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Dominated semigroups of operators and evolution processes

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Abstract. A semigroup S acting on a Banach lattice E is said to be dominated if there exists a positive semigroup T such that $|S(t)x| \leq T(t)|x|$ for all $x \in E$ and $t > 0$. It is shown that a semigroup on L^p is dominated if and only if it is associated with a family of operator valued measures.

Key words: dominated semigroup, evolution equation, modulus, regular operator, variation.

Introduction

The problem of determining when a semigroup S of operators acting on a Dedekind complete Banach lattice is dominated by a semigroup T of positive operators is an old one. Because $|S(t)x| \leq T(t)|x|$ for each element x, it follows that $S(t)$ must be a regular operator for each $t \geq 0$. A related question is when does the smallest such semigroup $|S|$ exist — the *modulus semigroup* of S. Although a C_0-*contraction* semigroup on L^1 is dominated by a positive semigroup (see [12], [14]), C. Kipnis [12, pp. 374–376] gives an example of a C_0-semigroup on ℓ^1 which is not dominated by any positive semigroup. Other examples are provided by the semigroups T_z mentioned below with $\Im z > 0$ and $\Re z \neq 0$. A sufficient condition that a semigroup S on an L^p-space be dominated by a positive semigroup is provided by I. Becker and G. Greiner [3, Proposition 2.3]: if there exists a real number ω such that $\| \, |S(t)| \, \|_{\mathcal{L}(L^p)} \leq e^{\omega t}$ for all $t \geq 0$ (that is, S is *quasicontractive with respect to the regular norm*), then S is dominated and if, in addition, S is a C_0-semigroup, then the modulus semigroup $|S|$ is also a C_0-semigroup.

Regular operators R acting on L^p-spaces were characterised in [10] in terms of an operator bound of the form

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$$\left\| \sum_{j=1}^{k} g_j R(f_j \psi) \right\|_p \leq C \left\| \sum_{j=1}^{k} f_j \otimes g_j \right\|_\infty \|\psi\|_p, \qquad \psi \in L^p. \tag{0.1}$$

Here C is a positive constant, f_j and g_j are bounded measurable functions for each $j = 1, \ldots, k$, and k is any positive integer. This has recently been generalised to regular operators on a Banach lattice [23].

Another way of viewing the bound (0.1) is that the additive vector valued set function $A \times B \mapsto \chi_B R(\chi_A \cdot \psi)$ is bounded in L^p on the algebra generated by all product sets $A \times B$ for each $\psi \in L^p$. In the present work, we adopt this viewpoint to characterise those semigroups S acting on L^p that are dominated by a semigroup of positive operators. The property in question is exactly that there should exist an associated family $\langle M_t \rangle_{t \geq 0}$ of uniformly bounded, additive operator valued set functions considered in the monograph [9] in relation to generalisations of the Feynman-Kac formula. Equivalently, a multilinear bound of the form (0.1) is both necessary and sufficient for the existence of a dominating semigroup acting on L^p.

The set functions $\langle M_t \rangle_{t \geq 0}$ are defined on the algebra generated by all *cylinder sets*

$$A = \{X_0 \in B_0,\, X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n,\, X_t \in B_{n+1}\} \tag{0.2}$$

with respect to the random process $\langle X_s \rangle_{0 \leq s \leq t}$. The operator valued set function M_t acts on L^p and is defined by the formula

$$M_t(A) = Q(B_{n+1})\, S(t - t_n) \cdots Q(B_2)\, S(t_2 - t_1)\, Q(B_1)\, S(t_1)\, Q(B_0); \tag{0.3}$$

see Section 3 for the precise statement. The spectral measure Q acting on $L^p(\mu)$ is multiplication by characteristic functions. The operator valued set function M_t represents the 'joint distribution' of the random process $\langle X_s \rangle_{s \geq 0}$ before time t.

If we know that S is dominated, then it is easy to see that M_t is uniformly bounded on the algebra generated by all cylinder sets A of the form (0.2). The technical question of proving that M_t is σ-additive and extending M_t to a suitable σ-algebra of subsets of a sample space Ω presents no problem if, say, μ is a regular Borel measure, so we may think of M_t as an operator valued 'measure' for each $t \geq 0$.

The results of Kipnis [12] and Kubokawa [14] may now be seen from this viewpoint. According to [9, Theorem 2.3.6], a *contraction* semigroup

S on L^1 produces operator valued measures $\langle M_t\rangle_{t\geq 0}$ acting on L^1 from formula (0.3) and so, according to Theorem 3.1 below, S has a dominating semigroup. Indeed, an estimate of the semivariation of an operator valued measure underlies the proofs given in [12] and [14] in the L^1 case. Furthermore, the C_0-semigroups constructed in [12] for ℓ^1 and the semigroups T_z mentioned below with $\Im z > 0$ and $\Re z \neq 0$ produce additive operator valued set functions $\langle M_t\rangle_{t\geq 0}$ acting on $L^1(\mathbb{R}^n)$ via formula (0.3). A calculation shows that these are *unbounded* on the algebra generated by the cylinder sets A of the form (0.2), that is, the range of each of the set functions M_t on the algebra is unbounded in the uniform operator norm. Such set functions cannot be the restriction of operator valued measures defined on a σ-algebra of subsets of a set and therefore according to Theorem 3.1 of the present paper, such semigroups cannot be dominated.

It is easy to see that the sufficient condition of I. Becker and G. Greiner [3, Proposition 2.3] for L^p spaces, yields operator valued measures $\langle M_t\rangle_{t\geq 0}$ acting on L^p from formula (0.3) and so, according to Theorem 3.1 below, S has a dominating semigroup.

Semigroups of positive operators acting on L^p-spaces have been the subject of recent interest in the mathematical physics literature in relation to *point interactions*. For example, by taking an appropriate selfadjoint extension of the Laplacian operator Δ restricted to $C_c^\infty(\mathbb{R}^3\setminus\{0\})$, we obtain a Hamiltonian operator $H = -\frac{1}{2}\Delta + c\delta$ for which $t \mapsto e^{-tH}$ is a C_0-semigroup of positive operators on L^p only for $\frac{3}{2} < p < 3$ (see [1], [4]). Semigroups such as these have associated positive operator valued measures acting on L^p by formula (0.3), but they are not given by the transition functions of a Markov process, that is, they are not *Markov semigroups*. Nevertheless, there is an associated random process measured by operator valued measures instead of a probability measure in the fashion described above.

On the other hand, the semigroups $T_z : t \mapsto \exp\left(\frac{it}{2z}\Delta\right)$ with $\Im(z) > 0$ arise in relation to analytic Feynman integrals [11]. Here Δ is the Laplacian in $\mathbb{R}^n$. Although T_z is a semigroup of regular operators on $L^p(\mathbb{R}^n)$, there is an associated operator valued measure acting on $L^p(\mathbb{R}^n)$ by formula (0.3) if and only if $\Re(z) = 0$. In this case, the operator valued measure can be written in terms of Wiener measure, see [9, Example 2.1.7] for the case $p = 2$. Thus, for any $1 \leq p \leq \infty$, it is only for $\Re z = 0$ that T_z is dominated by a semigroup of positive operators on $L^p(\mathbb{R}^n)$, and if $\Re z$ is equal to zero, then T_z is actually a semigroup of positive operators. The semigroup T_z

is a contraction on $L^2(\mathbb{R}^n)$, but it does not give rise to an operator valued measure by formula (0.3) if $\Re z \neq 0$ — otherwise Feynman integration would be much simpler! The result of Kipnis [12] and Kubokawa [14] is therefore special to L^1 and, by duality, to L^∞.

Perturbation theory for dominated semigroups in L^p-spaces has been the subject of recent study in [16]. The related question of showing that a positive C_0-semigroup dominates another C_0-semigroup by prescribing conditions on the generators is treated in [18, Part C-II, §4] and more recently, in [19].

In Section 1, we set the notation and terminology concerning Banach lattices used throughout the present work. The scalar variation of set functions defined on finite product sets by operator products is computed in Section 2. Special care has to be exercised to treat the case $p = \infty$, where we impose the additional assumption that the underlying measure space is localisable, [21, pp. 157–158], otherwise, there is no restriction on the measure space. The main result, Theorem 3.1, characterising dominated semigroups in terms of the uniform boundedness of each of the associated set functions $\langle M_t\rangle_{t\geq 0}$ is proved in Section 3. No continuity assumption is needed for the semigroups S. In Proposition 3.3, we show that in the case when the modulus semigroup $|S|$ does exist the operator valued measure N_t associated with $|S|$ is actually the smallest positive operator valued set function dominating M_t on the algebra of cylinder sets generated by $\langle X_s\rangle_{0\leq s\leq t}$. As a byproduct of the proof, we show in Proposition 3.5 that N_t and M_t, $t \geq 0$, share the same path spaces. Finally, a simple example shows that the sufficient condition of [3, Proposition 2.3] for semigroup domination on L^p-spaces is not a necessary condition. An illustration of semigroup domination and its relationship with operator valued measures is given in the context of the motion of a quantum particle in a magnetic field.

