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Rank-one commutators on invariant subspaces of the Hardy space on the bidisk III

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Abstract. We study a special type of invariant subspaces $\mathcal{M}$ on the bidisk which are studied in the previous papers. We determine the rank of cross commutators on $H^2 \ominus \mathcal{M}$, and study when $\mathcal{M}$ is generated by $\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})$ as an invariant subspace of H^2.

Key words: invariant subspaces, backward shift invariant subspaces, rank-one commutators.

1. Introduction

Let $\mathbb{D}$ and Γ be the open unit disk and the unit circle in the complex plane $\mathbb{C}$, respectively. We denote by $H^2 = H^2(\Gamma^2)$ the Hardy space over the torus Γ^2, and we denote two variables by z and w. For $\psi \in L^\infty = L^\infty(\Gamma^2)$, we define the Toeplitz operator on H^2 by $T_\psi f = P_{H^2}\psi f$, where P_{H^2} is the orthogonal projection from $L^2 = L^2(\Gamma^2)$ onto H^2. A closed subspace M of H^2 with $M \neq \{0\}$ and $M \neq H^2$ is said to be invariant if M is invariant under T_z and T_w. In one variable case, Beurling [Beu] represented a well known theorem that an invariant subspace M has a form $M = qH^2(\Gamma)$, where q is an inner function. But in two variables case, the structure of invariant subspaces of H^2 is very complicated; see [CG], [DY], [Rud], [Ya1], [Ya2], [Ya3].

For a fixed invariant subspace M of H^2, let R_z and R_w be the compression operators on M defined by $R_z = P_M T_z|_M$ and $R_w = P_M T_w|_M$, respectively, where P_M is the orthogonal projection from L^2 onto M. Write $N = H^2 \ominus M$. Then N is a backward shift invariant subspace, that is, $T_z^* N \subset N$ and $T_w^* N \subset N$. Let S_z and S_w be the compression operators on N defined by $S_z = P_N T_z|_N$ and $S_w = P_N T_w|_N$, respectively. We denote the cross commutators by $[R_z, R_w^*] = R_z R_w^* - R_w^* R_z$ and $[S_z, S_w^*] = S_z S_w^* - S_w^* S_z$, where R_w^* and S_z^* are the adjoint operators

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of R_w and S_z on M and N, respectively. We note that $S_z^* = T_z^*|_N$ and $T_z^* f = (f - f(0,w))/z$.

In 1988, Mandrekar [Man] showed that an invariant subspace M has a form $M = qH^2$, where q is an inner function, if and only if $[R_z, R_w^*] = 0$. On the other hand, in [INS1] Izuchi, Nakazi, and Seto proved that $[S_z, S_w^*] = 0$ if and only if M has one of the following forms:

$$M = q_1(z)H^2; \quad M = q_2(w)H^2; \quad M = q_1(z)H^2 + q_2(w)H^2$$

for some non-constant inner functions $q_1(z)$ and $q_2(w)$. We write $rank\,[R_z, R_w^*] = dim\,[R_z, R_w^*]M$ and $rank\,[S_z, S_w^*] = dim\,[S_z, S_w^*]N$.

In [II4], Izuchi and the author proved that $rank\,[S_z, S_w^*] = 1$ if and only if M has one of the following forms:

(i) $M = \varphi H^2$, where φ is a non-constant inner function and is not a one variable function,

(ii) $M = q_1(z)q_2(z)H^2 + q_2(z)q_3(w)H^2 + q_3(w)q_4(w)H^2$, where q_1, q_2, q_3, q_4 satisfy one of the following:

- (α) q_1, q_2, q_3, q_4 are one variable non-constant inner functions,
- (β) $q_1 = 0$, and q_2, q_3, q_4 are one variable non-constant inner functions,
- (γ) $q_4 = 0$, and q_1, q_2, q_3 are one variable non-constant inner functions.

Since $[R_z, R_w^*] = 0$ on wM, generally a cross commutator $[R_z, R_w^*]$ is small. In [Ya3, Theorem 2.3], Yang showed that the operator $[R_z, R_w^*]$ is Hilbert-Schmidt under a mild condition on M; see also [Ya1]. In [II1], [II3], Izuchi and the author studied M under the condition that $rank\,[R_z, R_w^*] = 1$, and found an interesting example of M satisfying $rank\,[R_z, R_w^*] = 1$. We denote by $ball\, H^\infty(\Gamma)$ the closed unit ball of $H^\infty(\Gamma)$ with the supremum norm. [Gar], [Hof] are nice references for the study of $H^\infty(\Gamma)$. We denote by $H^2(\Gamma_z)$ the Hardy space in variable z.

