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Grassmann geometry on the 3-dimensional unimodular Lie groups I

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Abstract. We study the Grassmann geometry of surfaces when the ambient space is a 3-dimensional unimodular Lie group with left invariant metric, that is, it is one of the 3-dimensional commutative Lie group, the 3-dimensional Heisenberg group, the groups of rigid motions on the Euclidean or the Minkowski planes, the special unitary group $SU(2)$, and the special real linear group $SL(2,\mathbb{R})$.

Key words: Grassmann geometry, unimodular Lie group, Heisenberg group, Euclidean plane, Minkowski plane, special unitary group, special linear group, totally geodesic surface, flat surface, minimal surface, surface of constant mean curvature

1. Introduction

Let (M,g) be an m-dimensional connected Riemannian homogeneous manifold and $I_o(M,g)$ the identity component of the isometry group of (M,g). Fix an integer r such that $1 \le r \le m$ and consider the Grassmann bundle $\mathrm{Gr}^r(TM)$ over M which consists of all r-dimensional linear subspaces of the tangent spaces of M. Then the Lie group $I_o(M,g)$ acts on $\mathrm{Gr}^r(TM)$ through the differentials of isometries and each $I_o(M,g)$-orbit $\mathcal{O}$ in $\mathrm{Gr}^r(TM)$ gives a homogeneous bundle over M. An r-dimensional connected submanifold S of M is called an *$\mathcal{O}$-submanifold* if all tangent spaces of S belong to $\mathcal{O}$, and the collection of $\mathcal{O}$-submanifolds is called an *$\mathcal{O}$-geometry*. Such an $\mathcal{O}$-geometry is collectively called the *Grassmann geometry of orbital type*. A typical example of $\mathcal{O}$-submanifold is an (extrinsic) homogeneous submanifold which is defined as an orbit $G(p)$ in M by a subgroup G of $I_o(M,g)$ where $p \in M$. In the study of Grassmann geometry of orbital type, the following two problems naturally arise: One is to consider whether a given $\mathcal{O}$-geometry is empty, or not; The other is to consider whether a nonempty $\mathcal{O}$-geometry has somewhat canonical $\mathcal{O}$-submanifolds such as totally geodesic submanifolds, minimal submanifolds, submanifolds of specific

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curvature and so on, or not. From the view of these points, the case when (M, g) is a Riemannian symmetric space has been studied in H. Naitoh [11] and J. Berndt-J. H. Eschenburg-H. Naitoh-K. Tsukada [2], so that the fundamental problems of Grassmann geometry have been settled for some kinds of orbital geometries and by the settlement all the symmetric submanifolds of Riemannian symmetric spaces have been completely classified.

Now in this paper we consider the case when (M, g) is a 3-dimensional unimodular Lie group with left invariant metric. The Lie algebras of 3-dimensional unimodular Lie groups have been all classified in J. Milnor [10] and there are the following six ones: $\mathfrak{su}(2)$, $\mathfrak{sl}(2, \mathbb{R})$, the Lie algebras $\mathfrak{e}(2)$, $\mathfrak{e}(1,1)$ of the groups of rigid motions on the Euclidean plane or the Minkowski plane, the Lie algebra $\mathfrak{h}_3$ of the Heisenberg group, and the commutative Lie algebra $\mathbb{R}^3$. Also, in the paper [12], V. Patrangenaru has determined the isometry groups of (G, g) where G is a simply connected 3-dimensional unimodular Lie group and g is an arbitrary left invariant metric on G. Using these results, we consider the Grassmann geometry on (G, g) of orbital type. Up to the present, the case of $\mathfrak{h}_3$ and the cases of $\mathfrak{e}(2)$, $\mathfrak{e}(1,1)$ have been already considered in J. Inoguchi-K. Kuwabara-H. Naitoh [5] and K. Kuwabara [9], respectively. We mainly consider the cases of $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{R})$ in this paper.

The contents of the paper are as follows. In Section 2, we recall Milnor's classification of 3-dimensional unimodular Lie algebras, and Patrangenaru's determination of the isometry groups of (G, g). In Section 3, we survey the Grassmann geometry on a 3-dimensional unimodular Lie group with left invariant metric, and overview the cases of $\mathfrak{h}_3$, $\mathfrak{e}(2)$ and $\mathfrak{e}(1,1)$.