1. Preliminaries

Let $\mathbb{F}$ denote either the real or complex number field. Let Y be a locally convex Hausdorff space over $\mathbb{F}$ with dual space Y' so that $\langle Y, Y'\rangle$ is a duality between the vector spaces Y and Y'. In this notation, we write $\langle y, y^*\rangle = y^*(y)$ for all $y \in Y$ and $y^* \in Z$, whenever Z is a linear subspace of the algebraic dual of Y such that Z separates points of Y. The space of all continuous linear maps from Y into Y is denoted by $\mathcal{L}(Y)$. The pointwise convergence topology on $\mathcal{L}(Y)$ is called the *strong operator topology.*

Now let Y be a Banach space, and assume that its dual Y' is equipped with the dual norm topology $\beta(Y', Y)$. Given $W \in \mathcal{L}(Y)$ define its dual operator $W \in \mathcal{L}(Y')$ by $\langle y, W'y' \rangle = \langle W(y), y' \rangle$ for all $y \in Y$ and $y' \in Y'$. Let $\sigma(Y', Y)$ denote the weak* topology on Y'. Then

$$\mathcal{L}(Y'_{\sigma(Y',Y)}) = \{W' : W \in \mathcal{L}(Y)\} \subseteq \mathcal{L}(Y'). \tag{1.1}$$

Let E be a Banach lattice over $\mathbb{F}$ with norm $\|\cdot\|$ and order relation $\leq$. In the case in which $\mathbb{F}$ is complex, see [21, §11], for example. The positive cone of E is defined to be $E_+ = \{x \in E : x \geq 0\}$. A continuous linear operator from E into a Banach lattice is called *positive* if W maps the positive cone E_+ of E into that of the codomain, and this is denoted by $W \geq 0$. Let $\mathcal{L}(E)_+$ denote the subset of $\mathcal{L}(E)$ consisting of all positive operators. Clearly, $\mathcal{L}(E)_+$ becomes an ordered set in which we can speak of increasing nets. We call an operator $W \in \mathcal{L}(E)$ *regular* if it is a linear combination of positive operators, and the linear subspace of $\mathcal{L}(E)$ consisting of all regular operators is denoted by $\mathcal{L}^{\mathrm{r}}(E)$. When E is Dedekind complete, the space $\mathcal{L}^{\mathrm{r}}(E)$ is a vector lattice such that each W has its modulus $|W|$ defined by $|W|(x) = \sup_{0 \leq |y| \leq x} |Wy|$ for all $x \in E_+$; see [21, Definition IV.1.7 and Theorem IV.1.8].

Remark 1.1 Suppose that the Banach lattice E is Dedekind complete. If operators $R \in \mathcal{L}(E)_+$ and $W \in \mathcal{L}(E)$ satisfy

$$|Wx| \leq R|x| \quad \text{for every} \quad x \in E, \tag{1.2}$$

then it easily follows that the operator W is regular and $|W| \leq R$. We can deduce the same conclusion even when (1.2) holds only for all *positive* elements x of E. Because this result does not seem to be available in the literature, we present a proof due to B. de Pagter.

To this end, assume that (1.2) holds for all $x \in E_+$. The operator W then becomes order bounded, and hence regular because E is Dedekind complete, [2, Theorem 1.13].

First consider the case when $\mathbb{F} = \mathbb{R}$. Let $W^+ = W \vee 0$ and $W^- = (-W) \vee 0$. Since $W^+x = \sup\{Wy : 0 \leq y \leq x\}$, [2, p. 13], we have $W^+ \leq R$. Moreover, $W^- \leq R$ because $W = (-W)^+$. Thus we have $W^+ \vee W^- \leq R$. Since

$$|W| = W^+ + W^- = W^+ \vee W^- + W^+ \wedge W^- = W^+ \vee W^-,$$

it follows that $|W| \leq R$.

Now consider the case when $\mathbb{F} = C$. There is a real Banach lattice $E_{\mathbb{R}}$ whose complexification equals E, [21, Definition II.11.3]. There exist operators W_1, $W_2 \in \mathcal{L}(E)$ such that $W_j(E_{\mathbb{R}}) \leq E_{\mathbb{R}}$ for $j = 1, 2$, the identity $W = W_1 + iW_2$ holds, and

$$|W| = \sup\big\{(\cos\theta)W_1 + (\sin\theta)W_2 : 0 \leq \theta < 2\pi\big\}; \tag{1.3}$$

see [21, Theorem IV.1.8]. Now fix $\theta \in \big[0, 2\pi\big[$ and $x \in E_+$. Since

$$|Wx| = \sup\big\{(\cos\varphi)W_1x + (\sin\varphi)W_2x : 0 \leq \varphi < 2\pi\big\}$$

(see [21, p. 134]), we have $\big((\cos\theta)W_1 + (\sin\theta)W_2\big)x \leq |Wx| \leq Rx$ for all $x \in E_+$. This together with (1.3) implies that $|W| \leq R$.

The dual space E' of E is a Banach lattice with positive cone E'_+ such that the modulus $|x'|$ of each $x' \in E'$ is defined as in the case of regular operators; see [21, Corollary 3, p. 235], for example.

A subset of $\mathcal{L}(E)$ is called *uniformly bounded* if it is bounded in the operator norm $\|\cdot\|$.

The Banach lattice E is called a KB-*space* if every norm bounded, increasing sequence in E is convergent. It follows from [24, Theorem 113.4] that every norm bounded, increasing *net* in a KB-space is convergent, from which we can easily derive the following result.

Proposition 1.2 *Let E be a* KB-*space. Then every uniformly bounded, increasing net $\{W_\alpha\}_{\alpha\in A}$ in $\mathcal{L}(E)_+$ has a limit $W \in \mathcal{L}(E)_+$ in the strong operator topology and* $\sup_{\alpha\in A} W_\alpha = W$ *in the vector lattice $\mathcal{L}^{\mathrm{r}}(E)$.*

Throughout the rest of this section, let $(\Sigma, \mathcal{E}, \mu)$ denote a measure space with $\mu(\mathcal{E}) \subseteq \big[0, \infty\big]$. When $1 \leq p < \infty$, the class of all equivalence classes of pth integrable functions on Σ is denoted by $L^p(\mu)$, and is equipped with the usual L^p-norm $\|\cdot\|_p$. Now let us consider the case when $p = \infty$. By $L^\infty(\mu)$ we denote the space of all equivalence classes of $\mathbb{F}$-valued μ-essentially bounded, $\mathcal{E}$-measurable functions on Σ, and by $\|\cdot\|_\infty$ the essential supremum norm [8, Definition (20.11) and Theorem (20.14)]. An equivalence class in $L^p(\mu)$ and its representative will not be distinguished when $1 \leq p \leq \infty$. The identity operator on $L^p(\mu)$ is denote by I_p.

If $1 \leq p < \infty$ then $L^p(\mu)$ is a Dedekind complete Banach lattice whose positive cone consists of all functions $f \in L^p(\mu)$ such that $f \geq 0$ (μ-almost everywhere), [21, Proposition II.8.3]. The Banach lattice $L^p(\mu)$ is a KB-space (for example, see [17, Corollary 2.4.13]).

When $p = \infty$, the space $L^\infty(\mu)$ is a Banach lattice whose positive cone consists of all $f \in L^\infty(\mu)$ such that $f \geq 0$ (locally μ-almost everywhere). The Banach lattice $L^\infty(\mu)$ is Dedekind complete if and only if $L^\infty(\mu)$ is the dual of $L^1(\mu)$ if and only if the measure space $(\Sigma, \mathcal{E}, \mu)$ is localisable, [21, Exercise II.23]. Before we go further, we shall recall the definition of localisable measure spaces. According to [21, pp. 157–158], the measure space $(\Sigma, \mathcal{E}, \mu)$ is called *localisable* if the following conditions are satisfied:

(a) $\mathcal{E}$ has no atoms of infinite measure; and

(b) with $\mathcal{E}_0$ denoting the ring of all sets in $\mathcal{E}$ of finite measure, if $\{f_A\}_{A\in\mathcal{E}_0}$ is a family of measurable functions on Σ such that $f_A = f_B$ (μ-almost everywhere) on $A \cap B$ whenever $A, B \in \mathcal{E}_0$, then there exists a locally measurable function $f : \Sigma \to C$ such that $f = f_A$ (μ-almost everywhere) on each set $A \in \mathcal{E}_0$.

Proposition 1.2 applies to the KB-space $L^p(\mu)$ if $1 \leq p < \infty$, but not to $L^\infty(\mu)$ which is not a KB-space except when $L^\infty(\mu)$ is finite-dimensional.

As usual, the space $L^\infty(\mu)$ poses special difficulties in our arguments. To deal with $L^\infty(\mu)$, we use the absolute weak* topology. Given x in a Banach lattice E, define a seminorm on E' by $q_x : x' \mapsto \langle |x|, |x'| \rangle$ for all $x' \in E'$. The locally convex Hausdorff topology $|\sigma|(E', E)$ given by the class of seminorms q_x, $x \in E$, is called the *absolute weak * topology* on the dual space E' of E, [2, p. 169].

The seminorms q_x, $x \in E$, are Riesz seminorms so that the lattice operations in E' are continuous in $|\sigma|(E', E)$. The convergence of an *increasing* net in E' with respect to $|\sigma|(E', E)$ is equivalent to that of the net with respect to the weak* topology $\sigma(E', E)$ on E'. The following proposition is a direct consequence of this remark.

Proposition 1.3 *The following statements hold on the dual Banach lattice E' of E.*

(i) *Every increasing net $\{x'_\alpha\}_{\alpha\in A}$ in E'_+ such that $\sup_{\alpha\in A} |\langle x'_\alpha, x\rangle| < \infty$ for every $x \in E$ has a limit in E' for the topology $|\sigma|(E', E)$.*

(ii) *If $\{W_\alpha\}_{\alpha\in A}$ is a uniformly bounded, increasing net in $\mathcal{L}(E')_+$ then there is an operator $W \in \mathcal{L}(E')_+$ such that $\lim_{\alpha\in A} W_\alpha x = Wx$ for every $x \in E'$ in the topology $|\sigma|(E', E)$.*

In Section 3 we shall apply the above proposition to $E' = L^\infty(\mu)$ when $(\Sigma, \mathcal{E}, \mu)$ is a localisable measure space. The following lemma, whose proof is straightforward, will also be needed in Section 3.