Let $G(z)$, $H(z) \in ball\, H^\infty(\Gamma_z)$ satisfying the following conditions:

(a) $G(z)$ is a non-extreme point in $ball\, H^\infty(\Gamma_z)$,

(b) $|H(z)|^2 = 1 - |G(z)|^2$ a.e. on Γ_z,

(c) $H_0(z)$ is an outer function with $|H_0(z)|^2 = 1 - |G(z)|^2$ a.e. on Γ_z.

(d) φ is an inner function with

$$\frac{\varphi H_0(z)}{w - G(z)} \in H^2.$$

It is known that $f(z)$ is an extreme point in *ball* $H^\infty(\Gamma_z)$ if and only if

$$\int_0^{2\pi} \log(1 - |f(e^{i\theta})|)\,\frac{d\theta}{2\pi} = -\infty;$$

see [Hof, pp. 138–139]. In [II3], it is proved that

$$\varphi H^2 \perp \frac{\varphi H(z)}{w - G(z)} H^2(\Gamma_z).$$

We write

$$\mathcal{M} = \varphi H^2 \oplus \frac{\varphi H(z)}{w - G(z)} H^2(\Gamma_z). \tag{1.1}$$

It is also proved that $\mathcal{M}$ is an invariant subspace with $rank\,[R_z, R_w^*] = 1$. In this paper, we study $\mathcal{M}$ more exactly. In this moment, there are no complete descriptions of M satisfying $rank\,[R_z, R_w^*] = 1$. We write $\mathcal{N} = H^2 \ominus \mathcal{M}$.

Recently in [II5], Izuchi and the author showed that for invariant subspaces M of H^2 with $rank\,[R_w^*, R_w][R_z^*, R_z] < \infty$,

$$rank\,[R_z, R_w^*] - 1 \le rank\,[S_z, S_w^*] \le rank\,[R_z, R_w^*] + 1.$$

We will study $rank\,[S_z, S_w^*]$ for $\mathcal{M}$.

About fifteen years ago, Hedenmalm [Hed] gave a very exciting theorem on the Bergman space $L_a^2(\mathbb{D})$ over the unit disk $\mathbb{D}$. He proved that there is an invariant subspace I of $L_a^2(\mathbb{D})$ satisfying $dim\,(I \ominus zI) = 2$. In [ARS], Aleman, Richter, and Sundberg proved that $[I \ominus zI] = I$ for every invariant subspace I of $L_a^2(\mathbb{D})$, where $[I \ominus zI]$ is the invariant subspace of $L_a^2(\mathbb{D})$ generated by $I \ominus zI$; see also [DS], [HKZ].

In H^2, we can consider similar issues. Let M be an invariant subspace of H^2. The space $M \ominus (zM + wM)$ is naturally considered as the corresponding space to $I \ominus zI$. The space $M \ominus (zM + wM)$ is one of the most important spaces for studying the structure of M. For a subset E of H^2, let $[E]$ denote the invariant subspace generated by all functions in E. In [Nak], Nakazi showed that for $f \in H^2$ $\dim([f] \ominus (z[f] + w[f])) = 1$, and posed a problem whether $[f] = [[f] \ominus (z[f] + w[f])]$ holds or not. Here our question is; when does $M = [M \ominus (zM + wM)]$ hold? We will answer this question for $\mathcal{M}$.

In Section 2, we treat the case that $G(z)$ is a constant. It is proved that if $G(0) \neq 0$ and $H(0) \neq 0$, then $dim\,(\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})) = 1$ and $\mathcal{M} \neq [\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})]$.

In Section 3, we study the case that $G(z)$ is non-constant. It is proved that if $G(0) \neq 0$ and $H(0) \neq 0$, then $dim\,(\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})) = 1$, and $\mathcal{M} = [\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})]$ for some cases. In the first glance, one thinks that $\mathcal{M}$ given by (1.1) is not a singly generated invariant subspace, but this is not true. We give an equivalent condition on $G(z)$ and $H(z)$ for which $\mathcal{M} = [\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})]$.

2. The case that $G(z)$ is constant

Let $\mathcal{M}$ be an invariant subspace given in (1.1) with conditions (a), (b), (c), and (d). In this section, we study the case that $G(z)$ is a constant function. Let $G(z) = a \in \mathbb{D}$. Then

$$\mathcal{M} = \varphi H^2 \oplus \frac{\varphi H(z)}{w - a} H^2(\Gamma_z).$$

By (b), we can write $H(z) = bI(z)$, where $b \in \mathbb{C}$ with $|b|^2 = 1 - |a|^2$ and $I(z)$ is inner. We note that

