the involutiveness as in (1.5),
(2) the property as stated in Proposition 1.17,
(3) the existence of submanifolds as in Corollary 2.10.

In the C^∞ case, (2) and (3) are equivalent and they imply (1), and under the condition called "locally of finite type" in [Ss], (1) implies (2) and hence (3) ([Ss], see also [St]). In the analytic case, this condition is always satisfied and thus (1) implies (3), which is result proved earlier in [N]. Their method should be applicable to prove Proposition 1.17 for non-reduced foliations, only assuming involutiveness, in the complex analytic case.

3. The local analytical triviality along the leaves

Hereafter all the foliations we consider are assumed to be reduced. In the preceding section we proved the existence of the leaves for a singular foliation E on M. The following theorem says that the structure of a singular foliation E is locally analytically trivial along the leaf containing each point p in M.

Theorem 3.1 (Local Analytical Triviality) *Let E $(\subset \Theta_M)$ be a reduced foliation of rank r on a complex manifold M. Let k be an integer with $0 \leq k \leq r$ and p a point in $L^{(k)}$ $(= S^{(k)} - S^{(k-1)})$. Then there exist a neighborhood D of 0 in $\mathbf{C}^{n-k}$, a singular foliation E' on D with $E'(0) = \{0\}$, a neighborhood U_p of p in M and a submersion $\pi\colon U_p \longrightarrow D$ with $\pi(p) = 0$ such that*

$$E|_{U_p} = \left(\pi^*(E'^a)\right)^a.$$

Proof. Take a coordinate neighborhood U of p with coordinates $(u_1, u_2, \ldots, u_n)$ on U such that $p = (0, 0, \ldots, 0)$. We denote by L_q the leaf of E containing each point $q \in U$ (the existence of the leaf has been proved in the preceding section). $p \in L^{(k)}$ implies $\dim_{\mathbf{C}} E(p) = k$, so L_p is a k-dimensional complex submanifold of M. Retaking the coordinates $(u_1, \ldots, u_n)$, we may assume

$$L_p \cap U = \{u_{k+1} = \cdots = u_n = 0\}.$$

Moreover, since $S^{(k-1)}$ is a closed subset of M and $L^{(k)} = S^{(k)} - S^{(k-1)}$, we may also assume that $U \cap S^{(k-1)} = \emptyset$.

At first, we take holomorphic vector fields $\gamma_1, \ldots, \gamma_k$ on U which satisfy the following two properties:

$$\begin{aligned} &\text{(i)} \quad \gamma_i = \frac{\partial}{\partial u_i} \quad \text{on} \quad L_p \cap U, \\ &\text{(ii)} \quad \gamma_i(q) \in E(q) \quad \text{for all} \quad q \in U. \end{aligned} \qquad (i = 1, 2, \ldots, k) \tag{3.2}$$

Using these vector fields $\gamma_1, \ldots, \gamma_k$, we define a holomorphic vector field V_x on U for each $x = (x_1, \ldots, x_k, 0, \ldots, 0) \in L_p \cap U$ as follows:

$$V_x = x_1\gamma_1 + x_2\gamma_2 + \cdots + x_k\gamma_k. \tag{3.3}$$

Let $\{\varphi_{x,t} = \exp tV_x\}$ be the local 1-parameter group of transformations induced by V_x. For $\varepsilon > 0$ we set

$$U_{(\varepsilon)} = \{(u_1, \ldots, u_n) \in U \mid |u_i| < \varepsilon \ (i = 1, 2, \ldots, n)\},$$
$$L_{(\varepsilon)} = L_p \cap U_{(\varepsilon)}.$$