Lemma 1.4 *Let $1 \le p \le \infty$. Then operators $W_1, W_2 \in \mathcal{L}(L^p(\mu))_+$ satisfy $W_1 \le W_2$ if and only if $\langle W_1 x, y\rangle \le \langle W_2 x, y\rangle$, for all $x \in L^p(\mu)_+$ and $y \in L^{p'}(\mu)_+$.*

2. Variation

By a *semialgebra* we mean a family $\mathcal{C}$ of subsets of a non-empty set such that

(i) the whole set is a member of $\mathcal{C}$;
(ii) $\mathcal{C}$ is closed under finite intersections; and
(iii) if $A, B \in \mathcal{C}$, then there exist a positive integer n and sets $U_j \in \mathcal{C}$, $j = 0, 1, \ldots, n$, such that $A \cap B = U_0$ and $A \backslash B = \bigcup_{j=1}^n U_j$, and such that $\bigcup_{j=0}^k U_j \in \mathcal{C}$ for all $k = 1, \ldots, n$.

The algebra generated by a semialgebra $\mathcal{C}$ is denoted by $a(\mathcal{C})$.

Typical examples of semialgebras are the family of all rectangles $\prod_{j=1}^n [a_j, b_j[$ in $\mathbb{R}^n$ and that of all cylinder sets $\{\omega : \omega(t_1) \in B_1, \ldots, \omega(t_k) \in B_k\}$ with $B_1, \ldots, B_k$ Borel subsets of $\mathbb{R}$, in the space $C([0, \infty[)$ of all continuous functions $\omega : [0, \infty[\to \mathbb{F}$.

An additive set function $m : \mathcal{A} \to Y$ from an algebra $\mathcal{A}$ of subsets of a set Ω to a locally convex space Y is said to be *bounded* or have *bounded range* if the set $\{m(A) : A \in \mathcal{A}\}$ is bounded in the topology of Y. In the case that X is a Banach space and $Y = \mathcal{L}(X)$, boundedness for the strong operator topology and for the uniform operator topology coincide by the uniform boundedness principle. In this case, the terminology *uniformly bounded* is used. Of course, an additive set function may be bounded on a semialgebra $\mathcal{C}$ but not on the algebra $a(\mathcal{C})$ it generates. In the examples which follow, this is precisely the phenomenon in which we are interested.

Throughout this section, let $(\Sigma, \mathcal{E}, \mu)$ be a measure space with $\mu(\mathcal{E}) \subseteq [0, \infty]$. By $\mathcal{L}^\infty(\mathcal{E})$ we denote the vector space of all $\mathbb{F}$-valued, bounded $\mathcal{E}$-measurable functions on Σ.

The family $\mathcal{E}^2 = \{B \times C : B, C \in \mathcal{E}\}$ of rectangles in the Cartesian product $\Sigma^2 = \Sigma \times \Sigma$ is a semialgebra. A vector-valued set function m on $\mathcal{E}^2$ is said to be *separately additive* if the set functions $m(\,\cdot\,, C)$ and $m(B, \,\cdot\,)$

on $\mathcal{E}$ are additive whenever $B, C \in \mathcal{E}$ are fixed. Such a set function m can easily be shown to be additive on the semialgebra $\mathcal{E}^2$. Then m is uniquely extended to an additive set function on $a(\mathcal{E}^2)$; see, for example, [13, Proposition 7.1].

Let $1 \le p \le \infty$. The adjoint index $p' \in [1, \infty]$ is the extended real number satisfying the equation $1/p + 1/p' = 1$, with the understanding that $1/\infty = 0$.

For each $f \in \mathcal{L}^\infty(\mathcal{E})$, let $Q_p(f) \in \mathcal{L}(L^p(\mu))$ denote the operator defined by

$$Q_p(f)x = fx, \qquad x \in L^p(\mu). \tag{2.1}$$

Given $B \in \mathcal{E}$, its characteristic function χ_B belongs to $\mathcal{L}^\infty(\mathcal{E})$ and we let $Q_p(B) = Q_p(\chi_B)$.

Remark 2.1 Let $1 \le p < \infty$. Let Q_p denote the $\mathcal{L}(L^p(\mu))$-valued additive set function $B \mapsto Q_p(B)$, $B \in \mathcal{E}$. Then Q_p is σ-additive in the strong operator topology on $\mathcal{L}(L^p(\mu))$, and satisfies $Q_p(\Sigma) = I_p$ and $Q_p(B \cap C) = Q_p(B)\,Q_p(C)$ for all $B, C \in \mathcal{E}$; in other words, Q_p is a *spectral measure*. The space of Q_p-integrable functions coincides with $\mathcal{L}^\infty(\mathcal{E})$ and the integral of each $f \in \mathcal{L}^\infty(\mathcal{E})$ over Σ is the operator $Q_p(f)$ defined in (2.1). For the details, see, for example, [5, Theorem XVIII.2.11 (c)] or [20, (1), p. 436]. In the case when $p = \infty$, the same result holds if $L^\infty(\mu)$ is equipped with the topology $\sigma(L^\infty(\mu), L^1(\mu))$.

Lemma 2.2 *Let $1 \le p \le \infty$ and let $R \in \mathcal{L}(L^p(\mu))$. Let $m : a(\mathcal{E}^2) \to \mathcal{L}(L^p(\mu))$ denote the additive set function satisfying*

$$m(B \times C) = Q_p(B)\,R\,Q_p(C), \qquad B, C \in \mathcal{E}. \tag{2.2}$$

(i) *The set function m has uniformly bounded range in $\mathcal{L}(L^p(\mu))$ if and only if there is a constant $c > 0$ such that*

$$\left\| \sum_{j=1}^{k} Q_p(g_j)\,R\,Q_p(f_j) \right\|_{\mathcal{L}(L^p(\mu))} \le c \cdot \left\| \sum_{j=1}^{k} f_j \otimes g_j \right\|_\infty$$

for all $f_j, g_j \in \mathcal{L}^\infty(\mathcal{E})$, $j = 1, \ldots, k$, and for all $k = 1, 2, \ldots$.

(ii) *If $p = 1$ or ∞, then m always has uniformly bounded range in $\mathcal{L}(L^p(\mu))$.*

(iii) *When* $1 < p < \infty$, *the set function* m *has uniformly bounded range in* $\mathcal{L}(L^p(\mu))$ *if and only if* R *is regular.*

Proof. (i) This has essentially been established in the second half of the proof of Proposition 1 in [10] as an application of [6, Proposition I.1.11 and Theorem I.1.13].

(ii) Apply (i) and [10, Proposition 1].

(iii) This is a direct consequence of [10, Theorem 1]. □

The equality $\mathcal{L}(L^1(\mu)) = \mathcal{L}^{\mathrm{r}}(L^1(\mu))$ holds; see Theorem IV.1.5 (ii) and the remark after Definition IV.4.2 in [21]. In the case when $p = \infty$ and $(\Sigma, \mathcal{E}, \mu)$ is localisable, we have $\mathcal{L}(L^\infty(\mu)) = \mathcal{L}^{\mathrm{r}}(L^\infty(\mu))$, [21, Theorem IV.1.5 (i)].

Let $1 \le p \le \infty$ and let $R \in \mathcal{L}^{\mathrm{r}}(L^p(\mu))$. Let m be the set function given in Lemma 2.2. Given $x \in L^p(\mu)$ and $y \in L^{p'}(\mu)$, define an additive set function $m_{x,y} : a(\mathcal{E}^2) \to \mathbb{F}$ by

$$m_{x,y}(A) = \langle m(A)x,\, y\rangle, \qquad A \in a(\mathcal{E}^2). \tag{2.3}$$

Since m has uniformly bounded range by Lemma 2.2, the range of $m_{x,y}$ is bounded in $\mathbb{F}$, and hence its variation $|m_{x,y}|_{\mathrm{v}}$ is bounded. By definition, the variation $|m_{x,y}|_{\mathrm{v}} : a(\mathcal{E}^2) \to [0, \infty[$ is the smallest, nonnegative-valued additive set function dominating $m_{x,y}$, or

$$|m_{x,y}|_{\mathrm{v}}(A) = \sup\left\{\sum_{B\in\pi} |m_{x,y}(B)|\right\},$$

where the supremum is taken over all finite partitions π of each set $A \in a(\mathcal{E}^2)$ by elements B of $\mathcal{E}$.

Lemma 2.3 *Let* $1 \le p \le \infty$ *and* $R \in \mathcal{L}^{\mathrm{r}}(L^p(\mu))$. *Let* $x \in L^p(\mu)$ *and* $y \in L^{p'}(\mu)$. *Let* $m_{x,y}$ *be the set function given by formulae* (2.2) *and* (2.3).

