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Singular perturbation of domains and semilinear elliptic equations III

Shuichi JIMBO

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Abstract. We consider a (parametrized) bounded domain, some portion of which degenerates and approaches a lower dimensional set when the parameter goes to zero. We consider semilinear elliptic equation with Neumann B.C. in this domain and the behavior of the solutions in the limit. We give a characterization for the solutions in the sense of uniform convergence.

Key words: partial degeneration, semilinear elliptic equation, domain perturbation.

1. Introduction

We deal with a domain $\Omega(\zeta)$ which partially degenerates to a lower dimensional set as $\zeta \to 0$. The shrinking subregion of $\Omega(\zeta)$ is denoted by $Q(\zeta)$. We will consider solutions of the semilinear elliptic equation

$$\Delta u + f(u) = 0 \quad \text{in} \quad \Omega(\zeta), \quad \partial u/\partial \nu = 0 \quad \text{on} \quad \partial\Omega(\zeta) \qquad \text{(Neumann B.C.)} \qquad (1.1)$$

for $\zeta \to 0$ and characterize their behaviors in this limitting process. In the previous work [10, 11, 12], we dealt with such problems for the Dumbbell shaped domain (cf. Fig. 1), which is obtained by connecting two disjoint domains by a thin cylindrical channel and gave a characterization of the solutions in the sense of "uniform convergence" when the channel part becomes thinner and approaches a 1-dimensional line segment. The methods of the proofs (scaling technique) used in that work, are restrictive and are not applicable to more general cases of partial degeneration of domains. In this paper we deal with this problem by a direct method based on concrete comparison (barrier) functions and generalize the previous results to the case where the limit set of $Q(\zeta)$ is higher-dimensional (cf. Example, Fig. 2).

There have been several significant studies on reaction-diffusion equations on variable domains since late 70's. Behavior of solutions and their

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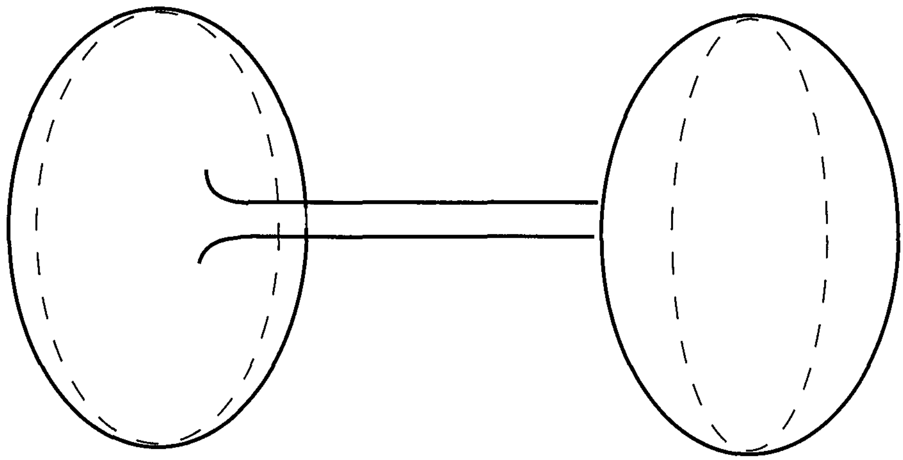

Fig. 1. Dumbbell, $n = 3$, $\ell = 1$.

structure in the process of domain varying, have been subjects of a great interest. Matano [19] dealt with (1.1) and proved an existence a non-constant stable solution in a Dumbbell domain, while there is no nontrivial stable solution in a convex domain (see Matano and Mimura [20] for competition system). By this result, he showed that the structure of solutions strongly depend on the shape of the domain. After this work, Hale and Vegas [9], Vegas [24] studied the detailed bifurcation structure of solutions in a certain situation of partial degeneration domains. Morita [21] studied the domain dependency of a reaction-diffusion system through inertial manifold theory and constructed a time-periodic solution with a spatially nontrivial pattern in a Dumbbell domain. Matano's method and the results were extended and generalized by the author's previous work [10, 11, 12], in which the behavior of solutions on the whole domain, including that on the shrinking region $Q(\zeta)$, is studied and characterized in the uniform convergence sense. Moreover, it was shown by an example that even the thin region $Q(\zeta)$ can have an important influence on the structure of the soluions. Such type of singularly perturbed domain is constructed through a surgical operation of domains, which is sometimes called a connected-sum of domains. For a given fixed domain (or several domains), a new domain is constructed by glueing small or thin regions to the fixed one. One natural issue in this situation is a structure of solutions (or a spectral problem) on the new do-

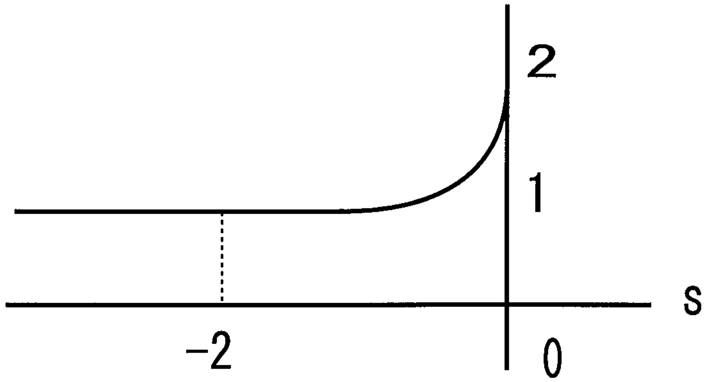

Fig. 2. The function $q = q(s)$ $(-\infty < s \leqq 0)$.

main and its relation to that on the fixed one. Such problems are of great interest in several different situations. One of them is a problem of vortex solutions in Ginzburg-Landau equation (cf. Dancer [7], Jimbo-Morita [14]), to which the results in this paper are applicable. Another related subject is the eigenvalue problem of the Neumann Laplacian on singularly perturbed domains See J.T. Beale [3], J. Arrieta [1, 2], S. Jimbo [13, 15] for characterization of eigenvalues for the the Dumbbell shaped domain. For other related topics of PDE's on connected-sum of domains, thin domains, network shaped domain and bumped domains, see [6], [16], [18], [22], [25] and the references therein.

Hereafter we formulate the problem and state the main results. We specify a bounded domain $\Omega(\zeta) \subset \mathbb{R}^n$ ($\zeta > 0$: parameter) in the following form:

$$\Omega(\zeta) = D \cup Q(\zeta) \tag{1.2}$$

where D and $Q(\zeta)$ are regions in $\mathbb{R}^n$ $(n \geqq 2)$ which are defined below. Let ℓ be a natural number such that $1 \leqq \ell < n$ and put $m = n - \ell$. For $x \in \mathbb{R}^n$, we can express it as $x = (x', x'') \in \mathbb{R}^n = \mathbb{R}^\ell \times \mathbb{R}^m$, where $x' = (x_1, x_2, \ldots, x_\ell) \in \mathbb{R}^\ell$ and $x'' = (x_{\ell+1}, \ldots, x_n) \in \mathbb{R}^m$. We denote the origin of $\mathbb{R}^\ell$ and $\mathbb{R}^m$ by o' and o'', respectively.

Hereafter we use the notation $B^{(p)}(y, \rho) = \{z \in \mathbb{R}^p \mid |z - y| < \rho\}$. which is the ball in $\mathbb{R}^p$ of radius ρ centered at y. We denote $B^{(\ell)}(o', \rho) = B^{(\ell)}(\rho)$,

$B^{(m)}(o'',\rho) = B^{(m)}(\rho)$ for simplicity.

Let $D \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^\ell$ be bounded domains (or disjoint union of bounded domains) with C^3 boundaries, respectively and we impose the following conditions on D and L.

Assumption

$$\begin{cases} \text{There exists } \zeta_0 > 0 \text{ such that } D \cup (\overline{L} \times B^{(m)}(3\zeta_0)) \\ \text{is connected in } \mathbb{R}^n \text{ and} \\ \{\overline{L} \times B^{(m)}(3\zeta_0)\} \cap \overline{D} = \partial L \times B^{(m)}(3\zeta_0) \subset \partial D. \end{cases} \tag{1.3}$$

We should note that $\overline{L}$ and ∂L are the closure and the boundary of L in $\mathbb{R}^\ell$. $\partial L \times \{o''\} \subset \partial D$ follows immediately. We use the following sets $D(r) \subset D$ and $L(r) \subset L$ defined by

$$\begin{cases} D(r) = \{x \in D \mid \operatorname{dist}(x, (\partial L \times \{o''\})) > r\}, \\ L(r) = \{x' \in L \mid \operatorname{dist}(x', \partial L) > r\}. \end{cases} \tag{1.4}$$

For the definition of the shrinking region $Q(\zeta)$, we prepare a positive continuous function $q = q(s)$ defined in $-\infty < s \leqq 0$ such that $q \in C^3((-\infty, 0))$ and

$$q(s) = 1 \quad (-\infty < s \leqq -1), \quad q'(s) > 0 \quad (-1 < s < 0), \quad q(0) = 2$$

and the inverse function q^{-1} satisfies $\lim_{t\uparrow 2} d^k q^{-1}(t)/dt^k = 0$ for $(1 \leqq k \leqq 3)$.

$Q(\zeta)$ is defined by $Q(\zeta) = Q_1(\zeta) \cup Q_2(\zeta)$ where

$$\begin{aligned} Q_1(\zeta) &= L(2\zeta) \times B^{(m)}(\zeta), \\ Q_2(\zeta) &= \{(\xi + s\,\mathbf{n}(\xi), \eta) \in \mathbb{R}^\ell \times \mathbb{R}^m \mid \\ &\qquad -2\zeta \leqq s \leqq 0,\ |\eta| < \zeta\, q(s/\zeta),\ \xi \in \partial L\}. \end{aligned}$$

Here $\mathbf{n}(\xi)$ is the unit outward normal vector on ∂L at ξ in $\mathbb{R}^\ell$. From the above conditions, $\Omega(\zeta)$ in (1.2) is a bounded domain in $\mathbb{R}^n$ with a C^3 boundary for $\zeta \in (0, \zeta_0)$.

In this paper, we consider the semilinear elliptic equation (1.1) for the domain $\Omega(\zeta)$. Here we note that $\Delta = \sum_{k=1}^n \partial^2/\partial x_k^2$, f is a real valued C^1 function in $\mathbb{R}$ and ν is the unit outward normal vector on $\partial\Omega(\zeta)$. We characterize the behavior of solutions of (1.1) when ζ tends to zero.

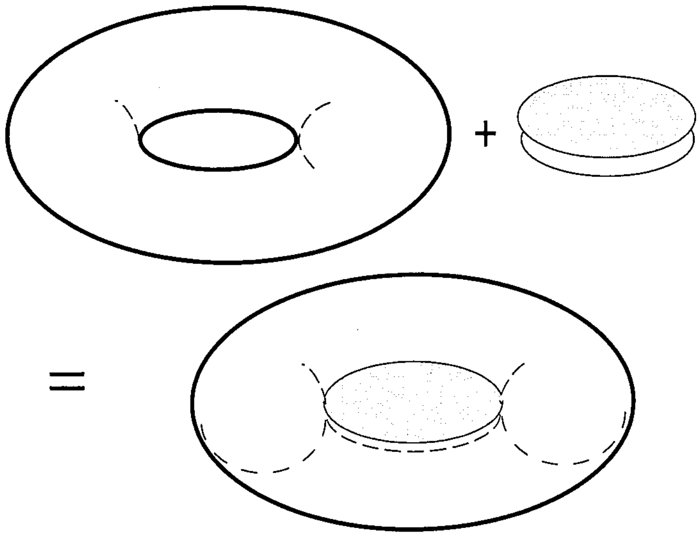

Fig. 3. $\Omega(\zeta) = D \cup Q(\zeta)$, $n = 3$, $\ell = 2$.

Theorem 1.1 *Let $\{\zeta_p\}_{p=1}^{\infty}$ be a positive sequence which converges to 0 as $p \to \infty$ and let $u_{\zeta_p} \in C^2(\overline{\Omega(\zeta_p)})$ be a solution of (1.1) for $\zeta = \zeta_p$ such that*

$$h \equiv \sup_{p \geqq 1} \sup_{x \in \Omega(\zeta_p)} |u_{\zeta_p}(x)| < \infty. \tag{1.5}$$

Then there exists a subsequence $\{\sigma_p\}_{p=1}^{\infty} \subset \{\zeta_p\}_{p=1}^{\infty}$ and functions $w \in C^2(\overline{D})$ and $V \in C^2(\overline{L})$ such that

$$\Delta w + f(w) = 0 \quad \textit{in } D, \ \frac{\partial w}{\partial \nu} = 0 \quad \textit{on } \partial D \quad (\textit{Neumann B.C.}), \tag{1.6}$$

$$\Delta' V + f(V) = 0 \quad \textit{in } L, \ V(x') = w(x', o'') \quad \textit{for } x' \in \partial L, \tag{1.7}$$

$$\lim_{p \to \infty} \sup_{x \in D} |u_{\sigma_p}(x) - w(x)| = 0, \tag{1.8}$$

$$\lim_{p \to \infty} \sup_{(x', x'') \in Q(\sigma_p)} |u_{\sigma_p}(x', x'') - V(x')| = 0, \tag{1.9}$$

where $\Delta' = \sum_{k=1}^{\ell} \partial^2 / \partial x_k^2$. Note that $\partial L \times \{o''\} \subset \partial D$ in (1.7).

Remark 1.2 One typical example of f is $f(u) = u - u^3$. In this case any solution u of (1.1) satisfies $|u(x)| \leqq 1$ (by the maximum principle). So Theorem 1.1 is applicable. If the nonlinearity f guarantees an upper bound and lower bound for solutions (such as the above), (1.5) is automatically true. For example, (1.5) can be repalced by the condition that $\xi f(\xi) < 0$ for large $|\xi|$.