In Section 4, we consider the case of $\mathfrak{su}(2)$. The Grassmann geometry of this case is divided into the following two cases; the case that the isotropy subgroup of $I_o(G, g)$ at the unity is trivial, and the case that the isotropy subgroup of $I_o(G, g)$ at the unity is $SO(2)$. We call the first case the Grassmann geometry of *trivial isotropy type*, and the second case the Grassmann geometry of *isotropy type* $SO(2)$. For the case of trivial isotropy type, we show that the Grassmann geometry is empty (Theorem 4.2). For the case of isotropy type $SO(2)$, we moreover divide the case into two cases and for the Grassmann geometry of each case we clarify the geometric states of $\mathcal{O}$-surfaces, such as the existence of totally geodesic $\mathcal{O}$-surfaces, flat $\mathcal{O}$-surfaces, and $\mathcal{O}$-surfaces of constant mean curvature (Theorem 4.12).

In Section 5, we consider the case of $\mathfrak{sl}(2,\mathbb{R})$. The Grassmann geometry of this case is, as the case of $\mathfrak{su}(2)$, firstly divided into two cases of trivial isotropy type and of isotropy type $SO(2)$. Moreover the second case is divided into 4 cases according to the difference of behaviors of Grassmann geometry. For each case of trivial isotropy type and of isotropy type $SO(2)$, we also clarify similar geometric states to the case of $\mathfrak{su}(2)$ (Theorem 5.8 and Theorem 5.19 respectively). But in the case of isotropy type $SO(2)$ there remain a unsolved problem about the existence of $\mathcal{O}$-surfaces of constant mean curvature. We will affirmatively solve this problem in the forthcoming paper II.

In Section 6, we first describe two common phenomena all over the Grassmann geometry of 3-dimensional unimodular Lie groups, *i.e.*, the relationship with a contact metric structure on (G, g) and the relationship with the sectional curvature of (G, g). Next we propose some problems related to the Grassmann geometry and lastly give the corrections for the paper [5] on the Grassmann geometry of Heisenberg group. The results of the paper [5] hold true, but there are some mistakes of calculations in it.

2. Left invariant metrics on unimodular Lie groups

2.1. Unimodular Lie groups

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over the field $\mathbb{R}$ of real numbers, furnished with the bracket product $[\cdot,\cdot]$. A Lie algebra $\mathfrak{g}$ is called *unimodular* if for any $X \in \mathfrak{g}$, the adjoint transformation $\mathrm{ad}(X)$ of $\mathfrak{g}$ has the trace 0, and a connected Lie group G associated with $\mathfrak{g}$ is also called unimodular. If a Lie algebra $\mathfrak{g}$ is 3-dimensional, we can naturally define the cross product $\times$ on $\mathfrak{g}$ by taking an inner product $\langle , \rangle$ on $\mathfrak{g}$ and by fixing an orientation on $\mathfrak{g}$. Then there exists a unique linear transformation L of $\mathfrak{g}$ such that $[X, Y] = L(X \times Y)$ for X, $Y \in \mathfrak{g}$. In the paper [10], J. Milnor showed that a 3-dimensional Lie algebra $\mathfrak{g}$ is unimodular if and only if the linear transformation L is symmetric with respect to $\langle , \rangle$, and he classified all the 3-dimensional unimodular Lie algebras $\mathfrak{g}$ by considering the signature of eigenvalues of L. Let λ_1, λ_2, λ_3 be the eigenvalues of L and take a positively oriented orthonomal basis $\{E_1, E_2, E_3\}$ of $\mathfrak{g}$ such that each E_i is an eigenvector with eigenvalue λ_i. Then it follows

$$[E_2, E_3] = \lambda_1 E_1, \quad [E_3, E_1] = \lambda_2 E_2, \quad [E_1, E_2] = \lambda_3 E_3 \tag{2.1}$$

and the classification of $\mathfrak{g}$ is described as follows:

Signature of $(\lambda_1, \lambda_2, \lambda_3)$	Unimodular Lie algebra	Description
$(+,+,+)$	$\mathfrak{su}(2)$	compact, simple
$(-,+,+)$	$\mathfrak{sl}(2,\mathbb{R})$	noncompact, simple
$(+,+,0)$	$\mathfrak{e}(2)$	solvable
$(-,+,0)$	$\mathfrak{e}(1,1)$	solvable
$(0,0,+)$	$\mathfrak{h}_3$	nilpotent
$(0,0,0)$	$\mathbb{R}^3$	commutative

