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Change of variables for weighted Hardy spaces on a domain

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Abstract. A generalized version of weighted Hardy spaces on a Euclidean domain is introduced and it is proved that the spaces are transformed to the same kind of spaces by certain smooth change of variables. Some related properties of the spaces, including a modified form of atomic decomposition, and some examples are also given.

Key words: weighted Hardy space, conformal mapping.

1. Introduction

After the pioneering work of Fefferman and Stein [FS], in which the Hardy space $H^p(\mathbb{R}^n)$ was introduced, several authors introduced variants of $H^p(\mathbb{R}^n)$. Goldberg [G] introduced the local Hardy space $h^p(\mathbb{R}^n)$. Strömberg and Torchinsky [ST] introduced the weighted Hardy spaces $H^p_w(\mathbb{R}^n)$. The present author introduced the Hardy space $H^p(\Omega)$ on a Euclidean domain in [M1] and its weighted version $H^p(\Omega, \lambda)$ in [M2] and [M3]. The purpose of the present paper is to introduce the Hardy spaces $H^p(\Omega, T, \lambda)$ for open subset Ω of $\mathbb{R}^n$ and show, among other things, that certain change of variables in the basic domain Ω transforms the space $H^p(\Omega, T, \lambda)$ into a space of the same kind. Our spaces $H^p(\Omega, T, \lambda)$ include, as their special cases, the spaces $h^p(\mathbb{R}^n)$, $H^p(\Omega)$, $H^p(\Omega, \lambda)$, and also the local version of $H^p_w(\mathbb{R}^n)$. Our result will show, for example, Goldberg's space $h^p(\mathbb{R})$ is transformed to the space $H^p((0,1), dx/x(1-x))$ of [M2] and [M3] by the change of variables $y = \log(x/(1-x))$. In this section, we shall explain the main results of this paper in some detail.

The following notation is used throughout this paper.

Notation The letters Ω and $\tilde{\Omega}$ always denote open subsets of $\mathbb{R}^n$. We write $\mathcal{D}'(\Omega)$ to denote the set of all distributions on Ω. If $\Omega \neq \mathbb{R}^n$, we write

$$d_\Omega(x) = \mathrm{dis}(x, \Omega^c) = \inf\{|x-y| \mid y \in \mathbb{R}^n \setminus \Omega\} \quad (x \in \Omega).$$

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For $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, we write the open Euclidean ball with center x and radius r as $B(x, r)$. If $B = B(x, r)$ is a ball and if $a \in (0, \infty)$, then we write $aB = B(x, ar)$. If λ is a Borel measure on Ω and $p \in (0, \infty)$, we write

$$\|f\|_{L^p(\Omega,\lambda)} = \left(\int_\Omega |f(x)|^p d\lambda(x) \right)^{1/p}.$$

If k is a nonnegative integer, then $\mathcal{P}_k$ denotes the set of all polynomial functions on $\mathbb{R}^n$ of degree not exceeding k. We write $\| \cdot \|_{L^\infty}$ and L^∞ to denote the L^∞-norm and the L^∞-space on $\mathbb{R}^n$ with respect to the Lebesgue measure. For $E \subset \mathbb{R}^n$, the closure of E in $\mathbb{R}^n$ is denoted by $\overline{E}$. If F and G are functions defined on a set X taking values in $[0, \infty) \cup \{\infty\}$ and if there exists a constant $A \in (0, \infty)$ such that $F(x) \leqq AG(x)$ for all $x \in X$, then we write '$F(x) \lesssim G(x)$ for $x \in X$' or '$G(x) \gtrsim F(x)$ for $x \in X$'. We write '$F(x) \approx G(x)$ for $x \in X$' if $F(x) \lesssim G(x)$ and $G(x) \lesssim F(x)$ for $x \in X$. We often omit to mention the set X if it is obviously recognized from the context. We use the letter c to denote positive constants, which may not be the same at different places. We write $c(\alpha, \beta, \dots)$, for example, to denote a positive constant which depends only on $\alpha, \beta, \dots$. If $P(x, y, \dots)$ is a proposition containing variables $x, y, \dots$, then we define $\mathbf{1}\{P(x, y, \dots)\}$ to be equal to 1 if the proposition $P(x, y, \dots)$ is true and to 0 if $P(x, y, \dots)$ is false. We write $\mathbf{1}_E$ to denote the defining function of a set E; thus $\mathbf{1}_E(x) = \mathbf{1}\{x \in E\}$.

Now the T and λ in $H^p(\Omega, T, \lambda)$ are the ones in the following definitions.

Definition 1.1 If $\Omega \neq \mathbb{R}^n$, then $\mathcal{T}(\Omega)$ denotes the set of all functions T on Ω that satisfy the following two conditions:
(i) $0 < T(x) \leqq \mathrm{dis}(x, \Omega^c)$ for all $x \in \Omega$;
(ii) $|T(x) - T(y)| \leqq |x - y|$ for all $x, y \in \Omega$.
If $\Omega = \mathbb{R}^n$, then $\mathcal{T}(\Omega) = \mathcal{T}(\mathbb{R}^n)$ denotes the set of all positive real-valued functions T that satisfy the condition (ii).

Notice that, if $T \in \mathcal{T}(\Omega)$ and $\alpha \in (0, 1]$, then $\alpha T \in \mathcal{T}(\Omega)$.

For $T \in \mathcal{T}(\Omega)$, $x \in \Omega$, and $\delta \in (0, \infty)$, we write

$$U^T(x, \delta) = B(x, \delta T(x)).$$

For $T \in \mathcal{T}(\Omega)$, we write $\mathcal{B}(\Omega, T)$ to denote the set of all balls $B(x, r)$ with $x \in \Omega$ and $0 < r < T(x)$.

Definition 1.2 Let $T \in \mathcal{T}(\Omega)$ and $\sigma \in (0,\infty)$. Then $\mathrm{Double}^*(\Omega, T, \sigma)$ denotes the set of all Borel measures λ on Ω for which there exist $\alpha \in (0,1)$ and $A \in [1,\infty)$ such that

$$0 < \lambda(B) \leqq At^\sigma \lambda(t^{-1}B) < \infty \tag{1.1}$$

for all balls $B \in \mathcal{B}(\Omega, \alpha T)$ and for all $t \in [1,\infty)$.

Some basic facts about the measures in $\mathrm{Double}^*(\Omega, T, \sigma)$ and about the related Hardy-Littlewood maximal operator will be given in Section 2, where, in particular, we shall show that, for $\Omega \neq \emptyset$, the class $\mathrm{Double}^*(\Omega, T, \sigma)$ is nonempty only if $\sigma \geqq n$ (see Lemma 2.2 (1)).

To define the space $H^p(\Omega, T, \lambda)$, we use the maximal functions for distributions that are defined as follows.

Definition 1.3 Let $T \in \mathcal{T}(\Omega)$ and $s \in (0,\infty)$.
(1) For C^∞ functions ψ on $\mathbb{R}^n$, we define

$$\|\psi\|_{\Lambda(s)} = \sup_B \left[\inf\left\{ |B|^{-1-s/n} \int_B |\psi(x) - P(x)|dx \,\middle|\, P \in \mathcal{P}_{[s]} \right\} \right],$$

where the sup is taken over all balls B of $\mathbb{R}^n$.
(2) For a ball B of $\mathbb{R}^n$, we define $\mathcal{A}_s(B)$ to be the set of all $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset B$ and $\|\psi\|_{\Lambda(s)} \leqq t^{-n-s}$, where t is the radius of B.
(3) For $\alpha \in (0,1]$ and $x \in \Omega$, we define $\mathcal{A}_s^{\alpha T}(x)$ to be the union of $\mathcal{A}_s(B)$ over all balls B satisfying $x \in B \in \mathcal{B}(\Omega, \alpha T)$.
(4) For $f \in \mathcal{D}'(\Omega)$ and $\alpha \in (0,1]$, we define the maximal function $f_s^{*,\alpha T}(x)$ $(x \in \Omega)$ by

$$f_s^{*,\alpha T}(x) = \sup\left\{ |\langle f, \psi\rangle| \mid \psi \in \mathcal{A}_s^{\alpha T}(x) \right\}.$$

The space $H^p(\Omega, T, \lambda)$ is defined as follows.

Definition 1.4 Let $p \in (0,\infty)$, $\sigma \in [n,\infty)$, and $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$. Take a positive real number α with $\alpha < 1/3$ and take a positive real number s satisfying $n + s > \max\{\sigma/p, \sigma\}$. Then we define, for $f \in \mathcal{D}'(\Omega)$,

$$\|f\|_{H^p(\Omega,T,\lambda)} = \|f_s^{*,\alpha T}\|_{L^p(\Omega,\lambda)}$$

and define $H^p(\Omega, T, \lambda)$ as the set of all $f \in \mathcal{D}'(\Omega)$ such that $\|f\|_{H^p(\Omega,T,\lambda)} < \infty$.

In Section 3, we shall show that the equivalence class of the quasinorm $\|\cdot\|_{H^p(\Omega,T,\lambda)}$ and the space $H^p(\Omega, T, \lambda)$ do not depend on the choice of α and s (see Theorem 3.2). In the same section, we shall also give other maximal functions which characterize $H^p(\Omega, T, \lambda)$ (see Theorems 3.4 and 3.7).

Remark 1.5 If $\Omega = \mathbb{R}^n$, $T(x) = 1$, and $\lambda =$ the Lebesgue measure, then $H^p(\Omega, T, \lambda)$ coincides with $h^p(\mathbb{R}^n)$ of [G]. If $\Omega = \mathbb{R}^n$, $T(x) = 1$, and $d\lambda(x) = w(x)dx$, then $H^p(\Omega, T, \lambda)$ is the local version of $H^p_w(\mathbb{R}^n)$ of [ST]. If $\Omega \neq \mathbb{R}^n$, $T(x) = \mathrm{dis}(x, \Omega^c)$, and $\lambda =$ the Lebesgue measure, then $H^p(\Omega, T, \lambda)$ coincides with $H^p(\Omega)$ of [M1]. If $\Omega \neq \mathbb{R}^n$ and $T(x) = \mathrm{dis}(x, \Omega^c)$, then $H^p(\Omega, T, \lambda)$ coincides with $H^p(\Omega, \lambda)$ of [M2] and [M3].

In order to state the result on the change of variables for $H^p(\Omega, T, \lambda)$, we use the following notation. Let $\Phi : \Omega \to \tilde{\Omega}$ be a C^∞ diffeomorphism. We write J_Φ and $J_{\Phi^{-1}}$ to denote the Jacobian determinant of Φ and Φ^{-1}, respectively. For $f \in \mathcal{D}'(\Omega)$, the distribution $f \circ \Phi^{-1} \in \mathcal{D}'(\tilde{\Omega})$ is defined by

$$\langle f \circ \Phi^{-1}, \varphi \rangle = \langle f, (\varphi \circ \Phi)|J_\Phi| \rangle \quad (\varphi \in C_0^\infty(\tilde{\Omega})).$$

If λ is a Borel measure on Ω, then $\Phi_* \lambda$ is defined to be the Borel measure on $\tilde{\Omega}$ that satisfy

$$\int_{\tilde{\Omega}} g(y) d(\Phi_* \lambda)(y) = \int_\Omega g(\Phi(x)) d\lambda(x)$$

for all nonnegative Borel functions g on $\tilde{\Omega}$.

Here is a remark. If λ is a Borel measure on Ω and if λ takes finite values for compact subsets of Ω, then λ could be considered as a distribution on Ω by

$$\langle \lambda, \varphi \rangle = \int_\Omega \varphi(x) d\lambda(x) \quad (\varphi \in C_0^\infty(\Omega)).$$

In this case, although $\lambda \circ \Phi^{-1}$ is also a Borel measure on $\tilde{\Omega}$, it is not equal to the Borel measure $\Phi_* \lambda$. The true relation is this:

$$d(\Phi_* \lambda)(y) = |J_{\Phi^{-1}}(y)| d(\lambda \circ \Phi^{-1})(y).$$

If the distribution f is a locally integrable function on Ω, then the distribution $f \circ \Phi^{-1}$ coincides with the composite function $f \circ \Phi^{-1}$, which is a locally integrable function on $\tilde{\Omega}$. In this case, their $L^p(\Omega, \lambda)$-quasinorms are related by the equality

$$\|f\|_{L^p(\Omega,\lambda)} = \|f \circ \Phi^{-1}\|_{L^p(\tilde{\Omega},\Phi_*\lambda)}.$$

The following theorem, which is the first main theorem of this paper, claims that the $H^p(\Omega, T, \lambda)$-quasinorms satisfy the similar relation if Φ satisfies certain conditions.

Theorem 1.6 *Let $T \in \mathcal{T}(\Omega)$, $\tilde{T} \in \mathcal{T}(\tilde{\Omega})$, and let $\Phi : \Omega \to \tilde{\Omega}$ be a C^∞ diffeomorphism. Assume the following*:
(a) *There exists a constant $G \in (0, \infty)$ such that*

$$|J_\Phi(x)| \geqq G^{-1}\tilde{T}(\Phi(x))^n T(x)^{-n}$$

for all $x \in \Omega$;
(b) *For each multi-index $\alpha \neq 0$, there exists $C_\alpha \in (0, \infty)$ such that*

$$|\partial_x^\alpha \Phi(x)| \leqq C_\alpha \tilde{T}(\Phi(x))T(x)^{-|\alpha|}$$

for all $x \in \Omega$.
Then the following hold for each $\sigma \in [n, \infty)$.
(1) *For Borel measures λ on Ω and the corresponding Borel measures $\Phi_*\lambda$ on $\tilde{\Omega}$, we have $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$ if and only if $\Phi_*\lambda \in \mathrm{Double}^*(\tilde{\Omega}, \tilde{T}, \sigma)$.*
(2) *If $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$ and $\Phi_*\lambda \in \mathrm{Double}^*(\tilde{\Omega}, \tilde{T}, \sigma)$ and if $p \in (0, \infty)$, then $\|f\|_{H^p(\Omega,T,\lambda)} \approx \|f \circ \Phi^{-1}\|_{H^p(\tilde{\Omega},\tilde{T},\Phi_*\lambda)}$ for all $f \in \mathcal{D}'(\Omega)$.*

This theorem will be proved in Section 4. In the same section, we also give some properties of the mappings Φ that satisfy the conditions of the theorem. Some examples of the mappings Φ will be given in Section 7, where we shall also prove that every conformal mapping of the 2-dimensional domain satisfies the conditions of Theorem 1.6 (see Proposition 7.7).