Before the proof of the main result, we carry out a preliminary argument. For the family $\{u_{\zeta_p}\}_{p=1}^{\infty}$ given in the Theorem, we can apply the arguments of regularity of solutions by the aid of the Schauder estimates of elliptic boundary value problems with the condition (1.5), we see that the family $\{u_{\zeta_p}\}_{p=1}^{\infty}$ is bounded in $C^{2+\alpha}(\overline{D(\rho)})$ for any $\rho > 0$, $\alpha \in [0,1)$ and so it is relatively compact in $C^2(\overline{D(\rho)})$. Hence Cantor's argument gives a subsequence $\{\sigma_p\}_{p=1}^{\infty}$ and $w \in C^2(\overline{D} \setminus (\partial L \times \{o''\}))$ such that

$$\lim_{p\to\infty} \sup_{x\in D(\rho)} |u_{\sigma_p}(x) - w(x)| = 0 \quad \text{for any} \quad \rho > 0. \tag{1.10}$$

It is easy to see w satisfies the equation in (1.6) and the Neumann boundary condition on $\partial D \setminus (\partial L \times \{o''\})$. Next, applying the removable singularity theorem with the boundedness condition $|w(x)| \leqq h$ in D, we see that w satisfies the Neumann boundary condition all over ∂D and $w \in C^{2+\alpha}(\overline{D})$ for $\alpha \in [0,1)$. However, the uniform convergence of $\{u_{\sigma_p}\}$ in $\overline{D}$ is not trivial and will be justified in §2. In the proof, we can assume without loss of generality, from the condition (1.5), that there exists $M > 0$ and $M_1 > 0$ such that

$$|f(\xi)| \leqq M_1, \quad |f'(\xi)| \leqq M \quad (\xi \in \mathbb{R}) \tag{1.11}$$

because we can modify ("cut off") the nonlinear term $f(u)$ in the region $|u| \geqq h$ adequately. We will start the proof of the theorem at this situation in the next section.

2. Preliminaries and uniform convergence in D

We deal with a varying domain and discuss the behavior of solutions in detail. We prepare notation for several subregions of $\Omega(\zeta)$ and construct certain auxiliary functions to estimate solutions.

Notation 2.1 For $\rho > 0$, $\kappa' \geqq \kappa \geqq 2\rho$, we define the following sets,

$$\widetilde{\Sigma}^+(\rho) = \{(s,\eta) \in \mathbb{R} \times \mathbb{R}^m \mid 0 < s,\ s^2 + |\eta|^2 < \rho^2\}$$
$$\widetilde{\Sigma}^-(\rho,\kappa) = \{(s,\eta) \in \mathbb{R} \times \mathbb{R}^m \mid -\kappa < s \leqq 0,\ |\eta| < \rho\, q(s/\rho)\}$$
$$\widetilde{\Sigma}^\circ(\rho,\kappa',\kappa) = \{(s,\eta) \in \mathbb{R} \times \mathbb{R}^m \mid -\kappa' \leqq s \leqq -\kappa,\ |\eta| < \rho\}$$
$$\widetilde{\Gamma}^+(\rho) = \{(s,\eta) \in \mathbb{R} \times \mathbb{R}^m \mid 0 < s,\ s^2 + |\eta|^2 = \rho^2\}$$
$$\widetilde{\Gamma}^-(\rho,\kappa) = \{(s,\eta) \in \mathbb{R} \times \mathbb{R}^m \mid s = -\kappa,\ |\eta| < \rho\}$$
$$\widetilde{\Lambda}(\rho) = \widetilde{\Sigma}^-(\rho, 2\rho) \cup \widetilde{\Sigma}^+(2\rho).$$

Notation 2.2 For $\rho > 0$, $\kappa > 0$, $\xi \in \partial L$, we define the following sets,

$$\Sigma^+(\rho,\xi) = \{(\xi + s\,\mathbf{n}(\xi), \eta) \in \mathbb{R}^\ell \times \mathbb{R}^m \mid (s,\eta) \in \widetilde{\Sigma}^+(\rho)\}$$
$$\Sigma^-(\rho,\kappa,\xi) = \{(\xi + s\,\mathbf{n}(\xi), \eta) \in \mathbb{R}^\ell \times \mathbb{R}^m \mid (s,\eta) \in \widetilde{\Sigma}^-(\rho,\kappa)\}$$
$$\Sigma^\circ(\rho,\kappa',\kappa,\xi) = \{(\xi + s\,\mathbf{n}(\xi), \eta) \in \mathbb{R}^\ell \times \mathbb{R}^m \mid (s,\eta) \in \widetilde{\Sigma}^\circ(\rho,\kappa',\kappa)\}$$
$$\Gamma^+(\rho,\xi) = \{(\xi + s\,\mathbf{n}(\xi), \eta) \in \mathbb{R}^\ell \times \mathbb{R}^m \mid (s,\eta) \in \widetilde{\Gamma}^+(\rho)\}$$
$$\Gamma^-(\rho,\kappa,\xi) = \{(\xi + s\,\mathbf{n}(\xi), \eta) \in \mathbb{R}^\ell \times \mathbb{R}^m \mid (s,\eta) \in \widetilde{\Gamma}^-(\rho,\kappa)\}$$
$$\Sigma^+(\rho) = \bigcup_{\xi\in\partial L} \Sigma^+(\rho,\xi), \quad \Sigma^-(\rho,\kappa) = \bigcup_{\xi\in\partial L} \Sigma^-(\rho,\kappa,\xi)$$
$$\Lambda(\rho) = \Sigma^-(\rho, 2\rho) \cup \Sigma^+(2\rho)$$
$$\Sigma^\circ(\rho,\kappa',\kappa) = \bigcup_{\xi\in\partial L} \Sigma^\circ(\rho,\kappa',\kappa,\xi)$$
$$\Gamma^+(\rho) = \bigcup_{\xi\in\partial L} \Gamma^+(\rho,\xi), \quad \Gamma^-(\rho,\kappa) = \bigcup_{\xi\in\partial L} \Gamma^-(\rho,\kappa,\xi).$$

It should be remarked that $\Lambda(\rho)$ is an open set in $\mathbb{R}^n$ for small $\rho > 0$.

Remark 2.3 Note that for any $\xi \in \partial L$,

$$(s,\eta) \in \widetilde{\Sigma}^+(\rho) \Longleftrightarrow x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Sigma^+(\rho,\xi)$$
$$(s,\eta) \in \widetilde{\Sigma}^-(\rho,\kappa) \Longleftrightarrow x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Sigma^-(\rho,\kappa,\xi)$$
$$(s,\eta) \in \widetilde{\Sigma}^\circ(\rho,\kappa',\kappa) \Longleftrightarrow x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Sigma^\circ(\rho,\kappa',\kappa,\xi)$$
$$(s,\eta) \in \widetilde{\Gamma}^+(\rho) \Longleftrightarrow x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Gamma^+(\rho,\xi)$$

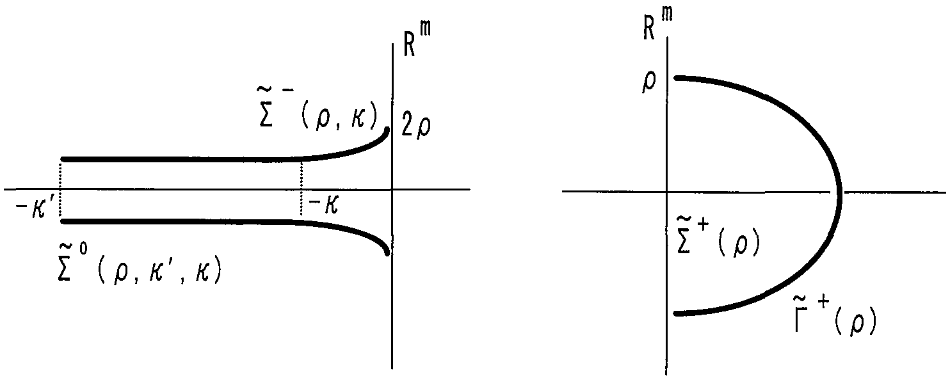

Fig. 4. $\widetilde{\Sigma}^+(\rho)$, $\widetilde{\Sigma}^-(\rho,\kappa)$, $\widetilde{\Sigma}^\circ(\rho,\kappa',\kappa)$.

holds if $|s|$ is small. The same relation is also true for $\widetilde{\Sigma}^-(\rho,\kappa,\xi)$, $\widetilde{\Sigma}^\circ(\rho,\kappa',\kappa)$, $\widetilde{\Gamma}^+(\rho)$, $\widetilde{\Gamma}^-(\rho,\kappa,\xi)$.

The coordinate system near ∂L

A point $x' \in \mathbb{R}^\ell$ near ∂L is uniquely expressed as

$$x' = \xi + s\,\mathbf{n}(\xi) \in \mathbb{R}^\ell \tag{2.1}$$

where $(\xi, s) \in \partial L \times (-r_0, r_0)$ provided that $r_0 > 0$ is small. Using a local coordinate $(\xi_1, \ldots, \xi_{\ell-1})$ of ∂L, we introduce the local coordinate $(\xi_1, \ldots, \xi_{\ell-1}, s)$ in this tubular neighborhood $\mathcal{O}$. Denote the metric tensor of $\mathbb{R}^\ell$ with respect to $(\xi, s) = (\xi_1, \ldots, \xi_{\ell-1}, s)$ by $g = (g_{i,j}(\xi, s))_{ij}$. Here s corresponds to the component for $i = \ell$. Let us remark that

$$g_{i\ell}(\xi,s) = g_{\ell i}(\xi,s) \equiv 0 \quad (1 \leqq i \leqq \ell-1), \quad g_{\ell\ell}(\xi,s) \equiv 1 \quad \text{in } \mathcal{O}$$

which follow from the relation (2.1).

Under this coordinate system and the metric tensor $g = (g_{ij}(\xi, s))$ we can express the Laplacian $\Delta' = \sum_{k=1}^{\ell} \partial^2/\partial x_k^2$ in $\mathbb{R}^\ell$ in terms of $(\xi_1, \ldots, \xi_{\ell-1}, s) \in \mathcal{O}$ as follows,

$$\Delta' u = \frac{1}{\sqrt{G}} \sum_{i,j=1}^{\ell-1} \frac{\partial}{\partial \xi_i}\left(\sqrt{G} g^{ij} \frac{\partial u}{\partial \xi_j}\right) + \frac{1}{\sqrt{G}} \frac{\partial}{\partial s}\left(\sqrt{G} \frac{\partial u}{\partial s}\right).$$

Here $G = \det(g_{ij}(\xi, s))$. $(g^{ij}(\xi, s))$ is the inverse matrix of $(g_{ij}(\xi, s))$. Remark that the functions $g^{ij} = g^{ij}(\xi, s)$, $g_{ij} = g_{ij}(\xi, s)$, $G = G(\xi, s)$ depend on the choice of the local coordinate $(\xi_1, \ldots, \xi_{\ell-1})$ on ∂L. On the other hand, $(1/\sqrt{G})(\partial\sqrt{G}/\partial s)$ does not depend on such choice, because it is equal to $\Delta' s$ and the function s is defined indepently of the choice of the coordinate on ∂L. Here we put

$$\alpha_0 \equiv \sup_{\xi\in\partial L, |s|\leqq r_0} \left| \frac{1}{\sqrt{G(\xi,s)}} \frac{\partial\sqrt{G(\xi,s)}}{\partial s} \right| + 1. \tag{2.2}$$

We prepare some functions to construct a barrier function to constrol the behavior of solutions around $\partial L \times \{o''\}$. We consider the following ODE,

$$\frac{d^2K}{dr^2} + \left(\frac{m}{r} - \alpha_0\right) \frac{dK}{dr} + (M+1)K = 0 \quad (r > 0). \tag{2.3}$$

By applying the standard Frobenius method, we can construct a formal power series solution of (2.3) and we can discuss its convergence by the aid of the method of a majorant series. Consequently we get solutions $K_1 = K_1(r)$ and $K_2 = K_2(r)$ which are convergent for all $r > 0$ and linearly independent. We summarize the results in the following proposition.

Proposition 2.4 *The equation* (2.3) *has two* (*linearly independent*) *solutions* $K_1(r)$, $K_2(r)$ *of the following form,*

$$\begin{cases} K_1(r) = \sum_{k=0}^{\infty} a_k r^k \quad (\textit{regular solution}), \\ K_2(r) = r^{-m+1}\left(\sum_{k=0}^{\infty} b_k r^k\right) + \left(\sum_{k=0}^{\infty} c_k r^k\right)\log r \\ \qquad\qquad\qquad\qquad\qquad\qquad (\textit{singular solution}), \end{cases} \tag{2.4}$$

and all the power series in (2.4) *are convergent for* $r > 0$. *We note*

$$\begin{cases} a_0 \neq 0 \quad \textit{for} \quad m \geqq 1, \\ b_0 \neq 0 \quad \textit{if} \quad m \geqq 2 \quad \textit{and} \quad c_0 \neq 0 \quad \textit{if} \quad m = 1. \end{cases}$$

Multiplying adequate constants to normalize the coefficient of the leading terms, we have the properties for some $r_1 \in (0, 3\zeta_0] \cap (0, r_0]$

$$\begin{cases} K_1 \text{ is regular at } r=0 \text{ and } K_1(0)=1 \quad (m \geqq 1), \\ \lim_{r\downarrow 0} K_2(r)/r^{-m+1} = 1 \quad (m \geqq 2), \\ \lim_{r\downarrow 0} K_2(r)/\log(1/r) = 1 \quad (m=1), \\ K_1(r)>0,\ K_1'(r)<0,\ K_1''(r)<0 \quad (0<r\leqq r_1), \\ K_2(r)>0,\ K_2'(r)<0 \quad (0<r\leqq r_1). \end{cases} \tag{2.5}$$

Next we consider the following equation for $N = N(z)$ $(z \in \widetilde{\Lambda}(1))$

$$\begin{cases} \Delta_z N = 0 \quad \text{in } \widetilde{\Lambda}(1), \\ N = 1 \quad \text{on } \widetilde{\Gamma}^-(1,2), \quad N=0 \quad \text{on } \widetilde{\Gamma}^+(2), \\ \partial N/\partial \widetilde{\mathbf{n}} = 0 \quad \text{on } \partial\widetilde{\Lambda}(1) \setminus (\widetilde{\Gamma}^-(1,2) \cup \widetilde{\Gamma}^+(2)), \end{cases} \tag{2.6}$$

where $z = (s,\eta) = (s,\eta_1,\ldots,\eta_m)$, $\Delta_z = \partial^2/\partial s^2 + \sum_{j=1}^m \partial^2/\partial\eta_j^2$ and $\widetilde{\mathbf{n}}$ is the outward unit normal vector on the boundary $\partial\widetilde{\Lambda}(1)$ in $\mathbb{R}^\ell$.