Here $\mathfrak{su}(2)$ denotes the Lie algebra of the special unitary group $SU(2)$, $\mathfrak{sl}(2,\mathbb{R})$ denotes the Lie algebra of the special real linear group $SL(2,\mathbb{R})$, $\mathfrak{e}(2)$ denotes the Lie algebra of the group of rigid motions of Euclidean 2-space, $\mathfrak{e}(1,1)$ denotes the Lie algebra of the group of rigid motions of Minkowski 2-space, $\mathfrak{h}_3$ denotes the Lie algebra of the Heisenberg group H_3 of all 3×3 real matrices of the form $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$, and $\mathbb{R}^3$ denotes the 3-dimensional commutative Lie algebra. Also, we note that in the above list, the order of signatures can be replaced by suitable changes of the orientation on $\mathfrak{g}$ or the numbering of E_i's.

Let $(\mathfrak{g}, \langle,\rangle)$ be a Lie algebra with inner product and G a connected Lie group with the Lie algebra $\mathfrak{g}$, and identify $\mathfrak{g}$ with the Lie algebra of left invariant vector fields on G which are vector fields preserved by the left translations of G. Then $\mathfrak{g}$ is also identified with the tangent space T_eG at the unit element e of G, and there exists a unique Riemannian metric g on G such that $g_e = \langle,\rangle$ and g is preserved by the left translations of G. This metric g is called a *left invariant metric* on G. The Riemannian manifold (G, g) is obviously a Riemannian homogeneous space, and the Riemannian connection ∇ can be calculated by Christoffel's formula as follows:

$$\nabla_{X_i} X_j = (1/2) \sum_k (\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) X_k$$

where $\{X_1, X_2, \cdots, X_n\}$ is an orthonormal basis of $\mathfrak{g}$ and α_{ijk}'s are the constants satisfying $[X_i, X_j] = \sum_k \alpha_{ijk} X_k$, which are called the *structure constants* of $[\cdot,\cdot]$. In particular, if (G, g) is a 3-dimensional unimodular Lie group, it holds by (2.1) that

$$
\begin{aligned}
&\nabla_{E_1}E_2 = \frac{1}{2}(-\lambda_1+\lambda_2+\lambda_3)E_3, \quad \nabla_{E_2}E_1 = \frac{1}{2}(-\lambda_1+\lambda_2-\lambda_3)E_3,\\
&\nabla_{E_2}E_3 = \frac{1}{2}(\lambda_1-\lambda_2+\lambda_3)E_1, \qquad \nabla_{E_3}E_2 = \frac{1}{2}(-\lambda_1-\lambda_2+\lambda_3)E_1,\\
&\nabla_{E_3}E_1 = \frac{1}{2}(\lambda_1+\lambda_2-\lambda_3)E_2, \qquad \nabla_{E_1}E_3 = \frac{1}{2}(\lambda_1-\lambda_2-\lambda_3)E_2,\\
&\nabla_{E_i}E_j = 0 \quad \text{for other } i,\ j.
\end{aligned}
\tag{2.2}
$$

Moreover, from (2.2), we can calculate the curvatures of (G, g) at e as follows. The *Ricci quadratic form* r is diagonalized by the eigenvectors E_1, E_2, E_3, together with the following *principal Ricci curvatures*

$$
r(E_1,E_1) = 2\mu_2\mu_3, \quad r(E_2,E_2) = 2\mu_3\mu_1, \quad r(E_3,E_3) = 2\mu_1\mu_2 \tag{2.3}
$$

where $\lambda_i = \mu_1+\mu_2+\mu_3-\mu_i$ for $i = 1, 2, 3$. The *scalar curvature* ρ is given by the equation $\rho = 2(\mu_2\mu_3+\mu_3\mu_1+\mu_1\mu_2)$. The *sectional curvature* $K(u, v)$ of the plane generated by linearly independent orthonormal vectors u, v can be calculated by the general formula

$$
K(u,v) = \|u\times v\|^2\rho/2 - r(u\times v, u\times v) \tag{2.4}
$$

for any 3-dimensional Riemannian manifold. (Refer to [10] for the detail). For each unimodular Lie algebra, these curvatures will be concretely given in the later case by case argument.

2.2. The special unitary group $SU(2)$

Next, for a 3-dimensional unimodular Lie group G, we give the set $\Lambda(\mathfrak{g})$ of isometry classes of left invariant metrics g on G and the dimension of isometry group $d(I(G,g))$ for each (G,g) when G is simply connected. These have been already given in [12].

