Hokkaido Mathematical Journal Vol. 39 (2010) p. 211–215

Solvable graphs and Fermat primes

Nobuo Iiyori

(Received March 18, 2009)

Abstract. A solvable graph of a finite group is one of generalized prime graphs of groups which are introduced as generalizations of a prime graph of a finite group in [1]. In this paper we will characterize a Fermat prime by a solvable graph of a finite group.

Key words: finite simple group.

1. Introduction

Let G be a finite group. The solvable graph $\Gamma_{sol}(G)$ of G is a graph whose vertex set $\pi(G)$ is the totality of primes which divide the order of G and a pair $(p, q) \in \pi(G) \times \pi(G)$ is an edge of $\Gamma_{sol}(G)$ if and only if $p \neq q$ and there exists a solvable group H of G whose order is divisible by pq. For example let G be the alternating group A_5 of degree 5. Then the solvable graph of A_5 is the following.

$$3 \text{———————} \overset{2}{\bullet} \text{———————} 5$$

This graph was defined in [1] as generalizations of a prime graph of a finite group and some of elementary properties were shown. The following is one of them.

Theorem *A solvable graph of a finite group is connected.*

When we use solvable graphs to analyze the structure of a finite group G, it is obviously important to determine the set $\{q \in \pi(G) | p$ and q are joined directly$\}$ for each $p \in \pi(G)$, that is, to determine the set of edges of $\Gamma_{sol}(G)$. For example, in [3] simple groups in whose solvable group any two odd primes are joined directly are classified and a condition of p-solvability for a finite group was given. In this paper we will consider when two primes p and q are joined directly in the solvable graph $\Gamma_{sol}(G)$ and the following theorem is the main theorem of this paper.

2000 Mathematics Subject Classification : 20E32.

Theorem 1 *A prime p is a Fermat prime if and only if p is joined to 2 directly in the solvable graphs of any finite groups whose order are divisible by $2p$.*

Our notation is standard (see [6]). We also use the following notations. For a non zero rational integer $n \in \mathbf{Z} - \{0\}$ and a prime p, n_p denotes the highest power of p which divides n. For a positive integer n, $\phi_n(q)$ is the n-th cyclotomic polynomial.

2. The Proof of Theorem 1

Proposition 1 ([1, Theorem 3]) *Let G be a finite group and $p, q \in \pi(G)$. p and q are not joined directly in $\Gamma_{sol}(G)$ if and only if there exists a series of normal subgroups of G*

$$G \trianglerighteq N \trianglerighteq M \trianglerighteq 1,$$

such that G/N and M are $\{p, q\}'-$group and N/M is a non abelian simple group such that p and q are not joined in $\Gamma_{sol}(N/M)$.

Lemma 1 *Let q be a power of a prime and let r be a prime such that $(q, r) = 1$. If n is the minimal integer of the set $\{m \in \mathbf{Z}_{>0} \,|\, r$ divides $\phi_m(q)\}$, then n is a divisor of $r - 1$.*

Proof. Straightforward from Fermat's Theorem. □

Lemma 2 *Let p be a prime such that $p = 2^{2^m} + 1$ for a non-negative integer m. If the order of a finite group G is divisible by $2p$, then 2 and p are joined in the solvable graph of G.*

Proof. Let G be a counter example for Lemma 2 of the smallest order, that is, 2 and a Fermat prime p are not joined directly in the solvable graph of G. Then G should be a non-abelian simple group by Proposition 1. We will verify that G is not isomorphic to any non-abelian simple group by the classification of finite simple groups. □

Case $G \simeq A_n$ $(n \geq 5)$.

It is easy to see the order of the normalizer of a cyclic subgroup of order p is even. Therefore 2 and p are joined directly. This contradicts our assumption.