(i) *The equalities*

$$\begin{aligned} |m_{x,y}|_{\mathrm{v}}(\Sigma^2) &= \sup\left\{\sum_{j=1}^{k} \left|\langle Q_p(g_j)\, R\, Q_p(f_j)x,\, y\rangle\right|\right\} \\ &= \sup\left\{\sum_{j=1}^{k} \left|\langle Q_p(f_j)\, R\, Q_p(g_j)x,\, y\rangle\right|\right\}, \end{aligned}$$

hold, where the supremum is taken over all $k = 1, 2, \ldots$ *and all* f_j, $g_j \in \mathcal{L}^\infty(\mathcal{E})$, $j = 1, \ldots, k$, *such that*

$$0 \leq |f_j| \leq 1 \quad \text{for all } j = 1, \ldots, k \quad \text{and} \quad \sum_{j=1}^{k} |g_j| \leq 1. \tag{2.4}$$

(ii) *Let* $B, C \in \mathcal{E}$. *If* $1 \leq p < \infty$, *or if* $p = \infty$ *and the measure space* $(\Sigma, \mathcal{E}, \mu)$ *is localisable, then*

$$|m_{x,y}|_{\mathrm{v}}(B \times C) = \big\langle Q_p(C)\, |R|\, Q_p(B)|x|,\, |y|\big\rangle \tag{2.5}$$

and

$$|m_{x,y}|_{\mathrm{v}}(B \times C) = \sup \left\{ \sum_{j=1}^{k} \big\langle Q_p(C_j)\, |R\, Q_p(B_j)\, x|,\, |y|\big\rangle \right\}, \tag{2.6}$$

where the supremum is taken over all $\mathcal{E}^2$*-partitions* $\{B_j \times C_j\}_{j=1}^k$ *of* $B \times C$. *In particular,*

$$|m_{x,y}|_{\mathrm{v}}(\Sigma^2) = \langle |R|(|x|),\, |y|\rangle. \tag{2.7}$$

Proof. (i) The $\mathbb{F}$-valued set function $m_{x,y}$ satisfies

$$|m_{x,y}|_{\mathrm{v}}(\Sigma^2) = \sup \left| \sum_{l=1}^{n} c_l \cdot m_{x,y}(B_l \times C_l) \right|$$

where the supremum is taken over all $n = 1, 2, \ldots$ and all scalars c_l with $|c_l| \leq 1$ and pairwise disjoint sets $B_l \times C_l \in \mathcal{E}^2$, $l = 1, \ldots, n$, (see [6, Proposition I.1.11] as the semivariation and variation of $m_{x,y}$ are the same). The first equality then follows from the fact that every $\mathcal{E}^2$-simple function ϕ on Σ^2 with $\|\phi\|_\infty \leq 1$ can be expressed in the form $\phi = \sum_{j=1}^k f_j \otimes g_j$ satisfying (2.4). The second equality in (i) is now obvious.

(ii) To establish (2.7), note that every family $\{B_l \times C_l\}_{l=1}^n$ of pairwise disjoint sets in $\mathcal{E}^2$ satisfies $\sum_{l=1}^n \big|\big\langle Q_p(C_l)\, R\, Q_p(B_l)x,\, y\big\rangle\big| \leq \big\langle |R|(|x|),\, |y|\big\rangle$, so $|m_{x,y}|_{\mathrm{v}}(\Sigma^2) \leq \big\langle |R|(|x|),\, |y|\big\rangle$. To prove the reverse inequality, let $\varepsilon > 0$. By [10, Corollary 1] there exist a number $k = 1, 2, \ldots$ and functions f_j, $g_j \in \mathcal{L}^\infty(\mathcal{E})$, $j = 1, \ldots, k$, satisfying $g_j \geq 0$ and the inequalities (2.4) such that

$$\big\langle |R|(|x|),\, |y|\big\rangle \leq \sum_{j=1}^{k} \big|\big\langle Q_p(g_j) R Q_p(f_j)|x|,\, |y|\big\rangle\big| + \varepsilon. \tag{2.8}$$

Choose x_0, $y_0 \in \mathcal{L}^\infty(\mathcal{E})$ such that $|x_0| = 1 = |y_0|$, $|x| = x_0 \cdot x$ and $|y| = y_0 \cdot y$ as functions on Σ. Then

$$\sum_{j=1}^{k} \big|\big\langle Q_p(g_j)\, R\, Q_p(f_j)|x|,\, |y|\big\rangle\big| = \sum_{j=1}^{k} \big|\big\langle Q_p(g_j \cdot y_0)\, R\, Q_p(f_j \cdot x_0)x,\, y\big\rangle\big|. \tag{2.9}$$

It now follows from statement (i), (2.8) and (2.9) that $\langle |R|(|x|), |y|\rangle \le |m_{x,y}|_{\mathrm{v}}(\Sigma^2) + \varepsilon$, which implies (2.7) as ε is arbitrary.

The identity (2.5) follows if we replace x and y by $x \cdot \chi_B$ and $y \cdot \chi_C$ respectively in (2.7).

The identity (2.6) follows from (2.5) and the definition of variation because

$$\begin{aligned}\big|\big\langle Q_p(C_0)\, R\, Q_p(B_0)x,\, y\big\rangle\big| &\le \big\langle Q_p(C_0)\, |R\, Q_p(B_0)x|,\, |y|\big\rangle \\ &\le \big\langle Q_p(C_0)\, |R|\, Q_p(B_0)|x|,\, |y|\big\rangle\end{aligned}$$

for all B_0, $C_0 \in \mathcal{E}$ satisfying $B_0 \subseteq B$ and $C_0 \subseteq C$. □

Lemma 2.4 *Let $p = 1$ and $R \in \mathcal{L}(L^1(\mu))$. Let $B \in \mathcal{E}$ be a set of σ-finite measure. If $y \in L^\infty(\mu)$, then there is a function $z \in L^\infty(\mu)$ such that $\langle x, R'y\rangle = \langle x, z\rangle$ for all $x \in L^1(\mu)$ vanishing outside B.*

Proof. The element $R'y$ of $(L^1(\mu))'$ may not be in $L^\infty(\mu)$ unless $(\Sigma, \mathcal{E}, \mu)$ is localisable. However, the restriction μ_B of μ to the σ-algebra $\{B \cap C : C \in \mathcal{E}\}$ of subsets of B has the property that $(L^1(\mu_B))' = L^\infty(\mu_B)$ because the measure μ_B is σ-finite. Thus the linear functional $R'y$ restricted to $L^1(\mu_B)$ which is naturally embedded into $L^1(\mu)$ can be represented by a function $z \in L^\infty(\mu_B)$. This proves the lemma. □

Lemma 2.5 *Let $p = \infty$, and assume that the measure space $(\Sigma, \mathcal{E}, \mu)$ is localisable so that $L^\infty(\mu) = (L^1(\mu))'$. Let $W \in \mathcal{L}^{\mathrm{r}}(L^1(\mu))$ and let $R = W' \in \mathcal{L}(L^\infty(\mu))$. If $x \in L^\infty(\mu)$ and $y \in L^1(\mu)$, then*

$$\big\langle |R|(|x|),\, |y|\big\rangle = \big\langle |x|,\, |W|(|y|)\big\rangle. \tag{2.10}$$

Consequently, the operators R and $|R|$ belong to $\mathcal{L}(L^\infty(\mu)_{|\sigma|})$ with $|\sigma|$ denoting the absolute weak topology $|\sigma|(L^\infty(\mu), L^1(\mu))$.*

Proof. Given f, $g \in \mathcal{L}^\infty(\mathcal{E})$, $x \in L^\infty(\mu)_+$ and $y \in L^1(\mu)_+$, we have

$$\big\langle Q_\infty(g)\, R\, Q_\infty(f)x,\, y\big\rangle = \big\langle x,\, Q_1(f)\, W\, Q_1(g)y\big\rangle,$$

and hence, statement (i) and (2.7) in Lemma 2.3 imply (2.10). The last statement of the lemma is a direct consequence of (2.10). □

When $n = 1, 2, \ldots$ and $1 \le p \le \infty$, each separately additive set function on the semialgebra $\mathcal{E}^{n+1} = \left\{\prod_{l=0}^{n} B_l : B_l \in \mathcal{E},\, l = 0, 1, \ldots, n\right\}$ is additive.

Proposition 2.6 *Let $n = 1, 2, \ldots$ and $1 \le p \le \infty$. Let $R_1, \ldots, R_n \in \mathcal{L}^{\mathrm{r}}(L^p(\mu))$. The unique additive extension to $a(\mathcal{E}^{n+1})$ of the additive map on $\mathcal{E}^{n+1}$:*

$$\prod_{l=0}^{n} B_l \longmapsto Q_p(B_n)\, R_n\, Q_p(B_{n-1}) \cdots Q_p(B_1) R_1 Q_p(B_0),$$

$$\prod_{l=0}^{n} B_l \in \mathcal{E}^{n+1},$$

is denoted by $m^{(n)} : a(\mathcal{E}^{n+1}) \to \mathcal{L}(L^p(\mu))$.

(i) *The set function $m^{(n)}$ has uniformly bounded range.*

(ii) *Suppose that $1 \le p < \infty$. Let $x \in L^p(\mu)$ and $y \in L^{p'}(\mu)$. Then the variation $|m^{(n)}_{x,y}|_{\mathrm{v}}$ of the additive set function $m^{(n)}_{x,y} : a(\mathcal{E}^{n+1}) \to \mathbb{F}$ given by $m^{(n)}_{x,y}(A) = \langle m^{(n)}(A)x,\, y\rangle$ for every $A \in a(\mathcal{E}^{n+1})$ satisfies*

$$|m^{(n)}_{x,y}|_{\mathrm{v}}\left(\prod_{l=0}^{n} B_l\right) = \big\langle Q_p(B_n)|R_n| \cdots Q_p(B_1)|R_1|Q_p(B_0)|x|,\, |y|\big\rangle \tag{2.11}$$

for all $\prod_{l=0}^{n} B_l \in \mathcal{E}^{n+1}$.

