Hokkaido Mathematical Journal Vol. 36 (2007) p. 365–381

Purifiability in pure subgroups

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(Received January 5, 2006; Revised May 29, 2006)

Abstract. Let G be an abelian group. A subgroup A of G is said to be *purifiable* in G if, among the pure subgroups of G containing A, there exists a minimal one. Suppose that A is purifiable in G and H is a pure subgroup of G containing A. Then A need not be purifiable in H. In this note, we ask for conditions that guarantee that A is purifiable in the intermediate group H. First, we prove that if A is a torsion-free purifiable subgroup of a group G and H is a direct summand of G containing A, then A is purifiable in H. Next, we characterize the pure subgroups K of a group G with the property that a torsion-free finite rank subgroup A of K is purifiable in K if A is purifiable in G.

Key words: purifiable subgroup, pure hull, strongly ADE decomposable group, mixed basic subgroup.

1. Introduction

Let G be an abelian group. A subgroup A of G is said *purifiable* in G if, among the pure subgroups of G containing A, there exists a minimal one. Such a minimal pure subgroup is called a *pure hull* of A.

Let G be a p-primary group, B a basic subgroup of G, and A a subgroup of B. In general, even if A is purifiable in G, then A need not be purifiable in B. This is shown by the following example from [3, Remark, p. 93].

Example 1.1 Let G be the maximal torsion subgroup of $\prod_{n=1}^{\infty}\langle x_n\rangle$ and $B=\bigoplus_{n=1}^{\infty}\langle x_n\rangle$ with $o(x)=p^n$. Set

$$y_i = x_{2i} + p^2x_{2i+1} - p^2x_{2i+2}.$$

Let $H=\langle y_i \mid i=1, 2, \ldots\rangle$ and let $\overline{H}$ be the p-adic closure of H in B. Then H is pure in G and $\overline{H}=\langle px_2\rangle\oplus H$ is not pure in G. Moreover $\overline{H}$ is purifiable in G but not in B.

Proof. By the proof of [3, Remark, p. 93], H is pure in G and $\overline{H}=\langle px_2\rangle\oplus H$ is not pure in G. Observe that $o(y_i)=p^{2i}$. Further, by [1, Proposition 3.4],

2000 Mathematics Subject Classification : 20K21, 20K27.

This work is supported by Grant–in–Aid for Science Research (c)

$\overline{H}$ is the unique maximal vertical subgroup of G supported by $\overline{H}[p]$ and contains H. If $\overline{H}$ is purifiable in B, then there exists a pure subgroup K of B such that $K \supseteq \overline{H}$ and $K[p] = \overline{H}[p]$. Since $\overline{H}$ is the unique maximal vertical subgroup of G supported by $\overline{H}[p]$ and K is vertical in B, we have $K = \overline{H}$ and this contradicts the fact that $\overline{H}$ is not purifiable in B. However, since G is torsion-complete and $\overline{H}$ is vertical in G, $\overline{H}$ is purifiable in G by [5, Theorem 4.8]. □

Suppose that $A \subseteq H \subseteq G$, and H is pure in G. Clearly, if A is purifiable in H, then it is purifiable in G. Now we pose the following problem.

Problem If a subgroup A of an abelian group G is purifiable in G, then, for which pure subgroups H containing A, is A purifiable in H?

In this note, we consider the Problem. First, we prove that if a torsion-free subgroup A of a group G is purifiable in G, then, for every direct summand H of G containing A, A is purifiable in H.

We characterized purifiable torsion-free finite rank subgroups in [9]. In Section 4, using the result, we characterize the pure subgroups K of an abelian group G with the property that a torsion-free finite rank subgroup A of K is purifiable in K if A is purifiable in G.

In the process, we consider $\dim(D/(E \oplus A))[p]$ where D/A is the maximal divisible subgroup of $(G/A)_p$ and E is the maximal divisible subgroup of G_p. Define $\dim(G, A, p) = \dim(D/(E \oplus A))[p]$. In fact, if the subgroup A of the group G is purifiable torsion-free finite rank in G and H is a pure subgroup of G containing A, then A is purifiable in H if and only if $\dim(G, A, p) = \dim(H, A, p)$ for all primes p. So the dimension $\dim(G, A, p)$ could play an important role in the study of purifiable torsion-free subgroups.

In Section 5, we consider pure subgroups of strongly ADE decomposable groups (see Definition 5.3) that are again strongly ADE decomposable. In Section 6, as an application of the main result, we give a sufficient condition for mixed basic subgroups (see Definition 6.1) of a group of torsion-free finite rank to be isomorphic. By [10, Example], not all mixed basic subgroups of a group are isomorphic.

All groups considered in this note are arbitrary abelian groups unless stated otherwise. The terminologies and notations not expressly introduced follow the usage of [2]. Throughout this note, $\mathbf{Z}*$ denotes the set of nonnegative integers, $\mathbf{P}$ the set of all prime integers, p always denotes a prime, T the maximal torsion subgroup, and G_p the p-component of an abelian group G.