It is easy to see that (2.6) has the unique smooth solution $N_0 = N_0(z)$. We define the constant α_1 by

$$\alpha_1 = \sup\{|\nabla_z N_0(z)| \mid z \in \widetilde{\Lambda}(1)\} + 1 > 0. \tag{2.7}$$

We can prove by the maximum principle and a regularity argument that there exist constants $\delta_2 > 0$, $\delta_1 > 0$ such that

$$-\frac{\delta_2}{2} \leqq \frac{\partial N_0}{\partial \widetilde{\mathbf{n}}} \leqq -2\delta_1 \quad \text{on } \widetilde{\Gamma}^+(2), \quad 2\delta_1 \leqq \frac{\partial N_0}{\partial \widetilde{\mathbf{n}}} \leqq \frac{\delta_2}{2} \quad \text{on } \widetilde{\Gamma}^-(1,2). \tag{2.8}$$

Next we define the constant α_2 by

$$\alpha_2 = \max\{2\alpha_0(4\delta_2+\alpha_1), 2\alpha_0(\alpha_1+1)\} > 0 \tag{2.9}$$

and consider the following equations,

$$\begin{cases} \Delta_z N + \zeta^2(M+1)N = -\alpha_2\zeta^2 \quad \text{in } \widetilde{\Lambda}(1), \\ N = 1 \quad \text{on } \widetilde{\Gamma}^-(1,2), \quad N=1 \quad \text{on } \widetilde{\Gamma}^+(2), \\ \partial N/\partial \widetilde{\mathbf{n}} = 0 \quad \text{on } \partial\widetilde{\Lambda}(1) \setminus (\widetilde{\Gamma}^-(1,2) \cup \widetilde{\Gamma}^+(2)), \end{cases} \tag{2.10}$$

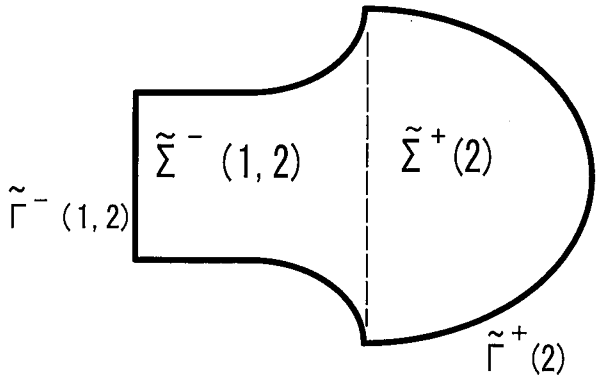

Fig. 5. $\widetilde{\Lambda}(1)=\widetilde{\Sigma}^-(1,2)\cup\widetilde{\Sigma}^+(2)$.

$$\begin{cases} \Delta_z N+\zeta^2(M+1)N=-\alpha_2\zeta \quad \text{in } \widetilde{\Lambda}(1), \\ N=1 \quad \text{on } \widetilde{\Gamma}^-(1,2), \quad N=0 \quad \text{on } \widetilde{\Gamma}^+(2), \\ \partial N/\partial\widetilde{\mathbf{n}}=0 \quad \text{on } \partial\widetilde{\Lambda}(1)\setminus(\widetilde{\Gamma}^-(1,2)\cup\widetilde{\Gamma}^+(2)). \end{cases} \tag{2.11}$$

Recall that the constant $M>0$ appeared in the condition (1.11). It is easy to see that each of (2.10) and (2.11) has a unique solution provided that $\zeta>0$ is small. We denote those solutions of (2.10), (2.11) by $N_1=N_1(z)$, $N_2=N_2(z)$, respectively. It should be noticed that N_1, N_2 depend on $\zeta>0$ while N_0 does not. Note that N_1 and N_2 smoothly approach the constant function 1 and N_0, respectively as $\zeta\to 0$.

Lemma 2.5 *There exist $\delta_3>0$ and $\zeta_1>0$ such that*

$$\begin{cases} |N_1(z)-1|+|\nabla_z N_1(z)|\leqq\delta_3\zeta^2 \quad \textit{in } \widetilde{\Lambda}(1), \\ -\delta_3\zeta^2\leqq\dfrac{\partial N_1}{\partial\widetilde{\mathbf{n}}}\leqq 0 \quad \textit{on } \widetilde{\Gamma}^+(2)\cup\widetilde{\Gamma}^-(1,2), \end{cases} \tag{2.12}$$

$$\begin{cases} |\nabla_z(N_2-N_0)|\leqq\delta_3\zeta, \\ -\delta_2\leqq\dfrac{\partial N_2}{\partial\widetilde{\mathbf{n}}}\leqq-\delta_1 \quad \textit{on } \widetilde{\Gamma}^+(2), \quad \delta_1\leqq\dfrac{\partial N_2}{\partial\widetilde{\mathbf{n}}}\leqq\delta_2 \quad \textit{on } \widetilde{\Gamma}^-(1,2), \end{cases} \tag{2.13}$$

Sketch of the Proof of Lemma 2.5. Consider the functions

$$\widehat{N}_1(z)\equiv(N_1(z)-1)/\zeta^2, \quad \widehat{N}_2(z)\equiv(N_2(z)-N_0(z))/\zeta$$

and we see that they satisfy

$$\begin{cases} \Delta_z \widehat{N}_1 + \zeta^2(M+1)\widehat{N}_1 = -\alpha_2 - M - 1 \quad \text{in} \quad \widetilde{\Lambda}(1), \\ \widehat{N}_1 = 0 \quad \text{on} \quad \widetilde{\Gamma}^-(1,2), \quad \widehat{N}_1 = 0 \quad \text{on} \quad \widetilde{\Gamma}^+(2), \\ \partial \widehat{N}_1/\partial \widetilde{\mathbf{n}} = 0 \quad \text{on} \quad \partial\widetilde{\Lambda}(1) \setminus (\widetilde{\Gamma}^-(1,2) \cup \widetilde{\Gamma}^+(2)), \end{cases}$$

$$\begin{cases} \Delta_z \widehat{N}_2 + \zeta^2(M+1)\widehat{N}_2 = -\alpha_2 - \zeta(M+1)N_0 \quad \text{in} \quad \widetilde{\Lambda}(1), \\ \widehat{N}_2 = 0 \quad \text{on} \quad \widetilde{\Gamma}^-(1,2), \quad \widehat{N}_2 = 0 \quad \text{on} \quad \widetilde{\Gamma}^+(2), \\ \partial \widehat{N}_2/\partial \widetilde{\mathbf{n}} = 0 \quad \text{on} \quad \partial\widetilde{\Lambda}(1) \setminus (\widetilde{\Gamma}^-(1,2) \cup \widetilde{\Gamma}^+(2)). \end{cases}$$

In these equations, we apply the Schauder estimate to $\widehat{N}_1$, $\widehat{N}_2$ and investigate those limit for $\zeta \to 0$ and obtain the conclusion. □

We also prepare the following equations, which depend on $\xi \in \partial L$ as a parameter,

$$\begin{cases} \Delta_z N = \mp \kappa_1 \zeta^2 \quad \text{for} \quad z = (s,\eta) \in \widetilde{\Lambda}(1), \\ N = w(\xi, o'') \mp \kappa_2 \zeta \quad \text{for} \quad z = (s,\eta) \in \widetilde{\Gamma}^-(1,2), \\ N = w(\xi + \zeta\, s\, \mathbf{n}(\xi), \zeta\eta) \quad \text{for} \quad z = (s,\eta) \in \widetilde{\Gamma}^+(2), \\ \partial N/\partial \widetilde{\mathbf{n}} = 0 \quad \text{for} \quad z \in \partial\widetilde{\Lambda}(1) \setminus (\widetilde{\Gamma}^-(1,2) \cup \widetilde{\Gamma}^+(2)). \end{cases} \tag{2.14}$$

We denote the unique solution of (2.14) by $N_3^\pm = N_3^\pm(\xi, z)$ (respectively), which depend on ξ, smoothly. Note also that both $N_3^\pm$ smoothly approaches $w(\xi, o'')$ for $\zeta \to 0$ and this convergence is uniform in $\xi \in \partial L$. We have the following properties for $N_3^\pm$.

Lemma 2.6 *There exists $\kappa_0 > 0$ such that for any $\kappa_1 > 0$, $\kappa_2 \geqq \kappa_0$ there exist $\zeta_2 = \zeta_2(\kappa_1, \kappa_2)$ and $0 < \delta_4 = \delta_4(\kappa_2) < \delta_5 = \delta_5(\kappa_2)$ with the following conditions*

$$\begin{cases} |\nabla_z N_3^\pm(\xi,z)| \leqq \delta_5 \zeta, \quad N_3^+(\xi,z) \leqq N_3^-(\xi,z) \quad \textit{in} \quad \widetilde{\Lambda}(1), \\ -\delta_5\zeta \leqq \dfrac{\partial N_3^-}{\partial \widetilde{\mathbf{n}}} \leqq -\delta_4\zeta \quad \textit{on} \quad \widetilde{\Gamma}^+(2), \quad \delta_4\zeta \leqq \dfrac{\partial N_3^-}{\partial \widetilde{\mathbf{n}}} \leqq \delta_5\zeta \quad \textit{on} \quad \widetilde{\Gamma}^-(1,2), \\ \delta_4\zeta \leqq \dfrac{\partial N_3^+}{\partial \widetilde{\mathbf{n}}} \leqq \delta_5\zeta \quad \textit{on} \quad \widetilde{\Gamma}^+(2), \quad -\delta_5\zeta \leqq \dfrac{\partial N_3^+}{\partial \widetilde{\mathbf{n}}} \leqq -\delta_4\zeta \quad \textit{on} \quad \widetilde{\Gamma}^-(1,2), \end{cases} \tag{2.15}$$

for $\xi \in \partial L$, $\zeta \in (0, \zeta_2]$. *Moreover we can take* $\delta_4(\kappa_2) > 0$ *so that*

$$\lim_{\kappa_2 \to \infty} \delta_4(\kappa_2) = \infty. \tag{2.16}$$

Notice that δ_4, δ_5 do not depend on $\kappa_1 > 0$, while ζ_2 does.

Sketch of the Proof of Lemma 2.6. We carry out a similar argument as in the previous lemma for the function $\widehat{N}_3^+(\xi, z) = (N_3^+(\xi, z) - w(\xi, o''))/\zeta$ which satisfies,

$$\begin{cases} \Delta_z \widehat{N}_3^+ = -\kappa_1 \zeta \quad \text{for} \quad z = (s, \eta) \in \widetilde{\Lambda}(1), \\ \widehat{N}_3^+ = -\kappa_2 \quad \text{on} \quad \widetilde{\Gamma}^-(1,2), \\ \widehat{N}_3^+ = (w(\xi + \zeta\, s\, \mathbf{n}(\xi), \zeta\eta) - w(\xi, o''))/\zeta \quad \text{on} \quad \widetilde{\Gamma}^+(2), \\ \partial \widehat{N}_3^+ / \partial \widetilde{\mathbf{n}} = 0 \quad \text{on} \quad \partial\widetilde{\Lambda}(1) \setminus (\widetilde{\Gamma}^-(1,2) \cup \widetilde{\Gamma}^+(2)). \end{cases}$$

By decomposing $\widehat{N}_3^+$ into two parts $\widehat{N}_{3,1}^+$ (depending on κ_1) and $\widehat{N}_{3,2}^+$ (depending on κ_2) by

$$\begin{cases} \Delta_z \widehat{N}_{3,1}^+ = -\kappa_1 \zeta \quad \text{in} \quad \widetilde{\Lambda}(1), \\ \widehat{N}_{3,1}^+ = 0 \quad \text{on} \quad \widetilde{\Gamma}^-(1,2), \\ \widehat{N}_{3,1}^+ = 0 \quad \text{on} \quad \widetilde{\Gamma}^+(2), \\ \partial \widehat{N}_{3,1}^+ / \partial \widetilde{\mathbf{n}} = 0 \quad \text{on} \quad \partial\widetilde{\Lambda}(1) \setminus (\widetilde{\Gamma}^-(1,2) \cup \widetilde{\Gamma}^+(2)), \end{cases}$$

$$\begin{cases} \Delta_z \widehat{N}_{3,2}^+ = 0 \quad \text{in} \quad \widetilde{\Lambda}(1), \\ \widehat{N}_{3,2}^+ = -\kappa_2 \quad \text{on} \quad \widetilde{\Gamma}^-(1,2), \\ \widehat{N}_{3,2}^+ = (w(\xi + \zeta\, s\, \mathbf{n}(\xi), \zeta\eta) - w(\xi, o''))/\zeta \quad \text{on} \quad \widetilde{\Gamma}^+(2), \\ \partial \widehat{N}_{3,2}^+ / \partial \widetilde{\mathbf{n}} = 0 \quad \text{on} \quad \partial\widetilde{\Lambda}(1) \setminus (\widetilde{\Gamma}^-(1,2) \cup \widetilde{\Gamma}^+(2)). \end{cases}$$

First restrict ζ_2 in the region $0 < \zeta_2 \leqq 1/\kappa_1$ and estimate $\widehat{N}_{3,1}^+$, $\widehat{N}_{3,2}^+$ for the limit for $\zeta \to 0$, we can obtain the conclusion the lemma. A similar argument applies to $\widehat{N}_3^-$. □

Barrier functions

We will define functions $\varphi_{1,\zeta}$, $\varphi_{2,\zeta}$, $\varphi_{3,\zeta}^{\pm}$ in the set

$$J(\zeta, r_2, r_1) \equiv \Sigma^\circ(\zeta, r_2, 2\zeta) \cup \Lambda(\zeta) \cup (\Sigma^+(r_1) \setminus \Sigma^+(2\zeta))$$

for $r_2 > 0$ ($r_1 > 0$ was fixed in Prop. 2.4). We remark that the choices of the constants α_0, α_1 and α_2 in (2.2), (2.7), (2.9) are determined to make these functions work to bound the solutions.