$$\varphi = \frac{w - a}{1 - \overline{a}w}\varphi_1$$

for some inner function φ_1. Thus we have

$$\begin{aligned} \mathcal{M} &= \varphi_1\left(\frac{w - a}{1 - \overline{a}w}H^2 \oplus \frac{H(z)}{1 - \overline{a}w}H^2(\Gamma_z)\right) \\ &= \varphi_1\left(\frac{w - a}{1 - \overline{a}w}H^2 + I(z)H^2\right) \\ &= \varphi_1\mathcal{M}_1, \end{aligned}$$

where

$$\mathcal{M}_1 = \frac{w - a}{1 - \overline{a}w}H^2 + I(z)H^2.$$

Proposition 2.1 *Let*

$$\mathcal{M} = \varphi_1 \mathcal{M}_1 = \varphi_1 \left(\frac{w-a}{1-\bar{a}w} H^2 + I(z) H^2 \right)$$

for some inner function φ_1, $a \in \mathbb{D}$, and one variable inner function $I(z)$. Then we have the following:

(i) *If φ_1 is constant, then $[S_z, S_w^*] = 0$.*
(ii) *Suppose that φ_1 is non-constant and $I(z)$ is constant. If φ_1 is one variable, then $[S_z, S_w^*] = 0$. If φ_1 is not one variable, then rank $[S_z, S_w^*] = 1$.*
(iii) *Suppose that φ_1 and $I(z)$ are non-constant. If φ_1 is one variable, then rank $[S_z, S_w^*] = 1$. If φ_1 is not one variable, then rank $[S_z, S_w^*] = 2$.*

To prove Proposition 2.1, we need the following lemma due to Yang [Ya4, p. 179].

Lemma 2.2 *rank $[S_z, S_w^*] \leq$ rank $[R_w^*, R_w][R_z^*, R_z]$.*

It is easy to see that

$$[R_w^*, R_w][R_z^*, R_z] = (I - R_w R_w^*)(I - R_z R_z^*) = P_{\mathcal{M} \ominus w\mathcal{M}} P_{\mathcal{M} \ominus z\mathcal{M}}.$$

Proof of Proposition 2.1. (i): By [INS1], we have that if φ_1 is constant, then $[S_z, S_w^*] = 0$.

(ii): Suppose that φ_1 is non-constant and $I(z)$ is constant. Then $\mathcal{M}_1 = H^2$ and $\mathcal{M} = \varphi_1 H^2$. By [INS1], we know that if φ_1 is one variable inner, then $[S_z, S_w^*] = 0$. Yang [Ya4] pointed out that if φ_1 is not one variable, then *rank* $[S_z, S_w^*] = 1$.

(iii): Suppose that φ_1 and $I(z)$ are non-constant inner functions. By [II4], if φ_1 is one variable, then *rank* $[S_z, S_w^*] = 1$, and if φ_1 is not one variable, then *rank* $[S_z, S_w^*] \geq 2$. We have

$$\mathcal{M} \ominus z\mathcal{M} = \varphi H^2(\Gamma_w) \oplus \mathbb{C} \cdot \frac{\varphi H(z)}{w-a}.$$

Hence

$$P_{\mathcal{M} \ominus w\mathcal{M}} P_{\mathcal{M} \ominus z\mathcal{M}} \mathcal{M} \subset \mathbb{C} \cdot \varphi \oplus \mathbb{C} \cdot \frac{\varphi H(z)}{w-a}.$$

By Lemma 2.2, we have $rank\,[S_z, S_w^*] \leq 2$. So, φ_1 is not one variable and $rank\,[S_z, S_w^*] = 2$. $\square$

In the following theorem, we study $[\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})]$. To study our problem, an inner factor φ_1 is not essential, so we may assume that $\varphi_1 = 1$. If $I(z)$ is constant, then $\mathcal{M} = H^2$. Hence $\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot 1$ and $[\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})] = \mathcal{M}$. So, we assume that $I(z)$ is non-constant.

Theorem 2.3 *Let*

$$\mathcal{M} = \frac{w-a}{1-\overline{a}w}H^2 + I(z)H^2$$

for some $a \in \mathbb{D}$ and a one variable non-constant inner function $I(z)$. Then we have the following:

(i) *If $a = 0$ and $I(0) = 0$, then*

$$\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot w + \mathbb{C} \cdot I(z)$$

and

$$[\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})] = \mathcal{M}.$$

(ii) *If $a = 0$ and $I(0) \neq 0$, then*

$$\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot I(z)$$

and

$$[\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})] = I(z)H^2 \neq \mathcal{M}.$$

(iii) *If $a \neq 0$ and $I(0) = 0$, then*

$$\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot \frac{w-a}{1-\overline{a}w}$$

and

$$[\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})] = \frac{w-a}{1-\overline{a}w}H^2 \neq \mathcal{M}.$$