Choosing $\varepsilon > 0$ sufficiently small, we may assume that $\varphi_{x,t}(q)$ stays in U for any $x \in L_{(\varepsilon)}$, $q \in U_{(\varepsilon)}$ and $t \in \mathbf{C}$ with $|t| < 2$. Moreover we set

$$\psi_x = \varphi_{x,1}.$$

The way of choice of ε tells us that $\psi_x\big(U_{(\varepsilon)}\big) \subset U$ for any $x \in L_{(\varepsilon)}$, thus we obtain a family of holomorphic maps $\{\psi_x \colon U_{(\varepsilon)} \to U\}_{x \in L_{(\varepsilon)}}$. We set

$$D = \{(u_1, \ldots, u_n) \in U \mid u_1 = u_2 = \cdots = u_k = 0\},$$

then (3.2) and (3.3) assures that ψ_x satisfies the following three properties:
(3.4) for any $x \in L_{(\varepsilon)}$, $\psi_x(p) = x$,
(3.5) for any $q \in U_{(\varepsilon)}$, $\psi_p(q) = q$,
(3.6) for any $x \in L_{(\varepsilon)}$ and $q \in U_{(\varepsilon)}$, $\psi_p(q) \in L_q$,

Let $h \colon L_{(\varepsilon)} \times D \to U$ be a map defined by $h(x, y) = \psi_x(y)$ for $x \in L_{(\varepsilon)}$ and $y \in D$. By the definition of $\psi_x(y)$, h is holomorphic. Moreover, if we consider $(u_1, \ldots, u_k)$ and $(u_{k+1}, \ldots, u_n)$ as coordinates on $L_{(\varepsilon)}$ and D respectively, h can be expressed explicitly as

$$h\big((u_1, \ldots, u_k), (u_{k+1}, \ldots, u_n)\big)$$
$$= \big(f_1(u_1, \ldots, u_n), \ldots, f_n(u_1, \ldots, u_n)\big),$$

where $f_i(u_1, \ldots, u_n)$ are holomorphic functions. Then (3.4) implies

$$h\big((u_1, \ldots, u_k), (0, \ldots, 0)\big) = (u_1, \ldots, u_k, 0, \ldots, 0),$$

in other words,

$$f_i(u_1, \ldots, u_k, 0, \ldots, 0) = \begin{cases} u_i & (1 \le i \le k) \\ 0 & (k+1 \le i \le n). \end{cases} \tag{3.7}$$

Similarly (3.5) implies

$$h\big((0, \ldots, 0), (u_{k+1}, \ldots, u_n)\big) = (0, \ldots, 0, u_{k+1}, \ldots, u_n),$$

in other words,

$$f_i(0, \ldots, 0, u_{k+1}, \ldots, u_n) = \begin{cases} 0 & (1 \leq i \leq k) \\ u_i & (k+1 \leq i \leq n). \end{cases} \tag{3.8}$$

(3.7) and (3.8) tell us $\big(\partial f_i/\partial u_j\big)(0) = \delta_{ij}$, so we have

$$\det\left(\frac{\partial(f_1, \ldots, f_n)}{\partial(u_1, \ldots, u_n)}(0)\right) = 1 \ (\neq 0).$$

Hence, if we take $\varepsilon > 0$ sufficiently small and set $U_p = h(L_{(\varepsilon)} \times D)$ then $h\colon L_{(\varepsilon)} \times D \to U_p$ is a biholomorphic map. We define new coordinates $(z_1, \ldots, z_n)$ on U_p as follows:

$$\text{for any } q \in U_p, \quad z_i(q) = \begin{cases} u_i\big(\mathrm{pr}_1 \circ h^{-1}(q)\big) & (1 \leq i \leq k) \\ u_i\big(\mathrm{pr}_2 \circ h^{-1}(q)\big) & (k+1 \leq i \leq n) \end{cases}$$

where pr_j denotes the projection to the j-th component. In other words,

for any $x \in L_{(\varepsilon)}$ and $y \in D$,

$$z_i\big(h(x, y)\big) = \begin{cases} u_i(x) & (1 \leq i \leq k) \\ u_i(y) & (k+1 \leq i \leq n). \end{cases}$$