2. Notation and basics

We recall definitions and properties mentioned in [5]. We frequently use them in this note.

From the definition [5, Definition 1.1] of p-almost-dense subgroups and its characterization [5, Proposition 1.3, Proposition 1.4], we can define p-almost-dense and almost-dense subgroups as follows.

Definition 2.1 A subgroup A of a group G is said to be *p-almost-dense* in G if, for all $n \in \mathbf{Z}*$,

$$p^n G[p] \subseteq A + p^{n+1} G.$$

Moreover, the subgroup A is said to be *almost–dense* in G if A is p-almost-dense in G for every $p \in \mathbf{P}$.

Recall definition of p-purifiable [purifiable] in a group G.

Definition 2.2 Let G be a group. A subgroup A of G is said to be *p-purifiable*[*purifiable*] in G if, among the p-pure[pure] subgroups of G containing A, there exists a minimal one. Such a minimal p-pure[pure] subgroup is called a *p-pure*[*pure*] *hull* of A.

Proposition 2.3 ([5, Theorem 1.8, Theorem 1.11]) *Let G be a group and A a subgroup of G. Let H be a p-pure* [*pure*] *subgroup of G containing A. Then H is a p-pure* [*pure*] *hull of A in G if and only if the following three conditions are satisfied*:

(1) *A is p-almost-dense* [*almost-dense*] *in H*;

(2) *H/A is p-primary* [*torsion*];

(3) [*for every $p \in \mathbf{P}$*,] *there exists $m_p \in \mathbf{Z}*$ such that*

$$p^{m_p} H[p] \subseteq A.$$

From Proposition 2.3, for purifiable torsion–free subgroups, we immediately obtain the following.

Corollary 2.4 *Let G be a group and A a subgroup of G. Suppose that A is purifiable in G. Let H be a pure hull of A in G. If A is torsion-free, then*

H_p is bounded for all $p \in \mathbf{P}$.

The following is a relationship between purifiability and p-purifiability.

Proposition 2.5 ([5, Theorem 1.12]) *Let G be a group. A subgroup A of G is purifiable in G if and only if, for every $p \in \mathbf{P}$, A is p-purifiable in G.*

Definition 2.6 Let G be a group and A a subgroup of G. For every $n \in \mathbf{Z}*$, we define the *nth p-overhang* of A in G to be the vector space

$$V_{p,n}(G,\, A) = \frac{(A + p^{n+1}G) \cap p^n G[p]}{(A \cap p^n G)[p] + p^{n+1}G[p]}.$$

Moreover, the set

$$O_A^G(p) = \{t \mid V_{p,t}(G,\, A) \neq 0\}.$$

is called the *p-overhang* set of A in G.

We immediately obtain the following properties.

Proposition 2.7 *Let G and A be as in* Definition 2.6. *Then the following hold.*
(1) *If $G_p = 0$, then $O_A^G(p) = \emptyset$.*
(2) *$V_{p,m+n}(G,\, A) = V_{p,n}(p^m G,\, A \cap p^m G)$ for all $n, m \in \mathbf{Z}*$.*

Proposition 2.8 ([5, Proposition 2.2]) *Let G be a group and A a subgroup of G. For a p-pure subgroup K of G containing A,*

$$V_{p,n}(G,\, A) \cong V_{p,n}(K,\, A)$$

for all $n \in \mathbf{Z}*$. Hence $O_A^G(p) = O_A^K(p)$.

Proposition 2.8 leads the following intrinsic necessary condition for p-purifiability.

Proposition 2.9 ([5, Theorem 2.3]) *If a subgroup A of a group G is p-purifiable in G, then the set $O_A^G(p)$ is finite.*

Proposition 2.9 and Proposition 2.7 (2) lead to the following useful property.

Corollary 2.10 *If a subgroup A of a group G is p-purifiable in G, then $O_{A \cap p^m G}^{p^m G}(p) = \emptyset$ for some $m \in \mathbf{Z}*$.*

The following is useful when we consider purifiable torsion-free subgroups. A proof was written in [9, Corollary 2.15].

Corollary 2.11 *Let G be a group and A a subgroup of G. Suppose that A is torsion-free and purifiable in G. Let H be a pure hull of A in G. Then the following are equivalent*:
(1) $O_A^G(p) = \emptyset$ *for all* $p \in \mathbf{P}$;
(2) H *is torsion-free.*

The following is used frequently in this note.

Proposition 2.12 ([5, Theorem 4.1]) *Let G be a group and A a subgroup of G. Then the following hold.*
(1) *If A is p-purifiable in G and H is a p-pure hull of A in G, then, for all $n \in \mathbf{Z}*$, $A \cap p^nG$ is p-purifiable in p^nG and p^nH is a p-pure hull of $A \cap p^nG$ in p^nG.*
(2) *If $A \cap p^mG$ is p-purifiable in p^mG for some $m \in \mathbf{Z}*$, then A is p-purifiable in G.*

We recall a characterization of purifiable torsion-free finite rank subgroups.