Let G be a connected Lie group with the Lie algebra $\mathfrak{su}(2)$. Then the set $\Lambda(\mathfrak{su}(2))$ of isometry classes is bijective to the set $\{(\lambda_1,\lambda_2,\lambda_3) : 0 < \lambda_1 \le \lambda_2 \le \lambda_3\}$, and when G is moreover simply connected, the dimension $d(I(G,g))$ is given in the following according to the isometry class of (G,g).

$$d(I(G,g)) = \begin{cases} 3 & \text{if } \lambda_1 < \lambda_2 < \lambda_3 \\ 4 & \text{if } \lambda_1 = \lambda_2 < \lambda_3 \text{ or } \lambda_1 < \lambda_2 = \lambda_3 \\ 6 & \text{if } \lambda_1 = \lambda_2 = \lambda_3 \end{cases}.$$

In the case when $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, the coresponding g is isometric to the standard metric on the 3-sphere S^3 of constant curvature $\lambda^2/4$.

2.3. The special real linear group $SL(2,\mathbb{R})$

Let G be a connected Lie group with the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Then the set $\Lambda(\mathfrak{sl}(2,\mathbb{R}))$ of isometry classes is bijective to the set $\{(\lambda_1,\lambda_2,\lambda_3) : \lambda_1 < 0 < \lambda_2 \leq \lambda_3\}$, and when G is moreover simply connected, the dimension $d(I(G,g))$ is given in the following according to the isometry class of (G,g).

$$d(I(G,g)) = \begin{cases} 3 & \text{if } \lambda_1 < 0 < \lambda_2 < \lambda_3 \\ 4 & \text{if } \lambda_1 < 0 < \lambda_2 = \lambda_3 \end{cases}.$$

Note that the metric g with $\lambda_2 = \lambda_3$ is isometric to the so-called *Sasaki lift metric* on the unit tangent sphere bundle of a hyperbolic 2-space. (*cf.* [14]).

2.4. The group of rigid motions on Euclidean 2-space

Let G be a connected Lie group with the Lie algebra $\mathfrak{e}(2)$. Then the set $\Lambda(\mathfrak{e}(2))$ of the isometry classes is bijective to the set $\{(\lambda_1,\lambda_2,\ 0) : 0 < \lambda_1 < \lambda_2 \text{ or } \lambda_1 = \lambda_2 = 1\}$, and when G is moreover simply connected, the dimension $d(I(G,g))$ is given in the following according to the isometry class of (G,g).

$$d(I(G,g)) = \begin{cases} 3 & \text{if } 0 < \lambda_1 < \lambda_2 \\ 6 & \text{if } \lambda_1 = \lambda_2 = 1 \end{cases}.$$

In the case when $\lambda_1 = \lambda_2 = 1$, the corresponding g is isometric to the Euclidean metric on the Euclidean 3-space $\mathbb{E}^3$.

2.5. The group of rigid motions on Minkowski 2-space

Let G be a connected Lie group with the Lie algebra $\mathfrak{e}(1,1)$. Then the set $\Lambda(\mathfrak{e}(1,1))$ of isometry classes is bijective to the set $\{(\lambda_1,\lambda_2,\ 0) : -\lambda_2 \leq \lambda_1 < 0 < \lambda_2\}$, and when G is moreover simply connected, the dimension $d(I(G,g))$ is equal to 3 for any g. The metric g with $\lambda_1 = -\lambda_2$ is of particular interest. In fact, the simply connected Riemannian homogeneous manifold (G,g) is isometric to the space Sol_3 of solvegeometry in the sense of W. Thurston [16].

2.6. The Heisenberg group H_3

Let G be a connected Lie group with the Lie algebra $\mathfrak{h}_3$. Then the set $\Lambda(\mathfrak{h}_3)$ of isometry classes is bijective to the set $\{(0,\ 0, \lambda_3) : 0 < \lambda_3\}$, and when G is moreover simply connected, the dimension $d(I(G,g))$ is equal to 4 for any g.

2.7. The commutative group $\mathbb{R}^3$

If G has the commutative Lie algebra $\mathbb{R}^3$, any left invariant metric g is locally isometric to the Euclidean metric on $\mathbb{E}^3$.