Definition For $x = (\xi + s\,\mathbf{n}(\xi), \eta) \in J(\zeta, r_2, r_1)$ $(z = (s, \eta), |z| = (s^2 + |\eta|^2)^{1/2})$, we put

$$\varphi_{1,\zeta}(x) = \begin{cases} K_1(|z|)/K_1(r_1) \quad \text{for} \ \ z = (s,\eta) \in \widetilde{\Sigma}^+(r_1) \setminus \widetilde{\Sigma}^+(2\zeta), \\ \left(\dfrac{K_1(2\zeta)}{K_1(r_1)}\right) N_1\left(\dfrac{z}{\zeta}\right) - \dfrac{\zeta}{2} N_2\left(\dfrac{z}{\zeta}\right) \quad \text{for} \ \ z = (s,\eta) \in \widetilde{\Lambda}(\zeta), \\ \left(\dfrac{K_1(2\zeta)}{K_1(r_1)} - \dfrac{\zeta}{2}\right) + 2\delta_2(s + 2\zeta) - \gamma_1(s+2\zeta)^2 \\ \qquad\qquad\qquad \text{for} \ \ z = (s,\eta) \in \widetilde{\Sigma}^\circ(\zeta, r_2, 2\zeta), \end{cases}$$

$$\varphi_{2,\zeta}(x) = \begin{cases} (-1/K_2'(2\zeta))K_2(|z|) \quad \text{for} \ \ z = (s,\eta) \in \widetilde{\Sigma}^+(r_1) \setminus \widetilde{\Sigma}^+(2\zeta), \\ -\dfrac{K_2(2\zeta)}{K_2'(2\zeta)} N_1\left(\dfrac{z}{\zeta}\right) + \dfrac{\zeta}{4\delta_2} N_2\left(\dfrac{z}{\zeta}\right) \quad \text{for} \ \ z = (s,\eta) \in \widetilde{\Lambda}(\zeta), \\ \left(-\dfrac{K_2(2\zeta)}{K_2'(2\zeta)} + \dfrac{\zeta}{4\delta_2}\right) - \dfrac{\delta_1}{16\delta_2}(s + 2\zeta) - \gamma_2(s+2\zeta)^2 \\ \qquad\qquad\qquad \text{for} \ \ z = (s,\eta) \in \widetilde{\Sigma}^\circ(\zeta, r_2, 2\zeta), \end{cases}$$

$$\varphi^+_{3,\zeta}(x) = \begin{cases} w(x) \quad \text{for} \ \ x \in \Sigma^+(r_1) \setminus \Sigma^+(2\zeta), \\ N_3^+(\xi, z/\zeta) \quad \text{for} \ \ x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Lambda(\zeta), \\ w(\xi, o'') - \kappa_2\zeta + \gamma_3(s+2\zeta) - \gamma_4(s+2\zeta)^2 \\ \qquad \text{for} \ \ x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Sigma^\circ(\zeta, r_2, 2\zeta), \end{cases}$$

$$\varphi^-_{3,\zeta}(x) = \begin{cases} w(x) \quad \text{for} \ \ x \in \Sigma^+(r_1) \setminus \Sigma^+(2\zeta), \\ N_3^-(\xi, z/\zeta) \quad \text{for} \ \ x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Lambda(\zeta), \\ w(\xi, o'') + \kappa_2\zeta - \gamma_3(s+2\zeta) + \gamma_4(s+2\zeta)^2 \\ \qquad \text{for} \ \ x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Sigma^\circ(\zeta, r_2, 2\zeta), \end{cases}$$

where $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$, $\gamma_4 > 0$ will be determined later.

It should be remarked that $x = (\xi + s\,\mathbf{n}(\xi), \eta)$ belongs to $\Sigma^\circ(\zeta, r_2, 2\zeta)$ or $\Lambda(\zeta)$ or $\Sigma^+(r_1) \setminus \Sigma^+(2\zeta)$ if and only if $z = (s, \eta)$ belongs to $\widetilde{\Sigma}^\circ(\zeta, r_2, 2\zeta)$, or $\widetilde{\Lambda}(\zeta)$ or $\widetilde{\Sigma}^+(r_1) \setminus \widetilde{\Sigma}^+(2\zeta)$, respectively.

It is easy to see that $\varphi_{1,\zeta}$, $\varphi_{2,\zeta}$, $\varphi_{3,\zeta}^{\pm}$ are C^0 and piecewise smooth. It is also true that

$$\varphi_{1,\zeta}, \varphi_{2,\zeta}, \varphi_{3,\zeta}^{\pm} \in H^1(J(\zeta, r_2, r_1)) \cap C^0(\overline{J(\zeta, r_2, r_1)}).$$

We are going to make an adequate linear combination of these functions which play a role of barrier to prove the uniform convergence in D. We see that there exists $r_2 > 0$ (independent of $\zeta > 0$) such that $\varphi_{1,\zeta}$, $\varphi_{2,\zeta}$ are positive in $J(\zeta, r_2, r_1)$ and

$$\begin{cases} 0 < \liminf_{\zeta \to 0} \Big(\inf_{x \in J(\zeta, r_2, r_1)} \varphi_{1,\zeta}(x) \Big) \leqq \limsup_{\zeta \to 0} \Big(\sup_{x \in J(\zeta, r_2, r_1)} \varphi_{1,\zeta}(x) \Big) < \infty, \\ 0 < \liminf_{\zeta \to 0} \Big(\inf_{x \in \Gamma^-(\zeta, r)} \varphi_{2,\zeta}(x) \Big) \leqq \limsup_{\zeta \to 0} \Big(\sup_{x \in \Gamma^-(\zeta, r)} \varphi_{2,\zeta}(x) \Big) < \infty, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r \in (0, r_2], \\ \lim_{\zeta \to 0} \Big(\sup_{\Sigma^+(r_1)} |\varphi_{2,\zeta}(x)| \Big) = 0. \end{cases} \tag{2.17}$$

Lemma 2.7 *We can take (smaller) r_1, $r_2 > 0$ and $\zeta_3 \in (0, \min(\zeta_1, \zeta_2)) > 0$ (depending on only D and function $q = q(s)$) such that the above two functions $\varphi_{1,\zeta}(x)$, $\varphi_{2,\zeta}(x)$ are positive and satisfy the following differential inequalities,*

$$\begin{cases} H_i(x) \equiv \Delta \varphi_{i,\zeta} + (M+1)\varphi_{i,\zeta} \leqq 0 \quad \text{in} \quad J(\zeta, r_2, r_1), \\ \partial \varphi_{i,\zeta} / \partial \nu = 0 \quad \text{on} \quad \partial J(\zeta, r_2, r_1) \setminus (\Gamma^+(r_1) \cup \Gamma^-(\zeta, r_2)), \end{cases} \tag{2.18}$$

for $i = 1, 2$, $0 < \zeta \leqq \zeta_3$. The meaning of the inequality in (2.18) *is taken in the generalized sense (cf. Gilbarg-Trudinger* [8 : *Chap.* 8]).

Proof of Lemma 2.7. First we calculate H_i in terms of the local coordinate in each subregion, $\Sigma^\circ(\zeta, r_2, 2\zeta)$, $\Lambda(\zeta)$ and $(\Sigma^+(r_1) \backslash \Sigma^+(2\zeta))$. By the property of the metric tensor $g = (g_{ij}(x))$, we have

$$\Delta \varphi(x) = \sum_{1 \leqq i,j \leqq \ell - 1} \frac{1}{\sqrt{G}} \frac{\partial}{\partial \xi_i} \left(\sqrt{G} g^{ij} \frac{\partial \varphi}{\partial \xi_j} \right) + \left(\frac{\partial^2}{\partial s^2} + \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial s} \frac{\partial}{\partial s} \right) \varphi + \sum_{j=1}^{m} \frac{\partial^2 \varphi}{\partial \eta_j^2}$$

and

$$\left(\frac{\partial^2}{\partial s^2}+\sum_{j=1}^{m}\frac{\partial^2}{\partial \eta_j^2}\right)\psi=\frac{\partial^2\psi}{\partial r^2}+\frac{m}{r}\frac{\partial\psi}{\partial r}$$

for a function $\psi=\psi(r)$, $(|z|=r=\sqrt{s^2+|\eta|^2})$. Noting that $\varphi_{1,\zeta}$, $\varphi_{2,\zeta}$ do not depend on ξ, we calculate $H_i(x)$ as follows,

$$H_i(x)=\left(\frac{\partial^2}{\partial s^2}+\sum_{j=1}^{m}\frac{\partial^2}{\partial \eta_j^2}\right)\varphi_{i,\zeta}+\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial s}\frac{\partial\varphi_{i,\zeta}}{\partial s}+(M+1)\varphi_{i,\zeta}.$$

(i) In $\Sigma^+(r_1)\setminus\Sigma^+(2\zeta)$, from (2.3), we have

$$H_1(x)=\left(\alpha_0+\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial s}\frac{s}{r}\right)\frac{K_1'(r)}{K_1(r_1)}\leqq 0\quad(2\zeta\leqq r\leqq r_1).$$

$$H_2(x)=\left(\alpha_0+\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial s}\frac{s}{r}\right)(-K_2'(|z|)/K_2'(2\zeta))\leqq 0\quad(2\zeta\leqq r\leqq r_1).$$

(ii) In $\Lambda(\zeta)$, from (2.10)–(2.11), we have

$$H_1(x)=\left(-\frac{K_1(2\zeta)}{K_1(r_1)}+\frac{1}{2}\right)\alpha_2 +\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial s}\left(\frac{K_1(2\zeta)}{\zeta K_1(r_1)}\frac{\partial N_1}{\partial s}\left(\frac{z}{\zeta}\right)-\frac{1}{2}\frac{\partial N_2}{\partial s}\left(\frac{z}{\zeta}\right)\right),$$

$$H_2(x)=\left(\frac{K_2(2\zeta)}{K_2'(2\zeta)}-\frac{1}{4\delta_2}\right)\alpha_2 +\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial s}\left(-\frac{K_2(2\zeta)}{K_2'(2\zeta)}\frac{1}{\zeta}\frac{\partial N_1}{\partial s}\left(\frac{z}{\zeta}\right)+\frac{1}{4\delta_2}\frac{\partial N_2}{\partial s}\left(\frac{z}{\zeta}\right)\right)$$

From $K_1(0)=1$, $K_1'(r)<0$, $K(r)>0$, $K_2'(r)<0$ for $0<r\leqq r_1$ and the definition of the constants α_0, α_1, α_2, both H_1, H_2 are negative in $\Lambda(\zeta)$ provided that $\zeta>0$ is small.

(iii) In $\Sigma^\circ(\zeta,r_2,2\zeta)$,

$$H_1(x)=-2\gamma_1+\frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial s}(2\delta_2-2\gamma_1(s+2\zeta)) +(M+1)\left[2\delta_2(s+2\zeta)-\gamma_1(s+2\zeta)^2+\left(\frac{K_1(2\zeta)}{K_1(r_1)}-\frac{\zeta}{2}\right)\right]$$

$$H_2(x) = -2\gamma_2 + \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial s}\left(-\frac{\delta_1}{16\delta_2} - 2\gamma_2(s+2\zeta)\right)$$
$$+ (M+1)\left[-\frac{\delta_1}{16\delta_2}(s+2\zeta) - \gamma_2(s+2\zeta)^2 + \left(-\frac{K_1(2\zeta)}{K_2'(2\zeta)} + \frac{\zeta}{4\delta_2}\right)\right].$$

Define γ_1, γ_2 by

$$2\gamma_1 = 2\alpha_0\delta_2 + \frac{M+1}{K_1(r_1)} + 1, \quad 2\gamma_2 = \frac{\alpha_0\delta_1}{16\delta_2} + 1.$$

From the properties of K_1, K_2, we can take $r_2 > 0$ small so that $H_1(x)$, $H_2(x)$ are negative in $\Sigma^\circ(\zeta, r_2, 2\zeta)$ for small $\zeta > 0$.

Next we consider the derivatives of $\varphi_{i,\zeta}$ $(i = 1, 2)$ in the normal direction on $\Gamma^+(2\zeta)$, $\Gamma^-(\zeta, 2\zeta)$ at which each $\varphi_{i,\zeta}$ is continous and is not C^1.

$$\frac{\partial\varphi_{1,\zeta}}{\partial r}_{|r=2\zeta+0} = \frac{K_1'(2\zeta)}{K_1(r_1)} \leqq 0 \quad \text{on } \Gamma^+(2\zeta),$$

$$\frac{\partial\varphi_{1,\zeta}}{\partial r}_{|r=2\zeta-0} = \frac{K_1(2\zeta)}{K_1(r_1)}\frac{1}{\zeta}\frac{\partial N_1}{\partial \tilde{\mathbf{n}}}\left(\frac{z}{\zeta}\right) - \frac{1}{2}\frac{\partial N_2}{\partial \tilde{\mathbf{n}}}\left(\frac{z}{\zeta}\right) \geqq \frac{\delta_1}{2} \quad \text{on } \Gamma^+(2\zeta),$$

$$\frac{\partial\varphi_{1,\zeta}}{\partial s}_{|s=-2\zeta+0} = -\frac{K_1(2\zeta)}{K_1(r_1)}\frac{1}{\zeta}\frac{\partial N_1}{\partial \tilde{\mathbf{n}}}(-2, \eta/\zeta) + \frac{1}{2}\frac{\partial N_2}{\partial \tilde{\mathbf{n}}}(-2, \eta/\zeta) \leqq \delta_2 \quad \text{on } \Gamma^-(\zeta, 2\zeta),$$

$$\frac{\partial\varphi_{1,\zeta}}{\partial s}_{|s=-2\zeta-0} = 2\delta_2 \quad \text{on } \Gamma^-(\zeta, 2\zeta),$$

for small $\zeta > 0$. From Lemma 2.5, Proposition 2.4, we see that

$$\frac{\partial\varphi_{1,\zeta}}{\partial r}_{|r=2\zeta+0} \leqq \frac{\partial\varphi_{1,\zeta}}{\partial r}_{|r=2\zeta-0} \quad \text{on } \Gamma^+(2\zeta), \tag{2.19}$$

$$\frac{\partial\varphi_{1,\zeta}}{\partial s}_{|s=-2\zeta+0} \leqq \frac{\partial\varphi_{1,\zeta}}{\partial s}_{|s=-2\zeta-0} \quad \text{on } \Gamma^-(\zeta, 2\zeta), \tag{2.20}$$

for small $\zeta > 0$. Next we calculate

$$\frac{\partial\varphi_{2,\zeta}}{\partial r}_{|r=2\zeta+0} = -\frac{K_2'(2\zeta)}{K_2'(2\zeta)} = -1 \quad \text{on } \Gamma^+(2\zeta),$$

$$\frac{\partial \varphi_{2,\zeta}}{\partial r}_{|r=2\zeta-0} = -\frac{K_2(2\zeta)}{K_2'(2\zeta)}\frac{1}{\zeta}\frac{\partial N_1}{\partial \widetilde{\mathbf{n}}}\left(\frac{z}{\zeta}\right) + \frac{1}{4\delta_2}\frac{\partial N_2}{\partial \widetilde{\mathbf{n}}}\left(\frac{z}{\zeta}\right) \geqq -\frac{1}{2} \quad \text{on } \Gamma^+(2\zeta),$$

$$\frac{\partial \varphi_{2,\zeta}}{\partial s}_{|s=-2\zeta+0} = \frac{K_2(2\zeta)}{K_2'(2\zeta)}\frac{1}{\zeta}\frac{\partial N_1}{\partial \widetilde{\mathbf{n}}}\left(-2, \frac{\eta}{\zeta}\right) - \frac{1}{4\delta_2}\frac{\partial N_2}{\partial \widetilde{\mathbf{n}}}\left(-2, \frac{\eta}{\zeta}\right) \leqq -\frac{\delta_1}{8\delta_2} \quad \text{on } \Gamma^-(\zeta, 2\zeta),$$

$$\frac{\partial \varphi_{2,\zeta}}{\partial s}_{|s=-2\zeta-0} = \frac{-\delta_1}{16\delta_2} \quad \text{on } \Gamma^-(\zeta, 2\zeta),$$

for small $\zeta > 0$. Hence we get

$$\frac{\partial \varphi_{2,\zeta}}{\partial r}_{|r=2\zeta+0} \leqq \frac{\partial \varphi_{2,\zeta}}{\partial r}_{|r=2\zeta-0} \quad \text{on } \Gamma^+(2\zeta), \tag{2.21}$$

$$\frac{\partial \varphi_{2,\zeta}}{\partial s}_{|s=-2\zeta+0} \leqq \frac{\partial \varphi_{2,\zeta}}{\partial s}_{|s=-2\zeta-0} \quad \text{on } \Gamma^-(\zeta, 2\zeta), \tag{2.22}$$

for small $\zeta > 0$. Each $\varphi_{i,\zeta}$ satisfies the Neumann B.C. on $\partial J(\zeta, r_2, r_1) \setminus (\Gamma^-(\zeta, r_2) \cup \Gamma^+(r_1))$. By using (i), (ii), (iii) and (2.19)–(2.22), we conclude H_i is positive in the generalized sense, i.e.