Clearly we have

$$\bigcup_{x \in L_{(\varepsilon)}} \psi_x(y) = \bigcup_{x \in L_{(\varepsilon)}} h(x, y)$$
$$= \{q \in U_p \mid z_i(q) = z_i(y) \quad (\text{for } i = k+1, \ldots, n)\}. \tag{3.9}$$

On the other hand, it follows from (3.6) that

$$\bigcup_{x \in L_{(\varepsilon)}} \psi_x(y) \subset L_y. \tag{3.10}$$

From (3.9) and (3.10) we have

$$\frac{\partial}{\partial z_1}(y), \frac{\partial}{\partial z_2}(y), \ldots, \frac{\partial}{\partial z_k}(y) \in E(y). \tag{3.11}$$

Next let us construct the submersion $\pi\colon U_p \longrightarrow D$ and the singular foliation E' on D. We identify D with a neighborhood W of 0 in $\mathbf{C}^{n-k} = \{(w_{k+1}, \ldots, w_n)\}$, and set $\pi = \mathrm{pr}_2 \circ h^{-1}$. Note that, by the definition of

$(z_1, \ldots, z_n)$, π is represented using the coordinates $(z_1, \ldots, z_n)$ on U_p and $(w_{k+1}, \ldots, w_n)$ on D as $\pi(z_1, \ldots, z_n) = (z_{k+1}, \ldots, z_n)$. It is clear that π is a holomorphic submersion from U_p onto D. Furthermore, let $\pi_* \colon TU_p \longrightarrow TD$ denote the push-forward of the vector fields form U_p to D. Then

$$\pi_*\left(\frac{\partial}{\partial z_i}\right) = \begin{cases} 0 & (1 \leq i \leq k) \\ \dfrac{\partial}{\partial w_i} & (k+1 \leq i \leq n). \end{cases}$$

Using π_* we define the coherent subsheaf $E' \subset \Theta_D$ by $(E')_y = \pi_*(E_y)$ for each point $y \in D$. Then we have $E'(p) = \{0\}$ since $E(p)$ is generated by $\partial/\partial z_1, \ldots, \partial/\partial z_k$.

In order to complete the proof, we must show that E' is integrable and $\left(\pi^*(E'^a)\right)^a = E|_{U_p}$. Let $\{v'_1, \ldots, v'_s\}$ be a system of local vector fields on D which generates E', and set

$$v'_j = \sum_{i=k+1}^{n} a^j{}_i(w_{k+1}, \ldots, w_n)\frac{\partial}{\partial w_i} \quad (1 \leq j \leq s), \tag{3.12}$$

where each $a^j{}_i$ is a holomorphic function on D. By the definition of E' and (3.11), it turns out that, for any $y = (0, \ldots, 0, y_{k+1}, \ldots, y_n) \in D$, $E(y)$ is spanned by following $k+s$ vectors:

$$\begin{aligned} &\frac{\partial}{\partial z_1}(y), \frac{\partial}{\partial z_2}(y), \ldots, \frac{\partial}{\partial z_k}(y), \\ &\tilde{v}_1 = \sum_{i=k+1}^{n} a^1{}_i(y_{k+1}, \ldots, y_n)\frac{\partial}{\partial z_i}(y), \\ &\quad\vdots \\ &\tilde{v}_s = \sum_{i=k+1}^{n} a^s{}_i(y_{k+1}, \ldots, y_n)\frac{\partial}{\partial z_i}(y). \end{aligned} \tag{3.13}$$