The next theorem concerns with the multiplication of the distributions of $H^p(\Omega, T, \lambda)$ by a smooth function.

Suppose w is a positive real-valued C^∞ function on Ω. Then we can multiply every distribution f on Ω by w and define wf as a distribution on Ω. If f is a locally integrable function, then wf is also a locally integrable function and the L^p-quasinorms of f and wf satisfy the relation

$$\|wf\|_{L^p(\Omega,\lambda)} = \|f\|_{L^p(\Omega,w^p\lambda)},$$

where, for a Borel measure λ on Ω, the Borel measure $w^p\lambda$ is defined in such a way that

$$\int_\Omega g(x)d(w^p\lambda)(x) = \int_\Omega g(x)w(x)^p d\lambda(x)$$

for all nonnegative Borel functions g on Ω. Our next theorem will claim that similar relation holds for the $H^p(\Omega,T,\lambda)$-quasinorms if w satisfies certain conditions. The conditions on w reads as follows.

Definition 1.7 Let $T \in \mathcal{T}(\Omega)$. A function w is said to be of class $W(\Omega,T)$ if it is a positive real-valued C^∞ function on Ω and if for each multi-index α there exists a constant $A_\alpha \in (0,\infty)$ such that

$$|\partial_x^\alpha w(x)| \leqq A_\alpha w(x) T(x)^{-|\alpha|} \tag{1.2}$$

for all $x \in \Omega$.

The following is the second main theorem of this paper.

Theorem 1.8 *Let $T \in \mathcal{T}(\Omega)$, $w \in W(\Omega,T)$, $\sigma \in [n,\infty)$, and $p \in (0,\infty)$. Then the following hold.*
(1) *A Borel measure λ on Ω is of the class* $\mathrm{Double}^*(\Omega,T,\sigma)$ *if and only if $w^p\lambda$ is of the class* $\mathrm{Double}^*(\Omega,T,\sigma)$.
(2) *If $\lambda \in$* $\mathrm{Double}^*(\Omega,T,\sigma)$, *then $\|wf\|_{H^p(\Omega,T,\lambda)} \approx \|f\|_{H^p(\Omega,T,w^p\lambda)}$ for all $f \in \mathcal{D}'(\Omega)$.*

This theorem will be proved in Section 5, where some properties of the functions in the class $W(\Omega,T)$ will also be given.

The third main theorem of this paper concerns with the atomic decomposition. One of the fundamental result in the theory of Hardy spaces is that they can be characterized in terms of atomic decomposition. In most cases, atoms h are required to satisfy the moment condition such as

$$\int h(x)P(x)dx = 0 \quad \text{for all } P \in \mathcal{P}_{m-1}.$$

We shall give an atomic decomposition theorem for $H^p(\Omega, T, \lambda)$ in terms of atoms h that satisfy the moment condition of the form

$$\int h(x)P(\Phi(x))w(x)dx = 0 \quad \text{for all } P \in \mathcal{P}_{m-1}, \tag{1.3}$$

where Φ is a mapping satisfying the conditions of Theorem 1.6 and w is a function of class $W(\Omega, T)$. The precise statement will be given in Theorem 6.1 in Section 6.

2. Doubling measures and maximal functions

In this section, we give some basic facts about the measures of the class $\mathrm{Double}^*(\Omega, T, \sigma)$ and the Hardy–Littlewood maximal operator related to the measure $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$. The results will be used in the next section to prove some fundamental facts about the spaces $H^p(\Omega, T, \lambda)$.

We shall first prove some properties of the measures in the class $\mathrm{Double}^*(\Omega, T, \sigma)$. We shall introduce the following notation.

Definition 2.1 Let $T \in \mathcal{T}(\Omega)$, $\sigma \in (0, \infty)$, and $A \in [1, \infty)$. If λ is a Borel measure on Ω and if the inequalities (1.1) hold for all balls $B \in \mathcal{B}(\Omega, T)$ and for all $t \in [1, \infty)$, then we write $\lambda \in \mathrm{Double}(\Omega, T, \sigma, A)$.

Using this notation, Definition 1.2 reads as follows: A Borel measure λ on Ω belongs to the class $\mathrm{Double}^*(\Omega, T, \sigma)$ if it belongs to the class $\mathrm{Double}(\Omega, \alpha T, \sigma, A)$ for some $\alpha \in (0, 1)$ and some $A \in [1, \infty)$.

We have the following.

Lemma 2.2 *Let $T \in \mathcal{T}(\Omega)$ and $\sigma \in (0, \infty)$.*
(1) *If $\Omega \neq \emptyset$ and $\mathrm{Double}^*(\Omega, T, \sigma) \neq \emptyset$, then $\sigma \geqq n$.*
(2) *If $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$, then, for each $\alpha \in (0, 1)$, there exists an $\tilde{A} \in [1, \infty)$ such that $\lambda \in \mathrm{Double}(\Omega, \alpha T, \sigma, \tilde{A})$.*

Proof. (1) Suppose $\Omega \neq \emptyset$ and $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$. Take $\alpha \in (0, 1)$ and $A \in [1, \infty)$ such that $\lambda \in \mathrm{Double}(\Omega, \alpha T, \sigma, A)$. Take a ball $B(x_0, t_0) \in \mathcal{B}(\Omega, 3^{-1}\alpha T)$. For each $\epsilon \in (0, 1/2)$, we take disjoint balls $B_j = B(x_j, \epsilon t_0)$ $(j = 1, \ldots, N)$ included in B_0. It is possible to take $N \approx \epsilon^{-n}$. We have $B_0 \subset B_j(x_j, 2t_0)$ and $B(x_j, 2t_0) \in \mathcal{B}(\Omega, \alpha T)$ since $\alpha T(x_j) \geqq \alpha T(x_0) - |x_j - x_0| > 2t_0$. Hence the doubling condition $\lambda \in \mathrm{Double}(\Omega, \alpha T, \sigma, A)$ implies that

$$\lambda(B_0) \leqq \lambda(B(x_i, 2t_0)) \leqq A(2/\epsilon)^\sigma \lambda(B_j)$$

and thus

$$N\lambda(B_0) \leqq \sum_{j=1}^{N} A(2/\epsilon)^\sigma \lambda(B_j) \leqq A(2/\epsilon)^\sigma \lambda(B_0). \tag{2.1}$$

Since $\lambda(B_0) \in (0, \infty)$ and since $N \approx \epsilon^{-n}$, (2.1) is possible only when $\sigma \geqq n$.

(2) Suppose $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$. Then there exist $\alpha_0 \in (0,1)$ and $A \in [1, \infty)$ such that $\lambda \in \mathrm{Double}(\Omega, \alpha_0 T, \sigma, A)$. Let $\alpha \in (0,1)$. We shall prove that $\lambda \in \mathrm{Double}(\Omega, \alpha T, \sigma, \tilde{A})$ for some $\tilde{A} \in [1, \infty)$. It is sufficient to consider only the case $\alpha > \alpha_0$. Take $\epsilon \in (0,1)$ so small that we have $\alpha^{-1} - 1 + \epsilon > 6\epsilon/\alpha_0$. Suppose $B_0 = B(x_0, r_0) \in \mathcal{B}(\Omega, \alpha T)$.

For each $x \in (1-\epsilon)B_0$, we set $Q_x = B(x, 2\epsilon r_0)$. Then, for $x \in (1-\epsilon)B_0$, we have $4^{-1}Q_x \subset (1 - 2^{-1}\epsilon)B_0$ and $3Q_x \in \mathcal{B}(\Omega, \alpha_0 T)$, the latter of which comes from

$$T(x) \geqq T(x_0) - |x - x_0| > \alpha^{-1} r_0 - (1-\epsilon) r_0 > 6\epsilon r_0/\alpha_0.$$

By elementary geometry for Euclidean metric, we see that the balls Q_x with $x \in (1-\epsilon)B_0$ cover $\overline{B}_0$. Then we can select a finite number of balls Q_{x_j} with $x_j \in (1-\epsilon)B_0$ ($j = 1, 2, \dots, m$) such that the balls Q_{x_j} are disjoint and the balls $3Q_{x_j}$ cover B_0. Since $3Q_{x_j} \in \mathcal{B}(\Omega, \alpha_0 T)$, the assumed doubling condition $\lambda \in \mathrm{Double}(\Omega, \alpha_0 T, \sigma, A)$ yields

$$\lambda(3Q_{x_j}) \leqq 12^\sigma A \lambda(4^{-1}Q_{x_j}).$$

Hence

$$\begin{aligned} \lambda(B_0) &\leqq \sum_{j=1}^{m} \lambda(3Q_{x_j}) \leqq 12^\sigma A \sum_{j=1}^{m} \lambda(4^{-1}Q_{x_j}) \\ &= 12^\sigma A \lambda\Bigl(\bigcup_{j=1}^{m} 4^{-1}Q_{x_j}\Bigr) \leqq 12^\sigma A \lambda((1 - 2^{-1}\epsilon)B_0). \end{aligned} \tag{2.2}$$

We write $A_1 = 12^\sigma A$. Then repeated application of (2.2) gives

$$\lambda(B_0) \leqq A_1^k \lambda((1 - 2^{-1}\epsilon)^k B_0) \tag{2.3}$$

for all positive integers k.

We take a positive integer N such that $(1-2^{-1}\epsilon)^N \leqq \alpha_0/\alpha$. If $1 \geqq t^{-1} \geqq \alpha_0/\alpha$, then (2.3) with $k=N$ yields

$$\lambda(B_0) \leqq A_1^N \lambda((1-2^{-1}\epsilon)^N B_0) \leqq A_1^N \lambda(t^{-1}B_0). \tag{2.4}$$

If $\alpha_0/\alpha > t^{-1} > 0$, then, since $(\alpha_0/\alpha)B_0 \in \mathcal{B}(\Omega, \alpha_0 T)$, the inequality (2.3) with $k=N$ and the assumed doubling condition $\lambda \in \mathrm{Double}(\Omega, \alpha_0 T, \sigma, A)$ yield

$$\begin{aligned}\lambda(B_0) &\leqq A_1^N \lambda((1-2^{-1}\epsilon)^N B_0) \leqq A_1^N \lambda((\alpha_0/\alpha)B_0)\\ &\leqq A_1^N A(\alpha_0 t/\alpha)^\sigma \lambda(t^{-1}B_0).\end{aligned} \tag{2.5}$$

From (2.4) and (2.5), we see that $\lambda \in \mathrm{Double}(\Omega, \alpha T, \sigma, \tilde{A})$ with $\tilde{A} = \max\{A_1^N, A_1^N A(\alpha_0/\alpha)^\sigma\}$. Lemma 2.2 is proved.

If $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$, then Lemma 2.2 (2) asserts that, for each $\alpha \in (0,1)$, there exists an $A_\alpha \in [1,\infty)$ such that $\lambda \in \mathrm{Double}(\Omega, \alpha T, \sigma, A_\alpha)$. We call any such function $(0,1) \ni \alpha \mapsto A_\alpha \in [1,\infty)$ the *doubling constant* of λ in $\mathrm{Double}^*(\Omega, T, \sigma)$.

We shall use the following variant of the Hardy–Littlewood maximal operator.

Definition 2.3 Let $T \in \mathcal{T}(\Omega)$, $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$ for some $\sigma \in [n, \infty)$, and let $\alpha \in (0,1)$. For nonnegative Borel functions f on Ω, we define the function $M_\lambda^{\alpha T}(f)(x)$ $(x \in \Omega)$ by

$$M_\lambda^{\alpha T}(f)(x) = \sup\left\{ \frac{1}{\lambda(B)} \int_B f(y)d\lambda(y) \;\middle|\; x \in B \in \mathcal{B}(\Omega, \alpha T) \right\}.$$

For $r \in (0,\infty)$, we define $M_{\lambda,r}^{\alpha T}(f) = [M_\lambda^{\alpha T}(f^r)(x)]^{1/r}$.

The following lemma gives the fundamental estimates for $M_\lambda^{\alpha T}(f)$.

Lemma 2.4 *Let $T \in \mathcal{T}(\Omega)$, $\sigma \in [n,\infty)$, and $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$. Let f and f_j denote nonnegative Borel functions on Ω.*
(1) *If $0 < \alpha < 1/3$ and $1 < p \leqq \infty$, then*

$$\left\| M_\lambda^{\alpha T}(f) \right\|_{L^p(\Omega,\lambda)} \leqq c\|f\|_{L^p(\Omega,\lambda)},$$

where c depends only on α, p, σ, and on the doubling constant of λ.
(2) *If $0 < \alpha < 1/9$ and $1 < p, q < \infty$, then*

$$\Big\| \Big[\sum_{j=1}^{\infty} \{ M_{\lambda}^{\alpha T}(f_j) \}^q \Big]^{1/q} \Big\|_{L^p(\Omega,\lambda)} \leqq c \Big\| \Big[\sum_{j=1}^{\infty} f_j^q \Big]^{1/q} \Big\|_{L^p(\Omega,\lambda)},$$

where c depends only on α, p, q, σ, and on the doubling constant of λ.