(iv) *If $a \neq 0$ and $I(0) \neq 0$, then*

$$\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot \left(\frac{-\overline{a}(w-a)}{(1-|a|^2)(1-\overline{a}w)} + \frac{\overline{I(0)}I(z)}{1-\overline{a}w} \right)$$

and

$$[\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})] \neq \mathcal{M}.$$

To describe $\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})$, we use Guo and Yang's result given in [GY]. Let M be an invariant subspace of H^2, and $K(\lambda, Z)$ be the reproducing kernel for M, $\lambda \in \Gamma^2$, and $Z = (z, w) \in \mathbb{D}^2$. Associated with M, Guo and Yang defined the core operator C on M by

$$C(f)(Z) = \int_{\Gamma^2} \left(1 - \overline{\lambda_1} z\right)\left(1 - \overline{\lambda_2} w\right) K(\lambda, Z) f(\lambda)\, dm(\lambda), \tag{2.1}$$

where $dm(\lambda)$ is the normarized Lebesgue measure on Γ^2, and they showed that

$$C = I - R_z R_z^* - R_w(1 - R_z R_z^*) R_w^*. \tag{2.2}$$

So, C is a bounded selfadjoint operator on M. Also they showed the following.

Lemma 2.4 *Let $f \in M$. Then $C(f) = f$ if and only if $f \in M \ominus (zM + wM)$.*

Proof of Theorem 2.3. It is not difficult to show (i), (ii), and (iii), so we shall show (iv). Write

$$q(w) = \frac{w-a}{1-\overline{a}w}.$$

By [Ya4, p. 176],

$$\begin{aligned} &\left(1 - \overline{\lambda_1} z\right)\left(1 - \overline{\lambda_2} w\right) K(\lambda, Z) \\ &\quad = \overline{I(\lambda_1)} I(z) + \overline{q(\lambda_2)} q(w) - \overline{I(\lambda_1) q(\lambda_2)} I(z) q(w). \end{aligned}$$

Hence by (2.1) and Lemma 2.4,

$$\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) \subset \mathbb{C} \cdot I(z) + \mathbb{C} \cdot q(w) + \mathbb{C} \cdot I(z)q(w).$$

Let $F = aI(z) + bq(w) + cI(z)q(w)$ satisfying $C(F) = F$. It is not difficult to see that

$$\begin{aligned} C(F) = &\big(a + b\overline{I(0)}q(0) + cq(0)\big)I(z) + \big(aI(0)\overline{q(0)} + b + cI(0)\big)q(w) \\ &- \big(a\overline{q(0)} + b\overline{I(0)} + c\big)I(z)q(w). \end{aligned}$$

Hence we get

$$b\overline{I(0)}q(0) + cq(0) = 0, \quad aI(0)\overline{q(0)} + cI(0) = 0, \quad -a\overline{q(0)} - b\overline{I(0)} = 2c.$$

Since $q(0) \neq 0$ and $I(0) \neq 0$, we have

$$a = -\frac{1}{\overline{q(0)}}c \quad \text{and} \quad b = -\frac{1}{\overline{I(0)}}c.$$

By Lemma 2.4, we have

$$-\frac{1}{\overline{q(0)}}I(z) - \frac{1}{\overline{I(0)}}q(w) + I(z)q(w) \in \mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}).$$

Therefore

$$\begin{aligned} &\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) \\ &\quad = \mathbb{C} \cdot \left(-\overline{I(0)}I(z) + \frac{\overline{a}(w-a)}{1-\overline{a}w} - \overline{aI(0)}I(z)\frac{w-a}{1-\overline{a}w} \right) \\ &\quad = \mathbb{C} \cdot \left(\frac{-\overline{a}(w-a)}{(1-|a|^2)(1-\overline{a}w)} + \frac{\overline{I(0)}I(z)}{1-\overline{a}w} \right). \end{aligned}$$

It remains to prove $[\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})] \neq \mathcal{M}$. Let $\mathcal{M}_1$ be the invariant subspace of H^2 generated by the function

$$\frac{-\overline{a}(w-a)}{(1-|a|^2)(1-\overline{a}w)} + \frac{\overline{I(0)}I(z)}{1-\overline{a}w}.$$

We note that $(1 - \overline{a}w)\mathcal{M}_1 = \mathcal{M}_1$. Then $\mathcal{M}_1$ is generated by

$$\frac{1 - \overline{a}w}{\overline{I(0)}}\left(\frac{-\overline{a}(w-a)}{(1-|a|^2)(1-\overline{a}w)} + \frac{\overline{I(0)}I(z)}{1-\overline{a}w}\right) = \frac{-\overline{a}(w-a)}{\overline{I(0)}(1-|a|^2)} + I(z).$$