Case $G \simeq \mathrm{PSL}_n(q)$ $(n \geq 2)$**.**

Suppose $G \simeq \mathrm{PSL}_2(q)$. Since $\{(q\pm 1)/(q\pm 1, 2)\} : 2$ and $q : \{(q-1)/(q-1, 2)\}$ are contained in $\mathrm{PSL}_2(q)$, 2 is joined to p directly. We may assume that $n \geq 3$. It is evident that $G \not\simeq \mathrm{PSL}_3(2)$ and $G \not\simeq \mathrm{PSL}_3(3)$. G has a subgroup H which is isomorphic to $(q^{n-1} : \mathrm{GL}_{n-1}(q))/\epsilon$, where $\epsilon|(q-1, n)$. By Lemma 1, we have $(q, p) = 1$. If p divides the order of H, then p divides the order of subgroup K of H which is isomorphic to $\mathrm{SL}_{n-1}(q)$. By the choice of G, 2 and p are joined directly in the solvable graph of G. It holds that p divides $\phi_n(q)$. By Lemma 1, n is a power of 2. This implies that 2 and p are joined directly in the solvable graph of G. This is a contradiction. It yields $G \not\simeq \mathrm{PSL}_n(q)$.

Case $G \simeq \mathrm{PSp}_n(q)$**.**

We may assume that $n \geq 4$. Then there exists a subgroup H such that $H \simeq (\mathrm{Sp}_{n-2}(q) \times \mathrm{SL}_2(q))/(q-1, 2)$. Since any odd prime in $\pi(H)$ joins to 2 directly in the solvable graph of G, $p|\phi_n(q) \cdot \phi_{n/2}(q)$ holds. G contains a subgroup which is isomorphic to $\mathrm{GL}_{n/2}(q)/(q-1, 2)$. By the case $G \simeq \mathrm{PSL}_n(q)$ $(n \geq 2)$, we have $p \nmid \phi_{n/2}(q)$, which implies $p|\phi_n(q)$. There exists a cyclic subgroup of order $\phi_n(q)$ of G and the order of its normalizer is $n\phi_n(q)$. By Lemma 1, n is a power of 2, we have a contradiction and it follows that $G \not\simeq \mathrm{PSp}_n(q)$.

Case $G \simeq \mathrm{PSU}_n(q)$**.**

We may assume that $n \geq 3$ and $p \geq 5$. Since $p \nmid q^2 - q + 1/(q+1, 3)$ by Burnside's transfer theorem, G is not isomorphic to $\mathrm{PSU}_3(q)$. G has a subgroup L which is isomorphic to $\mathrm{SU}_{n-1}(q)$. If p divides the order of L, then 2 and p are joined directly by the choice of G. It follows that $p|a$ where $a = \phi_n(q)/(q+1, n)$ if n is even and $a = \phi_{2n}(q)/(q+1, n)$ if n is odd. There exists a cyclic subgroup A of G whose order is a and A contains exactly one Sylow p-subgroup P of G. By Burnside's transfer theorem, 2 and p are joined directly in the solvable graph of G.

Case $G \simeq \mathrm{P}\Omega_{2n+1}(q)$**.**

We may assume that $n \geq 3$. By the choice of G, p should divide $\phi_{2n}(q)$. By the same manner in the case $G \simeq \mathrm{PSU}_n(q)$, we have $G \not\simeq \mathrm{P}\Omega_{2n+1}(q)$.

Case $G \simeq \mathrm{P}\Omega^+_{2n}(q)$**.**

We may assume that $n \geq 4$. By the case $G \simeq \mathrm{P}\Omega_{2n+1}(q)$, p should divide $\phi_n(q)$. This implies that n is a power of 2 and that p is joined to 2 directly. Hence we have $G \not\simeq \mathrm{P}\Omega^+_{2n}(q)$.

Case $G \simeq \mathrm{P}\Omega^-_{2n}(q)$.

We may asumme that $n \geq 4$. By the case of $G \simeq \mathrm{P}\Omega_{2n+1}(q)$, p should divide $\phi_{2n}(q)$. By the same manner in the case $G \simeq \mathrm{PSU}_n(q)$, we have $G \not\simeq \mathrm{P}\Omega^-_{2n}(q)$.

Case G is a exceptional simple group of Lie type.

If G is isomorphic to ${}^2B_2(q)$ or ${}^3D_4(q)$, then 2 is joined to every primes in $\pi(G)$ (See [1]), which implies that G is isomorphic to neither ${}^2B_2(q)$ nor ${}^3D_4(q)$. If G is isomorphic to $E_6(q)$, then p divides $\phi_9(q)$ [1]. This implies that $p|q-1$ and that p is joined to 2 directly. G is not isomorphic to $E_6(q)$. If G is isomorphic to ${}^2E_6(q)$, then p divides $\phi_{18}(q)$ [1]. This implies $p|q^2-1$ and p is joined to 2 directly. G is not isomorphic to ${}^2E_6(q)$ For other simple groups, we can see that p is joined to 2 directly by the tables in Abe-Iiyori [3], Iiyori-Yamaki [4] and Williams [5].