(iii) *Consider the case when $p = \infty$. Suppose that the measure space $(\Sigma, \mathcal{E}, \mu)$ is localisable and that $R_l \in \mathcal{L}(L^\infty(\mu)_\sigma)$ for every $l = 1, \ldots, n$ with σ standing for the weak* topology $\sigma(L^\infty(\mu), L^1(\mu))$. Then the equality (2.11) holds for every $\prod_{l=0}^{n} B_l \in \mathcal{E}^{n+1}$.*

Proof. (i) Since the operators $R_1, \ldots, R_n$ are regular, we may assume that they are positive. Then $m^{(n)}$ is a positive operator valued additive set function on $a(\mathcal{E}^{n+1})$ so that

$$0 \le m^{(n)}(A) \le m^{(n)}(\Sigma^{n+1}) = R_n \cdots, R_1,$$

which proves statement (i).

(ii) When $n = 1$, this statement has already been given in Lemma 2.3 (ii). So assume that (ii) holds for some positive integer n. Given $C \in \mathcal{E}$, define an additive set function $\nu^{(C)} : a(\mathcal{E}^{n+1}) \to \mathbb{F}$ by $\nu^{(C)}(\prod_{l=1}^{n+1} B_l) = m_{x,y}^{(n+1)}(C \times \prod_{l=1}^{n+1} B_l)$ for all $B_1, \ldots, B_{n+1} \in \mathcal{E}$. By the assumption we have

$$|\nu^{(C)}|_{\mathrm{v}}\left(\prod_{l=1}^{n+1} B_l\right) = \langle Q_p(B_{n+1})\,|R_{n+1}|\,Q_p(B_n) \cdots Q_p(B_1)\,|R_1\,Q_p(C)\,x|,\,|y|\rangle \tag{2.12}$$

for all $B_1, \ldots, B_{n+1} \in \mathcal{E}$.

Now fix a set $\prod_{l=0}^{n+1} B_l \in \mathcal{E}^{n+1}$. For simplicity let

$$U = Q_p(B_{n+1})\,|R_{n+1}|\,Q_p(B_n) \cdots Q_p(B_2)\,|R_2|.$$

It is then clear that

$$|m_{x,y}^{(n+1)}|_{\mathrm{v}}\left(\prod_{l=0}^{n+1} B_l\right) \le \langle U\,Q_p(B_1)\,|R_1|\,Q_p(B_0)\,|x|,\,|y|\rangle. \tag{2.13}$$

Now we show the reverse inequality. By the definition of dual operators

$$\langle UQ_p(B_1)\,|R_1|\,Q_p(B_0)\,|x|,\,|y|\rangle = \langle Q_p(B_1)\,|R_1|\,Q_p(B_0)\,|x|,\,U'(|y|)\rangle \tag{2.14}$$

when $1 < p < \infty$. If $p = 1$, then Lemma 2.4 allows us to assume that $U'(|y|) \in L^\infty(\mu)$ so that (2.14) still holds. Let $\varepsilon > 0$. Lemma 2.3 (ii) implies that there is an $\mathcal{E}^2$-partition $\{C_j \times D_j\}_{j=1}^k$ of $B_0 \times B_1$ ($k \in \mathbb{N}$) for which

$$\begin{aligned}\langle Q_p(B_1)\,|R_1|\,Q_p(B_0)\,|x|,\,U'(|y|)\rangle - \varepsilon &\le \sum_{j=1}^k \langle Q_p(D_j)\,|R_1 Q_p(C_j)\,x|,\,U'(|y|)\rangle \\ &= \sum_{j=1}^k \langle U\,Q_p(D_j)\,|R_1 Q_p(C_j)\,x|,\,|y|\rangle.\end{aligned} \tag{2.15}$$

Substituting C_j and D_j for C and B_1 in (2.12) gives that

$$\begin{aligned}&\langle U\,Q_p(D_j)\,|R_1 Q_p(C_j)\,x|,\,|y|\rangle \\ &= |\nu^{(C_j)}|_{\mathrm{v}}\left(D_j \times \prod_{l=2}^{n+1} B_l\right) \le |m_{x,y}^{(n+1)}|_{\mathrm{v}}\left(C_j \times D_j \times \prod_{l=2}^{n+1} B_l\right),\end{aligned} \tag{2.16}$$

for every $j = 1, \dots, k$. Thus, it follows from (2.14), (2.15) and (2.16) that

$$\begin{aligned}&\langle U\, Q_p(B_1)\, |R_1|\, Q_p(B_0)|x|,\, |y|\rangle - \varepsilon \\ &\le \sum_{j=1}^{k} |m_{x,y}^{(n+1)}|_{\mathrm{v}}\Big(C_j \times D_j \times \prod_{l=2}^{n+1} B_l\Big) = |m_{x,y}^{(n+1)}|_{\mathrm{v}}\Big(\prod_{l=0}^{n+1} B_l\Big),\end{aligned}$$

which establishes the reverse inequality of (2.13) because ε is arbitrary.

(iii) When $n = 1$, this statement has been given in Lemma 2.3 (ii). Assume that (iii) holds for a positive integer n. Let $B_0, B_1, \dots, B_{n+1} \in \mathcal{E}$. For each $l = 2, \dots, n+1$, we have $R_l = (W_l)'$ for some $W_l \in \mathcal{L}(L^1(\mu))$; see (1.1). An appeal to Lemma 2.3 (ii) ensures (2.11) with $(n+1)$ in place of n and with $p = \infty$. In fact, in the proof of statement (ii) replace U' by $|W_2|\, Q_1(B_2) \cdots Q_1(B_n)\, |W_{n+1}|\, Q_1(B_{n+1})$. □

3. Dominated semigroups

Let $(\Sigma, \mathcal{E}, \mu)$ be a measure space with $\mu(\mathcal{E}) \subseteq [0,\infty]$. Let $\Omega = \Sigma^{[0,\infty[}$, that is, Ω is the set of all paths $\omega : [0,\infty[\to \Sigma$. When $t \ge 0$, define a function $X_t : \Omega \to \Sigma$ by $X_t(\omega) = \omega(t)$ for each $\omega \in \Omega$. Given a finite subset K of $[0,\infty[$, let $X_K : \Omega \to \Sigma^K$ denote the function

$$\omega \mapsto (X_t(\omega))_{t\in K}, \qquad \omega \in \Omega.$$

Let $t > 0$ and let $\mathcal{P}_t$ denote the collection of all finite partitions of the interval $[0,t]$. When $\alpha, \beta \in \mathcal{P}_t$ and β is a refinement of α, we write $\alpha \le \beta$. This defines an order relation in $\mathcal{P}_t$ with which $\mathcal{P}_t$ becomes a directed set. For each $\alpha = \{0, t_1, \dots, t_n, t\} \in \mathcal{P}_t\, (n \in \mathbb{N})$, let

$$\mathcal{E}^\alpha = \left\{ \prod_{l=0}^{n+1} B_l : B_l \in \mathcal{E},\ l = 0, 1, \dots, n+1 \right\}$$

which is a semialgebra of subsets of the product space Σ^α. The class $\mathcal{A}_t^* = \bigcup_{\alpha\in\mathcal{P}_t} X_\alpha^{-1}(\mathcal{E}^\alpha)$ is a semialgebra of subsets of Ω. Those sets belonging to $\mathcal{A}_t^*$ are *cylinder sets* in Ω. Let $\mathcal{A}_t = a(\mathcal{A}_t^*)$, which contains the subalgebra $\mathcal{A}_t^{(\alpha)} = a(X_\alpha^{-1}(\mathcal{E}^\alpha))$ for every $\alpha \in \mathcal{P}_t$. It is then clear that $\mathcal{A}_t = \bigcup_{\alpha\in\mathcal{P}_t} \mathcal{A}_t^{(\alpha)}$. For $t = 0$, let $\mathcal{A}_0 = a(X_0^{-1}(\mathcal{E}))$.

Let $1 \le p \le \infty$. Let S be a (one-parameter) semigroup of continuous linear operators on the Banach lattice $L^p(\mu)$. By this we mean that $S : [0,\infty[\to \mathcal{L}(L^p(\mu))$ is a map such that $S(s+t) = S(s)\, S(t)$ for all $s,\, t \ge$

0 and $S(0) = I_p$. No assumption is made about the continuity of the semigroup S. Let $Q_p : \mathcal{E} \to \mathcal{L}(L^p(\mu))$ denote the additive set function given in Remark 2.1.

Let $t > 0$. We shall define an additive set function $M_t : \mathcal{A}_t \to \mathcal{L}(L^p(\mu))$ as follows. Let $A \in \mathcal{A}_t^*$ and choose an $\alpha = \{0, t_1, \ldots, t_n, t\} \in \mathcal{P}_t$ such that $A \in X_\alpha^{-1}(\mathcal{E}^\alpha)$, and hence $A = X_\alpha^{-1}\left(\prod_{l=0}^{n+1} B_l\right)$ for some $B_0, B_1, \ldots, B_{n+1} \in \mathcal{E}$.

Define an operator $M_t(A) \in \mathcal{L}(L^p(\mu))$ by

$$M_t(A) = Q_p(B_{n+1})\, S(t - t_n) \cdots S(t_2 - t_1)\, Q_p(B_1)\, S(t_1)\, Q_p(B_0).$$

Since S is a semigroup and $Q_p(\Sigma) = I_p$, the definition of $M_t(A)$ does not depend on the choice of $\alpha \in \mathcal{P}_t$ satisfying $A \in X_\alpha^{-1}(\mathcal{E}^\alpha)$. This is the basis of [9, Definition 2.1.1]. An explicit proof has been written, for example, in [13, Proposition 7.1]. It is clear that the set function

$$M_t : A \longmapsto M_t(A), \qquad A \in \mathcal{A}_t^*,$$

is additive because $\mathcal{P}_t$ is directed, and hence it can uniquely be extended to an additive set function on $\mathcal{A}_t$. This extension to $\mathcal{A}_t$ is denoted also by M_t.