On the other hand, for any $x = (x_1, \ldots, x_k, 0, \ldots, 0) \in L_{(\varepsilon)}$ we have

$$\psi_x(z_1, \ldots, z_n) = (z_1 + x_1, \ldots, z_k + x_k, z_{k+1}, \ldots, z_n),$$

which implies

$$(\psi_x)_*\left(\frac{\partial}{\partial z_i}(y)\right) = \frac{\partial}{\partial z_i}(\psi_x(y)) \quad (1 \leq i \leq n) \tag{3.14}$$

for any $y \in D$. Moreover, Proposition (1.17) says that there exists $\varepsilon > 0$ such that

$$(\psi_x)_*\big(E(y)\big) = E\big(\psi_x(y)\big) \tag{3.15}$$

holds for any $x \in L_{(\varepsilon)}$ and any $y \in D$. Then it follows from (3.13), (3.14) and (3.15) that the space $E\big(\psi_x(y)\big)$ is spanned by following $k+s$ vectors:

$$\begin{aligned}
&\frac{\partial}{\partial z_1}\big(\psi_x(y)\big),\ \frac{\partial}{\partial z_2}\big(\psi_x(y)\big),\ \ldots,\ \frac{\partial}{\partial z_k}\big(\psi_x(y)\big),\\
&(\psi_x)_*(\tilde{v}_1) = \sum_{i=k+1}^{n} a^1{}_i(y_{k+1},\ \ldots,\ y_n)\frac{\partial}{\partial z_i}\big(\psi_x(y)\big),\\
&\qquad\vdots\\
&(\psi_x)_*(\tilde{v}_s) = \sum_{i=k+1}^{n} a^s{}_i(y_{k+1},\ \ldots,\ y_n)\frac{\partial}{\partial z_i}\big(\psi_x(y)\big).
\end{aligned} \tag{3.16}$$

This means that for every point $q \in U_p$ a system of generators of the space $E(q)$ is given by (3.16), therefore $E|_{U_p}$ is generated by the following $k+s$ vector fields:

$$\begin{aligned}
&\frac{\partial}{\partial z_1},\ \frac{\partial}{\partial z_2},\ \ldots,\ \frac{\partial}{\partial z_k},\\
&v_1 = \sum_{i=k+1}^{n} a^1{}_i(z_{k+1},\ \ldots,\ z_n)\frac{\partial}{\partial z_i},\\
&\qquad\vdots\\
&v_s = \sum_{i=k+1}^{n} a^s{}_i(z_{k+1},\ \ldots,\ z_n)\frac{\partial}{\partial z_i}.
\end{aligned} \tag{3.17}$$

(3.12) and (3.17) tell us that if E' is not integrable the $E|_{U_p}$ is not integrable either, hence E' is integrable. Similarly, it turns out that E' is reduced from (3.12), (3.17) and the reducedness of E.

Now all we have to do to complete the proof is to show that

$$E|_{U_p} = \big(\pi^*(E'^a)\big)^a.$$

We can easily check that

$$(E')^a = \left\{ \omega' = \sum_{i=k+1}^{n} b_i(w_{k+1}, \dots, w_n) dw_i \;\middle|\; \sum_{i=k+1}^{n} a^j{}_i b_i \equiv 0 \quad (\text{for } 1 \leq \forall j \leq s) \cdots (*) \right\}. \tag{3.18}$$

By the definition of π and the coordinates $(z_1, \dots, z_n)$, $\pi^*(dw_i) = dz_i$ for all i with $k+1 \leq i \leq n$, hence we have

$$\pi^*\big((E')^a\big) = \left\{ \pi^*(\omega') = \sum_{i=k+1}^{n} b_i(z_{k+1}, \dots, z_n) dz_i \;\middle|\; (b_{k+1}, \dots, b_n) \text{ satisfies } (*) \right\}.$$