This lemma can be proved by slightly modifying the argument for the classical case where $\Omega = \mathbb{R}^n$ and T is replaced with ∞. We shall omit the proof. As for the argument for the classical case, see, e.g., [S, Chapter I, Section 3] and [ST, Chapter I, Theorem 3]. As for the assumptions on the parameter α, cf. the comments in [M3, Lemmas 2.1 and 2.2].

As an application of Lemma 2.4 (2), we can prove the following.

Lemma 2.5 *Let $T \in \mathcal{T}(\Omega)$, $\sigma \in [n, \infty)$, and $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$. Let $\alpha \in (0, 1/9)$, $p, q \in (0, \infty)$, and let μ be a real number satisfying $\mu > \max\{\sigma/p, \sigma/q\}$. Then there exists a constant c such that the following inequalities hold for all $B_j = B(x_j, r_j) \in \mathcal{B}(\Omega, \alpha T)$, all $a_j \in [0, \infty]$, and all $\epsilon \in (0, 1]$:*

$$\begin{aligned} &\Big\| \Big[\sum_{j=1}^{\infty} a_j^q \big(1 + r_j^{-1} | \cdot - x_j| \big)^{-\mu q} \mathbf{1}\{ | \cdot - x_j | < \alpha T(x_j) \} \Big]^{1/q} \Big\|_{L^p(\Omega,\lambda)} \\ &\qquad \leqq c \epsilon^{-\mu} \Big\| \Big[\sum_{j=1}^{\infty} a_j^q \mathbf{1}_{\epsilon B_j} \Big]^{1/q} \Big\|_{L^p(\Omega,\lambda)}, \end{aligned} \tag{2.6}$$

$$\Big\| \Big[\sum_{j=1}^{\infty} a_j^q \mathbf{1}_{B_j} \Big]^{1/q} \Big\|_{L^p(\Omega,\lambda)} \leqq c \epsilon^{-\mu} \Big\| \Big[\sum_{j=1}^{\infty} a_j^q \mathbf{1}_{\epsilon B_j} \Big]^{1/q} \Big\|_{L^p(\Omega,\lambda)}. \tag{2.7}$$

The constant c depends only on σ, α, p, q, μ, and the doubling constant of $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$.

In fact, (2.6) follows if we apply the inequality of Lemma 2.4 (2) to $f_j = a_j^{\sigma/\mu} \mathbf{1}_{\epsilon B_j}$ with $\mu p/\sigma$ and $\mu q/\sigma$ in place of p and q, and (2.7) follows from (2.6) since the function in the left hand side of (2.6) is pointwise larger than a constant multiple of the function in the left hand side of (2.7). Details are left to the reader (cf. the similar argument in [M3, Section 2]).

3. The space $H^p(\Omega, T, \lambda)$

In this section, we give some basic properties of the space $H^p(\Omega, T, \lambda)$. In particular, we shall prove that the equivalence class of the quasinorm $\|\cdot\|_{H^p(\Omega,T,\lambda)}$ and the space $H^p(\Omega, T, \lambda)$ defined in Definition 1.4 do not depend on the choice of α and s.

First, we give the atomic decomposition theorem.

Theorem 3.1 *Let $T \in \mathcal{T}(\Omega)$, $\sigma \in [n, \infty)$, $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$, and $p \in (0, \infty)$. Let s be a positive real number satisfying $n + s > \max\{\sigma/p, \sigma\}$ and let m be a positive integer satisfying $n + m > \max\{\sigma/p, \sigma\}$.*
(1) *Suppose $0 < \delta \leqq 1/20$ and suppose $B_i = U^T(x_i, \delta)$ with $x_i \in \Omega$. Suppose $g_i \in L^\infty$, $\operatorname{supp} g_i \subset \overline{B}_i$, and $\sum_i \|g_i\|_{L^\infty} \mathbf{1}_{B_i} \in L^p(\Omega, \lambda)$. Then the series $\sum_i g_i$ converges unconditionally in $\mathcal{D}'(\Omega)$ and we have*

$$\left\| \sum_i (g_i)_s^{*,T/40} \right\|_{L^p(\Omega,\lambda)} \leqq c \left\| \sum_i \|g_i\|_{L^\infty} \mathbf{1}_{B_i} \right\|_{L^p(\Omega,\lambda)},$$

where the constant c depends only on n, σ, p, s, δ, and the doubling constant of λ.
(2) *Suppose $Q_j \in \mathcal{B}(\Omega, T/20)$, $h_j \in L^\infty$, $\operatorname{supp} h_j \subset \overline{Q}_j$,*

$$\int h_j(x) P(x) dx = 0 \quad \text{for all } P \in \mathcal{P}_{m-1}, \tag{3.1}$$

and $\sum_j \|h_j\|_{L^\infty} \mathbf{1}_{Q_j} \in L^p(\Omega, \lambda)$. Then the series $\sum_j h_j$ converges unconditionally in $\mathcal{D}'(\Omega)$ and

$$\left\| \sum_j (h_j)_s^{*,T/40} \right\|_{L^p(\Omega,\lambda)} \leqq c \left\| \sum_j \|h_j\|_{L^\infty} \mathbf{1}_{Q_j} \right\|_{L^p(\Omega,\lambda)},$$

where the constant c depends only on n, σ, p, s, m, and the doubling constant of λ.
(3) *Suppose $f \in \mathcal{D}'(\Omega)$, $\alpha \in (0, 1)$, $f_s^{*,\alpha T} \in L^p(\Omega, \lambda)$, and $0 < \delta \leqq \alpha/4$. We write $v = \alpha/3\delta$. Then there exist g_i, B_i, h_j, and Q_j such that the following* (i)–(v) *hold:*

(i) *$f = \sum_i g_i + \sum_j h_j$ with the two series converging unconditionally in $\mathcal{D}'(\Omega)$;*

(ii) $g_i \in L^\infty$ *and* $\operatorname{supp} g_i \subset B_i = U^T(x_i, \delta)$;
(iii) $h_j \in L^\infty$, $\operatorname{supp} h_j \subset Q_j \in \mathcal{B}(\Omega, 3\delta T)$, *and* h_j *satisfies* (3.1);
(iv) *For each* $r \in (0, \infty)$, *we have*

$$\left(\sum_i \|g_i\|_{L^\infty}^r \mathbf{1}_{vB_i}(x) + \sum_j \|h_j\|_{L^\infty}^r \mathbf{1}_{vQ_j}(x) \right)^{1/r} \leqq c_r f_s^{*,\alpha T}(x)$$

for all $x \in \Omega$, *where* c_r *is a constant depending only on* n, s, m, v, *and* r;
(v) *For each* B_i, *there exists a ball* $B = U^T(y, \delta)$ *such that* $B_i \subset 2B$ *and* $B \cap \operatorname{supp} f \neq \emptyset$; *the same holds for the balls* Q_j.

For the case $T = d_\Omega$, $\Omega \neq \mathbb{R}^n$, an essentially same theorem as Theorem 3.1 is given in [M3, Theorem 1.2 and 1.3]. Although the exact statement of Theorem 3.1 for this special case is slightly different from that of [M3, loc. cit.] and although the assertion (v) is only implicitly given in the latter, the proof given in [M3, Section 4] can be easily modified and generalized to the case $T \in \mathcal{T}(\Omega)$. We omit the details of the proof of Theorem 3.1.

Here we give a remark concerning the assertion (3) of Theorem 3.1. In the proof of (3) as given in [M3, Section 4], with necessary modification, g_i, B_i, h_j, and Q_j are explicitly constructed from f, Ω, and αT with the aid of the function $f_s^{*,\alpha T}$, and the doubling condition on λ and the assumption $f_s^{*,\alpha T} \in L^p(\Omega, \lambda)$ are used only to prove the convergence assertion of (i). Hence the constant c_r of Theorem 3.1 (3)(iv) does not depend on λ and p. (Similar argument for the space $H^p(\mathbb{R}^n)$ can also be found in [U, Section IV].)

Using Theorem 3.1, we shall prove the following theorem.

Theorem 3.2 *Let* $T \in \mathcal{T}(\Omega)$, $\sigma \in [n, \infty)$, $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$, *and* $p \in (0, \infty)$. *Suppose* α_i *and* s_i $(i = 1, 2)$ *are positive real numbers such that* $\alpha_i < 1/3$ *and* $n + s_i > \max\{\sigma/p, \sigma\}$. *Then* $\|f_{s_1}^{*,\alpha_1 T}\|_{L^p(\Omega,\lambda)} \approx \|f_{s_2}^{*,\alpha_2 T}\|_{L^p(\Omega,\lambda)}$ *for all* $f \in \mathcal{D}'(\Omega)$. *The constants contained in this* $\approx$ *can be taken depending only on* n, σ, p, α_1, α_2, s_1, s_2, *and the doubling constant of* λ.

Proof. It is enough to show the inequality

$$\left\| f_{s_1}^{*,\alpha_1 T} \right\|_{L^p(\Omega,\lambda)} \leqq c \left\| f_{s_2}^{*,\alpha_2 T} \right\|_{L^p(\Omega,\lambda)}. \tag{3.2}$$

We shall prove the following two estimates:

$$\left\|f_{s_1}^{*,\alpha_1 T}\right\|_{L^p(\Omega,\lambda)} \leqq c\left\|f_{s_1}^{*,T/40}\right\|_{L^p(\Omega,\lambda)}, \tag{3.3}$$

$$\left\|f_{s_1}^{*,T/40}\right\|_{L^p(\Omega,\lambda)} \leqq c\left\|f_{s_2}^{*,\alpha_2 T}\right\|_{L^p(\Omega,\lambda)}, \tag{3.4}$$

which together will imply (3.2).

First we prove (3.3). To prove this, we take positive real numbers r and ϵ such that

$$r < p, \quad (1+\epsilon)\alpha_1 < 1/3, \quad \alpha_1^{-1} - 1 > 40\epsilon$$

and prove the following pointwise inequality

$$f_{s_1}^{*,\alpha_1 T}(x) \leqq cM_{\lambda,r}^{(1+\epsilon)\alpha_1 T}\big(f_{s_1}^{*,T/40}\big)(x). \tag{3.5}$$

Once this inequality is proved, (3.3) will readily follow by the use of Lemma 2.4 (1). To prove (3.5), suppose $B = B(y,t) \in \mathcal{B}(\Omega, \alpha_1 T)$ and $\psi \in \mathcal{A}_{s_1}(B)$. By using appropriate partition of unity, we can write ψ as $\psi = \sum_{j=1}^N \psi_j$ in such a way that $c^{-1}\psi_j \in \mathcal{A}_{s_1}(B_j)$, $B_j = B(y_j, \epsilon t)$, $y_j \in B$, $N = c(n,\epsilon)$, and $c = c(n, s_1, \epsilon)$. Since

$$T(y_j) \geqq T(y) - |y - y_j| > (\alpha_1^{-1} - 1)t > 40\epsilon t,$$

we have $B_j \in \mathcal{B}(\Omega, T/40)$. Hence

$$|\langle f, \psi_j\rangle| \leqq c \inf_{B_j} f_{s_1}^{*,T/40} \leqq c\left[\frac{1}{\lambda(B_j)}\int_{B_j}\big(f_{s_1}^{*,T/40}\big)^r d\lambda\right]^{1/r}. \tag{3.6}$$

Notice that $B_j \subset (1+\epsilon)B \subset B(y_j, (2+\epsilon)t)$. Since $T(y_j) > (\alpha_1^{-1} - 1)t$ and $\alpha_1^{-1} - 1 > 2 + \epsilon$, we can use the doubling condition on λ to obtain

$$\lambda((1+\epsilon)B) \leqq \lambda(B(y_j, (2+\epsilon)t)) \leqq c\lambda(B_j).$$

Hence from (3.6) we obtain

$$|\langle f, \psi\rangle| \leqq \sum_{j=1}^N |\langle f, \psi_j\rangle| \leqq c\left[\frac{1}{\lambda((1+\epsilon)B)}\int_{(1+\epsilon)B}\big(f_{s_1}^{*,T/40}\big)^r d\lambda\right]^{1/r}.$$

Notice that $(1+\epsilon)B \in \mathcal{B}(\Omega, (1+\epsilon)\alpha_1 T)$. Hence, taking supremum over ψ and B, we obtain (3.5). Thus we proved (3.3).

Next we prove (3.4). We may assume $\|f_{s_2}^{*,\alpha_2 T}\|_{L^p(\Omega,\lambda)} < \infty$. Then using Theorem 3.1 (3) with $s = s_2$, $\alpha = \alpha_2$, and with sufficiently small δ, we obtain the decomposition of f as mentioned there. In particular we have

$$\sum_i \|g_i\|_{L^\infty} \mathbf{1}_{B_i}(x) + \sum_j \|h_j\|_{L^\infty} \mathbf{1}_{Q_j}(x) \leqq c f_{s_2}^{*,\alpha_2 T}(x). \tag{3.7}$$

Then using the obvious inequality

$$f_{s_1}^{*,T/40}(x) = \Big(\sum_i g_i + \sum_j h_j\Big)_{s_1}^{*,T/40}(x) \leqq \sum_i (g_i)_{s_1}^{*,T/40}(x) + \sum_j (h_j)_{s_1}^{*,T/40}(x),$$

using Theorem 3.1 (1) and (2) with $s = s_1$, and using (3.7), we obtain

$$\begin{aligned}
\big\|f_{s_1}^{*,T/40}\big\|_{L^p(\Omega,\lambda)} &\leqq \Big\| \sum_i (g_i)_{s_1}^{*,T/40} + \sum_j (h_j)_{s_1}^{*,T/40} \Big\|_{L^p(\Omega,\lambda)} \\
&\leqq c\Big\| \sum_i \|g_i\|_{L^\infty} \mathbf{1}_{B_i} \Big\|_{L^p(\Omega,\lambda)} + c\Big\| \sum_j \|h_j\|_{L^\infty} \mathbf{1}_{Q_j} \Big\|_{L^p(\Omega,\lambda)} \\
&\leqq c\big\|f_{s_2}^{*,\alpha_2 T}\big\|_{L^p(\Omega,\lambda)}.
\end{aligned}$$

Thus we obtain (3.4). Theorem 3.2 is proved.