Proposition 2.13 ([9, Theorem 3.6]) *Let G be a group, A a torsion-free finite rank subgroup of G, and let $G^{(p)}/A = (G/A)_p$. Then A is p-purifiable in G if and only if the following two conditions are satisfied.*
(1) $O_{A\cap p^sG}^{p^sG}(p) = \emptyset$ *for some* $s \in \mathbf{Z}*$.
(2) *Let r be the least integer satisfying the condition (1). Then there exist an integer $m \geqq r$ and a divisible subgroup $D/(A \cap p^rG)$ of $p^r(G/(A \cap p^rG))_p = p^rG^{(p)}/(A \cap p^rG)$ such that*

$$p^nG^{(p)} + (A \cap p^rG) = D \oplus p^nG_p \quad \textit{for all integers} \quad n \geqq m.$$

We conclude this section with the following useful lemma.

Lemma 2.14 *Let G be a group and H a pure subgroup of G such that G/H is torsion. Then the following hold.*
(1) $G = H + T$.
(2) *If $G_p = H_p \oplus U_p$ for every $p \in \mathbf{P}$, then $G = H \oplus U$ where $U = \oplus_{p\in\mathbf{P}} U_p$.*
(3) *Suppose that A is purifiable torsion-free in G. Let K be a pure hull of A in G. If G/A is torsion, then $G = K \oplus T'$ for some subgroup T' of T.*

Proof. (1)(2) By [9, Lemma 3.1].

(3) By Proposition 2.5 and Corollary 2.4, for every $p \in \mathbf{P}$, K_p is bounded and so K_p is a direct summand of G_p. If G/A is torsion, then, by (2), the assertion is confirmed. □

3. Divisible subgroups of quotient groups

Let G be a group and A a torsion-free subgroup of G. In this section, we consider the maximal divisible subgroup of the quotient $(G/A)_p$.

First we give a convenient reduction theorem.

Theorem 3.1 *Let G be a group and A a torsion-free subgroup of G. Suppose that A is p-purifiable in G and $G = M \oplus N$ with $M \supseteq A$. Then A is p-purifiable in M.*

Proof. Since A is p-purifiable in G, by Corollary 2.10, $O^{p^mG}_{A\cap p^mG}(p) = \emptyset$ for some $m \in \mathbf{Z}*$. Let r be the least integer such that

$$O^{p^rG}_{A\cap p^rG}(p) = \emptyset. \tag{3.2}$$

Let H be a p-pure hull of A in G and $G^{(p)}/A = (G/A)_p$. Then, by Proposition 2.12(1), p^rH is a p-pure hull of $A\cap p^rG$ in $p^rG^{(p)}$. Then, by Lemma 2.14,

$$p^rG^{(p)} = p^rH \oplus K \quad \text{and} \quad K < p^rG_p. \tag{3.3}$$

By (3.2) and Corollary 2.11,

$$p^rH \quad \text{is torsion–free.} \tag{3.4}$$

Further, by (3.3) and (3.4), we have

$$K = p^rG_p \quad \text{and} \quad p^rG^{(p)} = p^rH \oplus p^rG_p. \tag{3.5}$$

Recall that $A \subseteq M$. Let $M^{(p)}/A = (M/A)_p$. Since $G = M \oplus N$, we have $G_p = M_p \oplus N_p$. Therefore, by (3.5), we have

$$p^rG^{(p)} = p^rH \oplus p^rM_p \oplus p^rN_p \quad \text{and} \quad p^rM_p \subseteq p^rM^{(p)}.$$

Intersecting with $p^rM^{(p)}$, we get

$$p^rM^{(p)} = L \oplus p^rM_p \quad \text{where} \quad L = p^rM^{(p)} \cap (p^rH \oplus p^rN_p). \tag{3.6}$$

By (3.6), L is torsion-free and p-pure in p^rM. Since $L/(A \cap p^rG)$ is a p-group, by Proposition 2.3, L is a p-pure hull of $A \cap p^rG$ in M. Hence, by Proposition 2.12(2), A is p-purifiable in M. □

Definition 3.7 Let G be a group and A a torsion-free subgroup of G. Let D/A be the maximal divisible subgroup of $(G/A)_p$ and E the maximal divisible subgroup of G_p. We define

$$\dim(G,\, A,\, p) = \dim(D/(E \oplus A))[p].$$

Remark 3.8 Let G, A, D, and E be as in Definition 3.7. Since A is torsion-free, $A \cap E = 0$ and there exists an E-high subgroup R of G containing A. Since E is divisible, $G = R \oplus E$. Let D'/A be the maximal divisible subgroup of $(R/A)_p$. Then $D/A = D'/A \oplus (E \oplus A)/A$ and

$$\begin{aligned}\dim(G,\, A,\, p) &= \dim(D/(E \oplus A))[p]\\ &= \dim(D'/A)[p] = \dim(R,\, A,\, p).\end{aligned}$$

Moreover, if A is p-purifiable in G, then A is p-purifiable in R by Theorem 3.1. Hence, when we consider purifiable torsion-free subgroups in the group G, without loss of generality, we may assume that G_p is reduced.