3. Grassmann geometry on simply connected unimodular Lie groups

In this section we consider the Grassmann geometry for surfaces of a 3-dimensional simply connected unimodular Lie group with a left invariant metric. Let G be a 3-dimensional simply connected unimodular Lie group and g a left invariant metric on G. Denote by $I_o(G,g)$ the connected component of the isometry group $I(G,g)$ containing the identity. We take an orbit $\mathcal{O}$ in $\mathrm{Gr}^2(TG)$ under the action of $I_o(G,g)$. As described in Introduction, $\mathrm{Gr}^2(TG)$ is the Grassmann bundle over G of all 2-planes tangent to G and $I_o(G,g)$ naturally acts on $\mathrm{Gr}^2(TG)$ through isometries of (G,g). Also, since (G,g) is a Riemannian homogeneous space, the orbit $\mathcal{O}$ is a homogeneous bundle over G with respect to $I_o(G,g)$. In this case an *$\mathcal{O}$-surface* is by definition a connected surface S in G such that $T_xS \in \mathcal{O}$ for any $x \in S$ and the *$\mathcal{O}$-geometry* is the collection of such the $\mathcal{O}$-surfaces.

In this study of Grassmann geometry, our aims are to determine whether a given $\mathcal{O}$-geometry is empty or not and next if it is not empty, to see whether it has somewhat canonical $\mathcal{O}$-surfaces, *e.g.*, minimal surfaces or parallel mean curvature surfaces, or not. To this end, we divide our argument into three cases: (i) dim $I_o(G,g) = 3$, (ii) dim $I_o(G,g) = 4$, and (iii) dim $I_o(G,g) = 6$.

In the case (i), $I_o(G,g)$ consists of all left translations of G. Hence an $I_o(G,g)$-orbit $\mathcal{O}$ induces a distribution on G which is constructed by the left translations of the unique plane $\Pi \subset T_eG$ which belongs to $\mathcal{O}$. So, the orbit space of $I_o(G,g)$-orbits is diffeomorphic to the Grassmann manifold $\mathrm{Gr}^2(T_eG)$ of all 2-planes in $T_e(G)$, which is also diffeomorphic to the real projective plane $\mathbb{R}P^2$. In this case an $\mathcal{O}$-geometry is not empty if and only if the distribution on G corresponding to $\mathcal{O}$ is involutive.

In the case (ii), $I_o(G,g)$ is the semi-direct of the group of left translations of G and the 1-dimensional isotropy subgroup K_o in $I_o(G,g)$ defined by putting $K_o = \{\varphi \in I_o(G,g) : \varphi(e) = e\}$, where K_o is isomorphic to $SO(2)$. Hence an $I_o(G,g)$-orbit $\mathcal{O}$ is a homogeneous bundle over G with a K_o-orbit in $\mathrm{Gr}^2(T_eG)$ as the fiber of e, and so the orbit space of $I_o(G,g)$-orbits is diffeomorphic to the quotient space $K_o \backslash \mathrm{Gr}^2(T_eG)$. Since the K_o-action on $T_e(G)$ is equivalent to a standard $SO(2)$-action on $\mathbb{R}^3$, we can see that this quotient space is also diffeomorphic to the interval $[0,1]$. In this case an $\mathcal{O}$-geometry is not empty if and only if there exists an involutive local section of the homogeneous bundle $\mathcal{O}$ over G. In fact, if there exists such an involutive local section, the leaves of its distribution are all $\mathcal{O}$-surfaces, and the converse also holds true by the following lemma.

Lemma 3.1 *Let (M,g) be a homogeneous Riemannian manifold and S a submanifold in M. Take a point $p \in S$. Then there exists an involutive local distribution $\mathcal{D}$ around p such that all the leaves of $\mathcal{D}$ are locally congruent to S.*

Proof. Define a mapping ϕ of $I_o(M,g)$ to M by putting $\phi(f) = f(p)$ for $f \in I_o(M,g)$. Since (M,g) is a homogeneous Riemannian manifold, the mapping ϕ is surjective and the differential ϕ_{*id} of ϕ at the identity map id is also a linear mapping of $T_{id}(I_o(M,g))$ onto T_pM. We can now take a local submanifold P of $I_o(M,g)$ through id such that the mapping $\Phi : P \times S \to M$ defined by putting $\Phi(f,q) = f(q)$ for $(f,q) \in P \times S$ is a local diffeomorphism around (id,p). Then, through Φ, the distribution TS on $P \times S$ induces the desired local distribution on M around p. $\square$

In the case (iii), $I_o(G,g)$ is the semi-direct of the group of left translations of G and the 3-dimensional isotropy subgroup K_o in $I_o(G,g)$, where K_o is isomorphic to $SO(3)$. Since K_o acts transitively on $\mathrm{Gr}^2(T_eG)$, an $I_o(G,g)$-orbit $\mathcal{O}$ is only one, thus, there exists only one $\mathcal{O}$-geometry on (G,g). In this case, any surface in G is an $\mathcal{O}$-surface.