For later purpose, we shall introduce another maximal function which characterizes the space $H^p(\Omega, T, \lambda)$.

Definition 3.3 Let $T \in \mathcal{T}(\Omega)$ and let k be a positive integer.
(1) For a ball B in $\mathbb{R}^n$, we define $\mathcal{A}_{(k)}(B)$ to be the set of all $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\psi \subset B$ and $|\partial_y^\nu \psi(ty)| \leqq t^{-n}$ for all multi-indices ν with $|\nu| \leqq k$ and for all $y \in \mathbb{R}^n$, where t denotes the radius of B.
(2) For $\alpha \in (0,1]$ and $x \in \Omega$, we define $\mathcal{A}_{(k)}^{\alpha T}(x)$ to be the union of $\mathcal{A}_{(k)}(B(x,t))$ over $t \in (0, \alpha T(x))$.
(3) For $f \in \mathcal{D}'(\Omega)$ and $\alpha \in (0,1]$, we define the maximal function $f_{(k)}^{*,\alpha T}(x)$ $(x \in \Omega)$ by

$$f_{(k)}^{*,\alpha T}(x) = \sup\big\{ |\langle f, \psi\rangle| \mid \psi \in \mathcal{A}_{(k)}^{\alpha T}(x) \big\}.$$

We have the following theorem.

Theorem 3.4 *Let $T \in \mathcal{T}(\Omega)$, $\sigma \in [n,\infty)$, $\lambda \in \mathrm{Double}^*(\Omega,T,\sigma)$, and $p \in (0,\infty)$. Let $\alpha \in (0,1/3)$ and let k be a positive integer satisfying $n+k > \max\{\sigma/p,\sigma\}$. Then $\|f\|_{H^p(\Omega,T,\lambda)} \approx \|f^{*,\alpha T}_{(k)}\|_{L^p(\Omega,\lambda)}$ for all $f \in \mathcal{D}'(\Omega)$.*

Proof. Take positive real numbers s_1 and s_2 such that $s_1 > k > s_2$ and $n + s_2 > \max\{\sigma/p,\sigma\}$. Then, for $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subset B(0,1)$, we have

$$\|\psi\|_{\Lambda(s_1)} \gtrsim \sum_{|\nu|\leqq k} \|\partial^\nu \psi\|_{L^\infty} \gtrsim \|\psi\|_{\Lambda(s_2)}.$$

By dilation and translation, we see that, if $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{supp}\psi \subset B(y,t)$, then

$$t^{s_1}\|\psi\|_{\Lambda(s_1)} \gtrsim \sum_{|\nu|\leqq k} \left\|\partial_x^\nu \psi(tx)\right\|_{L^\infty} \gtrsim t^{s_2}\|\psi\|_{\Lambda(s_2)}.$$

Observe also that, if $x \in B(y,t) \in \mathcal{B}(\Omega, 3^{-1}\alpha T)$, then $B(y,t) \subset B(x,2t) \in \mathcal{B}(\Omega,\alpha T)$. Using these facts, we see the pointwise inequality

$$f^{*,3^{-1}\alpha T}_{s_1}(x) \lesssim f^{*,\alpha T}_{(k)}(x) \lesssim f^{*,\alpha T}_{s_2}(x) \tag{3.8}$$

holds for all $f \in \mathcal{D}'(\Omega)$ and all $x \in \Omega$. Hence the desired result follows from Theorem 3.2. This completes the proof.

Finally we shall give a result concerning the 'radial' maximal functions. This result will not be used in this paper but will be of independent interest.

Definition 3.5 Let ϕ be a function in $C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\phi \subset B(0,1)$ and $\int \phi(x)dx \neq 0$. For $t \in (0,\infty)$, we write $(\phi)_t(x) = t^{-n}\phi(t^{-1}x)$. Let $T \in \mathcal{T}(\Omega)$ and $\alpha \in (0,1]$. For $f \in \mathcal{D}'(\Omega)$, we define the function $f^{+,(\phi),\alpha T}(x)$ $(x \in \Omega)$ by

$$f^{+,(\phi),\alpha T}(x) = \sup\{|\langle f, (\phi)_t(x-\cdot)\rangle| \mid 0 < t < \alpha T(x)\}.$$

Theorem 3.6 *Let $T \in \mathcal{T}(\Omega)$, $\sigma \in [n,\infty)$, $\lambda \in \mathrm{Double}^*(\Omega,T,\sigma)$. Let ϕ be a function as mentioned in Definition 3.5, $\alpha \in (0,1)$, and let s be a positive real number satisfying $n+s \geqq \sigma$. Then*

$$f_s^{*,3^{-1}\alpha T}(x) \leqq c M_{\lambda,\sigma/(n+s)}^{3^{-1}\alpha T}(f^{+,(\phi),\alpha T})(x)$$

for all $f \in \mathcal{D}'(\Omega)$ and all $x \in \Omega$, where c depends only on n, ϕ, α, s, σ, and the doubling constant of λ.

A proof of this theorem for the case $T(x) = d_\Omega(x)$, $\Omega \neq \mathbb{R}^n$, is given in [M3, Section 3, Lemma 3.3]. The proof can be modified to the general case without any difficulty.

Using Theorem 3.6 and the obvious converse inequality

$$f^{+,(\phi),\alpha T}(x) \leqq c_s f_s^{*,\alpha T}(x),$$

and also using Lemma 2.4 (1), we obtain the following.

Theorem 3.7 *Let $T \in \mathcal{T}(\Omega)$, $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$ for some $\sigma \in [n, \infty)$, and $p \in (0, \infty)$. Let ϕ be a function as mentioned in Definition 3.5 and let $\alpha \in (0, 1/3)$. Then $\|f\|_{H^p(\Omega,T,\lambda)} \approx \|f^{+,(\phi),\alpha T}\|_{L^p(\Omega,\lambda)}$ for all $f \in \mathcal{D}'(\Omega)$.*

4. Change of variables

In this section, we shall prove Theorem 1.6. We also prove some properties the mappings Φ that satisfy the conditions of this theorem.

Theorem 1.6 is a direct consequence of the following two propositions.

Proposition 4.1 *Let $T \in \mathcal{T}(\Omega)$, $\tilde{T} \in \mathcal{T}(\tilde{\Omega})$, and let $\Phi : \Omega \to \tilde{\Omega}$ be a C^∞ diffeomorphism. Assume there exist constants ϵ and K satisfying $0 < \epsilon < 1 \leqq K < \infty$ and $\epsilon K \leqq 1$ for which the following* (i)–(iv) *hold*:

(i) $\Phi(U^T(x,t)) \subset U^{\tilde{T}}(\Phi(x), Kt)$ *for all $x \in \Omega$ and all $t \in (0, \epsilon]$;*
(ii) $\Phi^{-1}(U^{\tilde{T}}(y,t)) \subset U^T(\Phi^{-1}(y), Kt)$ *for all $y \in \tilde{\Omega}$ and all $t \in (0, \epsilon]$;*
(iii) *For each multi-index $\alpha \neq 0$, there exists a constant $B_\alpha \in (0, \infty)$ such that*

$$\left|\partial_x^\alpha \Phi(x)\right| \leqq B_\alpha \tilde{T}(\Phi(x_0)) T(x_0)^{-|\alpha|}$$

whenever $x_0 \in \Omega$ and $x \in U^T(x_0, \epsilon)$;
(iv) *For each multi-index $\alpha \neq 0$, there exists a constant $B_\alpha \in (0, \infty)$ such that*

$$\left|\partial_y^\alpha \Phi^{-1}(y)\right| \leqq B_\alpha T(\Phi^{-1}(y_0)) \tilde{T}(y_0)^{-|\alpha|}$$

whenever $y_0 \in \tilde{\Omega}$ *and* $y \in U^{\tilde{T}}(y_0, \epsilon)$.

Then the conclusions of Theorem 1.6 hold.

Proposition 4.2 *The assumptions of Proposition* 4.1 *are satisfied if and only if the assumptions of Theorem* 1.6 *are satisfied.*

Proof of Proposition 4.1. (1) Suppose $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$. By Lemma 2.2 (2), there exists A such that $\lambda \in \mathrm{Double}(\Omega, 2^{-1}T, \sigma, A)$. We set $\delta = \min\{\epsilon, 1/2K\}$. Suppose $y \in \tilde{\Omega}$, $x = \Phi^{-1}(y)$, $t \in (0, \delta)$, and $s \in [1, \infty)$. Then the doubling condition $\lambda \in \mathrm{Double}(\Omega, 2^{-1}T, \sigma, A)$ implies that

$$\lambda(U^T(x, Kt)) \leqq A(K^2 s)^\sigma \lambda(U^T(x, K^{-1}s^{-1}t)).$$

By (i) and (ii), we have

$$\begin{aligned} \Phi^{-1}(U^{\tilde{T}}(y, t)) &\subset U^T(x, Kt), \\ \Phi(U^T(x, K^{-1}s^{-1}t)) &\subset U^{\tilde{T}}(y, s^{-1}t). \end{aligned}$$

Hence

$$\begin{aligned} (\Phi_*\lambda)(U^{\tilde{T}}(y, t)) &= \lambda(\Phi^{-1}(U^{\tilde{T}}(y, t))) \leqq \lambda(U^T(x, Kt)) \\ &\leqq A(K^2 s)^\sigma \lambda(U^T(x, K^{-1}s^{-1}t)) \\ &= A(K^2 s)^\sigma (\Phi_*\lambda)\big(\Phi(U^T(x, K^{-1}s^{-1}t))\big) \\ &\leqq A(K^2 s)^\sigma (\Phi_*\lambda)\big(U^{\tilde{T}}(y, s^{-1}t)\big). \end{aligned}$$

This implies that $\Phi_*\lambda \in \mathrm{Double}(\tilde{\Omega}, \delta\tilde{T}, \sigma, AK^{2\sigma})$ and thus $\Phi_*\lambda \in \mathrm{Double}^*(\tilde{\Omega}, \tilde{T}, \sigma)$. The converse also holds by the symmetry of the assumptions (i)–(iv).

(2) Let $x_0 \in \Omega$ and $y_0 = \Phi(x_0)$. Using (iii), we obtain the estimate

$$|\partial^\alpha J_\Phi(x)| \leqq c_\alpha \tilde{T}(y_0)^n T(x_0)^{-n-|\alpha|} \quad \text{for } x \in U^T(x_0, \epsilon). \tag{4.1}$$

Since $J_\Phi(x)$ is locally of constant sign, the same estimate holds also for $\partial^\alpha |J_\Phi(x)|$. Using this estimate and using (ii) and (iii), we can easily deduce the following: For each positive integer k, there exists a constant c_k

such that, if $t \in (0, \epsilon/K)$ and $\psi \in \mathcal{A}_{(k)}(U^{\tilde{T}}(y_0, t))$, then $c_k^{-1}(\psi \circ \Phi)|J_\Phi| \in \mathcal{A}_{(k)}(U^T(x_0, Kt))$. Hence taking the supremum of

$$|\langle f \circ \Phi^{-1}, \psi\rangle| = |\langle f, (\psi \circ \Phi)|J_\Phi|\rangle|$$

over $\psi \in \mathcal{A}_{(k)}(U^{\tilde{T}}(y_0, t))$ and $t \in (0, \epsilon/K)$, we obtain

$$(f \circ \Phi^{-1})^{*, \epsilon K^{-1}\tilde{T}}_{(k)}(y_0) \leqq c_k f^{*, \epsilon T}_{(k)}(x_0) = c_k f^{*, \epsilon T}_{(k)}(\Phi^{-1}(y_0)). \tag{4.2}$$

Assume $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$, $\Phi_* \lambda \in \mathrm{Double}^*(\tilde{\Omega}, \tilde{T}, \sigma)$, and $p \in (0, \infty)$. We take a positive integer k satisfying $n + k > \max\{\sigma/p, \sigma\}$. We may assume $\epsilon < 1/3$. Then, taking the $L^p(\tilde{\Omega}, \Phi_*\lambda)$-quasinorm of (4.2) and using Theorem 3.4, we obtain

$$\begin{aligned} \|f \circ \Phi^{-1}\|_{H^p(\tilde{\Omega}, \tilde{T}, \Phi_*\lambda)} &\approx \big\|(f \circ \Phi^{-1})^{*, \epsilon K^{-1}\tilde{T}}_{(k)}\big\|_{L^p(\tilde{\Omega}, \Phi_*\lambda)} \\ &\lesssim \big\|f^{*, \epsilon T}_{(k)} \circ \Phi^{-1}\big\|_{L^p(\tilde{\Omega}, \Phi_*\lambda)} = \big\|f^{*, \epsilon T}_{(k)}\big\|_{L^p(\Omega, \lambda)} \approx \|f\|_{H^p(\Omega, T, \lambda)}. \end{aligned}$$

The converse inequality $\|f\|_{H^p(\Omega, T, \lambda)} \lesssim \|f \circ \Phi^{-1}\|_{H^p(\tilde{\Omega}, \tilde{T}, \Phi_*\lambda)}$ also holds by the symmetry of the assumptions (i)–(iv). Proposition 4.1 is proved.

In order to prove Proposition 4.2, we use the following lemma, which gives a qualitative assertion for the implicit function theorem. We shall omit the proof of this lemma; a proof can be found, maybe implicit, in an undergraduate text on analysis. We shall write $\|\cdot\|$ to denote the operator norm for linear mappings $\mathbb{R}^n \to \mathbb{R}^n$.