In order to calculate $\big(\pi^*((E')^a)\big)^a$, let us consider the condition for a holomorphic vector field ξ on U_p to belong to $\big(\pi^*((E')^a)\big)^a$. We set $\xi = \sum_{l=1}^{n} c_l(z_1, \dots, z_n)\partial/\partial z_l$ where c_l are holomorphic functions on U_p. Then we have

$$\xi \in \Big(\pi^*\big((E')^a\big)\Big)^a \iff \begin{array}{l} \displaystyle\sum_{l=k+1}^{n} b_l(z_{k+1}, \dots, z_n) \\ \qquad \times\, c_l(z_1, \dots, z_n) \equiv 0 \\ \text{for any } (b_{k+1}, \dots, b_n) \text{ satisfying } (*). \end{array} \tag{3.19}$$

Let

$$c_l(z_1, \dots, z_n) = \sum_{(\alpha_1, \dots, \alpha_k) \geq (0, \dots, 0)} h_l^{(\alpha_1, \dots, \alpha_k)}(z_{k+1}, \dots, z_n) {z_1}^{\alpha_1} \cdots {z_k}^{\alpha_k} \tag{3.20}$$

be the series expansion of c_l with respect to $z_1, \dots, z_k$ (all $h_l^{(\alpha_1, \dots, a_k)}$ are holomorphic functions of $z_{k+1}, \dots, z_n$). Substituting (3.20) to (3.19),

$$\sum_{(\alpha_1, \dots, \alpha_k)} \left(\sum_{l=k+1}^{n} b_l h_l^{(\alpha_1, \dots, \alpha_k)} \right) {z_1}^{\alpha_1} \cdots {z_k}^{\alpha_k} \equiv 0,$$

thus we have

$$\sum_{l=k+1}^{n} b_l(z_{k+1}, \ldots, z_n) h_l^{(\alpha_1, \ldots, \alpha_k)}(z_{k+1}, \ldots, z_n) \equiv 0 \tag{3.21}$$

for every $(\alpha_1, \ldots, \alpha_k) \geq (0, \ldots, 0)$. For each $(\alpha_1, \ldots, \alpha_k)$ we define a holomorphic vector field $\xi^{(\alpha_1, \ldots, \alpha_k)}$ on W by

$$\xi^{(\alpha_1, \ldots, \alpha_k)} = \sum_{l=k+1}^{n} h_l^{(\alpha_1, \ldots, \alpha_k)}(x_{k+1}, \ldots, w_n) \frac{\partial}{\partial w_l}, \tag{3.22}$$

then (3.18) and (3.21) imply $\xi^{(\alpha_1, \ldots, \alpha_k)} \in \left((E')^a\right)^a = E'$. Since E' is generated by $v_1', \ldots, v_s'$, we can express $\xi^{(\alpha_1, \ldots, \alpha_k)}$ as

$$\xi^{(\alpha_1, \ldots, \alpha_k)} = \sum_{j=1}^{s} f_j^{(\alpha_1, \ldots, \alpha_k)}(w_{k+1}, \ldots, w_n) v_j', \tag{3.23}$$

where $f_j^{(\alpha_1, \ldots, \alpha_k)}$ are holomorphic functions on W. From (3.22) and (3.23),

$$\sum_{l=k+1}^{n} h_l^{(\alpha_1, \ldots, \alpha_k)}(z_{k+1}, \ldots, z_n) \frac{\partial}{\partial z_l} = \sum_{j=1}^{s} f_j^{(\alpha_1, \ldots, \alpha_k)}(z_{k+1}, \ldots, z_n) v_j$$