$$\int_{J(\zeta,r_2,r_1)} (-\nabla\varphi_{i,\zeta}\nabla\varphi - (M+1)\varphi_{i,\zeta}\,\varphi)\,dx \leqq 0 \quad (i = 1, 2)$$

for any $\varphi \in H^1(J(\zeta, r_2, r_1))$ such that $\varphi(x) \geqq 0$ in $J(\zeta, r_2, r_1)$ and $\varphi(x) = 0$ on $\Gamma^-(\zeta, r_2) \cup \Gamma^+(r_1)$. □

Lemma 2.8 *There exist $\kappa_1 > 0$, $\kappa_2 > 0$, $\gamma_3 > 0$, $\gamma_4 > 0$ and $r_3 \in (0, r_2)$ (which depend only on L and M, M_1) and $\zeta_4 \in (0, \zeta_3)$ such that $\varphi^{\pm}_{3,\zeta}(x)$ satisfy the differential inequalities*

$$\begin{cases} \varphi^+_{3,\zeta}(x) \leqq \varphi^-_{3,\zeta}(x) \quad (x \in J(\zeta, r_3, r_1)), \\ H_3^+ \equiv \Delta\varphi^+_{3,\zeta} + f(\varphi^+_{3,\zeta}) \leqq 0 \quad \text{in } J(\zeta, r_3, r_1), \\ \partial\varphi^+_{3,\zeta}/\partial\nu = 0 \quad \text{on } \partial J(\zeta, r_3, r_1) \setminus (\Gamma^+(r_1) \cup \Gamma^-(\zeta, r_3)), \end{cases} \tag{2.23}$$

in the generalized sense, in $J(\zeta, r_3, r_1)$ for $0 < \zeta \leqq \zeta_4$ and

$$\begin{cases} H_3^- \equiv \Delta\varphi^-_{3,\zeta} + f(\varphi^-_{3,\zeta}) \geqq 0 \quad \text{in } J(\zeta, r_3, r_1), \\ \partial\varphi^-_{3,\zeta}/\partial\nu = 0 \quad \text{on } \partial J(\zeta, r_3, r_1) \setminus (\Gamma^+(r_1) \cup \Gamma^-(\zeta, r_3)), \end{cases} \tag{2.24}$$

in the generalized sense, in $J(\zeta, r_3, r_1)$ for $0 < \zeta \leqq \zeta_4$.

Proof of Lemma 2.8. It is easy to see

$$H_3^\pm(x) = \Delta\varphi_{3,\zeta}^\pm + f(\varphi_{3,\zeta}^\pm) = \Delta w + f(w) = 0 \quad \text{in} \ \Sigma^+(r_1) \setminus \Sigma^+(2\zeta).$$

Next we consider the normal derivative of $\varphi_3^\pm$ at $\Gamma^+(2\zeta)$.

$$\left| \frac{\partial \varphi_{3,\zeta}^\pm}{\partial r}_{|r=2\zeta+0} \right| \leqq \sup_D |\nabla w|, \quad \frac{\partial \varphi_{3,\zeta}^+}{\partial r}_{|r=2\zeta-0} \geqq \delta_4 \quad \text{on} \ \Gamma^+(2\zeta),$$

$$\frac{\partial \varphi_{3,\zeta}^-}{\partial r}_{|r=2\zeta-0} \leqq -\delta_4 \quad \text{on} \ \Gamma^+(2\zeta),$$

By (2.16) in Lemma 2.6, we can take $\kappa_2 > 0$ large (to control $\delta_4 = \delta_4(\kappa_2)$) and fix it so that

$$\frac{\partial \varphi_{3,\zeta}^+}{\partial r}_{|r=2\zeta+0} \leqq \frac{\partial \varphi_{3,\zeta}^+}{\partial r}_{|r=2\zeta-0} \quad \text{on} \ \Gamma^+(2\zeta),$$

$$\frac{\partial \varphi_{3,\zeta}^-}{\partial r}_{|r=2\zeta+0} \geqq \frac{\partial \varphi_{3,\zeta}^-}{\partial r}_{|r=2\zeta-0} \quad \text{on} \ \Gamma^+(2\zeta),$$

for $0 < \zeta \leqq \zeta_2 = \zeta_2(\kappa_1, \kappa_2)$.

In $\Lambda(\zeta)$, we calculate $H_3^\pm$ in terms of the local coordinate system (ξ, s, η).

$$H_3^\pm(x) = \sum_{1 \leqq i,j \leqq \ell-1} \frac{1}{\sqrt{G}} \frac{\partial}{\partial \xi_i} \left(\sqrt{G} g^{ij}(\xi, s) \left(\frac{\partial N_3^\pm}{\partial \xi_j} \right) \left(\xi, \frac{z}{\zeta} \right) \right)$$
$$+ (\Delta_z N_3^\pm)(\xi, z/\zeta) + \frac{1}{\sqrt{G}} \frac{\partial \sqrt{G}}{\partial s} \frac{1}{\zeta} \frac{\partial N_3^\pm}{\partial s}(\xi, z/\zeta) + f(\varphi_{3,\zeta}^\pm).$$

Note that $\Delta_z N_3^\pm(z) \equiv \mp\kappa_1$ in $\Lambda(\zeta)$ and $|(1/\zeta)(\partial N_3^\pm/\partial s)(\xi, z/\zeta)| \leqq \delta_5$, the first and third terms in $H_3^\pm$, are bounded independent of ξ and $\zeta \in (0, \zeta_2(\kappa_1, \kappa_2))$. We should note that their bounds are independent of $\kappa_1 > 0$ and so we take κ_1 large so that we have $H_3^+(x) < 0$, $H_3^-(x) > 0$ in $\Lambda(\zeta)$.

Next we consider the derivative of $\varphi_{3,\zeta}^\pm$ on $\Gamma^-(\zeta, 2\zeta)$. Put $\gamma_3 = \delta_5 + 1$ and we have (with the aid of Lemma 2.6),

$$\frac{\partial \varphi_{3,\zeta}^+}{\partial s}_{|s=-2\zeta+0} \leqq \delta_5 < \gamma_3 = \frac{\partial \varphi_{3,\zeta}^+}{\partial s}_{|s=-2\zeta-0} \quad \text{on} \ \Gamma^-(\zeta, 2\zeta),$$

$$\frac{\partial \varphi_{3,\zeta}^-}{\partial s}_{|s=-2\zeta+0} \geqq -\delta_5 > -\gamma_3 = \frac{\partial \varphi_{3,\zeta}^-}{\partial s}_{|s=-2\zeta-0} \quad \text{on} \ \Gamma^-(\zeta, 2\zeta).$$

In $\Sigma^\circ(\zeta, r_2, 2\zeta)$, we calculate $H_3^\pm$ as follows

$$H_3^\pm = \mp 2\gamma_4 \pm \frac{1}{\sqrt{G}}\frac{\partial\sqrt{G}}{\partial s}(\gamma_3 - 2\gamma_4(s+2\zeta)) + \frac{1}{\sqrt{G}}\sum_{i,j=1}^{\ell-1}\frac{\partial}{\partial\xi_j}\left(g^{ij}\sqrt{G}\frac{\partial w(\xi,o'')}{\partial\xi_j}\right) + f(\varphi_{3,\zeta}^\pm)$$

By taking $r_3 \in (0, r_2)$ small and $\gamma_4 > 0$ large (independent of ζ), we have $H_3^+ < 0$ and $H_3^- > 0$ in $\Sigma^\circ(\zeta, r_3, 2\zeta)$.

Taking into account that $\varphi_{3,\zeta}^\pm$ satisfies the Neumann B.C. on $\partial J(\zeta, r_3, r_1)\backslash(\Gamma^-(\zeta, r_3)\cup\Gamma^+(r_1))$, we conclude the inequalities in Lemma 2.8 in the generalized sense. □

Uniform convergence of the solution in D

We will prove the uniform convergence of u_{σ_p} to $w(x)$ in D (cf. (1.10)) by making a barrier of $u_{\sigma_p}(x) - \varphi_{3,\zeta_p}^+(x)$ from above and one of $u_{\sigma_p}(x) - \varphi_{3,\zeta_p}^-(x)$ from below. Here the barrier function is a certain linear combination of $\varphi_{1,\zeta}(x)$ and $\varphi_{2,\zeta}(x)$.

As is seen before, we see that $\varphi_{1,\zeta}$, $\varphi_{2,\zeta}$ are positive in $J(\zeta, r_3, r_1)$ for small $\zeta > 0$ and

$$\liminf_{\zeta\to 0}\Big(\inf_{x\in\Gamma^-(\zeta,r_3)}\varphi_{1,\zeta}(x)\Big) > 0, \quad \liminf_{\zeta\to 0}\Big(\inf_{x\in\Gamma^-(\zeta,r_3)}\varphi_{2,\zeta}(x)\Big) > 0. \tag{2.25}$$

Lemma 2.9 *Let u be any solution of* (1.1) *in $\Omega(\zeta)$ such that $|u(x)| \leqq h$ in $\Omega(\zeta)$. Put*

$$c_1 = c_1(\zeta) = \Big(\sup_{x\in\Gamma^+(r_1)}|u(x) - w(x)|\Big) \Big/ \Big(\inf_{x\in\Gamma^+(r_1)}\varphi_{1,\zeta}(x)\Big), \tag{2.26}$$

$$c_2 = c_2(\zeta) = \Big(\sup_{x\in\Gamma^-(\zeta,r_3)}(|\varphi_{3,\zeta}^+(x)| + |\varphi_{3,\zeta}^-(x)|) + h\Big) \Big/ \Big(\inf_{x\in\Gamma^-(\zeta,r_3)}\varphi_{2,\zeta}(x)\Big). \tag{2.27}$$

Then it holds that

$$\begin{aligned}&\varphi_{3,\zeta}^{-}(x) - (c_1\varphi_{1,\zeta}(x) + c_2\varphi_{2,\zeta}(x))\\ &\quad \leqq u(x) \leqq \varphi_{3,\zeta}^{+}(x) + c_1\varphi_{1,\zeta}(x) + c_2\varphi_{2,\zeta}(x)\\ &\qquad\qquad \textit{in } J(\zeta, r_3, r_1) \quad \textit{for } 0 < \zeta \leqq \zeta_4. \end{aligned} \tag{2.28}$$

Recall that h appeared in the condition (1.5) in §1.

Proof of Lemma 2.9. From the inequalities in Lemma 2.7, Lemma 2.8 and the equation for u, we have,

$$\begin{aligned}&\Delta(c_1\varphi_{1,\zeta} + c_2\varphi_{2,\zeta} - (u - \varphi_{3,\zeta}^{+}))\\ &\quad + A(x)\big(c_1\varphi_{1,\zeta}(x) + c_2\varphi_{2,\zeta}(x) - (u - \varphi_{3,\zeta}^{+}(x))\big)\\ &\leqq c_1 g_1 + c_2 g_2 - (M + 1 - A(x))(c_1\varphi_{1,\zeta} + c_2\varphi_{2,\zeta}) < 0 \quad \text{in} \quad J(\zeta, r_3, r_1),\end{aligned}$$

$$\begin{aligned}&\Delta(c_1\varphi_{1,\zeta} + c_2\varphi_{2,\zeta} + (u - \varphi_{3,\zeta}^{-}))\\ &\quad + B(x)\big(c_1\varphi_{1,\zeta}(x) + c_2\varphi_{2,\zeta}(x) + (u - \varphi_{3,\zeta}^{-}(x))\big)\\ &\leqq c_1 g_1 + c_2 g_2 - (M + 1 + B(x))(c_1\varphi_{1,\zeta} + c_2\varphi_{2,\zeta}) < 0 \quad \text{in} \quad J(\zeta, r_3, r_1),\end{aligned}$$

where

$$A(x) = \int_0^1 f'(tu(x) + (1-t)\varphi_{3,\zeta}^{+}(x))\, dt,$$

$$B(x) = \int_0^1 f'(tu(x) + (1-t)\varphi_{3,\zeta}^{-}(x))\, dt.$$

By the definition of c_1, c_2, we have

$$-(c_1\varphi_{1,\zeta} + c_2\varphi_{2,\zeta}) \leqq u(x) - \varphi_{3,\zeta}^{-}(x) \quad \text{on} \quad \Gamma^{-}(\zeta, r_3) \cup \Gamma^{+}(r_1),$$

$$u(x) - \varphi_{3,\zeta}^{+}(x) \leqq c_1\varphi_{1,\zeta} + c_2\varphi_{2,\zeta} \quad \text{on} \quad \Gamma^{-}(\zeta, r_3) \cup \Gamma^{+}(r_1).$$

Note also $c_1\varphi_{1,\zeta} + c_2\varphi_{2,\zeta}$ (positive) and $u - \varphi_{3,\zeta}^{\pm}$ satisfy the Neumann boundary condition on $\partial J(\zeta, r_3, r_1) \setminus (\Gamma^{-}(\zeta, r_3) \cup \Gamma^{+}(r_1))$. Therefore by the maximum principle in the generalized version, we get the inequality (2.28). □

Put $\zeta = \sigma_p$ and $u = u_{\sigma_p}$ in Lemma 2.9, we have

$$\begin{aligned}&\varphi_{3,\sigma_p}^{-}(x) - (c_1(\sigma_p)\varphi_{1,\sigma_p}(x) + c_2(\sigma_p)\varphi_{2,\sigma_p}(x))\\ &\quad \leqq u_{\sigma_p}(x) \leqq \varphi_{3,\sigma_p}^{+}(x) + c_1(\sigma_p)\varphi_{1,\sigma_p}(x) + c_2(\sigma_p)\varphi_{2,\sigma_p}(x) \quad \text{in} \quad D\end{aligned}$$

for large p, where

$$c_1(\sigma_p) = \Big(\sup_{x\in\Gamma^+(r_1)} |u_{\sigma_p}(x) - w(x)|\Big) \Big/ \Big(\inf_{x\in\Gamma^+(r_1)} \varphi_{1,\sigma_p}(x)\Big),$$

$$c_2(\sigma_p) = \Big(\sup_{x\in\Gamma^-(\sigma_p,r_3)} (|\varphi^+_{3,\sigma_p}(x)| + |\varphi^-_{3,\sigma_p}(x)|) + h\Big) \Big/ \Big(\inf_{x\in\Gamma^-(\sigma_p,r_3)} \varphi_{2,\sigma_p}(x)\Big).$$

Note $\lim_{p\to\infty} c_1(\sigma_p) = 0$ from (1.10) and $\lim_{\zeta\to 0}\sup_{\Sigma^+(r_1)} |\varphi_{2,\zeta}(x)| = 0$ and $c_2(\sigma_p)$ is bounded from above and below by positive numbers as $p \to \infty$. We conclude the uniform convergence of u_{σ_p} to $w(x)$ in D for $p \to \infty$. This implies (1.8) in the main theorem.