When $t = 0$, the set function $M_0 : \mathcal{A}_0 \to \mathcal{L}(L^p(\mu))$ is defined by $M_0(X_0^{-1}(B)) = Q_p(B)$ for every $B \in \mathcal{E}$.

In the notation of [9, Definition 2.1.1] we have obtained a temporally homogeneous Markov evolution process $(\Omega, \langle \mathcal{A}_t \rangle_{t \geq 0}, \langle M_t \rangle_{t \geq 0}; \langle X_t \rangle_{t \geq 0})$ with time set $[0, \infty[$, stochastic space $(\Sigma, \mathcal{E})$ and state space $L^p(\mu)$, provided $1 \leq p < \infty$. The terminology emphasises the relationship with Markov processes in probability theory, where S is a semigroup of operators acting on the space of probability measures λ representing the initial distribution of the process. The action is given in terms of the transition function $p_t(x, dy)$ so that

$$[S(t)\lambda](dy) = \int_\Sigma \lambda(dx)\, p_t(x, dy).$$

Now consider the case in which $p = \infty$. In the present setting, we need to assume that

(A1) $(\Sigma, \mathcal{E}, \mu)$ is localisable, and

(A2) $S(t) \in \mathcal{L}(L^\infty(\mu)_\sigma)$ for each $t \geq 0$,

where σ stands for the weak* topology $\sigma(L^\infty(\mu), L^1(\mu))$. The corresponding state space is $L^\infty(\mu)_\sigma$. Condition (A2) is equivalent to the requirement that each bounded linear operator $S(t) : L^\infty(\mu) \to L^\infty(\mu)$ is the dual of a bounded linear operator in $\mathcal{L}(L^1(\mu))$; see (1.1).

A semigroup $T : [0, \infty[\to \mathcal{L}(L^p(\mu))$ is called *positive* if $T(t) \geq 0$ for every $t \geq 0$. Such a semigroup T is said to *dominate* S if

$$T(t)|x| \geq |S(t)x|, \qquad x \in L^p(\mu), \tag{3.1}$$

for every $t \geq 0$. Clearly (3.1) is equivalent to that $S(t)$ is regular and $T(t) \geq |S(t)|$ for every $t \geq 0$ (see Section 1). Here, if $p = \infty$ then we need to assume (A1) which guarantees the existence of $|S(t)|$. A semigroup S is said to be *dominated* if such a dominating semigroup T exists. When S is dominated and $|S|$ is a semigroup of positive operators with the property that if T dominates S then $|S|(t) \leq T(t)$ for all $t \geq 0$, we call $|S|$ the *modulus semigroup* of S. We are now ready to present our main result.

Theorem 3.1 *Let $1 \leq p \leq \infty$, and assume the extra conditions* (A1) *and* (A2) *when $p = \infty$. Then the operator-valued, additive set function $M_t : \mathcal{A}_t \to \mathcal{L}(L^p(\mu))$ has uniformly bounded range for every $t \geq 0$ if and only if S is dominated. In this case, the modulus semigroup $|S|$ of S exists.*

The following lemma will be used in the proof of the above theorem.

Lemma 3.2 *Let $1 \leq p \leq \infty$, and assume* (A1) *and* (A2) *for $p = \infty$. Let $t > 0$ and suppose that there is a constant $c > 0$ satisfying $\|M_t(A)\| \leq c$ for all $A \in \mathcal{A}_t$.*

(i) *For every $t \geq 0$, the operator $S(t)$ is regular.*

(ii) *Given $\alpha = \{0, t_1 \ldots, t_n, t\} \in \mathcal{P}_t$, let*

$$W_t^{(\alpha)} = |S(t - t_n)|\,|S(t_n - t_{n-1})| \cdots |S(t_2 - t_1)|\,|S(t_1)|.$$

Then, there exists an operator $|S|(t) \in \mathcal{L}(L^p(\mu))$ such that given $x \in L^p(\mu)$

$$|S|(t)x = \lim_{\alpha \in \mathcal{P}_t} W_t^{(\alpha)} x,$$

in the norm $\|\cdot\|_p$ when $1 \leq p < \infty$ and in the absolute weak topology $|\sigma|(L^\infty(\mu), L^1(\mu))$ when $p = \infty$. Moreover*

$$|S|(t) = \sup_{\alpha \in \mathcal{P}_t} W_t^{(\alpha)} \quad \text{and} \quad |S|(t) \geq |S(t)| \tag{3.2}$$

in the order of the vector lattice $\mathcal{L}^r(L^p(\mu))$.

(iii) *The equality* $|S|(t) = |S|(t-s)\,|S|(s)$ *holds whenever* $0 < s < t$.

(iv) *Put* $|S|(0) = I_p$. *Then* $|S| : [0,\infty[\to \mathcal{L}(L^p(\mu))$ *is the modulus semigroup of* S.

Proof. (i) If $p = 1$ or ∞, then every operator in $\mathcal{L}(L^p(\mu))$ is regular, so consider the case when $1 < p < \infty$. By assumption M_t has uniformly bounded range, and hence, $M_t(\mathcal{A}_t^{(\alpha)})$ is also uniformly bounded for the coarsest partition $\alpha = \{0, t\} \in \mathcal{P}_t$. Now statement (i) is a consequence of Lemma 2.2 (iii) and the fact that

$$M_t\big(X_\alpha^{-1}(B \times C)\big) = Q_p(B)\, S(t)\, Q_p(C), \qquad B, C \in \mathcal{E}.$$

(ii) Let $\alpha \in \mathcal{P}_t$. Let $x \in L^p(\mu)$ and $y \in L^{p'}(\mu)$. Define an $\mathbb{F}$-valued, additive set function $M_{t,x,y}$ by

$$M_{t,x,y}(A) = \langle M_t(A)x,\, y\rangle, \qquad A \in \mathcal{A}_t, \tag{3.3}$$

and its restriction to $\mathcal{A}_t^{(\alpha)}$ is denoted by $M_{t,x,y}^{(\alpha)}$. Applying Proposition 2.6 to the $\mathbb{F}$-valued, additive set function $M_{t,x,y}^{(\alpha)} \circ X_\alpha^{-1}$ on $a(\mathcal{E}^\alpha)$ yields

$$\Big|M_{t,x,y}^{(\alpha)}\Big|_{\mathrm{v}}(\Omega) = \big\langle W_t^{(\alpha)}(|x|),\, |y|\big\rangle. \tag{3.4}$$

Moreover, by [6, Proposition I.1.11],

$$\Big|M_{t,x,y}^{(\alpha)}\Big|_{\mathrm{v}}(\Omega) \le |M_{t,x,y}|_{\mathrm{v}}(\Omega) \le 4c \cdot \|x\|_p \cdot \|y\|_{p'}. \tag{3.5}$$

It follows from (3.4) and (3.5) that $\big|\big\langle W_t^{(\alpha)}x,\, y\big\rangle\big| \le 4c \cdot \|x\|_p \cdot \|y\|_{p'}$ for all $x \in L^p(\mu)$ and $y \in L^{p'}(\mu)$, and hence, $\big\|W_t^{(\alpha)}\big\| \le 4c$ for all $\alpha \in \mathcal{P}_t$. In other words, the increasing net $\{W_t^{(\alpha)}x\}_{\alpha\in\mathcal{P}_t}$ is uniformly bounded in $\mathcal{L}(L^p(\mu))_+$. Hence, statement (i) is a consequence of Proposition 1.2 and [17, Corollary 2.4.13] when $1 \le p < \infty$, and that of Proposition 1.3 when $p = \infty$ because $L^\infty(\mu) = (L^1(\mu))'$.

(iii) Let $\alpha \in \mathcal{P}_t$. Take an $\alpha' \in \mathcal{P}_t$ such that $\alpha \le \alpha'$ and $\{s\} \in \alpha'$. There exist $\beta \in \mathcal{P}_s$ and $\gamma \in \mathcal{P}_{t-s}$ satisfying $W_t^{(\alpha')} = W_{t-s}^{(\gamma)} W_s^{(\beta)}$. Since $W_{t-s}^{(\gamma)} \le |S|(t-s)$ and $W_s^{(\beta)} \le |S|(s)$ and since $\alpha \in \mathcal{P}_t$ is arbitrary, it follows that $|S|(t) \le |S|(t-s)\,|S|(s)$.

If $\beta \in \mathcal{P}_s$ and $\gamma \in \mathcal{P}_{t-s}$ then $|S|(t) \ge W_{t-s}^{(\beta)} W_s^{(\gamma)}$ because $W_{t-s}^{(\beta)} W_s^{(\gamma)} = W_t^{(\alpha)}$ for some $\alpha \in \mathcal{P}_t$. When $1 \le p < \infty$ it follows, from statement (i) applied to $|S|(t-s)$ and $|S|(s)$ separately, that $|S|(t) \geq \lim_{\gamma\in\mathcal{P}_{t-s}} \lim_{\beta\in\mathcal{P}_s} W^{(\gamma)}_{t-s} W^{(\beta)}_s = |S|(t-s)\,|S|(s)$.