holds for every $(\alpha_1, \ldots, \alpha_k)$. Hence we obtain

$$\begin{aligned}
\xi &= \sum_{l=1}^{n} c_l(z_1, \ldots, z_n) \frac{\partial}{\partial z_l} \\
&= \sum_{l=k}^{n} c_l \frac{\partial}{\partial z_l} + \sum_{l=k+1}^{n} c_l(z_1, \ldots, z_n) \frac{\partial}{\partial z_l} \\
&= \sum_{l=k}^{n} c_l \frac{\partial}{\partial z_l} \\
&\quad + \sum_{l=k+1}^{n} \left(\sum_{(\alpha_1, \ldots, \alpha_k)} h_l^{(\alpha_1, \ldots, \alpha_k)}(z_{k+1}, \ldots, z_n) \cdot {z_1}^{\alpha_1} \cdots {z_k}^{\alpha_k} \right) \frac{\partial}{\partial z_l} \\
&= \sum_{l=k}^{n} c_l \frac{\partial}{\partial z_l} \\
&\quad + \sum_{(\alpha_1, \ldots, \alpha_k)} \left({z_1}^{\alpha_1} \cdots {z_k}^{\alpha_k} \left(\sum_{l=k+1}^{n} h_l^{(\alpha_1, \ldots, \alpha_k)}(z_{k+1}, \ldots, z_n) \frac{\partial}{\partial z_l} \right) \right)
\end{aligned}$$

$$
\begin{aligned}
&=\sum_{l=k}^{n} c_l \frac{\partial}{\partial z_l} \\
&\quad + \sum_{(\alpha_1, \dots, \alpha_k)} \left({z_1}^{\alpha_1} \cdots {z_k}^{\alpha_k} \left(\sum_{j=1}^{s} f_j^{(\alpha_1, \dots, \alpha_k)}(z_{k+1}, \dots, z_n) v_j \right) \right) \\
&=\sum_{l=k}^{n} c_l \frac{\partial}{\partial z_l} \\
&\quad + \sum_{j=1}^{s} \left(\sum_{(\alpha_1, \dots, \alpha_k)} f_j^{(\alpha_1, \dots, \alpha_k)}(z_{k+1}, \dots, z_n) \cdot {z_1}^{\alpha_1} \cdots {z_k}^{\alpha_k} \right) v_j. \cdots (**)
\end{aligned}
$$

Note that each $\sum_{(\alpha_1, \dots, \alpha_k)} f_j^{(\alpha_1, \dots, \alpha_k)}(z_{k+1}, \dots, z_n) \cdot {z_1}^{\alpha_1} \cdots {z_k}^{\alpha_k}$ appearing in $(**)$ can represent an arbitrary holomorphic function on U_p. This implies

$$
\begin{aligned}
\xi \in \left(\pi^*((E')^a)\right)^a \Longleftrightarrow & \xi \text{ can be express as a linear combination} \\
& \text{of } \partial/\partial z_1, \dots, \partial/\partial z_1, v_1, \dots, v_s \text{ like } (**) \\
\Longleftrightarrow & \xi \in E|_{U_p},
\end{aligned}
$$

thus we have $\left(\pi^*((E')^a)\right)^a = E|_{U_p}$. □

Remark 3.24 As mentioned before, the above process of obtaining (3.17) gives an alternative proof of the normal form theorem of [C] (cf. Remark 1.19). We may prove the "local analytical triviality", expressed in one form or another, without assuming that E be reduced, only assuming the involutiveness (1.5). However, if we wish to express the local triviality as in Theorem 3.1, E out to be reduced.

As an application of Theorem (3.1), we can show the following proposition.

Proposition 3.25 *If a singular foliation E $(\in \Theta_M)$ is reduced, then* $\operatorname{codim} S(E) \geq 2$.

Remark 3.26 For the converse of this proposition, we have counterexamples. However, under the assumption that E is locally free, the converse is also true (cf. Remark (1.6)).

Proof of (3.25). Suppose that E is reduced and $\operatorname{codim} S(E) = 1$. Set $\dim_{\mathbf{C}} M = n$ and $\operatorname{rank} E = r$. First we chose a point $p \in S(E)$ such that $p \notin \operatorname{Sing}(S(E))$ and $\dim_p S(E) = n-1$. Take a sufficiently small neighborhood

U of p and coordinates $(z_1, \ldots, z_n)$ on U such that $U \cap S(E) = \{z_n = 0\}$ and $p = (0, \ldots, 0)$. We set $k = \max\{\dim_{\mathbf{C}} E(q) \mid q \in U \cap S(E)\}$, then clearly $0 \leq k \leq r-1$.