Lemma 4.3 *Let $x_0 \in \mathbb{R}^n$, $t \in (0, \infty)$, $M \in (0, \infty)$, and let $F : B(x_0, t) \to \mathbb{R}^n$ be a C^1 mapping. Assume the Fréchet derivative $F'(x_0)$ is invertible and $\|F'(x_0)^{-1}\| \leqq M$. Also assume $\|F'(x_0)^{-1}F'(x) - I\| \leqq 1/2$ for all $x \in B(x_0, t)$. Then $F(B(x_0, t)) \supset B(F(x_0), t/2M)$.*

Proof of Proposition 4.2. First we assume (a) and (b) of Theorem 1.6 and deduce (i)–(iv) of Proposition 4.1.

By the condition (b), there exists a constant c_1 such that

$$\|\Phi'(x)\| \leqq c_1 \tilde{T}(\Phi(x)) T(x)^{-1} \tag{4.3}$$

for all $x \in \Omega$. We set $\eta = \min\{2^{-1}, (9c_1)^{-1}\}$. We shall first prove that

$$|\Phi(x) - \Phi(x_0)| \leqq 2^{-1}\tilde{T}(\Phi(x_0)) \quad \text{if } x_0 \in \Omega \text{ and } x \in U^T(x_0, \eta). \tag{4.4}$$

Suppose on the contrary there exist $x_0,\ x_1 \in \Omega$ such that $|x_1 - x_0| < \eta T(x_0)$ and $|\Phi(x_1) - \Phi(x_0)| > 2^{-1}\tilde{T}(\Phi(x_0))$. We write $x_t = (1-t)x_0 + tx_1$ ($t \in [0,1]$). Then by continuity there exists $s \in (0,1)$ such that

$$|\Phi(x_s) - \Phi(x_0)| = 2^{-1}\tilde{T}(\Phi(x_0)) \tag{4.5}$$

and

$$|\Phi(x_t) - \Phi(x_0)| < 2^{-1}\tilde{T}(\Phi(x_0)) \quad \text{for } t \in (0,s). \tag{4.6}$$

Using (4.3), we see that

$$\begin{aligned}\left|\frac{d}{dt}\Phi(x_t)\right| &= |\Phi'(x_t)(x_1 - x_0)| \\ &\leqq \|\Phi'(x_t)\||x_1 - x_0| \leqq c_1\tilde{T}(\Phi(x_t))T(x_t)^{-1}|x_1 - x_0|.\end{aligned}$$

For $t \in (0,s)$, we use the Lipschitz continuity of $\tilde{T}$ and T and use (4.6) to see that

$$\tilde{T}(\Phi(x_t)) \leqq \tilde{T}(\Phi(x_0)) + |\Phi(x_t) - \Phi(x_0)| < (3/2)\tilde{T}(\Phi(x_0))$$

and

$$T(x_t) \geqq T(x_0) - |x_t - x_0| > (1-\eta)T(x_0) \geqq 2^{-1}T(x_0)$$

and thus

$$\left|\frac{d}{dt}\Phi(x_t)\right| \leqq 3c_1\tilde{T}(\Phi(x_0))T(x_0)^{-1}|x_1 - x_0|.$$

Hence

$$\begin{aligned}|\Phi(x_s) - \Phi(x_0)| &\leqq \int_0^s \left|\frac{d}{dt}\Phi(x_t)\right| dt \\ &\leqq 3c_1\tilde{T}(\Phi(x_0))T(x_0)^{-1}|x_1 - x_0|s < 3c_1\tilde{T}(\Phi(x_0))\eta \\ &\leqq 3^{-1}\tilde{T}(\Phi(x_0)),\end{aligned}$$

which contradicts (4.5). Thus we proved (4.4).

Using (4.4), we can improve (b) as follows:
(c) If $x_0 \in \Omega$ and $x \in U^T(x_0, \eta)$, then $|\partial_x^\alpha \Phi(x)| \leqq 3 \cdot 2^{|\alpha|-1} C_\alpha \tilde{T}(\Phi(x_0)) T(x_0)^{-|\alpha|}$ for each $\alpha \neq 0$.
In fact, if $x_0 \in \Omega$ and $x \in U^T(x_0, \eta)$, then, by the Lipschitz continuity of $\tilde{T}$ and T and by (4.4), we have

$$\tilde{T}(\Phi(x)) \leqq \tilde{T}(\Phi(x_0)) + |\Phi(x) - \Phi(x_0)| \leqq (3/2)\tilde{T}(\Phi(x_0))$$

and

$$\tilde{T}(x) \geqq T(x_0) - |x - x_0| > (1/2)T(x_0),$$

and hence (c) follows from (b).

We shall now prove (i)–(iv). Although we shall not refer to the constants ϵ and K in the following argument, the existence of them will be apparent.

(i) If $0 < t \leqq \eta$ and $x \in U^T(x_0, t)$, then using (c) with $|\alpha| = 1$ we see that

$$|\Phi(x) - \Phi(x_0)| \leqq c\tilde{T}(\Phi(x_0))T(x_0)^{-1}|x - x_0| < c\tilde{T}(\Phi(x_0))t.$$

This implies $\Phi(U^T(x_0, t)) \subset U^{\tilde{T}}(\Phi(x_0), ct)$ for $t \in (0, \eta]$.

(ii) From (a) and (b) with $|\alpha| = 1$, we see that there exists a constant c_2 such that

$$\|\Phi'(x_0)^{-1}\| \leqq c_2 \tilde{T}(\Phi(x_0))^{-1} T(x_0). \tag{4.7}$$

On the other hand, using (c) with $|\alpha| = 2$, we see the following: If $t \in (0, \eta]$ and $x \in U^T(x_0, t)$, then

$$\|\Phi'(x) - \Phi'(x_0)\| \leqq c\tilde{T}(\Phi(x_0))T(x_0)^{-2}|x - x_0| < c\tilde{T}(\Phi(x_0))T(x_0)^{-1}t. \tag{4.8}$$

If $t \in (0, \eta]$ and $x \in U^T(x_0, t)$, then (4.7) and (4.8) imply $\|\Phi'(x_0)^{-1}\Phi'(x) - I\| \leqq ct$. Hence we can take an $\eta' \in (0, \eta]$ so small that we have

$$\|\Phi'(x_0)^{-1}\Phi'(x) - I\| \leqq 1/2 \quad \text{if } x \in U^T(x_0, \eta'). \tag{4.9}$$

By Lemma 4.3 and by (4.7) and (4.9), we see that if $0 < t \leqq \eta'$ then

$$\begin{aligned}\Phi(U^T(x_0,t)) &\supset B\big(\Phi(x_0), tT(x_0)/2c_2\tilde{T}(\Phi(x_0))^{-1}T(x_0)\big)\\ &= U^{\tilde{T}}(\Phi(x_0), t/2c_2). \end{aligned} \tag{4.10}$$

(iii) The condition (iii) immediately follows from (c).

(iv) Let c_2 and η' be the constants as mentioned above. We may assume $c_2 \geqq 1$. We shall prove the following: If $y_0 \in \tilde{\Omega}$ and $y \in U^{\tilde{T}}(y_0, \eta'/2c_2)$, then

$$\big|\partial_y^\alpha \Phi^{-1}(y)\big| \leqq c_\alpha T(\Phi^{-1}(y_0))\tilde{T}(y_0)^{-|\alpha|} \tag{4.11}$$

for each $\alpha \neq 0$. We shall prove this by induction on $|\alpha|$. For the rest of the argument we assume $y_0 \in \tilde{\Omega}$ and $y \in U^{\tilde{T}}(y_0, \eta'/2c_2)$ and we write $x_0 = \Phi^{-1}(y_0)$ and $x = \Phi^{-1}(y)$. Notice that, by (4.10),

$$x \in \Phi^{-1}\big(U^{\tilde{T}}(y_0, \eta'/2c_2)\big) \subset U^T(x_0, \eta') \subset U^T(x_0, \eta).$$

We first prove (4.11) for $|\alpha| = 1$. Since $y \in U^{\tilde{T}}(y_0, \eta'/2c_2) \subset U^{\tilde{T}}(y_0, 1/4)$ and since $x \in U^T(x_0, \eta') \subset U^T(x_0, 1/2)$, the Lipschitz continuity of $\tilde{T}$ and T yield

$$\begin{aligned}(3/4)\tilde{T}(y_0) < \tilde{T}(y_0) - |y - y_0| \leqq \tilde{T}(y) \leqq \tilde{T}(y_0) + |y - y_0| < (5/4)\tilde{T}(y_0),\\ (1/2)T(x_0) < T(x_0) - |x - x_0| \leqq T(x) \leqq T(x_0) + |x - x_0| < (3/2)T(x_0).\end{aligned}$$

Combining these inequalities with (a), we have

$$|J_\Phi(x)| \geqq G^{-1}\tilde{T}(y)^n T(x)^{-n} \geqq c\tilde{T}(y_0)^n T(x_0)^{-n}. \tag{4.12}$$

From (4.12) and (c) with $|\alpha| = 1$, we see that

$$\|(\Phi^{-1})'(y)\| = \|\Phi'(x)^{-1}\| \leqq cT(x_0)\tilde{T}(y_0)^{-1}. \tag{4.13}$$

This shows the case $|\alpha| = 1$ of (4.11).

Next suppose k is a positive integer and suppose (4.11) holds for $1 \leqq |\alpha| \leqq k$. We shall show (4.11) for $|\alpha| = k + 1$. Let β be a multi-index with $|\beta| = k$ and apply ∂_y^β to the equality $\Phi'(x)(\Phi^{-1})'(y) = I$. Then we have

$$\Phi'(x)\partial_y^\beta(\Phi^{-1})'(y) + \sum_{\substack{\beta'+\beta''=\beta,\\ \beta'\neq 0}} \binom{\beta}{\beta'} \big(\partial_y^{\beta'}\Phi'(x)\big)\big(\partial_y^{\beta''}(\Phi^{-1})'(y)\big) = 0. \quad (4.14)$$

For $|\beta'| \leqq |\beta| = k$, we use (c) and the induction hypothesis to see that

$$\big\|\partial_y^{\beta'}\Phi'(x)\big\| \leqq cT(x_0)^{-1}\tilde{T}(y_0)^{1-|\beta'|}.$$

For $|\beta''| < |\beta| = k$, the induction hypothesis implies that

$$\big\|\partial_y^{\beta''}(\Phi^{-1})'(y)\big\| \leqq cT(x_0)\tilde{T}(y_0)^{-1-|\beta''|}.$$

Combining these estimates with (4.14) and (4.13), we obtain

$$\begin{aligned}\big\|\partial_y^{\beta}(\Phi^{-1})'(y)\big\| &\leqq c\|\Phi'(x)^{-1}\| \sum_{\substack{\beta'+\beta''=\beta,\\ \beta'\neq 0}} \big\|\partial_y^{\beta'}\Phi'(x)\big\| \cdot \big\|\partial_y^{\beta''}(\Phi^{-1})'(y)\big\| \\ &\leqq cT(x_0)\tilde{T}(y_0)^{-1-|\beta|}.\end{aligned}$$

This shows (4.11) for $|\alpha| = k+1$. Thus we proved (iv).

Next we shall prove the converse assertion that (i)–(iv) of Proposition 4.1 imply (a) and (b) of Theorem 1.6. Assume (i)–(iv) of Proposition 4.1. Then (b) is obvious from (iii). By (iv) for $|\alpha| = 1$ we have

$$|J_{\Phi^{-1}}(y)| \leqq cT(\Phi^{-1}(y))^n\tilde{T}(y)^{-n},$$

which implies (a) since $J_{\Phi^{-1}}(y) = J_\Phi(\Phi^{-1}(y))^{-1}$. This completes the proof of Proposition 4.2 and thus the proof of Theorem 1.6 is also complete.

Remark 4.4 A close look at the above proof shows the following relations between the constants of Theorem 1.6 and Proposition 4.1. If (a) and (b) of Theorem 1.6 holds, then the constants in (i)–(iv) of Proposition 4.1 can be taken as follows:

$$\begin{aligned}\epsilon &= c(n, G, \max\{C_\alpha \mid |\alpha| = 1,\, 2\}),\\ K &= c(n, G, \max\{C_\alpha \mid |\alpha| = 1\}),\\ B_\alpha &= c(n, \alpha, G, \max\{C_\beta \mid 1 \leqq |\beta| \leqq |\alpha|\}).\end{aligned}$$

If (i)–(iv) of Proposition 4.1 hold, then the constant c contained in the relation $\approx$ of (2) of Theorem 1.6 can be taken as

$$c = c(n, p, \sigma, A, \epsilon, K, \max\{B_\alpha \mid 1 \leqq |\alpha| \leqq m_0 + 1\}),$$

where A is the constant such that $\lambda \in \mathrm{Double}(\Omega, 2^{-1}T, \sigma, A)$ and m_0 is the smallest positive integer such that $n + m_0 > \max\{\sigma/p, \sigma\}$.

Finally we give some results concerning the mapping Φ of Theorem 1.6. We shall simply say that $\Phi : (\Omega, T) \to (\tilde{\Omega}, \tilde{T})$ satisfies the conditions of Theorem 1.6 if Ω, $\tilde{\Omega}$, T, $\tilde{T}$, and Φ satisfy the conditions of Theorem 1.6.

Proposition 4.5 (1) *If $\Phi : (\Omega, T) \to (\tilde{\Omega}, \tilde{T})$ satisfies the conditions of Theorem* 1.6*, then so does the inverse mapping $\Phi^{-1} : (\tilde{\Omega}, \tilde{T}) \to (\Omega, T)$.*
(2) *If $\Phi : (\Omega_0, T_0) \to (\Omega_1, T_1)$ and $\Psi : (\Omega_1, T_1) \to (\Omega_2, T_2)$ satisfy the conditions of Theorem* 1.6*, then so does the composite mapping $\Psi \circ \Phi : (\Omega_0, T_0) \to (\Omega_2, T_2)$.*
(3) *Suppose $\Phi : (\Omega, T) \to (\tilde{\Omega}, \tilde{T})$ satisfies the conditions of Theorem* 1.6*. Let V be a proper open subset of Ω, let $\tilde{V} = \Phi(V)$, and let*

$$S(x) = \min\{T(x), d_V(x)\} \quad (x \in V),$$

$$\tilde{S}(y) = \min\{\tilde{T}(y), d_{\tilde{V}}(y)\} \quad (y \in \tilde{V}).$$

Then $S \in \mathcal{T}(V)$, $\tilde{S} \in \mathcal{T}(\tilde{V})$, and $\Phi|V : (V, S) \to (\tilde{V}, \tilde{S})$ satisfies the conditions of Theorem 1.6*.*

Proof. The assertion (1) immediately follows from Proposition 4.2 and from the symmetry of the conditions of Proposition 4.1. The assertion (2) can be proved by elementary calculations.