Therefore it is enough to show that

$$\left[\frac{-\overline{a}(w-a)}{\overline{I(0)}(1-|a|^2)} + I(z)\right] \neq \frac{w-a}{1-\overline{a}w}H^2 + I(z)H^2.$$

To prove this, we suppose that the equality holds. Since $1 - \overline{a}w$ is an invertible function in $H^\infty(\Gamma_w)$, we have

$$\left[\frac{-\overline{a}(w-a)}{\overline{I(0)}(1-|a|^2)} + I(z)\right] = (w-a)H^2 + I(z)H^2.$$

Since $I(z)$ is non-constant inner, $I(\mathbb{D})$ is dense in $\mathbb{D}$. Since the range of the function

$$\frac{-\overline{a}(w-a)}{\overline{I(0)}(1-|a|^2)}$$

contains small open disks with center 0, one sees that the common zero set in $\mathbb{D}^2$ of

$$\left[\frac{-\overline{a}(w-a)}{\overline{I(0)}(1-|a|^2)} + I(z)\right]$$

has a nonempty connected component. On the other hand, the common zero set in $\mathbb{D}^2$ of $(w-a)H^2 + I(z)H^2$ is $\{(\zeta, a) \in \mathbb{D}^2 | I(\zeta) = 0\}$ and this set is either empty or a discrete set. This is a contradiction. This completes the proof. □

3. The case that $G(z)$ is non-constant

Write

$$h_0 = \frac{\varphi H(z)}{w - G(z)} \in H^2. \tag{3.1}$$

Then $\mathcal{M} = \varphi H^2 + h_0 H^2(\Gamma_z)$. By (3.1),

$$h_0 = \varphi H(z) \sum_{n=0}^{\infty} \overline{w}^{(n+1)} G^n(z), \tag{3.2}$$

so for $i \neq j$ we have

$$\begin{aligned}
\langle h_0 z^i, h_0 z^j \rangle &= \sum_{n=0}^{\infty} \left\langle z^i H(z) G^n(z), z^j H(z) G^n(z) \right\rangle \\
&= \left\langle |H(z)|^2 \sum_{n=0}^{\infty} |G(z)|^{2n}, z^{j-i} \right\rangle \\
&= \langle 1, z^{j-i} \rangle \qquad \text{by condition (b)} \\
&= 0.
\end{aligned}$$

Hence

$$h_0 z^i \perp h_0 z^j, \quad i \neq j, \tag{3.3}$$

so we have $\mathcal{M} \subset H^2$. By (3.2), for $i, j, k \geq 0$

$$\left\langle \varphi z^i w^j, h_0 z^k \right\rangle = \left\langle z^i w^j, z^k H(z) \sum_{n=0}^{\infty} \overline{w}^{(n+1)} G^n(z) \right\rangle = 0.$$

Thus we have

$$\varphi H^2 \perp h_0 H^2(\Gamma_z),$$

so $\mathcal{M} = \varphi H^2 \oplus h_0 H^2(\Gamma_z)$. By condition (b) and (3.2), we have

$$\|h_0\|^2 = \sum_{n=0}^{\infty} \|H(z) G^n(z)\|^2 = \int_0^{2\pi} \frac{|H(e^{i\theta})|^2}{1 - |G(e^{i\theta})|^2} \frac{d\theta}{2\pi} = 1. \tag{3.4}$$

Since $\mathcal{M} = \varphi H^2 \oplus h_0 H^2(\Gamma_z)$, by (3.3) we have

$$\mathcal{M} \ominus z\mathcal{M} = \varphi H^2(\Gamma_w) \oplus \mathbb{C} \cdot h_0. \tag{3.5}$$

Proposition 3.1 *If $G(z)$ is non-constant, then rank* $[S_z, S_w^*] = 2$.

Proof. By (3.5), we have

$$\begin{aligned}[R_w^*, R_w][R_z^*, R_z]\mathcal{M} &= P_{\mathcal{M}\ominus w\mathcal{M}}(\mathcal{M}\ominus z\mathcal{M})\\ &= P_{\mathcal{M}\ominus w\mathcal{M}}(\varphi H^2(\Gamma_w)\oplus \mathbb{C}\cdot h_0)\\ &= P_{\mathcal{M}\ominus w\mathcal{M}}(\mathbb{C}\cdot\varphi\oplus\mathbb{C}\cdot h_0).\end{aligned}$$

Therefore by Lemma 2.2, *rank* $[S_z, S_w^*] \le 2$. By [INS1], we have $[S_z, S_w^*] \neq 0$. By [II4], *rank* $[S_z, S_w^*] \neq 1$. Thus we get *rank* $[S_z, S_w^*] = 2$. □