Next, choose a point q in $U \cap S(E)$ such that $\dim_{\mathbf{C}} E(q) = k$. We 'shift', for simplicity, the coordinates $(z_1, \ldots, z_n)$ on U so that $q = (0, \ldots, 0)$. Since $S^{(k-1)} = \{x \in M \mid \dim_{\mathbf{C}} E(x) \leq k-1\}$ is a closed set, we can take a neighborhood U_q $(\subset U)$ of q so that $U_q \cap S^{(k-1)} = \emptyset$. Then we have $U_q \cap S(E) = U_q \cap L^{(k)}$ (for the definition of $L^{(k)}$, see (1.14)). Applying Theorem (3.1) (or (3.17) in the proof), we can retake U_q and $(z_1, \ldots, z_n)$ so that $E|_{U_q}$ is generated by the following $k+l$ holomorphic vector fields:

$$\begin{aligned} &\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_k}, \\ &v_1 = \sum_{i=k+1}^{n} a^1{}_i(z_{k+1}, \ldots, z_n)\frac{\partial}{\partial z_i}, \\ &\quad\vdots \\ &v_s = \sum_{i=k+1}^{n} a^s{}_i(z_{k+1}, \ldots, z_n)\frac{\partial}{\partial z_i}. \end{aligned} \tag{3.27}$$

If $s = 0$ then E gives a non-singular foliation on U_q. This contradicts $q \in S(E)$, so we have $s \geq 1$. On the other hand, $U_q \cap S(E) = U_q \cap L^{(k)}$ implies that $\dim_{\mathbf{C}} E(x) = k$ holds for every point $x \in U_q \cap S(E)$, therefore all $a^j{}_i$ appearing in (3.27) satisfy $a^j{}_i(z_{k+1}, \ldots, z_{n-1}, 0) \equiv 0$. For $i = k+1, \ldots, n$, we represent $a^1{}_i$ as

$$a^1{}_i(z_{k+1}, \ldots, z_n) = {z_n}^{\alpha_i} \cdot b_i(z_{k+1}, \ldots, z_n)$$

where $\alpha_i \in \mathbf{Z}$ and b_i are holomorphic functions such that $b_i(z_{k+1}, \ldots, z_{n-1}, 0) \not\equiv 0$. Note that α_i and b_i are uniquely determined and $\alpha_i \geq 1$. We set $\alpha = \min\{\alpha_i\}$, and define a holomorphic vector field $\tilde{v_1}$ on U_q by

$$\tilde{v_1} = \sum_{i=k+1}^{n} {z_n}^{\alpha_i - \alpha} b_i(z_{k+1}, \ldots, z_n)\frac{\partial}{\partial z_i} \; \left(= \frac{1}{{z_n}^{\alpha}} v_1\right).$$

Then we have $\tilde{v_1}|_{U_q - S(E)}$, but $\tilde{v_1} \notin E|_{U_q}$ since $\tilde{v_1} \not\equiv 0$. This contradicts that E is reduced. $\square$

Let us close this paper by giving some examples about singular foliations and its local analytical triviality.

Example 3.28 Let f be the holomorphic function on $M = \mathbf{C}^3$ defined by

$$f(x, y, z) = z(x^2 - y^2),$$

and ω the holomorphic 1-form on $\mathbf{C}^3$ defined by

$$\omega = df = 2xzdx - 2yzdy + (x^2 - y^2)dz.$$

The coherent subsheaf F ($\subset \Omega_M$) generated by ω is integrable since $d\omega = ddf = 0$, so F defines a singular foliation on $\mathbf{C}^3$. $E = F^a$ ($\subset \Theta_M$) is generated by the following two vector fields:

$$\begin{cases} v_1 = y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y} \\ v_2 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} - 2z\dfrac{\partial}{\partial z}. \end{cases} \tag{3.29}$$