We shall prove (3). The assertion that $S \in \mathcal{T}(V)$ and $\tilde{S} \in \mathcal{T}(\tilde{V})$ is readily checked. By Proposition 4.2, $\Phi : (\Omega, T) \to (\tilde{\Omega}, \tilde{T})$ satisfies the conditions of Proposition 4.1. Let ϵ and K be the constants of Proposition 4.1. Suppose $x \in V$ and write $y = \Phi(x)$. By (ii) of Proposition 4.1, we have

$$B(y, \epsilon S(x)T(x)^{-1}\tilde{T}(y)) \subset \Phi(B(x, K\epsilon S(x))) \subset \Phi(B(x, S(x))) \subset \Phi(V) = \tilde{V}.$$

This implies $\epsilon S(x)T(x)^{-1}\tilde{T}(y) \leqq d_{\tilde{V}}(y)$. We also have $S(x)T(x)^{-1}\tilde{T}(y) \leqq \tilde{T}(y)$ since $S(x) \leqq T(x)$. Hence $\epsilon S(x)T(x)^{-1}\tilde{T}(y) \leqq \min\{d_{\tilde{V}}(y), \tilde{T}(y)\} =$

$\tilde{S}(y)$ or, equivalently,

$$\epsilon \frac{\tilde{T}(y)}{T(x)} \leqq \frac{\tilde{S}(y)}{S(x)}.$$

By the symmetry of the conditions of Proposition 4.1, we have the similar inequality with x and y interchanged. Hence we obtain

$$\epsilon \frac{\tilde{T}(y)}{T(x)} \leqq \frac{\tilde{S}(y)}{S(x)} \leqq \frac{1}{\epsilon} \frac{\tilde{T}(y)}{T(x)}.$$

Using this inequality, we easily see that $\Phi|V : (V, S) \to (\tilde{V}, \tilde{S})$ satisfies the conditions of Theorem 1.6. This completes the proof of Proposition 4.5.

5. Multiplication by smooth functions

In this section, we prove Theorem 1.8. Before we proceed to the proof of Theorem 1.8, we shall give some properties of the functions in the class $W(\Omega, T)$.

Proposition 5.1 *Let $T \in \mathcal{T}(\Omega)$ and $w \in W(\Omega, T)$. Then there exists a constant $\delta \in (0,1)$ such that $1/2 \leqq w(x)/w(x_0) \leqq 2$ whenever $x_0 \in \Omega$ and $x \in U^T(x_0, \delta)$.*

Proof. Let A_α be the constants in (1.2) and set

$$\delta = \min\left\{4^{-1}, (16\sqrt{n} \max\{A_\alpha \mid |\alpha| = 1\})^{-1}\right\}.$$

We shall prove that this δ has the desired property.

We first prove that

$$w(x) \leqq 2w(x_0) \quad \text{if } x_0 \in \Omega \text{ and } x \in U^T(x_0, 2\delta). \tag{5.1}$$

Suppose on the contrary there exist x_0, $x_1 \in \Omega$ such that $x_1 \in U^T(x_0, 2\delta)$ and $w(x_1) > 2w(x_0)$. We write $x_t = (1-t)x_0 + tx_1$. Then, by continuity, there exists $s \in (0,1)$ such that $w(x_s) = 2w(x_0)$ and $w(x_t) < 2w(x_0)$ for $t \in (0, s)$. For $t \in (0, s)$, we have

$$T(x_t) \geqq T(x_0) - |x_t - x_0| > T(x_0) - 2\delta T(x_0) \geqq 2^{-1} T(x_0)$$

and hence

$$\begin{aligned}\left|\frac{d}{dt}w(x_t)\right| &= \left|\sum_{j=1}^{n}\frac{\partial w}{\partial x_j}(x_t)(x_{1,j}-x_{0,j})\right| \\ &\leqq \sum_{j=1}^{n}\max\{A_\alpha \mid |\alpha|=1\}w(x_t)T(x_t)^{-1}|x_{1,j}-x_{0,j}| \\ &\leqq \sum_{j=1}^{n}4\max\{A_\alpha \mid |\alpha|=1\}w(x_0)T(x_0)^{-1}|x_{1,j}-x_{0,j}| \\ &< 4\max\{A_\alpha \mid |\alpha|=1\}w(x_0)\sqrt{n}\cdot 2\delta \\ &\leqq 2^{-1}w(x_0).\end{aligned}$$

Hence

$$w(x_s) = w(x_0) + \int_0^s \frac{d}{dt}w(x_t)dt \leqq (3/2)w(x_0),$$

which contradicts the equality $w(x_s) = 2w(x_0)$. Thus (5.1) is proved.

Now suppose $x_0 \in \Omega$ and $x \in U^T(x_0, \delta)$. Then (5.1) gives $w(x) \leqq 2w(x_0)$. Moreover, by the Lipschitz continuity of T, we have $|T(x) - T(x_0)| \leqq |x - x_0| < \delta T(x_0) \leqq 4^{-1}T(x_0)$ and hence

$$(3/4)T(x_0) < T(x) < (5/4)T(x_0).$$

Thus

$$|x - x_0| < \delta T(x_0) < (4/3)\delta T(x) < 2\delta T(x)$$

and thus $x_0 \in U^T(x, 2\delta)$. Hence (5.1) gives $w(x_0) \leqq 2w(x)$. Proposition 5.1 is proved.

We omit the proof of the next proposition, which will be quite elementary.

Proposition 5.2 *If w and v are functions of class $W(\Omega, T)$, then so are the functions $w + v$, wv, w/v, and w^a with $a \in \mathbb{R}$.*

The next proposition will be used in Section 6.

Proposition 5.3 *Suppose* Ω, $\tilde{\Omega}$, T, $\tilde{T}$, *and* Φ *satisfy the assumptions of Theorem* 1.6 *and suppose* $w \in W(\Omega, T)$. *Then*:
(1) *The function* $|J_\Phi|$ *is of class* $W(\Omega, T)$;
(2) *The composite function* $w \circ \Phi^{-1}$ *is of class* $W(\tilde{\Omega}, \tilde{T})$.

Proof. The claim (1) can be seen from (4.1) and (a) of Theorem 1.6. The claim (2) can be seen by the use of (iv) of Proposition 4.1 (cf. Proposition 4.2). Proposition 5.3 is proved.

No we shall prove Theorem 1.8.

Proof of Theorem 1.8. Let δ be the constant as mentioned in Proposition 5.1. We may assume $\delta < 1/3$. Notice that $T(x) \approx T(x_0)$ for x and x_0 satisfying $x \in U^T(x_0, \delta)$.

(1) For balls $B(x_0, t)$ with $t < \delta T(x_0)$, the inequality of Proposition 5.1 yields

$$(w^p \lambda)(B(x_0, t)) \approx (w(x_0))^p \lambda(B(x_0, t)).$$

From this inequality, it is obvious that the doubling conditions on $w^p \lambda$ and on λ are mutually equivalent.

(2) Let $\lambda \in \mathrm{Double}^*(\Omega, T, \sigma)$ and $p \in (0, \infty)$. Take a positive integer k such that $n + k > \max\{\sigma/p, \sigma\}$. Let $x_0 \in \Omega$, $t \in (0, \delta)$, and $\psi \in \mathcal{A}_{(k)}(U^T(x_0, t))$. Then, using the inequalities of Proposition 5.1, we easily see that there exists a constant c_0 such that $c_0^{-1} w(x_0)^{-1} w\psi \in \mathcal{A}_{(k)}(U^T(x_0, t))$ and $c_0^{-1} w(x_0) w^{-1} \psi \in \mathcal{A}_{(k)}(U^T(x_0, t))$. Hence, for each $f \in \mathcal{D}'(\Omega)$, we have

$$|\langle wf, \psi \rangle| = |\langle f, w\psi \rangle| \leqq c_0 w(x_0) f^{*,\delta T}_{(k)}(x_0)$$

and

$$|\langle f, \psi \rangle| = |\langle wf, w^{-1}\psi \rangle| \leqq c_0 w(x_0)^{-1} (wf)^{*,\delta T}_{(k)}(x_0).$$

Taking the supremum over ψ and t, we obtain

$$(wf)^{*,\delta T}_{(k)}(x_0) \approx w(x_0) f^{*,\delta T}_{(k)}(x_0). \tag{5.2}$$

Hence, using Theorem 3.4, we have

$$\begin{aligned}\|wf\|_{H^p(\Omega,T,\lambda)} &\approx \left\|(wf)^{*,\delta T}_{(k)}\right\|_{L^p(\Omega,\lambda)} \\ &\approx \left\|wf^{*,\delta T}_{(k)}\right\|_{L^p(\Omega,\lambda)} = \left\|f^{*,\delta T}_{(k)}\right\|_{L^p(\Omega,w^p\lambda)} \approx \|f\|_{H^p(\Omega,T,w^p\lambda)}.\end{aligned}$$

Theorem 1.8 is proved.

Remark 5.4 From the above proof we see the following: If A_α is the constant in (1.2), $\sigma \in [n,\infty)$, $\lambda \in \mathrm{Double}^*(\Omega,T,\sigma)$, and if m_0 is the smallest positive integer satisfying $n+m_0 > \max\{\sigma/p,\sigma\}$, then the constants contained in the relation $\approx$ of Theorem 1.8 (2) can be taken depending only on n, p, σ, $\max\{A_\alpha \mid |\alpha| \leqq m_0\}$, and on the doubling constant of λ.

6. Modified form of atomic decomposition

An atomic decomposition for $H^p(\Omega,T,\lambda)$ has been given in Theorem 3.1. The purpose of this section is to give a modified atomic decomposition that involves a moment condition of the form (1.3). The result reads as follows.

Theorem 6.1 *Suppose Ω, $\tilde{\Omega}$, T, $\tilde{T}$, and Φ satisfy the assumptions of Proposition* 4.1; *let K be the constant as mentioned in the proposition. Let $w \in W(\Omega,T)$. Let $\sigma \in [n,\infty)$, $\lambda \in \mathrm{Double}^*(\Omega,T,\sigma)$, $p \in (0,\infty)$, and let k and m be positive integers satisfying $n+k > \max\{\sigma/p,\sigma\}$ and $n+m > \max\{\sigma/p,\sigma\}$. Then the following* (1) *and* (2) *hold.*
(1) *Suppose $Q_j \in \mathcal{B}(\Omega,T/20)$, $h_j \in L^\infty$, $\operatorname{supp} h_j \subset \overline{Q}_j$,*

$$\int h_j(x)P(\Phi(x))w(x)dx = 0 \quad \textit{for all } P \in \mathcal{P}_{m-1}, \tag{6.1}$$

and suppose $\sum_j \|h_j\|_{L^\infty}\mathbf{1}_{Q_j} \in L^p(\Omega,\lambda)$. Then the series $\sum_j h_j$ converges unconditionally in $\mathcal{D}'(\Omega)$ and

$$\left\|\sum_j h_j\right\|_{H^p(\Omega,T,\lambda)} \leqq c\left\|\sum_j \|h_j\|_{L^\infty}\mathbf{1}_{Q_j}\right\|_{L^p(\Omega,\lambda)}.$$

(2) *If $f \in H^p(\Omega,T,\lambda)$, if α is a sufficiently small positive real number, and if $0 < \delta \leqq \alpha/12$, then there exist g_i, B_i, h_j, and Q_j such that the following* (i)–(v) *hold:*

(i) $f = \sum_i g_i + \sum_j h_j$ *with the two series converging unconditionally in* $\mathcal{D}'(\Omega)$;
(ii) $g_i \in L^\infty$ *and* $\operatorname{supp} g_i \subset B_i = U^T(x_i, \delta)$;
(iii) $h_j \in L^\infty$, $\operatorname{supp} h_j \subset Q_j \in \mathcal{B}(\Omega, 3\delta T)$, *and* h_j *satisfies* (6.1);
(iv) *For each* $r \in (0, \infty)$, *there exists a constant* $c_r \in (0, \infty)$ *such that*

$$\left(\sum_i \|g_i\|_{L^\infty}^r \mathbf{1}\{x \in K^{-2} v B_i\} + \sum_j \|h_j\|_{L^\infty}^r \mathbf{1}\{x \in K^{-2} v Q_j\} \right)^{1/r} \leqq c_r f_{(k)}^{*,\alpha T}(x)$$

for all $x \in \Omega$, *where* $v = \alpha/9\delta$;
(v) *For each* B_i, *there exists a ball* $B = U^T(y, \delta)$ *such that* $B_i \subset 2K^2 B$ *and* $B \cap \operatorname{supp} f \neq \emptyset$; *the same holds for the balls* Q_j.