Next, we study $[\mathcal{M}\ominus(z\mathcal{M}+w\mathcal{M})]$. Recall that, by (2.2)

$$C = I - R_zR_z^* - R_w(I - R_zR_z^*)R_w^* = P_{\mathcal{M}\ominus z\mathcal{M}} - R_wP_{\mathcal{M}\ominus z\mathcal{M}}R_w^*.$$

Now it is easy to see that $C = 0$ on $w\varphi H^2$. We have

$$\mathcal{M}\ominus w\varphi H^2 = \varphi H^2(\Gamma_z)\oplus h_0H^2(\Gamma_z). \tag{3.6}$$

Lemma 3.2 *For $f(z)\in H^2(\Gamma_z)$, we have the following*:

(i) $C(\varphi f(z)) = f(0)\varphi - \langle f(z), H(z)\rangle wh_0$.
(ii) $C(h_0f(z)) = f(0)h_0 - \langle f(z), G(z)\rangle wh_0$.
(iii) $C(wh_0) = H(0)\varphi + G(0)h_0 - wh_0$.
(iv) $C(\mathcal{M}) = \mathbb{C}\cdot\varphi + \mathbb{C}\cdot h_0 + \mathbb{C}\cdot wh_0$.

Proof. (i): We have $P_{\mathcal{M}\ominus z\mathcal{M}}\varphi f(z) = f(0)\varphi$. Since $R_w^*(\varphi f(z)) \perp \varphi H^2$, we have

$$\begin{aligned}R_wP_{\mathcal{M}\ominus z\mathcal{M}}R_w^*(\varphi f(z)) &= w\big\langle R_w^*\varphi f(z), h_0\big\rangle h_0 \qquad \text{by (3.4) and (3.5)}\\ &= \langle \varphi f(z), wh_0\rangle wh_0\\ &= \Big\langle f(z), \sum_{n=0}^{\infty}\overline{w}^nG(z)^nH(z)\Big\rangle wh_0\\ &= \langle f(z), H(z)\rangle wh_0.\end{aligned}$$

Thus we get (i).

(ii): We have $P_{\mathcal{M}\ominus z\mathcal{M}}(h_0f(z)) = f(0)h_0$. Since $R_w^*(h_0f(z)) \perp \varphi H^2$, we have

$$
\begin{aligned}
&R_w P_{\mathcal{M}\ominus z\mathcal{M}} R_w^*(h_0 f(z)) \\
&\quad = \left\langle R_w^* h_0 f(z), h_0 \right\rangle w h_0 \\
&\quad = \langle h_0 f(z), w h_0 \rangle w h_0 \\
&\quad = \left\langle \sum_{n=0}^{\infty} f(z) H(z) G(z)^n \overline{w}^{(n+1)}, \sum_{n=-1}^{\infty} H(z) G(z)^{n+1} \overline{w}^{(n+1)} \right\rangle w h_0 \\
&\quad = \sum_{n=0}^{\infty} \langle f(z) H(z) G(z)^n, H(z) G(z)^{n+1} \rangle w h_0 \\
&\quad = \left\langle f(z), \sum_{n=0}^{\infty} |H(z)|^2 |G(z)|^{2n} G(z) \right\rangle w h_0 \\
&\quad = \langle f(z), G(z) \rangle w h_0.
\end{aligned}
$$

(iii): Since $wh_0 = \varphi H(z) + h_0 G(z)$, by (i) and (ii) we have

$$
\begin{aligned}
C(wh_0) &= H(0)\varphi + G(0)h_0 - (\|H\|^2 + \|G\|^2) w h_0 \\
&= H(0)\varphi + G(0)h_0 - wh_0 \qquad \text{by condition (b).}
\end{aligned}
$$

(iv): This follows from (3.6), (i) and (ii). □

Theorem 3.3 *Suppose that $G(z)$ is non-constant. Then we have the following*:

(i) *If $G(0) = H(0) = 0$, then*

$$\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot \varphi \oplus \mathbb{C} \cdot \frac{\varphi H(z)}{w - G(z)}$$

and

$$\mathcal{M} = [\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})].$$

(ii) *If $G(0) = 0$ and $H(0) \neq 0$, then*

$$\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot \frac{\varphi H(z)}{w - G(z)},$$

and $\mathcal{M} = [\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})]$ *if and only if* $H(z) = aH_0(z)$ *for some* $a \in \mathbb{C}$ *with* $|a| = 1$.