Case G is isomorphic to one of 26 sporadic simple groups.

In this case it is sufficient to consider about primes 3,5 and 17. So we can easily verify that p is joined to 2 directly in G by [2]. It is easy to know orders of centralizers of involutions in G and p is joined to 2 directly if p divides the order. If the order is prime to p, then we can easily see that p is joined to 2 directly from the following argument. If the number of conjugacy classes of elements of order p in G is less than $p-1$, then the order of the normalizer of a cyclic subgroup of order p is not coprime to $p-1$. In our case we assume $p = 2^{2^n}+1$ for a positive integer n. Therefore if G satisfies this condition, then p is joined to 2 directly. For example suppose that a Sylow 17-subgroup of G is cyclic of order 17 (We note that $|G|_{17} = 17$ if $17 \in \pi(G)$). If G has a 16 conjugacy classes of elements of order 17, then Burnside's transfer theorem leads that G has a normal $17'$-complement, which is a contradiction. Hence 17 is joined to 2 directly. If $p = 3$ or 5, then we can see that p is joined to 2 directly by the same manner. This completes the proof of the lemma.

Lemma 3 *Let p be a prime such that $p \neq 2^{2^n}+1$ for any non-negative integer n. Then there exists a non abelian simple group G in whose solvable graph 2 and p are not joined directly.*

Proof. By our hypothesis, there exist non-negative integers a, b such that $p-1 = 2^a \cdot b$, $b \equiv 1(2)$ and $b \geq 3$. Choose a prime c which divides b. Since $c|p-1$, there exists a positive integer r such that $r < p$ and the order of $\bar{r}$ in the unit group of the ring $\mathbf{Z}/p\mathbf{Z}$ is c. We choose r the smallest number among such integers. There exists a non-negative integer m such that $s = r+mp$ is a prime by Dirichlet's theorem. Then we have $p|(s^c-1)$ and $p \not| s-1$. This implies that the order of the normalizer of a Hall cyclic subgroup M of order $\phi_c(s)/(s-1,c)$ of $\mathrm{PSL}_c(s)$ divides $c \cdot (s^c-1)/(s-1)(s-1,c)$. Since this subgroup M is a TI-set, no subgroup of order 2 of $\mathrm{PSL}_c(s)$ normalizes a non trivial p-subgroup of $\mathrm{PSL}_c(s)$. If $s = 2$, then we can see directly that the order of any 2-local subgroup of $\mathrm{PSL}_c(2)$ is not divisible by p. If $s \neq 2$, then Maschke's theorem implies that an elementary 2-subgroup of $\mathrm{PSL}_c(s)$ is conjugate to a subgroup of the diagonal subgroup of $\mathrm{PSL}_c(s)$. It yields that no non trivial p-subgroup of $\mathrm{PSL}_c(s)$ normalizes no non trivial elementary 2-subgroup of it. 2 and p are not joined directly. □

A Proof of Theorem 1. Theorem 1 follows from Lemma 2 and Lemma 3. □

References

[1] Abe S. and Iiyori N., *A gengeraization of prime graphs of finite groups.* Hokkaido. Math. J. **29** (2000), 391–407.

[2] Conway J. H. *et al*, "*Atlas of Finite Groups*", Oxford Univ. Press (Cleandon), London/New York 1985.

[3] Iiyori N., *p-Solvability and a generalization of prime graphs of finite groups.* Comm. Algebra **30** 4 (2002), 1679–1692.

[4] Iiyori N. and Yamaki H., *Prime graph components of the simple group of Lie type over the field of even characteristic.* J. Algebra **155** (1993), 335–343.

[5] Williams J. S., *Prime graph components of finite groups.* J. Algebra **69** (1981), 487–513.

[6] Suzuki M., "Group Theory I, II". Springer, Berlin-Heidelberg- New York, 1982.

Faculty of Education
Yamaguchi University
Yamaguchi, 753-8512 Japan
E-mail: iiyori@yamaguchi-u.ac.jp