Lemma 3.9 *Let F be a torsion-free group and B a subgroup of F. Suppose that F/B is torsion. Then*

$$\dim(F/B)[p] \leqq rk(F).$$

Proof. Since $B \subseteq F \subseteq \mathbf{Q}F$, we have

$$(F/B)[p] = (F \cap p^{-1}B)/B \subseteq p^{-1}B/B \cong B/pB.$$

Hence $\dim(F/B)[p] \leqq rk(B/pB) \leqq rk_p(B) \leqq rk(B) \leqq rk(F)$. □

Using Proposition 2.13, we can obtain the following result.

Proposition 3.10 *Let G be a group, A a subgroup of G, and $G^{(p)}/A = (G/A)_p$. Then the following hold.*

(1) *For every integer $n \geqq 0$, let $D_n/(A \cap p^nG)$ be the maximal divisible subgroup of $p^n(G/(A \cap p^nG))_p = p^nG^{(p)}/(A \cap p^nG)$. Then*

$$D_0/A \cong D_n/(A \cap p^nG)$$

for all integers $n \geqq 0$.

(2) *If A is p-purifiable subgroup of G and A is torsion-free of finite rank, then*

$$\dim(G,\, A,\, p) = \dim(D_r/(A \cap p^rG))[p]$$

is finite.

Proof. (1) Since $p^n(G^{(p)}/A) = (p^nG^{(p)} + A)/A \cong p^nG^{(p)}/(A \cap p^nG)$, the maximal divisible subgroup D_0/A of $G^{(p)}/A$ is isomorphic to $D_n/(A \cap p^nG)$.
(2) By Remark 3.8, without loss of generality, we may assume that

$$G_p \quad \text{is reduced.} \tag{3.11}$$

Suppose that A is purifiable torsion-free finite rank subgroup of G. Then, by Corollary 2.10, $O^{p^sG}_{A\cap p^sG}(p) = \emptyset$ for some $s \in \mathbf{Z}$. Let r be the least integer such that $O^{p^rG}_{A\cap p^rG}(p) = \emptyset$. Further, by Proposition 2.13, there exists an integer $m \geqq r$ such that

$$p^nG^{(p)} + (A \cap p^rG) = D_r \oplus p^nG_p \quad \text{for all} \quad n \geqq m \tag{3.12}$$

Then we have

$$\begin{aligned}\dim(G,\, A,\, p) &\overset{(3.12)(3.9)}{=} \dim(D_0/A)[p] \\ &\overset{(1)}{=} \dim(D_r/(A \cap p^rG))[p].\end{aligned} \tag{3.13}$$

By the definition of D_r, $D_r \subseteq p^rG^{(p)}$ and so $T(D_r) \subseteq p^rG_p$. Hence, by (3.12), D_r is torsion-free. Further $rk(A \cap p^rG)$ is finite and $D_r/(A \cap p^rG)$ is a p-group. Therefore, by Lemma 3.9, $\dim(D_r/(A \cap p^rG))[p]$ is finite and by (3.13), (2) is confirmed. □

4. Purifiability in pure subgroups

In this section, we consider purifiable torsion-free finite rank subgroups A in a group G.

Lemma 4.1 *Let G be a group, A a torsion-free subgroup of G and H a pure subgroup of G containing A. For every $p \in \mathbf{P}$, let $G^{(p)}/A = (G/A)_p$ and $H^{(p)}/A = (H/A)_p$. Then the following hold.*
(1) *$G^{(p)}$ is p-pure in G and $T(G^{(p)}) = G_p$.*
(2) *$G^{(p)} = H^{(p)} + G_p$.*
(3) *Suppose that A is p-purifiable torsion-free in G and let L be a pure hull of A in G. Then $G^{(p)} = L^{(p)} \oplus K$ where $L^{(p)}/A = (L/A)_p$ and K is a subgroup of G_p.*

Proof. (1) is easily checked.
(2) Note that $H^{(p)}$ is p-pure in G and $G^{(p)}/H^{(p)}$ is a p-group. Let $g \in G^{(p)}$.

Then $p^n g \in H^{(p)} \cap p^n G = p^n H^{(p)}$ for some $n \in \mathbf{Z}$. Then we have $p^n g = p^n h$ for some $h \in H$ and $g - h \in G_p$. Hence $G^{(p)} = H^{(p)} + G_p$.
(3) By Proposition 2.3(3), L_p is bounded and so L_p is a direct summand of G_p. Hence, by (2) with $H = L$, the assertion is confirmed. □

We recall the so-called Dedekind short exact sequence.