(iii) *If* $G(0) \neq 0$ *and* $H(0) = 0$, *then*

$$\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot \varphi$$

and

$$\mathcal{M} \neq \varphi H^2(\Gamma^2) = [\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})].$$

(iv) *If* $G(0) \neq 0$ *and* $H(0) \neq 0$, *then*

$$\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot \varphi\left(1 - \frac{\overline{H(0)}H(z)}{\overline{G(0)}(w - G(z))}\right),$$

and $\mathcal{M} = [\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})]$ *if and only if*

$$1 \leq \left|G(z) + \frac{\overline{H(0)}H(z)}{\overline{G(0)}}\right|$$

for every $z \in \mathbb{D}$.

Proof. By (3.5),

$$\mathcal{M} \ominus z\mathcal{M} = \varphi H^2(\Gamma_w) \oplus \mathbb{C} \cdot h_0.$$

Then we can get easily

$$\{0\} \neq \mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) \subset \mathbb{C} \cdot \varphi \oplus \mathbb{C} \cdot h_0.$$

Let's start to prove Theorem 3.3.

(i): Since $G(0) = H(0) = 0$, by Lemmas 2.4 and 3.2 we have

$$\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot \varphi \oplus \mathbb{C} \cdot h_0,$$

In this case, it is easy to see that $\mathcal{M} = [\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})]$.

(ii): By Lemmas 2.4 and 3.2, we have $\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot h_0$. We can write $H(z) = I(z)H_0(z)$ for some inner function $I(z)$. Since $H_0(z)$ is outer, by [Gar, p. 85] there exists a sequence of polynomials $\{p_n(z)\}_n$

such that $|p_n(z)H_0(z)| \leq 1$ a.e. on Γ_z and $p_n(z)H_0(z) \to 1$ a.e. on Γ_z as $n \to \infty$. Since $(w - G(z))h_0 = \varphi H(z)$, by the Lebesgue dominated convergence theorem

$$p_n(z)(w - G(z))h_0 = p_n(z)H_0(z)\varphi I(z) \to \varphi I(z) \qquad \text{in } H^2.$$

Hence we get $\varphi I(z) \in [h_0]$.

Now we prove that $\mathcal{M} = [\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M})]$ if and only if $H(z) = aH_0(z)$ for some $a \in \mathbb{C}$ with $|a| = 1$.

($\Leftarrow$) If $I(z) = a$, then we have $\varphi \in [h_0]$ and $\mathcal{M} = [\varphi, h_0] = [h_0]$.

($\Rightarrow$) To prove this by the contradiction, suppose that $I(z)$ is non-constant. Then $\varphi T_z^* I(z) \neq 0$ and $\varphi T_z^* I(z) \in \mathcal{M}$. For every non-negative integers i, j, we have

$$\begin{aligned} \left\langle \varphi T_z^* I(z), z^i w^j h_0 \right\rangle &= \left\langle T^*{}_z I(z), z^i w^j I(z) H_0(z) \sum_{k=0}^{\infty} \overline{w}^{(k+1)} G^k(z) \right\rangle \\ &= \left\langle 1, \sum_{k=0}^{\infty} z^{i+1} \overline{w}^{(k+1-j) H_0(z) G^k(z)} \right\rangle \\ &= 0. \end{aligned}$$

Hence we get $\varphi T_z^* I(z) \perp [h_0]$, so that $\mathcal{M} \neq [h_0]$. Thus we get (ii).

(iii): By Lemmas 2.4 and 3.2, we have $\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot \varphi$. Then we easily get (iii).

(iv): Let $F = a\varphi + bh_0$ satisfying $C(F) = F$. By Lemma 3.2,

$$C(F) = a\varphi - a\overline{H(0)}wh_0 + bh_0 - b\overline{G(0)}wh_0.$$

Hence we get $b = -\overline{H(0)}a/\overline{G(0)}$. By Lemma 2.4,

$$\mathcal{M} \ominus (z\mathcal{M} + w\mathcal{M}) = \mathbb{C} \cdot (\varphi + \alpha h_0),$$

where $\alpha = -\overline{H(0)}/\overline{G(0)}$. By (3.1),

$$\varphi + \alpha h_0 = \varphi\left(1 + \frac{\alpha H(z)}{w - G(z)}\right).$$

Since

$$\mathcal{M} = \varphi\left(H^2 \oplus \frac{H(z)}{w - G(z)} H^2(\Gamma_z)\right),$$

we have

$$\left[1 + \frac{\alpha H(z)}{w - G(z)}\right] \subset H^2 \oplus \frac{H(z)}{w - G(z)} H^2(\Gamma_z), \tag{3.7}$$

and it holds that

$$\mathcal{M} = \varphi\left[1 + \frac{\alpha H(z)}{w - G(z)}\right] \tag{3.8}$$

if and only if

$$\left[1 + \frac{\alpha H(z)}{w - G(z)}\right] = H^2 \oplus \frac{H(z)}{w - G(z)} H^2(\Gamma_z). \tag{3.9}$$