Now let $p = \infty$. Let $|\sigma|$ stand for the topology $|\sigma|(L^\infty(\mu), L^1(\mu))$. Let $x \in L^\infty(\mu)_+$. By (i) the net $\{W^{(\beta)}_s x\}_{\beta\in\mathcal{P}_s}$ converges to $|S|(s)x$ in the topology $|\sigma|$. If $\gamma \in \mathcal{P}_{t-s}$ then $W^{(\gamma)}_{t-s} \in \mathcal{L}(L^\infty(\mu)_{|\sigma|})$ by Lemma 2.5, and hence $|S|(t)x \geq \lim_{\beta\in\mathcal{P}_s} W^{(\gamma)}_{t-s} W^{(\beta)}_s x = W^{(\gamma)}_{t-s}\,|S|(s)\,x$, where the limit is with respect to $|\sigma|$. Again by (i), we have

$$|S|(t)x \geq \lim_{\gamma\in\mathcal{P}_{t-s}} W^{(\gamma)}_{t-s}\,|S|(s)\,x = |S|(t-s)\,|S|(s)\,x,$$

from which the inequality $|S|(t) \geq |S|(t-s)\,|S|(s)$ follows as $x \in L^\infty(\mu)_+$ is arbitrary.

(iv) It follows from statements (i) and (ii) that $|S|$ is a semigroup dominating S. Let $T : [0,\infty[\to \mathcal{L}(L^p(\mu))$ be a positive semigroup dominating S. If $t \geq 0$, then $T(t) \geq W^{(\alpha)}_t$ for every $\alpha \in \mathcal{P}_t$ and hence $T(t) \geq |S|(t)$. In other words, $|S|$ is the smallest positive semigroup dominating S. □

Proof of Theorem 3.1. Assume first that M_t has uniformly bounded range for every $t \geq 0$. Let $|S|(t) \in \mathcal{L}(L^p(\mu))$, $t > 0$, denote the operators obtained in Lemma 3.2. Let $|S|(0) = I_p$. Then according to Lemma 3.2, the semigroup $|S| : [0,\infty[\to \mathcal{L}(L^p(\mu))$ is the smallest positive semigroup dominating S.

To prove the converse statement, let $t \geq 0$. Define an additive set function $N_t : \mathcal{A}_t \to \mathcal{L}(L^p(\mu))$ by replacing S with $|S|$ in the definition of M_t. Then

$$\|M_t(A)\| \leq \|N_t(A)\| \leq \|N_t(\Omega)\| = \|\,|S|(t)\|, \qquad A \in \mathcal{A}_t,$$

which implies that M_t has uniformly bounded range. □

Let $t \geq 0$. The existence of a modulus semigroup $|S|$ is established in Theorem 3.1. The natural question arises as to whether or not the additive set function $N_t : \mathcal{A}_t \to \mathcal{L}(L^p(\mu))$ given in the above proof is the smallest one *dominating* M_t in the sense that $|M_t(A)| \leq N_t(A)$ for all $A \in \mathcal{A}_t$. This question has the affirmative answer as is seen from the following result.

Proposition 3.3 *Let $t \geq 0$ and $1 \leq p \leq \infty$. Let the assumption be as in* Theorem 3.1. *Suppose that the additive set function $M_t : \mathcal{A}_t \to \mathcal{L}(L^p(\mu))$ has uniformly bounded range. Let $N_t : \mathcal{A}_t \to \mathcal{L}(L^p(\mu))$ denote the additive* *set function defined by the replacement of S with $|S|$ in the definition of M_t. Then N_t is the smallest, positive operator valued additive set function on $\mathcal{A}_t$ dominating M_t. Moreover, the equality*

$$\langle N_t(\,\cdot\,)x, y\rangle = |M_{t,x,y}|_{\mathrm{v}} \quad on \;\; \mathcal{A}_t \tag{3.6}$$

holds for every $x \in L^p(\mu)_+$ and $y \in L^{p'}(\mu)_+$, where $M_{t,x,y} : \mathcal{A}_t \to \mathbb{F}$ is the additive set function given by (3.3). In particular,

$$\langle |S|(t)x, y\rangle = \langle N_t(\Omega)x, y\rangle = |M_{t,x,y}|_{\mathrm{v}}(\Omega), \quad for\ all \;\; t \geq 0. \tag{3.7}$$

Proof. It suffices to consider the case in which $t > 0$. Fix $x \in L^p(\mu)_+$ and $y \in L^{p'}(\mu)_+$, and let $N_{t,x,y}$ denote the $[0,\infty[$-valued, additive set function on $\mathcal{A}_t$ defined by

$$N_{t,x,y}(A) = \langle N_t(A)x,\, y\rangle, \qquad A \in \mathcal{A}_t. \tag{3.8}$$

By the definition of variation

$$|M_{t,x,y}|_{\mathrm{v}} \leq N_{t,x,y} \quad \text{on} \;\; \mathcal{A}_t. \tag{3.9}$$

Now let $A \in \mathcal{A}_t^*$ and choose an $\alpha = \{0, t_1, \dots, t_1, t\} \in \mathcal{P}_t$ such that $A = X_\alpha^{-1}\big(\prod_{l=0}^{n+1} B_l\big)$ for some $B_0, B_1, \dots, B_{n+1} \in \mathcal{E}$. Let $\beta_0 \in \mathcal{P}_{t_1}$, $\beta_1 \in \mathcal{P}_{t_2-t_1}, \dots, \beta_{n-1} \in \mathcal{P}_{t_n-t_{n-1}}$ and $\beta_n \in \mathcal{P}_{t-t_n}$. By Proposition 2.6 there exists a refinement $\gamma \in \mathcal{P}_t$ of α such that $A \in X_\gamma^{-1}(\mathcal{E}^\gamma)$ and

$$\begin{aligned}\big\langle Q_p(B_{n+1})\, W_{t-t_n}^{(\beta_n)}\, Q_p(B_n) \cdots W_{t_2-t_1}^{(\beta_1)}\, Q_p(B_1)\, W_{t_1}^{(\beta_0)}\, Q_p(B_0)\, x,\, y\big\rangle \\ = \big|M_{t,x,y}^{(\gamma)}\big|_{\mathrm{v}}(A),\end{aligned} \tag{3.10}$$

where $M_{t,x,y}^{(\gamma)}$ denotes the restriction of $M_{t,x,y}$ to $\mathcal{A}_t^{(\gamma)}$. Since $|M_{t,x,y}^{(\gamma)}|_{\mathrm{v}}(A) \leq |M_{t,x,y}|_{\mathrm{v}}(A)$, it follows that

$$\begin{aligned}\big\langle Q_p(B_{n+1})\, W_{t-t_n}^{(\beta_n)}\, Q_p(B_n) \cdots W_{t_2-t_1}^{(\beta_1)}\, Q_p(B_1)\, |S|(t_1)\, Q_p(B_0)\, x,\, y\big\rangle \\ \leq |M_{t,x,y}|_{\mathrm{v}}(A).\end{aligned} \tag{3.11}$$

In fact, Lemma 3.2 (i) implies that the net $\big\{W_{t_1}^{(\beta_0)}\, Q_p(B_0)\, x\big\}_{\beta_0 \in \mathcal{P}_{t_1}}$ converges to $|S|(t_1)\, Q_p(B_0)\, x$ in the norm $\|\cdot\|_p$ when $1 \leq p < \infty$ and in the topology $|\sigma|(L^\infty(\mu),\, L^1(\mu))$ when $p = \infty$. So (3.11) follows at once if $1 \leq p < \infty$. In the case in which $p = \infty$, the inequality (3.11) is a consequence of the fact that the operator $Q_\infty(B_{n+1})\, W_{t-t_n}^{(\beta_n)}\, Q_\infty(B_n) \cdots W_{t_2-t_1}^{(\beta_1)}\, Q_\infty(B_1)$ is continuous from $L^\infty(\mu)$ into itself when $L^\infty(\mu)$ is equipped with

$|\sigma|(L^\infty(\mu), L^1(\mu))$. By continuing this process n times in (3.11) we can prove that

$$N_{t,x,y}(A) \le \langle N_t(A)x, y\rangle \le |M_{t,x,y}|_{\rm v}(A).$$

Since these inequalities hold for every $A \in \mathcal{A}_t^*$, the additive set functions $N_{t,x,y}$ and $|M_{t,x,y}|_{\rm v}$ satisfy the inequality $N_{t,x,y} \le |M_{t,x,y}|_{\rm v}$ setwise on the algebra $\mathcal{A}_t$. This, together with (3.9), implies the equality (3.6). Then (3.7) follows by evaluating the set functions in the equality (3.6) at Ω.

Now, let $P_t : \mathcal{A}_t \to \mathcal{L}(L^p(\mu))$ be an arbitrary positive operator valued additive set function dominating M_t. The definition of variation yields $\langle P_t(\,\cdot\,)x, y\rangle \ge |M_{t,x,y}|_{\rm v} = \langle N_t(\,\cdot\,)x, y\rangle$ as set functions on $\mathcal{A}_t$. Hence, by Lemma 1.4 we have $P_t(A) \ge N_t(A)$ for every $A \in \mathcal{A}_t$, which completes the proof of the proposition. □

From Proposition 3.3 arises a natural question as to whether or not N_t on $\mathcal{A}$ has a σ-additive extension to the smallest σ-algebra $\mathcal{S}_t$ generated by $\mathcal{A}_t$ whenever M_t has such an extension $\widetilde{M_t}$. Moreover, we also expect that if $\widetilde{M_t}$ is 'concentrated' on a subset Λ then so is $\widetilde{N}_t$. Up to this point, the path space $\Omega = \Sigma^{[0,\infty[}$ has played a formal role. It may be possible to take Ω to be much smaller.