Remark 6.2 A close look at the proof of this theorem to be given below will show the following facts about the constants in the theorem. Let ϵ and B_α be the constants in the conditions of Proposition 4.1. Let A_α be the constant in (1.2). Let m_0 be the smallest positive integer such that $n + m_0 > \max\{\sigma/p, \sigma\}$. Then the following hold.
(1) The constant c of Theorem 6.1 (1) can be taken depending only on n, p, σ, ϵ, K, $\max\{A_\alpha \mid |\alpha| \leqq m_0\}$, $\max\{B_\alpha \mid 1 \leqq |\alpha| \leqq m_0 + 1\}$, and on the doubling constant of λ in $\mathrm{Double}^*(\Omega, T, \sigma)$.
(2) The constant α of Theorem 6.1 (2) can be taken arbitrarily in the range $0 < \alpha \leqq \alpha_0$, where α_0 is a constant that depends only on n, ϵ, K, $\max\{B_\alpha \mid 1 \leqq |\alpha| \leqq 2\}$, and $\max\{A_\alpha \mid |\alpha| = 1\}$.
(3) The constant c_r of Theorem 6.1 (2) (iv) can be taken depending only on n, k, m, v, r, K, $\max\{A_\alpha \mid |\alpha| \leqq k\}$, and $\max\{B_\alpha \mid 1 \leqq |\alpha| \leqq k + 1\}$.

Proof of Theorem 6.1. (1) We write $Q_j = U^T(x_j, s_j)$. Notice that $s_j < 1/20$ by our assumption that $Q_j \in \mathcal{B}(\Omega, T/20)$. We take a sufficiently small number $\xi > 0$ and divide the indices j into the two sets $\mathrm{I} = \{j \mid s_j < \xi\}$ and $\mathrm{II} = \{j \mid \xi \leqq s_j < 1/20\}$.

First we consider the series over $j \in \mathrm{II}$. We shall prove that, for any $\xi > 0$ fixed, the series $\sum_{j \in \mathrm{II}} h_j$ converges unconditionally in $\mathcal{D}'(\Omega)$ and

$$\left\| \sum_{j \in \mathrm{II}} h_j \right\|_{H^p(\Omega, T, \lambda)} \leqq c_\xi \left\| \sum_{j \in \mathrm{II}} \|h_j\|_{L^\infty} \mathbf{1}_{Q_j} \right\|_{L^p(\Omega, \lambda)}. \tag{6.2}$$

To prove this, we set $Q'_j = U^T(x_j, 1/20)$ for $j \in \mathrm{II}$. Then $Q'_j \supset Q_j \supset 20\xi Q'_j$ and hence, by Lemma 2.5,

$$\Big\| \sum_{j \in \mathrm{II}} \|h_j\|_{L^\infty} \mathbf{1}_{Q'_j} \Big\|_{L^p(\Omega,\lambda)} \leqq c_\xi \Big\| \sum_{j \in \mathrm{II}} \|h_j\|_{L^\infty} \mathbf{1}_{Q_j} \Big\|_{L^p(\Omega,\lambda)} < \infty. \tag{6.3}$$

Since $\operatorname{supp} h_j \subset \overline{Q}_j \subset Q'_j$, by virtue of Theorem 3.1 (1), the series $\sum_{j \in \mathrm{II}} h_j$ converges unconditionally in $\mathcal{D}'(\Omega)$ and

$$\Big\| \sum_{j \in \mathrm{II}} h_j \Big\|_{H^p(\Omega,T,\lambda)} \leqq c \Big\| \sum_{j \in \mathrm{II}} \|h_j\|_{L^\infty} \mathbf{1}_{Q'_j} \Big\|_{L^p(\Omega,\lambda)}. \tag{6.4}$$

Thus (6.2) follows from (6.4) and (6.3).

Next we consider the series over $j \in \mathrm{I}$. We shall show that, if $\xi > 0$ is chosen sufficiently small, then the series $\sum_{j \in \mathrm{I}} h_j$ converges unconditionally in $\mathcal{D}'(\Omega)$ and

$$\Big\| \sum_{j \in \mathrm{I}} h_j \Big\|_{H^p(\Omega,T,\lambda)} \leqq c \Big\| \sum_{j \in \mathrm{I}} \|h_j\|_{L^\infty} \mathbf{1}_{Q_j} \Big\|_{L^p(\Omega,\lambda)}. \tag{6.5}$$

In order to prove this, we define, for $j \in \mathrm{I}$,

$$f_j = w|J_\Phi|^{-1} h_j, \quad \tilde{f}_j = f_j \circ \Phi^{-1}, \quad R_j = U^{\tilde{T}}(\Phi(x_j), K s_j).$$

We also define the Borel measures μ on Ω and μ_* on $\tilde{\Omega}$ by

$$\mu = (w^{-1}|J_\Phi|)^p \lambda, \quad \mu_* = \Phi_* \mu. \tag{6.6}$$

Then $\tilde{f}_j \in L^\infty$, $\operatorname{supp} \tilde{f}_j = \Phi(\operatorname{supp} h_j) \subset \Phi(\overline{Q}_j) \subset \overline{R}_j$, $R_j \in \mathcal{B}(\tilde{\Omega}, K\xi\tilde{T})$, and

$$\begin{aligned}\int \tilde{f}_j(y) P(y) dy &= \int \tilde{f}_j(\Phi(x)) P(\Phi(x)) |J_\Phi(x)| dx \\ &= \int h_j(x) w(x) P(\Phi(x)) dx = 0\end{aligned}$$

for all $P \in \mathcal{P}_{m-1}$. We shall prove

$$\Big\| \sum_{j \in \mathrm{I}} \|\tilde{f}_j\|_{L^\infty} \mathbf{1}_{R_j} \Big\|_{L^p(\tilde{\Omega}, \mu_*)} \leqq c \Big\| \sum_{j \in \mathrm{I}} \|h_j\|_{L^\infty} \mathbf{1}_{Q_j} \Big\|_{L^p(\Omega, \lambda)}. \tag{6.7}$$

Suppose for the moment (6.7) is proved. Then, by Theorem 3.1 (2), the series $\sum_{j \in \mathrm{I}} \tilde{f}_j$ converges unconditionally in $\mathcal{D}'(\tilde{\Omega})$ and

$$\Big\| \sum_{j \in \mathrm{I}} \tilde{f}_j \Big\|_{H^p(\tilde{\Omega}, \tilde{T}, \mu_*)} \leqq c \Big\| \sum_{j \in \mathrm{I}} \|\tilde{f}_j\|_{L^\infty} \mathbf{1}_{R_j} \Big\|_{L^p(\tilde{\Omega}, \mu_*)}.$$

Hence the series $\sum_{j \in \mathrm{I}} h_j = \sum_{j \in \mathrm{I}} w^{-1} |J_\Phi| (\tilde{f}_j \circ \Phi)$ also converges unconditionally in $\mathcal{D}'(\Omega)$ and, by Theorem 1.8 and Theorem 1.6 (2),

$$\begin{aligned} \Big\| \sum_{j \in \mathrm{I}} h_j \Big\|_{H^p(\Omega, T, \lambda)} &= \Big\| w^{-1} |J_\Phi| \sum_{j \in \mathrm{I}} f_j \Big\|_{H^p(\Omega, T, \lambda)} \\ &\approx \Big\| \sum_{j \in \mathrm{I}} f_j \Big\|_{H^p(\Omega, T, \mu)} \approx \Big\| \sum_{j \in \mathrm{I}} \tilde{f}_j \Big\|_{H^p(\tilde{\Omega}, \tilde{T}, \mu_*)}. \end{aligned}$$

Combining the above inequalities, we obtain (6.5). Thus it is sufficient to prove (6.7).

We shall prove (6.7). Suppose $j \in \mathrm{I}$. By Propositions 5.1 and 5.3, we see that

$$\|\tilde{f}_j\|_{L^\infty} = \|f_j\|_{L^\infty} \approx w(x) |J_\Phi(x)|^{-1} \|h_j\|_{L^\infty} \quad \text{for all } x \in Q_j. \tag{6.8}$$

Since $\Phi^{-1}(K^{-2} R_j) \subset Q_j$, (6.8) implies

$$\|\tilde{f}_j\|_{L^\infty} \mathbf{1}_{K^{-2} R_j}(y) \leqq c w(\Phi^{-1}(y)) |J_\Phi(\Phi^{-1}(y))|^{-1} \|h_j\|_{L^\infty} \mathbf{1}_{Q_j}(\Phi^{-1}(y)).$$

Hence, by Lemma 2.5,

$$\begin{aligned} \Big\| \sum_{j \in \mathrm{I}} \|\tilde{f}_j\|_{L^\infty} \mathbf{1}_{R_j} \Big\|_{L^p(\tilde{\Omega}, \mu_*)} &\leqq c \Big\| \sum_{j \in \mathrm{I}} \|\tilde{f}_j\|_{L^\infty} \mathbf{1}_{K^{-2} R_j} \Big\|_{L^p(\tilde{\Omega}, \mu_*)} \\ &\leqq c \Big\| \sum_{j \in \mathrm{I}} w(\Phi^{-1}(y)) |J_\Phi(\Phi^{-1}(y))|^{-1} \|h_j\|_{L^\infty} \mathbf{1}_{Q_j}(\Phi^{-1}(y)) \Big\|_{L^p(\tilde{\Omega}, \mu_*)} \end{aligned}$$

$$= c\left\| \sum_{j\in\mathrm{I}} w(x)|J_\Phi(x)|^{-1}\|h_j\|_{L^\infty}\mathbf{1}_{Q_j}(x)\right\|_{L^p(\Omega,\mu)}$$

$$= c\left\| \sum_{j\in\mathrm{I}} \|h_j\|_{L^\infty}\mathbf{1}_{Q_j}\right\|_{L^p(\Omega,\lambda)},$$

which proves (6.7).

The claim (1) now follows from (6.2) and (6.5).

(2) We assume $f \in H^p(\Omega, T, \lambda)$, $\alpha > 0$ is sufficiently small, and $0 < \delta \leqq \alpha/12$. We take a real number s such that $s > k$.

We define $F \in \mathcal{D}'(\Omega)$, $\tilde{F} \in \mathcal{D}'(\tilde{\Omega})$, $\tilde{\alpha}$, and $\tilde{\delta}$ by

$$F = w|J_\Phi|^{-1}f, \quad \tilde{F} = F \circ \Phi^{-1},$$

$$\tilde{\alpha} = \alpha/3K, \quad \tilde{\delta} = \delta/K.$$

We also define μ_* by (6.6). Using the inequalities (3.8), (4.2), and (5.2), and using Propositions 5.3 and 5.2, we have

$$\begin{aligned}\tilde{F}_s^{*,\tilde{\alpha}\tilde{T}}(\Phi(x)) &\leqq c\tilde{F}_{(k)}^{*,3\tilde{\alpha}\tilde{T}}(\Phi(x)) \leqq cF_{(k)}^{*,\alpha T}(x) \\ &\leqq cw(x)|J_\Phi(x)|^{-1}f_{(k)}^{*,\alpha T}(x). \end{aligned} \tag{6.9}$$

From the assumption $f \in H^p(\Omega, T, \lambda)$, we have $f_{(k)}^{*,\alpha T} \in L^p(\Omega, \lambda)$ and hence (6.9) implies that $\tilde{F}_s^{*,\tilde{\alpha}\tilde{T}} \in L^p(\tilde{\Omega}, \mu_*)$. We see that $\mu_* \in \mathrm{Double}^*(\tilde{\Omega}, \tilde{T}, \sigma)$ by using Propositions 5.3 and 5.2, Theorem 1.8 (1), and Theorem 1.6 (1). Hence, by Theorem 3.1 (3), $\tilde{F}$ can be decomposed as follows:

(a) $\tilde{F} = \sum_i a_i + \sum_j b_j$ with the two series converging unconditionally in $\mathcal{D}'(\tilde{\Omega})$;

(b) $a_i \in L^\infty$ and $\operatorname{supp} a_i \subset \tilde{B}_i = U^{\tilde{T}}(y_i, \tilde{\delta})$;

(c) $b_j \in L^\infty$, $\operatorname{supp} b_j \subset \tilde{Q}_j = U^{\tilde{T}}(z_j, s_j)$, $s_j < 3\tilde{\delta}$, and

$$\int b_j(y)P(y)dy = 0 \quad \text{for all } P \in \mathcal{P}_{m-1};$$

(d) For $v = \tilde{\alpha}/3\tilde{\delta}$ and for each $r \in (0, \infty)$, we have

$$\Big(\sum_i \|a_i\|_{L^\infty}^r \mathbf{1}\{y \in v\tilde{B}_i\} + \sum_j \|b_j\|_{L^\infty}^r \mathbf{1}\{y \in v\tilde{Q}_j\}\Big)^{1/r} \leqq c_r \tilde{F}_s^{*,\tilde{\alpha}\tilde{T}}(y)$$

for all $y \in \tilde{\Omega}$;
(e) For each $\tilde{B}_i$, there exists a ball $\tilde{B} = U^{\tilde{T}}(y, \tilde{\delta})$ such that $\tilde{B}_i \subset 2\tilde{B}$ and $\tilde{B} \cap \operatorname{supp} \tilde{F} \neq \emptyset$; the same holds for the balls $\tilde{Q}_j$.
We define g_i, B_i, h_j, and Q_j by

$$\begin{aligned} g_i &= w^{-1}|J_\Phi|(a_i \circ \Phi), \quad B_i = U^T(\Phi^{-1}(y_i), \delta), \\ h_j &= w^{-1}|J_\Phi|(b_j \circ \Phi), \quad Q_j = U^T(\Phi^{-1}(z_i), Ks_j). \end{aligned}$$

We shall prove that these satisfy the conditions (i)–(v) of the theorem.

The condition (i) immediately follows from (a).

The conditions (ii) and (iii) can be seen as follows. We have

$$\begin{aligned} \operatorname{supp} g_i &= \Phi^{-1}(\operatorname{supp} a_i) \subset \Phi^{-1}(U^{\tilde{T}}(y_i, \tilde{\delta})) \subset U^T(\Phi^{-1}(y_i), \delta) = B_i, \\ \operatorname{supp} h_j &= \Phi^{-1}(\operatorname{supp} b_j) \subset \Phi^{-1}(U^{\tilde{T}}(z_j, s_j)) \subset U^T(\Phi^{-1}(z_j), Ks_j) = Q_j. \end{aligned}$$

Since $Ks_j < 3K\tilde{\delta} = 3\delta$, we have $Q_j \in \mathcal{B}(\Omega, 3\delta T)$. Since, by Propositions 5.1 and 5.3 (1), the functions w and $|J_\Phi|$ are roughly equal to constants on the balls B_i and Q_j, we see that g_i and h_j are functions of L^∞. The moment condition (6.1) follows from the moment condition on b_j as given in (c).