Lemma 4.2 *Let T, S, K be submodules of a module G over some ring such that $T \subseteq S$. Then there exists the following short exact sequence*:

$$0 \longrightarrow \frac{S \cap K}{T \cap K} \longrightarrow \frac{S}{T} \longrightarrow \frac{S + K}{T + K} \longrightarrow 0.$$

In particular, if $(S \cap K) + T = S$, then $(S \cap K)(T \cap K) \cong S/T$.

Lemma 4.3 *Let G be a group, A a torsion-free subgroup of G, H a p-pure subgroup of G containing A, $G^{(p)}/A = (G/A)_p$, $H^{(p)}/A = (H/A)_p$, and $r \in \mathbf{Z}*$. Then we have*

$$\frac{p^n H^{(p)} + (A \cap p^r G)}{p^n H_p \oplus (A \cap p^r G)} \cong \frac{p^n G^{(p)} + (A \cap p^r G)}{p^n G_p \oplus (A \cap p^r G)}$$

for all integers $n \geqq r$.

Proof. Let $S = p^n G^{(p)} + (A \cap p^r G)$, $T = p^n G_p \oplus (A \cap p^r G)$, and $K = H^{(p)}$. Then

$$\begin{aligned} S \cap K &= (p^n G^{(p)} + (A \cap p^r G)) \cap H^{(p)} \\ &= (p^n G^{(p)} \cap H^{(p)}) + (A \cap p^r G) = p^n H^{(p)} + (A \cap p^r G), \\ T \cap K &= (p^n G_p + (A \cap p^r G)) \cap H^{(p)} \\ &= (p^n G_p \cap H^{(p)}) + (A \cap p^r G) = p^n H_p + (A \cap p^r G), \end{aligned}$$

and

$$\begin{aligned} (S \cap K) + T &= p^n H^{(p)} + (A \cap p^r G) + (p^n G_p \oplus (A \cap p^r G)) \\ &= p^n H^{(p)} + p^n G_p + (A \cap p^r G) \\ &\overset{(4.1(2))}{=} p^n G^{(p)} + (A \cap p^r G) = S. \end{aligned}$$

Hence, by Lemma 4.2, the assertion is confirmed. □

Theorem 4.4 *Let G be a group, A a torsion-free subgroup of finite rank of G, L a p-pure subgroup of G containing A, $G^{(p)}/A = (G/A)_p$, and $L^{(p)}/A = (L/A)_p$. Suppose that A is p-purifiable in G. Then A is p-purifiable in L if* *and only if*

$$\dim(G,\ A,\ p)=\dim(L,\ A,\ p). \tag{4.5}$$

Proof. By Remark 3.8, without loss of generality, we may assume that G_p is reduced. By Proposition 2.13, the following two conditions are satisfied.

(1) $O^{p^sG}_{A\cap p^sG}(p)=\emptyset$ for some $s\in \mathbf{Z}*$.

(2) Let r be the least integer satisfying the condition (1) and let $D/(A\cap p^rG)$ be the maximal divisible subgroup of $p^r(G/(A\cap p^rG))_p = p^rG^{(p)}/(A\cap p^rG)$. Then there exists a nonnegative integer $m \geqq r$ such that

$$p^nG^{(p)}+(A\cap p^rG)=D\oplus p^nG_p \quad \text{for all} \quad n\geqq m.$$

By the definition of r in (2) and Proposition 2.8, r is the least integer such that

$$O^{p^rL}_{A\cap p^rL}(p)=\emptyset. \tag{4.6}$$

Further, by the above condition (2), we have

$$\frac{p^nG^{(p)}+(A\cap p^rG)}{A\cap p^rG}=\frac{D}{A\cap p^rG}\oplus\frac{p^nG_p\oplus(A\cap p^rG)}{A\cap p^rG} \quad \text{for all} \quad n\geqq m. \tag{4.7}$$

($\Rightarrow$) Suppose that A is purifiable in L. Note that $A\cap p^rG=A\cap p^rL$. By Proposition 2.13 and (4.6), there exist a nonnegative integer $m'\geqq r$ and a divisible subgroup $D'/(A\cap p^rG)$ of $p^r(L/(A\cap p^rG))_p=p^rL^{(p)}/(A\cap p^rG)$ such that

$$p^nL^{(p)}+(A\cap p^rG)=D'\oplus p^nL_p \quad \text{for all} \quad n\geqq m'$$

and hence, for all $n\geqq m'$,

$$\frac{p^nL^{(p)}+(A\cap p^rG)}{A\cap p^rG}=\frac{D'}{A\cap p^rG}\oplus\frac{p^nL_p\oplus(A\cap p^rG)}{A\cap p^rG}. \tag{4.8}$$

By Lemma 4.3, (4.7) and (4.8),

$$D'/(A\cap p^rG)\cong D/(A\cap p^rG). \tag{4.9}$$

Therefore, by Proposition 3.10, $\dim(G,\ A,\ p)=\dim(L,\ A,\ p)$.