3. Uniform convergence in $Q(\zeta)$

In this section we consider the behavior of solutions in the shrinking part $Q(\zeta)$. We recall the following estimate,

$$\begin{aligned}&\varphi^-_{3,\zeta}(x) - (c_1(\zeta)\varphi_{1,\zeta}(x) + c_2(\zeta)\varphi_{2,\zeta}(x))\\ &\quad\leqq u_\zeta(x) \leqq \varphi^+_{3,\zeta}(x) + c_1(\zeta)\varphi_{1,\zeta}(x) + c_2(\zeta)\varphi_{2,\zeta}(x),\\ &\qquad\qquad \text{in}\ \ J(\zeta, r_3, r_1) \quad (0 < \zeta = \sigma_p \leqq \zeta_4).\end{aligned}$$

which was obtained in the previous section. From this estimate we derive a behavior of u_{σ_p} near ∂L. Taking the limit $p \to \infty$, we have

$$\begin{aligned}&\limsup_{p\to\infty} \sup_{x=(\xi+s\,\mathbf{n}(\xi),\eta)\in\Sigma^-(\sigma_p,r_3)} |u_{\sigma_p}(\xi + s\,\mathbf{n}(\xi), \eta) - w(\xi, o'')|\\ &\quad\leqq |\gamma_3 s - \gamma_4 s^2| + |-(\delta_1/16\delta_2)s - \gamma_2 s^2| \cdot \limsup_{p\to\infty} c_2(\sigma_p)\\ &\qquad\qquad (-r_3 \leqq s \leqq 0).\end{aligned} \tag{3.1}$$

We will obtain estimates on the derivatives of the solutions and deduce a certain compactness of solutions. For that purpose, the following two lemmas are key estimates to prove the convergence of the solutions in $Q(\zeta)$. Let u be any solution of (1.1) such that $|u(x)| \leqq h$ in $\Omega(\zeta)$. We have the following properties for this solution.

Lemma 3.1 *For any $\rho > 0$ there exists $c_3(\rho) > 0$ such that*

$$|\nabla_{x'} u(x', x'')| \leqq c_3(\rho) \quad in \quad \overline{Q(\zeta) \setminus \Sigma^-(\zeta, 2\rho)} \quad (0 < \zeta \leqq \zeta_0). \tag{3.2}$$

Lemma 3.2 *For any $\rho > 0$ there exists $c_4(\rho) > 0$ such that*

$$|\langle \tau \cdot \nabla_{x''} u(x', x'') \rangle| \leqq c_4(\rho)\zeta \quad \textit{for} \quad x = (x', x'') \in L(2\rho) \times \partial B^{(m)}(\zeta), \tag{3.3}$$

and any unit vector $\tau \in \mathbb{R}^m$ such that $\langle x'' \cdot \tau \rangle = 0$ $(0 < \zeta \leqq \zeta_0)$.

Proof of Lemma 3.1. We estimate $\nabla_{x'} u(x)$ by reflection with a barrier function (cf. Gilbarg-Trudinger [8; Chap. 14]). We consider the derivative of u in x'-direction. Let $0 < 2\zeta < \rho$. Take any $x_0 = (x_0', x_0'') \in Q(\zeta) \setminus \Sigma^-(\zeta, 2\rho)$ and take any unit vector $\tau \in \mathbb{R}^\ell$ and fix them. Then

$$\mathcal{V}(\zeta) \equiv B^{(\ell)}(x_0', \rho) \times B^{(m)}(\zeta) \subset Q(\zeta)$$

holds. We give an upper bound to $|\tau \cdot \nabla_{x'} u(x_0)|$ by constructing a certain barrier function. We use the reflection map $x \longmapsto \widehat{x}$ defined by

$$\widehat{x} = (x' - 2\langle x' - x_0' \cdot \tau \rangle \tau, x'') \quad \text{for} \quad x = (x', x'').$$

This reflection is with respect to the hyperplane $P_1 = \{(x', x'') \mid \langle x' - x_0' \cdot \tau \rangle = 0\}$ which contains x_0 and is orthogonal to the vector τ. Here $\langle \cdot \rangle$ denotes the inner product of two vectors. Let us consider the domain

$$\mathcal{V}_1(\zeta) = \{x = (x', x'') \in \mathcal{V}(\zeta) \mid \langle x' - x_0' \cdot \tau \rangle \geqq 0,$$
$$|x' - x_0' - \langle x' - x_0' \cdot \tau \rangle \tau| < \rho/2\}.$$

Note that if u is a solution, $u(\widehat{x})$ also satisfies the equation in $\mathcal{V}(\zeta)$. Hence if we put $U(x) = u(x) - u(\widehat{x})$, then

$$\Delta U + f(u(x)) - f(u(\widehat{x})) = 0 \quad \text{in} \quad \mathcal{V}(\zeta)$$

with $U = 0$ on $\mathcal{V}(\zeta) \cap P_1$. We restrict U on $\mathcal{V}_1(\zeta)$ and discuss its derivative in the τ direction on $\mathcal{V}_1(\zeta) \cap P_1$. We will use the relation $2\langle \tau \cdot \nabla_{x'} u \rangle = \langle \tau \cdot \nabla_{x'} U \rangle$ on $\mathcal{V}_1(\zeta) \cap P_1$ to get the estimate on $\partial u / \partial \tau$.

Define a function

$$\phi(x) = a\,(2\rho\langle x' - x_0' \cdot \tau \rangle - \langle x' - x_0' \cdot \tau \rangle^2) + b\,\phi'(x),$$

where

$$\phi'(x) = \begin{cases} 0 \quad \text{for} \quad x \in \mathcal{V}_1(\zeta), \; |x' - x_0' - \langle x' - x_0' \cdot \tau \rangle \tau| \leqq \rho/4, \\ (|x' - x_0' - \langle x' - x_0' \cdot \tau \rangle \tau| - \rho/4)^2 \\ \qquad \text{for} \quad x \in \mathcal{V}_1(\zeta), \; \rho/4 < |x' - x_0' - \langle x' - x_0' \cdot \tau \rangle \tau| \leqq \rho/2. \end{cases}$$

It is easy to see that ϕ is C^1 and piecewise smooth in the closure of $\mathcal{V}_1(\zeta)$. Put

$$a = (M_1/2) + 32(\ell h/\rho^2), \quad b = 32h/\rho^2,$$

then we have the differential inequalities

$$\Delta(\phi \pm U) < 0 \ \text{ in } \ \mathcal{V}_1(\zeta), \quad -\phi(x) \leqq U(x) \leqq \phi(x) \ \text{ on } \ \partial\mathcal{V}_1(\zeta) \setminus \partial Q(\zeta).$$

Note that $\phi(x)$, $U(x)$ satisfy the Neumann B.C. on $\partial\mathcal{V}_1(\zeta) \cap \partial Q(\zeta)$. Hence we apply the maximum principle and obtain

$$-\phi(x) \leqq U(x) \leqq \phi(x) \quad \text{for } \ x \in \mathcal{V}_1(\zeta).$$

Therefore we get

$$|\tau \cdot \nabla_{x'} u(x_0)| = (1/2)|\tau \cdot \nabla_{x'} U(x_0)| \leqq a = (M/2) + 4(\ell h/\rho^2).$$

This bound does not depend on the choice of $x_0 \in Q(\zeta) \setminus \Sigma^-(\zeta, 2\rho)$ and unit vector $\tau \in \mathbb{R}^m$, we have the conclusion of the lemma. □

Proof of Lemma 3.2. Similarly as the proof of Lemma 3.1, we prove the estimate by the reflection. Take any unit vector $\tau \in \mathbb{R}^m$. Define the reflection with respect to the hyperplane $P_2 = \{(x', x'') \mid \langle x'' \cdot \tau \rangle = 0\}$ by $\widehat{x} = (x', x'' - 2\langle x'' \cdot \tau \rangle \tau)$. By using this, we define a function $U(x) = u(x) - u(\widehat{x})$ in the domain

$$\mathcal{V}_2(\zeta) = \{(x', x'') \in L(\rho) \times B^{(m)}(\zeta) \mid \langle \tau \cdot x'' \rangle \geqq 0\} \subset Q(\zeta).$$

Note that $U(x) = 0$ on $P_2 \cap \mathcal{V}_2(\zeta)$. We construct a comparison function

$$\phi(x) = a\,(2\zeta\langle x'' \cdot \tau \rangle - \langle x'' \cdot \tau \rangle^2) + b\,\phi'(x),$$

where

$$\phi'(x) = \begin{cases} 0 \quad \text{for } \ x = (x', x'') \in \mathcal{V}_2(\zeta), \ x' \in L(2\rho) \\ (s + 2\rho)^2 \quad \text{for } \ x = (\xi + s\,\mathbf{n}(\xi), x'') \in \mathcal{V}_2(\zeta), \\ \qquad\qquad\qquad\qquad -2\rho \leqq s \leqq -\rho, \ \xi \in \partial L \end{cases}$$

Similarly as in Lemma 3.1, we can choose $a > 0$, $b > 0$ which depend only on $\rho > 0$, by which we can prove

$$-\phi(x) \leqq U(x) \leqq \phi(x) \quad x \in \mathcal{V}_2(\zeta).$$

This implies

$$|\tau\cdot\nabla_{x''}U(x)| \leqq |\tau\cdot\nabla_{x''}\phi(x)| = 2a\zeta$$
$$\text{if } x'\in L(2\rho),\ x=(x',x'')\in P_2\cap\partial\mathcal{V}_2(\zeta).$$

Particularly it is true for $x=(x',x'')\in L(2\rho)\times\partial B^{(m)}(\zeta)$ such that $\langle x'',\tau\rangle = 0$. By using the relation $2\langle\tau\cdot\nabla_{x''}u(x)\rangle = \langle\tau\cdot\nabla_{x''}U(x)\rangle$ for $x\in P_2\cap\partial\mathcal{V}_2(\zeta)$. We complete the proof of Lemma 3.2. □

Let u be any solution of (1.1) with $|u(x)|\leqq h$ in $\Omega(\zeta)$. To discuss the convergence of solutions in $Q(\zeta)$, we use the following change of the variable

$$y'=x',\quad y''=x''/\zeta,\quad \widehat{u}_\zeta(y',y'')=u(y',\zeta\, y'') \tag{3.4}$$

and we define the domain $\widetilde{Q}(\zeta)$ by

$$\widetilde{Q}(\zeta)=\{(y',y'')\in\mathbb{R}^\ell\times\mathbb{R}^m \mid (y',\zeta y'')\in Q(\zeta)\}. \tag{3.5}$$

We also denote some portions of $\widetilde{Q}(\zeta)$ by $\widetilde{Q}'(\zeta,\rho)$ and $\widetilde{S}(\zeta,\rho)$ which are defined as follows,

$$\widetilde{Q}'(\zeta,\rho)=\{(y',y'')\in\widetilde{Q}(\zeta)\mid y'\in L(\rho)\},$$
$$\widetilde{S}'(\zeta,\rho)=\{(y',y'')\in\partial\widetilde{Q}(\zeta)\mid y'\in L(\rho)\}.$$

Remark that $\widetilde{Q}'(\zeta,\rho)$ and $\widetilde{S}'(\zeta,\rho)$ do not depend on ζ if $0<2\zeta\leqq\rho$. That is,

$$\widetilde{Q}'(\zeta,\rho)=L(\rho)\times B^{(m)}(1),\quad \widetilde{S}'(\zeta,\rho)=L(\rho)\times\partial B^{(m)}(1).$$

So we denote them by $\widetilde{Q}'(\rho)$ and $\widetilde{S}'(\rho)$ if $0<2\zeta\leqq\rho$. It is easy to see that $\widehat{u}_\zeta$ satisfies the equation

$$\left(\Delta_{y'}+\frac{1}{\zeta^2}\Delta_{y''}\right)\widehat{u}_\zeta+f(\widehat{u}_\zeta)=0\quad\text{in}\ \ \widetilde{Q}'(\rho)\quad(0<2\zeta\leqq\rho), \tag{3.6}$$

and the Neumann boundary condition on $\widetilde{S}'(\rho)$.