We shall prove (iv). Notice that $v = \tilde{\alpha}/3\tilde{\delta} = \alpha/9\delta$. By Proposition 5.1, we have

$$\begin{aligned} \|g_i\|_{L^\infty} &\approx w(x)^{-1}|J_\Phi(x)|\|a_i\|_{L^\infty} \quad \text{for all } x \in vB_i, \\ \|h_j\|_{L^\infty} &\approx w(x)^{-1}|J_\Phi(x)|\|b_j\|_{L^\infty} \quad \text{for all } x \in vQ_j. \end{aligned}$$

We also have

$$\begin{aligned} \Phi(K^{-2}vB_i) &= \Phi\big(U^T(\Phi^{-1}(y_j), K^{-2}v\delta)\big) \subset U^{\tilde{T}}(y_i, v\tilde{\delta}) = v\tilde{B}_i, \\ \Phi(K^{-2}vQ_j) &= \Phi\big(U^T(\Phi^{-1}(z_j), K^{-1}vs_j)\big) \subset U^{\tilde{T}}(z_j, vs_j) = v\tilde{Q}_j. \end{aligned}$$

Hence, for each $r \in (0, \infty)$, we use (d) and (6.9) to see that

$$
\begin{aligned}
&\Big(\sum_i \|g_i\|_{L^\infty}^r \mathbf{1}\{x \in K^{-2}vB_i\} + \sum_j \|h_j\|_{L^\infty}^r \mathbf{1}\{x \in K^{-2}vQ_j\}\Big)^{1/r} \\
&\leqq cw(x)^{-1}|J_\Phi(x)|\Big(\sum_i \|a_i\|_{L^\infty}^r \mathbf{1}\{\Phi(x) \in v\tilde{B}_i\} \\
&\qquad\qquad + \sum_j \|b_j\|_{L^\infty}^r \mathbf{1}\{\Phi(x) \in v\tilde{Q}_j\}\Big)^{1/r} \\
&\leqq c_r w(x)^{-1}|J_\Phi(x)|\tilde{F}_s^{*,\tilde{\alpha}\tilde{T}}(\Phi(x)) \\
&\leqq c_r f_{(k)}^{*,\alpha T}(x),
\end{aligned}
$$

which proves (iv).

Finally we shall prove (v). If $\tilde{B}_i$ and $\tilde{B} = U^{\tilde{T}}(y,\tilde{\delta})$ satisfy the conditions as mentioned in (e), then

$$
\begin{aligned}
B_i = U^T(\Phi^{-1}(y_i),\delta) &\subset \Phi^{-1}\big(U^{\tilde{T}}(y_i, K^2\tilde{\delta})\big) = \Phi^{-1}(K^2\tilde{B}_i) \\
&\subset \Phi^{-1}(2K^2\tilde{B}) = \Phi^{-1}\big(U^{\tilde{T}}(y, 2K^2\tilde{\delta})\big) \\
&\subset U^T(\Phi^{-1}(y), 2K^3\tilde{\delta}) = 2K^2U^T(\Phi^{-1}(y),\delta)
\end{aligned}
$$

and

$$
\begin{aligned}
U^T(\Phi^{-1}(y),\delta) \cap \operatorname{supp} f &= U^T(\Phi^{-1}(y), K\tilde{\delta}) \cap \operatorname{supp} F \\
&\supset \Phi^{-1}\big(U^{\tilde{T}}(y,\tilde{\delta}) \cap \operatorname{supp} \tilde{F}\big) \neq \emptyset.
\end{aligned}
$$

Hence $B = U^T(\Phi^{-1}(y),\delta)$ satisfies the conditions of (v). The same holds for Q_j. Theorem 6.1 is proved.

7. Examples

In this section, we give some examples of the mapping Φ of Theorem 1.6 and Proposition 4.1. We shall also prove a proposition which will give plenty of examples in the case of 2-dimension.

The Examples 7.1–7.5 are the examples in the 1-dimensional case. For these examples, to check the conditions of Theorem 1.6 or Proposition 4.1 is an elementary task and the details are left to the reader. In these examples, we use the letters x and y to denote the variables in Ω and $\tilde{\Omega}$, respectively; $x \in \Omega$ and $y \in \tilde{\Omega}$.

Example 7.1 $\Omega = (0, \pi)$, $\tilde{\Omega} = (-1, 1)$, $T(x) = d_\Omega(x)$, $\tilde{T}(y) = d_{\tilde{\Omega}}(y)$, $\Phi(x) = \cos x$. In this case, $\tilde{T}(y) \approx T(x)^2$. This case was already considered in [M4].

Example 7.2 $\Omega = (-\pi/2, \pi/2)$, $\tilde{\Omega} = \mathbb{R}$, $T(x) = d_\Omega(x)$, $\tilde{T}(y) = 1 + |y|$, $\Phi(x) = \tan x$. In this case, $\tilde{T}(y) \approx 1/T(x)$.

Example 7.3 $\Omega = (0, \infty)$, $\tilde{\Omega} = \mathbb{R}$, $T(x) = d_\Omega(x) = x$, $\tilde{T}(y) = 1$, $\Phi(x) = \log x$.

Example 7.4 $\Omega = (0, 1)$, $\tilde{\Omega} = (0, \infty)$, $T(x) = d_\Omega(x)$, $\tilde{T}(y) = d_{\tilde{\Omega}}(y) = y$, $\Phi(x) = x/(1-x)$.

Example 7.5 $\Omega = (0, 1)$, $\tilde{\Omega} = \mathbb{R}$, $T(x) = d_\Omega(x)$, $\tilde{T}(y) = 1$, $\Phi(x) = \log\{x/(1-x)\}$. This Φ is the composition of the Φ's of Examples 7.4 and 7.3.

The next is an example for arbitrary dimension.

Example 7.6 $\Omega = \tilde{\Omega} = \mathbb{R}^n \setminus \{0\}$, $T(x) = \tilde{T}(x) = d_\Omega(x) = |x|$, $\Phi(x) = x/|x|^2$.

Proof. If $a \in \mathbb{R}^n \setminus \{0\}$, $b = \Phi(a) = a/|a|^2$, and $t \in (0, 1)$, then

$$\begin{aligned}\Phi(U^T(a,t)) &= \Phi(B(a, t|a|)) = B\big(b/(1-t^2), t|b|/(1-t^2)\big)\\ &\subset B(b, t|b|/(1-t)) = U^{\tilde{T}}(b, t/(1-t)).\end{aligned}$$

From this we easily see that $\Phi = \Phi^{-1}$ satisfy the conditions (i) and (ii) of Proposition 4.1. To check the conditions (iii) and (iv) of Proposition 4.1 is easy.

Finally we shall give a result in the 2-dimensional case. In order to state the result, we identify $\mathbb{R}^2$ with the complex number field $\mathbb{C}$ by identifying $(x, y) \in \mathbb{R}^2$ with $x + iy \in \mathbb{C}$. For open subsets Ω and $\tilde{\Omega}$ of $\mathbb{R}^2 = \mathbb{C}$, we say that Φ is a *conformal* mapping of Ω onto $\tilde{\Omega}$ if $\Phi(x, y)$ is a holomorphic function of $x + iy \in \Omega$ and if the mapping $\Phi : \Omega \to \tilde{\Omega}$ is a bijection.

The result reads as follows.

Proposition 7.7 *If Ω and $\tilde{\Omega}$ are proper open subsets of $\mathbb{R}^2 = \mathbb{C}$ and if Φ*

is a conformal mapping of Ω *onto* $\tilde{\Omega}$*, then* Ω, $\tilde{\Omega}$, $T = d_\Omega$, $\tilde{T} = d_{\tilde{\Omega}}$*, and* Φ *satisfy the conditions of Theorem* 1.6*.*

In order to prove this proposition, we use the lemmas to be given below. In these lemmas, we use the letter z to denote a complex variable and write the complex derivatives as $\Phi'(z) = d\Phi(z)/dz$ and $\Phi^{(k)}(z) = d^k\Phi(z)/dz^k$. In the first lemma, we shall say that a holomorphic function on a domain $D \subset \mathbb{C}$ is *univalent* in D if the mapping $D \ni z \mapsto f(z)$ is one to one; thus the conformal mapping and the univalent function are in fact the same thing.

Lemma 7.8 *Suppose* f *is a univalent function on* $B(0,1) \subset \mathbb{C}$ *and* $f(0) = 0$ *and* $f'(0) = 1$*. Then:*
(1) $f(B(0,1)) \supset B(0,1/4)$;
(2) $|z|/(1+|z|)^2 \leqq |f(z)| \leqq |z|/(1-|z|)^2$ *for all* $z \in B(0,1)$*.*

The facts given in the above lemma is well known as the distortion theorems for univalent functions. For a proof, see, e.g., [H, Theorems 1.2 and 1.3].

Lemma 7.9 *Let* Ω *and* $\tilde{\Omega}$ *be proper open subsets of* $\mathbb{C}$ *and let* Φ *be a conformal mapping of* Ω *onto* $\tilde{\Omega}$*. Let* $z_0 \in \Omega$ *and* $w_0 = \Phi(z_0) \in \tilde{\Omega}$*. Then the following hold.*
(1) $\frac{d_{\tilde{\Omega}}(w_0)}{4d_\Omega(z_0)} \leqq |\Phi'(z_0)| \leqq \frac{4d_{\tilde{\Omega}}(w_0)}{d_\Omega(z_0)}$.
(2) *If* $\delta \in (0,1)$*, then* $\Phi(B(z_0, \delta d_\Omega(z_0))) \supset B(w_0, 4^{-1}\delta(1+\delta)^{-2} d_{\tilde{\Omega}}(w_0))$*.*
(3) *If* $\delta \in (0,1)$*, then* $\Phi(B(z_0, \delta d_\Omega(z_0))) \subset B(w_0, 4\delta(1-\delta)^{-2} d_{\tilde{\Omega}}(w_0))$*.*
(4) *If* $0 < \epsilon < \delta < 1$ *and* k *is a positive integer, then, for all* $z \in B(z_0, \epsilon d_\Omega(z_0))$*, we have*

$$|\Phi^{(k)}(z)| \leqq \frac{4k!\delta}{(1-\delta)^2(\delta-\epsilon)^k} \cdot \frac{d_{\tilde{\Omega}}(w_0)}{d_\Omega(z_0)^k}.$$

Proof. We define the function f by

$$f(z) = \frac{\Phi(z_0 + d_\Omega(z_0)z) - w_0}{d_\Omega(z_0)\Phi'(z_0)} \quad (z \in B(0,1)).$$

This function satisfies the assumptions of Lemma 7.8.

(1) Applying Lemma 7.8 (1) to f, we have

$$\Phi(B(z_0, d_\Omega(z_0)) \supset B\big(w_0, 4^{-1} d_\Omega(z_0)|\Phi'(z_0)|\big).$$

This implies $d_{\tilde{\Omega}}(w_0) \geqq 4^{-1} d_\Omega(z_0)|\Phi'(z_0)|$ or, equivalently, the second inequality of (1). We obtain the first inequality of (1) by applying the second one to Φ^{-1}.

(2) Let $\delta \in (0,1)$. By the first inequality of Lemma 7.8 (2), we have $|f(z)| \geqq \delta(1+\delta)^{-2}$ for $|z| = \delta$. Hence, by Rouché's theorem, $f(B(0,\delta)) \supset B(0, \delta(1+\delta)^{-2})$, which is equivalent to

$$\Phi\big(B(z_0, \delta d_\Omega(z_0))\big) \supset B\big(w_0, \delta(1+\delta)^{-2} d_\Omega(z_0)|\Phi'(z_0)|\big).$$

This and the first inequality of (1) implies the desired result.

(3) Let $\delta \in (0,1)$. By the second inequality of Lemma 7.8 (2), we have $f(B(0,\delta)) \subset B(0, \delta(1-\delta)^{-2})$, which is equivalent to

$$\Phi\big(B(z_0, \delta d_\Omega(z_0))\big) \subset B\big(w_0, \delta(1-\delta)^{-2} d_\Omega(z_0)|\Phi'(z_0)|\big).$$

This and the second inequality of (1) implies the desired result.

(4) Let $0 < \epsilon < \delta < 1$. By the second inequality of Lemma 7.8 (2), we have $|f(z)| \leqq \delta(1-\delta)^{-2}$ for $|z| = \delta$. Hence Cauchy's inequality gives, for $|z| < \epsilon$,

$$|f^{(k)}(z)| \leqq k!\delta(1-\delta)^{-2}(\delta-\epsilon)^{-k}$$

or, equivalently,

$$d_\Omega(z_0)^k |\Phi^{(k)}(z_0 + d_\Omega(z_0)z)| \leqq k!\delta(1-\delta)^{-2}(\delta-\epsilon)^{-k} d_\Omega(z_0)|\Phi'(z_0)|.$$

This and the second inequality of (1) give the desired inequality. Lemma 7.9 is proved.

Now we shall prove Proposition 7.7.

Proof of Proposition 7.7. By Proposition 4.2, it is sufficient to show that Ω, $\tilde{\Omega}$, $T = d_\Omega$, $\tilde{T} = d_{\tilde{\Omega}}$, and Φ satisfy the conditions (i)–(iv) of Proposition 4.1. The conditions (i) and (ii) follow from Lemma 7.9 (3) and (2), respectively. The condition (iii) follows from Lemma 7.9 (4). The condition (iv) follows if we apply Lemma 7.9 (4) to Φ^{-1}. This completes the proof of Proposition 7.7.

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