($\Leftarrow$) Suppose that (4.5) is satisfied. Let $D''/(A\cap p^rG)$ be the maximal divisible subgroup of $p^r(L/(A\cap p^rG))_p = p^rL^{(p)}/(A\cap p^rG)$. Note that

$D'' \subseteq D$. Since

$$\dim \frac{D}{A\cap p^r G}[p] \overset{(3.10(2))}{=} \dim(G,\, A,\, p) \overset{(4.5)}{=} \dim(L,\, A,\, p) \overset{(3.10(2))}{=} \dim \frac{D''}{A\cap p^r G}[p],$$

we have

$$D/(A\cap p^r G)) \cong D''/(A\cap p^r G). \tag{4.10}$$

By Proposition 3.10, $\dim(D/(A\cap p^r G))[p]$ is finite. Therefore $D = D''$ and we have

$$p^n L^{(p)} + (A\cap p^r G) = L\cap (p^n G^{(p)} + (A\cap p^r G)) \overset{(2)}{=} L\cap (D\oplus p^n G_p) = D''\oplus p^n L_p$$

for all $n \geqq m$. Therefore, by (4.6) and Proposition 2.13, A is p-purifiable in L. □

Theorem 4.4 and Proposition 2.5 combined lead to the following.

Corollary 4.11 *Let G be a group, A a torsion-free finite rank subgroup of G, and H a pure subgroup of G containing A. Suppose that A is purifiable in G. Then A is purifiable in H if and only if, for every $p\in \mathbf{P}$, we have*

$$\dim(G,\, A,\, p) = \dim(H,\, A,\, p).$$

5. Strongly ADE decomposable groups

First we recall definition of full free subgroups of a group G.

Definition 5.1 Let G be a group. A subgroup of A of G is said to be *full free* in G if A is free and G/A is torsion.

Definition 5.2 A group G is said to be an *ADE group* if there exists a torsion-free subgroup N such that N is almost-dense and T-high in G. Such a subgroup N is called a *moho* subgroup of G.

We studied ADE groups of torsion-free rank 1 in [4]. By Definition 2.1, we can easily see that all torsion-free groups are ADE groups.

We also recall the definition of strongly ADE decomposable groups.

Definition 5.3 A group G is said to be a *strongly ADE decomposable group* if there exists a purifiable T-high subgroup of G.

From Definition 5.3 and Lemma 2.14, we have the following.

Proposition 5.4 *Let G be a strongly ADE decomposable group. Then there exist an ADE subgroup H of G and a subgroup U of T such that $G = H \oplus U$.*

Proof. By Definition 5.3, there exists a purifiable T-high subgroup N of G. Note that G/N is torsion. Let H be a pure hull of N in G. Then, by Lemma 2.14, we have $G = H \oplus U$ for some subgroup U of T. By Proposition 2.3, N is almost-dense in H. Hence H is an ADE group with N as a moho subgroup. □

Let G be a strongly ADE decomposable group. Then there exist a T-high subgroup N of G. If $O_N^G(p) = \emptyset$ for every $p \in \mathbf{P}$, then, by Corollary 2.11, H is torsion-free. Hence G is splitting. We know that non-splitting ADE groups exist (see [4, Example]). So strongly ADE decomposable is weakening of splitting and splitting is an extreme case of strongly ADE decomposable.

Strongly ADE decomposable groups of torsion-free rank 1 were investigated in [6], [7], and [9].

Recall [10, Example 4.2]. $\mathbf{Z}$ denotes the ring of integers and $\mathbf{N}$ the set of all positive integers.

Example 5.5 Let $G = A \oplus B$ where $A = \mathbf{Z}[p^{-1}] = \{m/p^n \mid m \in \mathbf{Z},\, n \in \mathbf{N}\}$ and $B = \bigoplus_{n=1}^{\infty} \langle x_n \rangle$ with $o(x_n) = p^n$. Let $a_n = 1/p^n$. Define

$$L = \langle a_n + x_n \mid n \geqq 1 \rangle.$$

Then L is pure in G and not strongly ADE decomposable.

Proof. By [10, Property 4.10], L is pure in G. Suppose that L is strongly ADE decomposable. Then there exists a purifiable T-high subgroup N of L. Let H be a pure hull of N in L. By Proposition 5.4, we have $L = H \oplus U$ for some subgroup U of B. Further, by Corollary 2.4, $T(H)$ is bounded and so $T(H)$ is a direct summand of H. Hence H is splitting and so L is splitting. This contradicts [10, Property 4.13]. Therefore the assertion is clear. □

By Example 5.5, not all pure subgroups of strongly ADE decomposable groups are strongly ADE decomposable. In the rest of this section, we examine pure subgroups of strongly ADE decomposable groups of finite torsion-free rank. Before doing this, the following is useful.

Proposition 5.6 ([9, Theorem 4.4]) *Let G be a group. Then G is strongly ADE decomposable group if and only if there exists a purifiable full free subgroup of G.*

Theorem 5.7 *Let G be a strongly ADE decomposable group of finite torsion-free rank and L a pure subgroup of G. Then L is a strongly ADE decomposable group if and only if there exists a full free subgroup A of L such that A is purifiable in G and*

$$\dim(G, A, p) = \dim(L, A, p)$$

for every $p \in \mathbf{P}$.