By multiplying the equation (3.6) by $\widehat{u}_\zeta$ and integration in $\widetilde{Q}'(\rho)$, we obtain

$$\int_{\widetilde{Q}'(\rho)}\left(|\nabla_{y'}\widehat{u}_\zeta|^2+(1/\zeta^2)|\nabla_{y''}\widehat{u}_\zeta|^2-f(\widehat{u}_\zeta)\,\widehat{u}_\zeta\right)\,dy'dy''$$
$$=\int_{\partial L(\rho)\times B^{(m)}(1)}\widehat{u}_\zeta\,(\partial\widehat{u}_\zeta/\partial\,\mathbf{n})\,dS_{y'}dy'$$

Using the upper bound of solution $|u| \leqq h$ and the estimate on the gradient of u in Lemma 3.1, we see that that there exists $c_5 = c_5(\rho) > 0$

$$\begin{cases} \displaystyle\int_{\widetilde{Q}'(\rho)} |\nabla_{y'}\widehat{u}_\zeta|^2\, dy'dy'' \leqq c_5(\rho) \\ \displaystyle\int_{\widetilde{Q}'(\rho)} |\nabla_{y''}\widehat{u}_\zeta|^2\, dy'dy'' \leqq c_5(\rho)\zeta^2 \end{cases} \tag{3.7}$$

for $0 < 2\zeta \leqq \rho$. From these estimates, we conclude $\{\widehat{u}_\zeta\}_{0<\zeta\leqq\rho/2}$ is relatively compact in $L^2(\widetilde{Q}'(\rho))$ for any small $\rho > 0$. Applying Cantor's argument, we get a subsequence σ_p' and a function u_0 on L such that $u_0(y', y'')$ is independent of y'' and

$$\lim_{p\to\infty} \|\widehat{u}_{\sigma_p'} - u_0\|_{L^2(\widetilde{Q}'(\rho))} = 0, \quad |u_0(y')| \leqq h \quad \text{in} \quad L,$$

$$\int_{\widetilde{Q}'(\rho)} \Big(\nabla_{y'}\widehat{u}_{\sigma_p'}\nabla_y'\phi(y') - f(\widehat{u}_{\sigma_p'})\,\phi(y')\Big)\, dy'dy'' = 0$$

for any $\phi = \phi(y')$ in $C_o^\infty(L)$ with $\mathrm{Supp}(\phi) \subset L(\rho)$. Taking $p \to \infty$, we get

$$\int_{L(\rho)\times B^{(m)}(1)} \left(\nabla_{y'}u_0\nabla_y'\phi(y') - f(u_0)\,\phi\right)\, dy'dy'' = 0$$

for any small $\rho > 0$. This implies

$$\Delta_{y'}u_0 + f(u_0) = 0 \quad \text{in} \quad L. \tag{3.8}$$

By the Schauder estimate, u_0 is a C^2 in the interior of L (classical solution) of this elliptic equation. Moreover from the estimate in (3.1), we have

$$\lim_{\rho\to 0} \sup\{|u_0(\xi + s\,\mathbf{n}(\xi)) - w(\xi, o'')| \mid \xi \in \partial L,\ -\rho \leqq s \leqq 0\} = 0 \tag{3.9}$$

This justifies the boundary condition $u_0(\xi) = w(\xi, o'')$ on ∂L and u_0 is C^2 up to the boundary of L. We will prove that $\widehat{u}_{\sigma_p'}$ "uniformly converges" to u_0. Hereafeter we denote σ_p' by σ_p.

Lemma 3.3

$$\lim_{p\to\infty} \sup_{x=(x',x'')\in Q(\sigma_p)} |u_{\sigma_p}(x', x'') - u_0(x')| = 0.$$

Proof of Lemma 3.3. Using Lemma 3.1 and Lemma 3.2, we see that $\{\widehat{u}_\zeta\}_{0<\zeta\leqq\rho/2}$ is precompact in $C^0(\widetilde{S}'(\rho))$ for any $\rho > 0$. So $\widehat{u}_{\sigma_p}$ uniformly converges to u_0 in $\widetilde{S}'(\rho)$ for any $\rho > 0$. This consideration with the estimate (3.1) implies that $\widehat{u}_{\sigma_p}$ uniformly converges to u_0 in the whole boundary of $\widetilde{Q}(\sigma_p)$. It is in other words,

$$\lim_{p\to\infty} \sup_{x=(x',x'')\in\partial Q(\sigma_p)} |u_{\sigma_p}(x',x'') - u_0(x')| = 0. \tag{3.10}$$

We estimate the convergence in the whole $Q(\sigma_p)$. Define

$$\kappa(p) = \sup_{x=(x',x'')\in\partial Q(\sigma_p)} |u_{\sigma_p}(x',x'') - u_0(x')| + \sup_{x=(x',x'')\in\Sigma^-(\sigma_p,2\sigma_p)} |u_{\sigma_p}(x',x'') - u_0(x')|.$$

It is also easy to see from (3.1) that $\lim_{p\to\infty}\kappa(p) = 0$. Let us define

$$u^{\pm}_{0,\sigma_p}(x) = u_0(x') \pm M_1(\sigma_p^2 - |x''|^2) + \kappa(\sigma_p) \quad (\text{cf. } M_1 \text{ in } (1.11)).$$

It is true that

$$u^-_{0,\sigma_p}(x) \leqq u_{\sigma_p}(x) \leqq u^+_{0,\sigma_p}(x) \quad \text{on } \partial(Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p)).$$

On the other hand, an easy calculation gives

$$\Delta(u^+_{0,\sigma_p} - u_{\sigma_p}) \leqq 0, \quad \Delta(u_{\sigma_p} - u^-_{0,\sigma_p}) \leqq 0, \text{ in } Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p)$$

for large p. By the maximum principle, we conclude

$$u^-_{0,\sigma_p}(x) \leqq u_{\sigma_p}(x) \leqq u^+_{0,\sigma_p}(x) \quad \text{in } Q(\sigma_p)\setminus\Sigma^-(\sigma_p,2\sigma_p).$$

□

This estimate proves (1.9) in the main theorem. Summing the results in §2 and §3, we complete the proof of the main result of this paper.

4. Generalization

In the previous sections, we have dealt with the scalar (stationary) reaction-diffusion equation and obtained the characterization for behaviors of solutions. We are going to generalize those results to the case of system (vector valued) reaction-diffusion equations. In this problem it is important to estimate solutions uniformly in the fixed region D (up to the interfacial region near ∂L) and for that purpose, we constructed barrier fuctions and apply a comparison technique to bound the difference between the perturbed solution ($\zeta > 0$) and the limit solution ($\zeta = 0$). Here, in this procedure we have to be careful because, in general, the maximum principle (or comparison technique) may not be applicable to system reaction-diffusion equations such as our case. But the situation where we apply the comparison technique, is very special. We construct and use the (positive) barrier functions in a very thin region around the interfacial region between the fixed domain D and $Q(\zeta)$, and so the comparison method is still applicable to our case. Now we formulate the problem.

We consider the following stationary reaction-diffusion system in $\Omega(\zeta)$ ($\zeta > 0$),

$$\Delta \boldsymbol{u} + C_\zeta(x)\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{0} \quad \text{in} \quad \Omega(\zeta), \quad \partial \boldsymbol{u}/\partial \nu = \boldsymbol{0} \quad \text{on} \quad \partial\Omega(\zeta), \tag{4.1}$$

where

$$\boldsymbol{u} = {}^t(u_1, u_2, \ldots, u_d) \quad \boldsymbol{f}(\boldsymbol{u}) = {}^t(f_1(\boldsymbol{u}), f_2(\boldsymbol{u}), \ldots, f_d(\boldsymbol{u})),$$
$$\boldsymbol{f} \in C^1(\mathbb{R}^d, \mathbb{R}^d),$$
$$C_\zeta = (C_{ij,\zeta})_{1 \leqq i,j \leqq d} \in C^1(\overline{\Omega(\zeta)}, M_d(\mathbb{R})),$$
$$C = (C_{ij})_{1 \leqq i,j \leqq d} \in C^1(\overline{D}, M_d(\mathbb{R})),$$
$$\overline{C} = (\overline{C}_{ij})_{1 \leqq i,j \leqq d} \in C^1(\overline{L}, M_d(\mathbb{R}))$$

and

$$\lim_{\zeta \to 0} \sup_{x \in D} |C_\zeta(x) - C(x)| = 0, \quad \lim_{\zeta \to 0} \sup_{x=(x',x'') \in Q(\zeta)} |C_\zeta(x', x'') - \overline{C}(x')| = 0.$$

Here we denoted the set of all $d \times d$ real matrices by $M_d(\mathbb{R})$. One important and simple example (our result is applicable) is the Ginzburg-Landau equation (see [7], [14], $\boldsymbol{f}(\boldsymbol{u}) = \lambda(1 - |\boldsymbol{u}|^2)\boldsymbol{u}$).

We have the following result.

Theorem 4.1 *Assume that* $\{\zeta_p\}_{p=1}^\infty$ *is a sequence of positive values with* $\lim_{p\to\infty} \zeta_p = 0$ *and* $\boldsymbol{u}_{\zeta_p}$ *is a solution of* (4.1) *for* $\zeta = \zeta_p$ *with*

$$|\boldsymbol{u}_{\zeta_p}(x)| \leqq h < +\infty \quad (x \in \Omega(\zeta_p), \ p \geqq 1). \tag{4.2}$$

Then there exist a subsequence $\{\sigma_p\}_{p=1}^\infty \subset \{\zeta_p\}_{p=1}^\infty$ *and* $\boldsymbol{w} \in C^2(\overline{D}, \mathbb{R}^d)$ *and* $\boldsymbol{V} \in C^2(\overline{L}, \mathbb{R}^d)$ *such that*

$$\Delta \boldsymbol{w} + C(x)\boldsymbol{f}(\boldsymbol{w}) = \mathbf{0} \quad in \ \ D, \quad \frac{\partial \boldsymbol{w}}{\partial \nu} = \mathbf{0} \quad on \ \ \partial D, \tag{4.3}$$

$$\Delta' \boldsymbol{V} + \overline{C}(x')\boldsymbol{f}(\boldsymbol{V}) = \mathbf{0} \quad in \ \ L, \quad \boldsymbol{V}(x') = \boldsymbol{w}(x', o'') \quad (x' \in \partial L), \tag{4.4}$$

$$\lim_{p\to\infty} \sup_{x\in D} |\boldsymbol{u}_{\sigma_p}(x) - \boldsymbol{w}(x)| = 0,$$

$$\lim_{p\to\infty} \sup_{(x',x'')\in Q(\sigma_p)} |\boldsymbol{u}_{\sigma_p}(x', x'') - \boldsymbol{V}(x')| = 0. \tag{4.5}$$

Sketch of the Proof of Theorem 4.1. We can carry out a similar argument as the scalar case except for the construction of the barrier function. From the boundedness of the solution (4.2) (cf. similar argument in §1), we can claim that there exist a subsequence $\{\sigma_p\} \subset \{\zeta_p\}$ and $\boldsymbol{w} = {}^t(w_1, w_2, \ldots, w_d) \in C^2(\overline{D}; \mathbb{R}^d)$ such that (4.3) holds and

$$\lim_{p\to\infty} \sup_{x\in D(\eta)} |\boldsymbol{u}_{\sigma_p}(x) - \boldsymbol{w}(x)| = 0 \quad \text{for any} \ \ \eta > 0. \tag{4.6}$$

From the condition of the uniform boundedness of $\{\boldsymbol{u}_{\zeta_p}\}$, we can modify $\boldsymbol{f}(\boldsymbol{\xi})$ for large $|\boldsymbol{\xi}|$ ("cut off") without changing the solutions. Thus we can assume, without loss of generality that

$$|\boldsymbol{f}(\boldsymbol{\xi})| \leqq M_1', \quad \sum_{k=1}^{d} |C_{ik,\zeta}(x) \frac{\partial f_k}{\partial \xi_j}(\boldsymbol{\xi})| \leqq M'$$
$$(x \in \Omega(\zeta), \ \zeta > 0, \ \boldsymbol{\xi} \in \mathbb{R}^d, \ 1 \leqq i, j \leqq d). \tag{4.7}$$

We will construct a barrier to bound the behavior of the solution in the region $J(\zeta, r_2, r_1)$ around the interfacial set $\partial L \times \{o''\}$. □

Barrier functions. Now we consider barrier functions to control $|\boldsymbol{u}_{\sigma_p}(x) - \boldsymbol{w}(x)|$. In the Section 2, $\varphi_{1,\zeta}$ and $\varphi_{2,\zeta}$ were defined and the important inequalities (cf. Lemma 2.7) are deduced. We should note that they depend on the constant $M > 0$ and D, the function $q = q(s)$, but not on the equation and solutions.

We use the same $\varphi_{1,\zeta}$ and $\varphi_{2,\zeta}$ by putting $M = d(M' + 1) - 1$, where M' is the constant in (4.7) and it is related with the nonlinear term $\boldsymbol{f}$. We state the inequalities for these barrier functions in Lemma 2.7 for our later use.

$$\begin{cases} \Delta\varphi_{i,\zeta} + d(M'+1)\varphi_{i,\zeta} \leqq 0 \quad \text{in} \ \ J(\zeta, r_2, r_1), \\ \partial\varphi_{i,\zeta}/\partial\nu = 0 \ \ \text{in} \ \ \partial J(\zeta, r_2, r_1) \setminus (\Gamma^+(r_1) \cup \Gamma^-(\zeta, r_2)), \end{cases} \quad (i = 1, 2) \tag{4.8}$$

for $0 < \zeta \leqq \zeta_3$.

Next we define $\varphi_{3,\zeta}^{i,\pm}$ $(1 \leqq i \leqq d)$, which is obtained by replacing w by w_i in the definition of $\varphi_{3,\zeta}^{\pm}$ in §2. Namely,

$$\varphi_{3,\zeta}^{i,+}(x) = \begin{cases} w_i(x) \quad \text{for} \ \ x \in \Sigma^+(r_1) \setminus \Sigma^+(2\zeta), \\ N_3^+(\xi, z/\zeta) \quad \text{for} \ \ x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Lambda(\zeta), \\ w(\xi, o'') - \kappa_2\zeta + \gamma_3(s + 2\zeta) - \gamma_4(s + 2\zeta)^2 \\ \qquad\quad \text{for} \ \ x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Sigma^\circ(\zeta, r_2, 2\zeta), \end{cases} \quad (1 \leqq i \leqq d),$$

$$\varphi_{3,\zeta}^{i,-}(x) = \begin{cases} w_i(x) \quad \text{for} \ \ x \in \Sigma^+(r_1) \setminus \Sigma^+(2\zeta), \\ N_3^-(\xi, z/\zeta) \quad \text{for} \ \ x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Lambda(\zeta), \\ w(\xi, o'') + \kappa_2\zeta - \gamma_3(s + 2\zeta) + \gamma_4(s + 2\zeta)^2 \\ \qquad\quad \text{for} \ \ x = (\xi + s\,\mathbf{n}(\xi), \eta) \in \Sigma^\circ(\zeta, r_2, 2\zeta), \end{cases} \quad (1 \leqq i \leqq d).$$

It includes parameters κ_1, κ_2, κ_3, κ_4, which are to be fixed depending on M', M_1'.

Denote $\boldsymbol{\varphi}_{3,\zeta}^{\pm} = (\varphi_{3,\zeta}^{1,\pm}, \ldots, \varphi_{3,\zeta}^{d,\pm})$. Similarly as Lemma 2.8, we have the following differential inequalities.