E is reduced, and $\operatorname{rank} E = 2$. By (3.29), $S(E) = S^{(1)} = \{xz = yz = x^2 - y^2 = 0\} = \{x = y = 0\} \cup \{z = x^2 - y^2 = 0\}$ and $S^{(0)} = \{(0, 0, 0)\}$. According to Theorem (3.1), the structure of E is locally analytically trivial along each leaves, in particular along $\{x = y = 0\} - \{0\}$, $\{z = x - y = 0\} - \{0\}$ and $\{z = x + y = 0\} - \{0\}$.

Example 3.30 Let ω be the holomorphic 1-form on $\mathbf{C}^4 = \{(x, y, z, w)\}$ defined by

$$\omega = x(z^2 - w^2)dx - y(z^2 - w^2)dy - z(x^2 - y^2)dz + w(x^2 - y^2)dw.$$

It is easy to check that $d\omega = 4(-xzdx \wedge dz + xwdx \wedge dw + yzdy \wedge dz - ywdy \wedge dw)$. The coherent subsheaf F ($\subset \Omega_M$) generated by ω is integrable since $\omega \wedge d\omega = 0$, so F defines a singular foliation on $\mathbf{C}^4$. $E = F^a$ ($\subset \Theta_M$) is generated by the following three vector fields:

$$\begin{cases} v_1 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z} + w\dfrac{\partial}{\partial w} \\ v_2 = y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y} \\ v_3 = \qquad\qquad\qquad w\dfrac{\partial}{\partial z} + z\dfrac{\partial}{\partial w}. \end{cases} \tag{3.31}$$

E is reduced, and $\operatorname{rank} E = 3$. By (3.31),

$$
\begin{aligned}
S(E)=&\\
S^{(2)}=&\{x(z^2-w^2)=y(z^2-w^2)=z(x^2-y^2)=w(x^2-y^2)=0\}\\
=&\{x=y=0\}\cup\{z=w=0\}\cup\{x^2-y^2=z^2-w^2=0\},\\
S^{(1)}=&\{x^2-y^2=z^2-w^2=xz=xw=yz=yw=0\}\\
=&\{x^2-y^2=z=w=0\}\cup\{z^2-w^2=x=y=0\},\\
S^{(0)}=&\{(0,0,0,0)\}.
\end{aligned}
$$

Example 3.32 Let ω be the holomorphic 1-form on $\mathbf{C}^3$ defined by

$$\omega = y(x+y)dx - x(x+y)dy + (x^3-y^3)dz.$$

It is easy to check that $d\omega = 3\{(-x-y)dx\wedge dy + x^2 dx\wedge dz - y^2 dy\wedge dz\}$. The coherent subsheaf F ($\subset \Omega_M$) generated by ω is integrable since $\omega\wedge d\omega = 0$, so F defines a singular foliation on $\mathbf{C}^3$. $E = F^a$ ($\subset \Theta_M$) is generated by the following two vector fields:

$$
\begin{cases}
v_1 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}\\
v_2 = y^2\dfrac{\partial}{\partial x} + x^2\dfrac{\partial}{\partial y} + (x+y)\dfrac{\partial}{\partial z}.
\end{cases}
\tag{3.33}
$$

E is reduced, and $\operatorname{rank} E = 2$. By (3.33),

$$
\begin{aligned}
S(E) = S^{(1)}=&\{y(x+y)=x(x+y)=x^3-y^3=0\}\\
=&\{x=y=0\}\cup\{x+y=x^3-y^3=0\}\\
=&\{x=y=0\},\\
S^{(0)}=&\{x=y=0\}=S^{(1)}.
\end{aligned}
$$

In this case, Theorem (3.1) means nothing particular about the structure of E along $S(E)=\{x=y=0\}$, since the leaf passing through each point $p\in S(E)$ consists of only one point.

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