First suppose that $1 \leq |G(z) - \alpha H(z)|$ for every $z \in \mathbb{D}$. Then by [IY, Corollary 2.7], we have

$$[w - (G(z) - \alpha H(z))] = H^2.$$

This shows that

$$\begin{aligned} \left[1 + \frac{\alpha H(z)}{w - G(z)}\right] &\supset \overline{(w - G(z))\left[1 + \frac{\alpha H(z)}{w - G(z)}\right]} \\ &= [w - (G(z) - \alpha H(z))] \\ &= H^2. \end{aligned}$$

Hence

$$1, \quad \frac{H(z)}{w - G(z)} \in \left[1 + \frac{\alpha H(z)}{w - G(z)}\right].$$

Therefore

$$H^2 \oplus \frac{H(z)}{w - G(z)} H^2(\Gamma_z) \subset \left[1 + \frac{\alpha H(z)}{w - G(z)}\right].$$

By (3.7), we have (3.9). So we get (3.8).

Finally we consider the case that $|G(z_0)-\alpha H(z_0)|<1$ for some $z_0 \in D$. By [IY, Corollary 2.7],

$$[w-(G(z)-\alpha H(z))] \neq H^2. \tag{3.10}$$

In this case, it is sufficient to prove that (3.9) does not hold. Suppose that (3.9) holds. Then we have

$$(w-G(z))\left[1+\frac{\alpha H(z)}{w-G(z)}\right]=(w-G(z))H^2+H(z)H^2(\Gamma_z). \tag{3.11}$$

Since

$$(w-G(z))\left[1+\frac{\alpha H(z)}{w-G(z)}\right]\subset[w-(G(z)-\alpha H(z))],$$

by (3.11) we have

$$[w-G(z)]\subset[w-(G(z)-\alpha H(z))].$$

Hence

$$\begin{aligned}\{0\} &\neq H^2\ominus[w-(G(z)-\alpha H(z))] \qquad \text{by (3.10)}\\ &\subset H^2\ominus[w-G(z)].\end{aligned}$$

By [IY, Theorem 2.5, Corollary 2.9 and its proof], we have

$$G(z)-\alpha H(z)=G(z).$$

This contradicts that $H(z)\neq 0$. Therefore (3.9) does not hold. This completes the proof. $\square$

In the case (iv) in Theorem 3.1, by (3.7) we have

$$\varphi\left[1+\frac{\alpha H(z)}{w-G(z)}\right]\subset\mathcal{M},$$

where $\alpha=-\overline{H(0)}/\overline{G(0)}$.

Example 3.4 There are invariant subspaces $\mathcal{M}$ such that

$$\varphi\left[1 + \frac{\alpha H(z)}{w - G(z)}\right] = \mathcal{M} \tag{3.12}$$

and

$$\varphi\left[1 + \frac{\alpha H(z)}{w - G(z)}\right] \subsetneqq \mathcal{M}, \tag{3.13}$$

respectively.

Let $G(z) = I(z)/2$ for some non-constant inner function $I(z)$ with $I(0) \neq 0$. Then $G(z)$ is a non-constant and non-extreme point in *ball* $H^\infty(\Gamma_z)$ with $G(0) \neq 0$.

First we give an example of (3.12). Let $H(z) = \sqrt{3}/2$. Then $|G(z)|^2 + |H(z)|^2 = 1$ a.e. on Γ_z and $H(0) \neq 0$. For each $z \in \mathbb{D}$, we have

$$|G(z) - \alpha H(z)| = \left|\frac{I(z)}{2} + \frac{3}{2\overline{I(0)}}\right| \geq \frac{1}{2}\left(\frac{3}{|I(0)|} - 1\right) > 1.$$

Thus by Theorem 3.3, we have (3.12).

Next we give an example of (3.13). Let $H(z) = \sqrt{3}I(z)/2$. Then we have $|G(z)|^2 + |H(z)|^2 = 1$ a.e. on Γ_z, $H(0) \neq 0$ and $G(z) - \alpha H(z) = 2I(z)$. Since $I(z)$ is non-constant inner, there exists $z_0 \in \mathbb{D}$ such that $|G(z_0) - \alpha H(z_0)| < 1$. Thus by Theorem 3.3, we get (3.13).

When $G(z)$ is contained in the disk algebra, the space of functions $f(z) \in C(\overline{\mathbb{D}})$ which are analytic in $\mathbb{D}$, the existence of inner function φ satisfying

$$\varphi\left[1 + \frac{\alpha H(z)}{w - G(z)}\right] \subset H^2$$

is known, see [II3, Theorem 2.3].

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