To give a precise statement addressing these problems in Proposition 3.5 below, let Y be a locally convex Hausdorff space. The vector space $\mathcal{L}(Y)$ equipped with the strong operator topology is denoted by $\mathcal{L}_s(Y)$. Those $\mathcal{L}_s(Y)$-valued σ-additive set functions are called *operator valued* measures. Let $t \ge 0$. Let ν be an $\mathbb{F}$-valued or operator valued measure. Let Λ be a *thick* subset of Ω; in other words, $\nu(E) = 0$ for every $E \in \mathcal{S}_t$ satisfying $E \cap \Lambda = \emptyset$; the scalar case can be found in [7, §17], for example. In this case, we may say that ν is *concentrated* on the thick subset Λ. For instance, Wiener measure defined on the measurable space $(\Omega, \mathcal{S}_t)$ is concentrated on the subspace of all $\mathbb{F}$-valued continuous functions on $[0,\infty[$ which vanish at 0 and which are constant on $[t,\infty[$; see [15, §3], for example. We omit the proof of the following lemma as it is routine in view of the definition of variation.

Lemma 3.4 *Let $t \ge 0$. Suppose that an additive set function $\nu : \mathcal{A}_t \to \mathbb{F}$ admits a σ-additive extension $\tilde{\nu}$ to $\mathcal{S}_t$ (that is, ν is the restriction of the measure $\tilde{\nu} : \mathcal{S}_t \to \mathbb{F}$ to $\mathcal{A}_t$). Then the variation $|\nu|_{\rm v} : \mathcal{A}_t \to [0,\infty[$ of ν is the restriction of the variation $|\tilde{\nu}|_{\rm v} : \mathcal{S}_t \to [0,\infty[$ (of $\tilde{\nu}$).*

In the following proposition we adopt the notation of Proposition 3.3.

Proposition 3.5 *Let $t \geq 0$ and $1 \leq p \leq \infty$. Let $\Omega \subseteq \Sigma^{[0,\infty[}$ and let $\mathcal{A}_t$ be the algebra generated by cylinder sets (0.2). The σ-algebra of subsets of Ω generated by $\mathcal{A}_t$ is denoted by $\mathcal{S}_t$. When $1 \leq p < \infty$ the space $L^p(\mu)$ is equipped with the usual norm topology, and when $p = \infty$ the space $L^\infty(\mu)$ is with the weak* topology $\sigma(L^\infty(\mu), L^1(\mu))$.*

Assume that the additive set function $M_t : \mathcal{A}_t \to \mathcal{L}(L^p(\mu))$ has uniformly bounded range and, furthermore, that M_t has an extension to an operator valued measure $\widetilde{M_t} : \mathcal{S}_t \to \mathcal{L}_s(L^p(\mu))$ on the σ-algebra $\mathcal{S}_t$ of subsets of Ω.

(i) *There is an operator valued measure $\widetilde{N}_t : \mathcal{S}_t \to \mathcal{L}_s(L^p(\mu))$ such that N_t is the restriction of $\widetilde{N}_t$ and*

$$\langle \widetilde{N}_t(\,\cdot\,)x,\, y\rangle = |\widetilde{M_{t,x,y}}|_{\mathrm{v}}, \quad x \in L^p(\mu)_+, \quad y \in L^{p'}(\mu)_+ \tag{3.12}$$

on $\mathcal{S}_t$. Here $\widetilde{M_{t,x,y}}$ is the measure $\langle \widetilde{M_t}(\,\cdot\,)x,\, y\rangle$ on $\mathcal{S}_t$.

(ii) *If $\widetilde{M_t}$ is concentrated on a subset Λ of Ω (that is, Λ is thick with respect to $\widetilde{M_t}$), then so is $\widetilde{N}_t$.*

Proof. (i) We can easily define an additive set function $\widetilde{N}_t : \mathcal{S}_t \to \mathcal{L}_s(L^p(\mu))$ satisfying (3.12) on $\mathcal{S}_t$, in which $|\widetilde{M_{t,x,y}}|_{\mathrm{v}}$ is σ-additive. Consequently, the set function $\langle \widetilde{N}_t(\,\cdot\,)x,\, y\rangle$ is σ-additive for *all* $x \in L^p(\mu)$ and $y \in L^{p'}(\mu)$. In other words, with a fixed $x \in L^p(\mu)$, the set function $\widetilde{N}_t(\,\cdot\,)x : \mathcal{S}_t \to L^p(\mu)$ is σ-additive by the Orlicz-Pettis theorem, [6, Corollary I.4.4], and hence, $\widetilde{N}_t : \mathcal{S}_t \to \mathcal{L}_s(L^p(\mu))$ is σ-additive.

Now fix $x \in L^p(\mu)_+$ and $y \in L^{p'}(\mu)_+$. It follows from Lemma 3.4 that $|\widetilde{M_{t,x,y}}|_{\mathrm{v}}$ is an extension of $|M_{t,x,y}|_{\mathrm{v}}$. This together with (3.6) and (3.12) implies that $\langle \widetilde{N}_t(\,\cdot\,)x,\, y\rangle = \langle N_t(\,\cdot\,)x,\, y\rangle$ on $\mathcal{A}_t$, and hence, $\widetilde{N}_t = N_t$ on $\mathcal{A}_t$ because $x \in L^p(\mu)_+$ and $y \in L^{p'}(\mu)_+$ are arbitrarily fixed.

(ii) This follows from (3.12). □

Remark 3.6 It is not assumed that the semigroup S in Theorem 3.1 is strongly continuous at zero, that is, a C_0-*semigroup*. Conditions under which the modulus semigroup $|S|$ is also a C_0-semigroup when S is a C_0-semigroup are considered by Becker and Greiner [3]. A necessary and sufficient condition in any Banach lattice with an order continuous norm is that S should be dominated by a positive C_0-semigroup [3, Theorem 2.1]. However, it is not so easy to detect if such a dominating semigroup exists.

On the other hand, if the C_0-semigroup S is *quasicontractive with respect to the regular norm* in the sense of [3, Proposition 2.3] (see the Introduction), then it is shown in [3, Proposition 2.3] that the modulus semigroup $|S|$ exists and is strongly continuous at zero. The following simple example shows that this is not a necessary condition for the case of C_0-semigroups acting on L^p-spaces.

Example 3.7 Let $1 \leq p < \infty$ and $\phi = \exp\left(\chi_{[0,\infty[}\right)$. Let T be the group of translations acting on $L^p(\mathbb{R})$. The operator of multiplication by a bounded measurable function ψ is written as M_ψ. Then the semigroup S of operators defined by $S(t) = M_{1/\phi}T(t)M_\phi$ for all $t \geq 0$ has the property that $\|S(t)\|_{\mathcal{L}(L^p)} = e$ for all $t > 0$. Although S is a C_0-semigroup of positive operators, it is not quasicontractive with respect to the regular norm as mentioned above, so the sufficient condition of [3, Proposition 2.3] for semigroup domination on L^p-spaces is not a necessary condition.

In addition to the examples mentioned in [3, §3], the following example illustrates the connection between the modulus semigroup and the semivariation of the associated measure.

Example 3.8 Let $\mathbf{P} = \frac{1}{i}\frac{d}{dx}$ be the momentum operator acting in $L^2(\mathbb{R})$ for a quantum mechanical particle with unit mass on the line. The Hamiltonian operator for the motion of the particle under the influence of a magnetic field corresponding to a magnetic vector potential $\mathbf{A} : \mathbb{R} \to \mathbb{R}$ is given by $H(\mathbf{A}) = \frac{1}{2}(\mathbf{P} - i\mathbf{A})^2$ (in an appropriate system of units) and $e^{-iH(\mathbf{A})t}$, $t \in \mathbb{R}$, is the corresponding dynamical group. In the imaginary-time picture, the semigroup $e^{-H(\mathbf{A})t}$, $t \geq 0$, is dominated and its modulus semigroup is $e^{-H(0)t}$, $t \geq 0$, corresponding to a zero magnetic vector potential.

Let $M_t^{\mathbf{A}}$ be the operator valued set function corresponding to the semigroup $S(t) = e^{-H(\mathbf{A})t}$, $t \geq 0$, in the formula (0.3), and set $M_t = M_t^0$ for all $t \geq 0$. Then M_t is related to Wiener measure $\mathbf{W}^x$ starting at $x \in \mathbb{R}$ by the formula

$$\langle M_t(E)\phi, \psi\rangle = \int_{\mathbb{R}} \left(\int_E \overline{\psi}(X_t)\, d\mathbf{W}^x \right) \phi(x)\, dx, \qquad E \in \mathcal{S}_t \text{ and } \phi, \psi \in L^2(\mathbb{R}).$$

The underlying path space Ω may be taken to be the space of all continuous functions $\omega : [0, \infty[\to \mathbb{R}$ with $X_s(\omega) = \omega(s)$ for all $s \geq 0$.

The operator valued measure $M_t^{\mathbf{A}}$ is related to the unperturbed measure M_t via the Feynman-Kac-Itô formula, [22, §15],

$$M_t^{\mathbf{A}}(E) = \int_E \exp\left(-i\left(\frac{1}{2}\int_0^t \mathbf{A}' \circ X_s\, ds + \int_0^t \mathbf{A} \circ X_s\, dX_s\right)\right) dM_t \tag{3.13}$$

for all $E \in \mathcal{S}_t$ and $t \geq 0$. That $e^{-H(0)t}$, $t \geq 0$, is actually the modulus semigroup of $e^{-H(\mathbf{A})t}$, $t \geq 0$, is easily seen from (3.7) and (3.13), because the integrand of the right-hand side of equation (3.13) has modulus one, so that

$$\langle e^{-H(0)t}\psi, \phi\rangle = M_{t,\psi,\phi}(\Omega) = |M_{t,\psi,\phi}^{\mathbf{A}}|(\Omega), \quad \text{for all } \psi, \phi \in L^2(\mathbb{R})_+.$$

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