Proof. ($\Rightarrow$) Suppose that L is strongly ADE decomposable. Then, by Proposition 5.6, there exists a purifiable full free subgroup A of L. Then A is purifiable in G. Further, by Corollary 4.11, $\dim(G, A, p) = \dim(L, A, p)$ for every $p \in \mathbf{P}$.
($\Leftarrow$) Suppose that there exists a full free subgroup A of L such that A is purifiable in G and $\dim(G, A, p) = \dim(L, A, p)$ for every $p \in \mathbf{P}$. Then, by Corollary 4.11, A is purifiable in L and by Proposition 5.6, L is strongly ADE decomposable. □

Since splitting is an extreme case of being strongly ADE decomposable on the basis of Proposition 5.6, Proposition 5.4, and Corollary 2.11, we can characterize splitting groups as follows.

Proposition 5.8 *Let G be a group. Then G is splitting if and only if there exists a purifiable full free subgroup A of G such that $O_A^G(p) = \emptyset$ for every $p \in \mathbf{P}$.*

Proof. ($\Rightarrow$) Suppose that G is splitting. Then $G = F \oplus T$ for some torsion-free subgroup F of G. Let A be a full free subgroup of F. Then, by Proposition 2.3, F is a pure hull of A in G and so A is purifiable in G. Further, by Proposition 2.7(2), $O_A^G(p) = O_A^F(p) = \emptyset$ for every $p \in \mathbf{P}$. Hence the assertion is confirmed.
($\Leftarrow$) Suppose that there exists a purifiable full free subgroup A of G such that $O_A^G(p) = \emptyset$ for every $p \in \mathbf{P}$. By Proposition 5.6 and an argument as in

the proof of Proposition 5.4, we have $G = H \oplus U$ for a pure hull H of A and some subgroup U of T. By Corollary 2.11, H is torsion-free and so G is splitting. □

From Corollary 4.11 and Proposition 5.8, we have the following.

Corollary 5.9 *Let G be a splitting group of finite torsion-free rank and L a pure subgroup of G. Then L is splitting if and only if there exists a full free subgroup A of L such that A is purifiable in G and*

$$\dim(G, A, p) = \dim(L, A, p) \quad \textit{and} \quad O_A^G(p) = \emptyset$$

for every $p \in \mathbf{P}$.

Proof. ($\Rightarrow$) Suppose that L is splitting. Then, by Proposition 5.8, there exists a purifiable full free subgroup A of L such that $O_A^L(p) = \emptyset$ for every $p \in \mathbf{P}$. Note that, by Proposition 2.8, $O_A^G(p) = \emptyset$ for every $p \in \mathbf{P}$. Since A is purifiable in G, by Corollary 4.11, $\dim(G, A, p) = \dim(L, A, p)$ for every $p \in \mathbf{P}$. Hence the assertion is confirmed.
($\Leftarrow$) Suppose that there exists a full free subgroup A of L such that A is purifiable in G, $\dim(G, A, p) = \dim(L, A, p)$, and $O_A^G(p) = \emptyset$ for every $p \in \mathbf{P}$. Then, by Corollary 4.11, A is purifiable in L. Further, by Proposition 2.8, $O_A^L(p) = \emptyset$ for every $p \in \mathbf{P}$. Hence, by Proposition 5.8, L is splitting. □

6. Isomorphy of mixed basic subgroups

We extended the concept of basic subgroups from p-primary abelian groups to arbitrary abelian groups in [10]. The basic subgroup extended to arbitrary abelian groups is called a *mixed basic subgroup*. By [10, Example], not all mixed basic subgroups of a group are isomorphic.

Now we consider isomorphy of mixed basic subgroups of strongly ADE decomposable groups of finite torsion-free rank. First we recall the definition of mixed basic subgroups.

Definition 6.1 A subgroup L of a group G is said to be a *mixed basic subgroup* of G if L satisfies the following three conditions:
(1) $T(L)$ is a direct sum of cyclic groups;
(2) L is pure in G;
(3) G/L is torsion divisible.

Recall that $f_t(G_p)$ is the tth Ulm-Kaplansky invariant of the p-component of a group G.

Lemma 6.2 *Let G be a group and A a torsion-free rank-k subgroup of G. Suppose that A is purifiable in G. Let H be a pure hull in G. Then*

$$f_n(H_p) \leqq k. \tag{6.3}$$

Furthermore, H_p is finite for all $p \in \mathbf{P}$.