Lemma 4.2 (cf. Lemma 2.8)

$$\begin{cases} \varphi_{3,\zeta}^{i,-} \geqq \varphi_{3,\zeta}^{i,+} \quad in \ \ J(\zeta, r_3, r_1), \\ \Delta\varphi_{3,\zeta}^{i,+} + \sum_{k=1}^{d} C_{ik} f_k(\boldsymbol{\varphi}_{3,\zeta}^{+}) \leqq 0 \quad in \ \ J(\zeta, r_3, r_1), \\ \Delta\varphi_{3,\zeta}^{i,-} + \sum_{k=1}^{d} C_{ik} f_k(\boldsymbol{\varphi}_{3,\zeta}^{-}) \geqq 0 \quad in \ \ J(\zeta, r_3, r_1), \\ \partial\varphi_{3,\zeta}^{i,\pm}/\partial\nu = 0 \quad on \ \ \partial J(\zeta, r_3, r_1) \setminus (\Gamma^+(r_1) \cup \Gamma^-(\zeta, r_3)), \end{cases} \quad (1 \leqq i \leqq d) \tag{4.9}$$

We define the following quantity,

$$c_1'(\sigma_p) = \Big(\sup_{x\in\Gamma^+(r_1)} |\boldsymbol{u}_{\sigma_p}(x) - \boldsymbol{w}(x)|\Big) \Big/ \Big(\inf_{x\in\Gamma^+(r_1)} \varphi_{1,\sigma_p}(x)\Big)$$
$$+ 2M_1'\Big(\sup_{x\in J(\sigma_p,r_3,r_1)} |C_{ik}(x) - C_{ik,\sigma_p}(x)|\Big) \Big/$$
$$\Big(\inf_{x\in J(\sigma_p,r_3,r_1)} \varphi_{1,\sigma_p}(x)\Big) \quad (4.10)$$

$$c_2'(\sigma_p) = \Big\{1 + h + \sup_{x\in\Gamma^+(r_1)} |\boldsymbol{\varphi}^+_{3,\sigma_p}(x)| + \sup_{x\in\Gamma^+(r_1)} |\boldsymbol{\varphi}^-_{3,\sigma_p}(x)|\Big\} \Big/$$
$$\Big(\inf_{x\in\Gamma^-(\sigma_p,r_3)} \varphi_{2,\sigma_p}(x)\Big). \quad (4.11)$$

Then we have the following estimate.

Lemma 4.3

$$\begin{cases} \varphi^{i,-}_{3,\sigma_p}(x) - c_1'(\sigma_p)\varphi_{1,\sigma_p}(x) - c_2'(\sigma_p)\varphi_{2,\sigma_p}(x) \leqq u_{i,\sigma_p}(x) \\ \quad \leqq \varphi^{i,+}_{3,\sigma_p}(x) + c_1'(\sigma_p)\varphi_{1,\sigma_p}(x) + c_2'(\sigma_p)\varphi_{2,\sigma_p}(x) \\ \qquad\qquad \textit{in} \;\; J(\sigma_p, r_3, r_1), \quad (1 \leqq i \leqq d). \end{cases} \quad (4.12)$$

for any $p \geqq 1$.

Proof of Lemma 4.3. From the definition of $c_1'(\sigma_p)$ and $c_2'(\sigma_p)$, we have, for $1 \leqq i \leqq d$,

$$\begin{cases} \varphi^{i,-}_{3,\sigma_p}(x) - u_{i,\sigma_p}(x) < c_1'(\sigma_p)\varphi_{1,\sigma_p}(x) + c_2'(\sigma_p)\varphi_{2,\sigma_p}(x) \\ u_{i,\sigma_p}(x) - \varphi^{i,+}_{3,\sigma_p}(x) < c_1'(\sigma_p)\varphi_{1,\sigma_p}(x) + c_2'(\sigma_p)\varphi_{2,\sigma_p}(x) \\ \qquad\qquad \text{on} \;\; \Gamma^+(r_1) \cup \Gamma^-(\sigma_p, r_3). \end{cases}$$

Noting that $\varphi^{i,+}_{3,\zeta}(x) \leqq \varphi^{i,-}_{3,\zeta}(x)$ (cf. (4.9)), we have

$$u_{i,\sigma_p}(x) - \varphi^{i,-}_{3,\sigma_p}(x) \leqq u_{i,\sigma_p}(x) - \varphi^{i,+}_{3,\sigma_p}(x)$$
$$\text{in} \;\; J(\sigma_p, r_3, r_1), \quad (1 \leqq i \leqq d). \quad (4.13)$$

We will prove that if $0 < \epsilon \leqq 1$,

$$\begin{cases} -\,(c_1'(\sigma_p)\varphi_{1,\sigma_p}(x) + c_2'(\sigma_p)\varphi_{2,\sigma_p}(x)) \leqq \epsilon\big(u_{i,\sigma_p}(x) - \varphi^{i,-}_{3,\sigma_p}(x)\big) \\ \epsilon\big(u_{i,\sigma_p}(x) - \varphi^{i,+}_{3,\sigma_p}(x)\big) \leqq c_1'(\sigma_p)\varphi_{1,\sigma_p}(x) + c_2'(\sigma_p)\varphi_{2,\sigma_p}(x) \\ \qquad\qquad \text{in} \;\; J(\sigma_p, r_3, r_1), \; 1 \leqq i \leqq d. \end{cases} \quad (4.14)$$

The second inequality follows from (4.13). We know that this is true for $x \in \Gamma^+(r_1) \cup \Gamma^-(\sigma_p, r_3)$ and $p \geqq 1$. Moreover, for any $p \geqq 1$, the inequality (4.14) is true for small $\epsilon > 0$. For the proof of (4.12), we assume that there is a finite supremum of $\epsilon > 0$ for which (4.14) is true. Denote it by $\epsilon_0 > 0$. If $\epsilon_0 \geqq 1$, the proof is finished. We consider the case $0 < \epsilon_0 < 1$.

From the equations for $\boldsymbol{u}_{\sigma_p}$, φ_{1,σ_p}, φ_{2,σ_p} and $\boldsymbol{\varphi}^{\pm}_{3,\sigma_p}$, we have

$$\begin{aligned}
&\Delta\big(c_1'(\sigma_p)\varphi_{1,\sigma_p} + c_2'(\sigma_p)\varphi_{2,\sigma_p} + \epsilon(\varphi^{i,+}_{3,\sigma_p} - u_{i,\sigma_p})\big) \\
&\qquad + dM'\left(c_1'(\sigma_p)\varphi_{1,\sigma_p} + c_2'(\sigma_p)\varphi_{2,\sigma_p}\right) + \epsilon\sum_{j=1}^{d} A_{ij,\sigma_p}(\varphi^{j,+}_{3,\sigma_p} - u_{j,\sigma_p}) \\
&\leqq -d(c_1'(\sigma_p)\varphi_{1,\sigma_p} + c_2'(\sigma_p)\varphi_{2,\sigma_p}) + \epsilon\sum_{k=1}^{d}(C_{i,k} - C_{ik,\sigma_p})f_k(\boldsymbol{\varphi}^{+}_{3,\sigma_p}) \\
&\qquad\qquad\qquad \text{in } \ J(\sigma_p, r_3, r_1), \qquad (4.15)
\end{aligned}$$

and

$$\begin{aligned}
&\Delta\big(c_1'(\sigma_p)\varphi_{1,\sigma_p} + c_2'(\sigma_p)\varphi_{2,\sigma_p} + \epsilon(u_{i,\sigma_p} - \varphi^{i,-}_{3,\sigma_p})\big) \\
&\qquad + dM'\left(c_1'(\sigma_p)\varphi_{1,\sigma_p} + c_2'(\sigma_p)\varphi_{2,\sigma_p}\right) + \epsilon\sum_{j=1}^{d} B_{ij,\sigma_p}(u_{j,\sigma_p} - \varphi^{j,-}_{3,\sigma_p}) \\
&\leqq -d(c_1'(\sigma_p)\varphi_{1,\sigma_p} + c_2'(\sigma_p)\varphi_{2,\sigma_p}) + \epsilon\sum_{k=1}^{d}(C_{i,k} - C_{ik,\sigma_p})f_k(\boldsymbol{\varphi}^{-}_{3,\sigma_p}) \\
&\qquad\qquad\qquad \text{in } \ J(\sigma_p, r_3, r_1), \qquad (4.16)
\end{aligned}$$

for $1 \leqq i \leqq d$, where

$$A_{ij,\sigma_p}(x) = \sum_{k=1}^{d} C_{ik,\sigma_p}(x)\int_0^1 \frac{\partial f_k}{\partial \xi_j}(\tau\boldsymbol{\varphi}^{+}_{3,\sigma_p}(x) + (1-\tau)\boldsymbol{u}_{\sigma_p}(x))\,d\tau$$

$$B_{ij,\sigma_p}(x) = \sum_{k=1}^{d} C_{ik,\sigma_p}(x)\int_0^1 \frac{\partial f_k}{\partial \xi_j}(\tau\boldsymbol{u}_{\sigma_p}(x) + (1-\tau)\boldsymbol{\varphi}^{-}_{3,\sigma_p}(x))\,d\tau.$$

From the condition (4.7), $|A_{ij,\sigma_p}(x)| \leqq M'$, $|A_{ij,\sigma_p}(x)| \leqq M'$ follow. Now (4.14) is valid for $0 \leqq \epsilon \leqq \epsilon_0$. Consequently, we have, from (4.13) and (4.14),

$$\begin{cases} \epsilon_0|u_{i,\sigma_p}(x) - \varphi^{i,-}_{3,\sigma_p}(x)| \leqq c'_1(\sigma_p)\varphi_{1,\sigma_p}(x) + c'_2(\sigma_p)\varphi_{2,\sigma_p}(x) \\ \epsilon_0|u_{i,\sigma_p}(x) - \varphi^{i,+}_{3,\sigma_p}(x)| \leqq c'_1(\sigma_p)\varphi_{1,\sigma_p}(x) + c'_2(\sigma_p)\varphi_{2,\sigma_p}(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } J(\sigma_p, r_3, r_1), \end{cases} \tag{4.17}$$

for $1 \leqq i \leqq d$. For simplicity of notation, we define

$$\Phi^i_{\epsilon,\sigma_p}(x) = c'_1(\sigma_p)\varphi_{1,\sigma_p}(x) + c'_2(\sigma_p)\varphi_{2,\sigma_p}(x) + \epsilon(u_{i,\sigma_p}(x) - \varphi^{i,-}_{3,\sigma_p}(x)),$$

$$\Psi^i_{\epsilon,\sigma_p}(x) = c'_1(\sigma_p)\varphi_{1,\sigma_p}(x) + c'_2(\sigma_p)\varphi_{2,\sigma_p}(x) + \epsilon(\varphi^{i,+}_{3,\sigma_p}(x) - u_{i,\sigma_p}(x)).$$

Then $\Phi^i_{\epsilon_0,\sigma_p}(x)$, $\Psi^i_{\epsilon_0,\sigma_p}(x)$ are non-negative in $J(\sigma_p, r_3, r_1)$ for $1 \leqq i \leqq d$. From the definition of ϵ_0, either $\Phi^i_{\epsilon_0,\sigma_p}$ or $\Psi^i_{\epsilon_0,\sigma_p}$ takes 0 in $\overline{J(\sigma_p, r_3, r_1)} \setminus (\Gamma^+(r_1) \cup \Gamma^-(\sigma_p, r_3))$ for some i.

We divide the situation into 4 cases.

Case I: There exists i such that $\Phi^i_{\epsilon_0,\sigma_p}$ takes zero at an interior point y of $J(\sigma_p, r_3, r_1)$. On the other hand, from (4.16) with (4.17) and the definition of $c'_1(\sigma_p)$, we have

$$\Delta\Phi^i_{\epsilon_0,\sigma_p} < 0 \quad \text{in } J(\sigma_p, r_3, r_1). \tag{4.18}$$

Since $\Phi_{\epsilon_0,\sigma_p}(x)$ in $J(\sigma_p, r_3, r_1)$, we apply the maximum principle and we get a contradiction.

Case II: There exists i such that $\Psi^i_{\epsilon_0,\sigma_p}$ takes zero at an interior point y of $J(\sigma_p, r_3, r_1)$. Similarly as in the case I, we get a contradiction in the argument on (4.15) and (4.17) with the maximum principle.

Case III: $\Phi^i_{\epsilon_0,\sigma_p}(1 \leqq i \leqq d)$ are positive in the interior of $J(\sigma_p, r_3, r_1)$ for $1 \leqq i \leqq d$ but some $\Phi^i_{\epsilon_0,\sigma_p}$ takes zero at a boundary point $y \in J(\sigma_p, r_3, r_1) \setminus (\Gamma^+(r_1) \cup \Gamma^-(\sigma_p, r_3))$. Applying the Hopf maximum principle at y to the differential inequality (4.18), we get the normal derivarive of $\Phi^i_{\epsilon_0,\sigma_p}$ is negative. It contradicts to the Neumann B.C. of $\Phi^i_{\epsilon_0,\sigma_p}$.

Case IV: $\Psi^i_{\epsilon_0,\sigma_p}(1 \leqq i \leqq d)$ are positive in the interior of $J(\sigma_p, r_3, r_1)$ for $1 \leqq i \leqq d$ but some $\Psi^i_{\epsilon_0,\sigma_p}$ takes zero at a boundary point $y \in J(\sigma_p, r_3, r_1) \setminus (\Gamma^+(r_1) \cup \Gamma^-(\sigma_p, r_3))$. A completely similar argument as in Case III applies and yields a contradiction. Thus we complete the proof of Lemma 4.3. □

From Lemma 4.3, we have

$$\limsup_{p\to\infty} \sup_{x=(\xi+s\,\mathbf{n}(\xi),\eta)\in\Sigma^-(\sigma_p,r_3)} |u_{i,\sigma_p}(\xi+s\,\mathbf{n}(\xi),\eta)-w_i(\xi,o'')|$$
$$\leqq |\gamma_3 s-\gamma_4 s^2| + |-(\delta_1/16\delta_2)s-\gamma_2 s^2|\cdot \limsup_{p\to\infty} c_2(\sigma_p)$$
$$(-r_3 \leqq s \leqq 0), \qquad (4.19)$$

$$\lim_{p\to\infty}\sup_{x\in D} |\boldsymbol{u}_{\sigma_p}(x)-\boldsymbol{w}(x)| = 0. \qquad (4.20)$$

It remains to prove that a subsequence of $\boldsymbol{u}_{\sigma_p|Q(\sigma_p)}$ converges (like Lemma 3.3). However this part is same as the case of the scalar equation in §3, because we only carry out the argument for each component of $\boldsymbol{u}_{\sigma_p}$ and get the conclusion of Theorem 4.1.

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Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan
E-mail: jimbo@math.sci.hokudai.ac.jp