Proof. Suppose, by way of contradiction, that

$$p^n H[p] = \Big(\bigoplus_{i=1}^{k+1} \langle x_i \rangle\Big) \bigoplus S_n \bigoplus p^{n+1} H[p]$$

where $h_p(x_i) = n$ and S_n is a subgroup of $p^n H[p]$. By Proposition 2.3(1), A is almost-dense in H. Hence there exist $a_i \in A$ and $g_i \in H$ for $1 \leqq i \leqq k+1$ such that $x_i = a_i + p^{n+1} g_i$. Since $rk(A) = k$, $\{a_i \mid i = 1, 2, \ldots, k+1\}$ is linearly dependent. Hence we have $\sum_{i=1}^{k+1} \alpha_i a_i = 0$ with $\alpha_i \in \mathbf{Z}$ and $\alpha_r a_r \neq 0$ for some $1 \leqq r \leqq k$. If p divides α_i for every $1 \leqq i \leqq k+1$, then $p(\sum_{i=1}^{k+1} \beta_i a_i) = 0$ with $\beta_i \in \mathbf{Z}$ and $\sum_{i=1}^{k+1} \beta_i a_i = 0$. Hence we may assume that p does not divide α_m for some $1 \leqq m \leqq k+1$. It follows that

$$\sum_{i=1}^{k+1} \alpha_i x_i = \sum_{i=1}^{k+1} \alpha_i a_i + \sum_{i=1}^{k+1} \alpha_i p^{n+1} g_i = \sum_{i=1}^{k+1} \alpha_i p^{n+1} g_i.$$

This is a contradiction. Hence (6.3) is confirmed.

Further, by Proposition 2.5 and Corollary 2.4, H_p is bounded. Hence, by (6.3), H_p is finite for every $p \in \mathbf{P}$. □

Theorem 6.4 *Let G be a strongly ADE decomposable group of torsion-free rank $k \geqq 1$. Suppose that pure subgroups L, M of G are strongly ADE decomposable of torsion-free rank k. Then $L \cong M$ if and only if $T(L) \cong T(M)$ and there exists a common full free subgroup B of L and M such that B is purifiable in G and* $\dim(L, B, p) = \dim(G, B, p) = \dim(M, B, p)$ *for all $p \in \mathbf{P}$.*

Proof. ($\Rightarrow$) Suppose that $L \cong M$. Then it is immediate that $T(L) \cong T(M)$. Since L is strongly ADE decomposable, by Proposition 5.6, there exists a purifiable full free subgroup A of L. Let H be a pure hull of A in L. Then, by Lemma 2.14 and Lemma 6.2, we have

$$L = H \oplus U, \quad L_p = H_p \oplus U_p, \quad H_p \quad \text{is finite for all } p \in \mathbf{P}. \tag{6.5}$$

Since $M \cong L$, we have

$$M = C \oplus W, \quad C \cong H, \quad \text{and} \quad W \cong U.$$

Then $A \cap C$ is a full free subgroup of C. Let $B = A \cap C$. Since H_p, C_p are bounded for all $p \in \mathbf{P}$, B is purifiable in both H and C by [5, Theorem 5.2] and hence so is B in all of G, L and M. Then, by Corollary 4.11, $\dim(L, B, p) = \dim(G, B, p) = \dim(M, B, p)$ for all $p \in \mathbf{P}$. Hence the assertion is confirmed.
($\Leftarrow$) Suppose that $T(L) \cong T(M)$ and there exists a full free subgroup B of both L and M such that B is purifiable in G and $\dim(L, B, p) = \dim(G, B, p) = \dim(M, B, p)$ for all $p \in \mathbf{P}$. By Corollary 4.11, B is purifiable in both L and M. Let K, N be pure hulls of B in L, M, respectively. Then, by Lemma 2.14 and Lemma 6.2, we have

$$L = K \oplus V, \quad L_p = K_p \oplus V_p, \quad K_p \quad \text{is finite for all} \quad p \in \mathbf{P} \tag{6.6}$$

and

$$M = N \oplus W, \quad M_p = N_p \oplus W_p, \quad N_p \quad \text{is finite for all} \quad p \in \mathbf{P} \tag{6.7}$$

Since K and N are pure hulls of B in G, by [8, Theorem],

$$K \cong N. \tag{6.8}$$

Since $T(L) \cong T(M)$, by (6.8), (6.6) and (6.7), we have $V_p \cong W_p$ for all $p \in \mathbf{P}$. Hence $L \cong M$. □

As an application of Theorem 6.4, we obtain the following result. Note that all basic subgroups of the maximal torsion subgroups of mixed basic subgroups are isomorphic.

Corollary 6.9 *Let G be a strongly ADE decomposable group of torsion-free rank $k \geqq 1$. Suppose that mixed basic subgroups L, M of G are strongly ADE decomposable of torsion-free rank k. Then $L \cong M$ if and only if there exists a common full free subgroup B of L and M such that B is purifiable in G and* $\dim(L, B, p) = \dim(G, B, p) = \dim(M, B, p)$ *for all* $p \in \mathbf{P}$.

Acknowledgment Part of this work was completed during the author's visit at the University of Hawaii. The author thanks the Department of Mathematics at the University of Hawaii and Professor Adolf Mader for their hospitality.

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