According to the cases (i), (ii), and (iii), we call each case the Grassmann geometry of *trivial isotropy type*, of *isotropy type* $SO(2)$, and of *isotropy type* $SO(3)$, respectively.

In the papers [9], the Grassmann geometry for the cases $\mathfrak{g} = \mathfrak{e}(2)$ and

$\mathfrak{e}(1,1)$ has been studied. The Grassmann geometry for these cases is of trivial isotropy type. Also, in the paper [5], the one for the case $\mathfrak{g} = \mathfrak{h}_3$ has been studied. In this case the Grassmann geometry is of isotropy type $SO(2)$. The Grassmann geometry of isotropy type $SO(3)$ is just the surface theory of $\mathbb{E}^3$ or S^3. We here overview the results in the papers [9] and [5].

3.1. Grassmann geometry for the case $\mathfrak{g} = \mathfrak{e}(2)$

Let G be a simply connected Lie group whose Lie algebra $\mathfrak{g} = \mathfrak{e}(2)$ and g a left invariant metric on G determined by the the orthonormal basis $\{E_1, E_2, E_3\}$ with $0 < \lambda_1 < \lambda_2$ and $\lambda_3 = 0$. In this case $\dim I_o(G,g) = 3$, thus, the Grassmann geometry on (G,g) is of trivial isotropy type. Hence the orbit space of $I_o(G,g)$-orbits is given by $\mathrm{Gr}^2(T_eG)$, denoted by $\mathrm{Gr}^2(\mathfrak{g})$.

Let $S^2(\mathfrak{g})$ be the unit sphere in $\mathfrak{g}$ centered at the origin and for $W \in S^2(\mathfrak{g})$, let $P(W)$ denote the linear plane orthogonal to W. The mapping $P : S^2(\mathfrak{g}) \to \mathrm{Gr}^2(\mathfrak{g})$ induces the bijection $P : \mathbb{R}P^2(\mathfrak{g}) \to \mathrm{Gr}^2(\mathfrak{g})$. Now for $W \in S^2(\mathfrak{g})$ let $\mathcal{O}(P(W))$ be the $I_o(G,g)$-orbit containing the 2-plane $P(W)$. Then we have the following theorem.

Theorem 3.2 (Kuwabara [9]) *Let G be a simply connected Lie group with the Lie algebra $\mathfrak{e}(2)$ and g a left invariant metric on G. Take λ_i, E_i $(i = 1, 2, 3)$ corresponding to g as in Section 2 and assume that $0 < \lambda_1 < \lambda_2$ and $\lambda_3 = 0$. Then, the $\mathcal{O}(P(W))$-geometry is non-empty if and only if $P(W) = P(E_3)$. Moreover, any $\mathcal{O}(P(E_3))$-surface S is a minimal flat surface in G without geodesic points and the tangent planes of S have constant positive sectional curvature $(\lambda_1 - \lambda_2)^2/4$ of (G,g).*

Remark 3.3 In this case, since the orbit $\mathcal{O}(P(E_3))$ gives a left invariant distribution on G, the maximal $\mathcal{O}(P(E_3))$-surfaces are homogeneous surfaces in G which are congruent to each other, in particular, they are complete. Moreover, they are diffeomorphic to $\mathbb{R}^2$ since G is the universal covering of $E(2)$ diffeomorphic to $\mathbb{R}^3$ and the distribution is generated by the left invariant vector fields E_1 and E_2 which satisfy $[E_1, E_2] = 0$. According to [6], the $\mathcal{O}(P(E_3))$-surfaces are the only surfaces in G with parallel second fundamental form. Thus in G, the class of nonempty $\mathcal{O}(W)$-surfaces is identical to the class of parallel surfaces.

Remark 3.4 For $W \in S^2(\mathfrak{g})$, the sectional curvature $K(P(W))$ of (G,g) is by the formulas (2.3) and (2.4) as follows:

$$\begin{aligned}K(P(W)) = &-\frac{(\lambda_1-\lambda_2)^2}{4}-\frac{\lambda_1^2-\lambda_2^2}{2}\langle W,E_1\rangle^2\\&-\frac{\lambda_2^2-\lambda_1^2}{2}\langle W,E_2\rangle^2+\frac{(\lambda_1-\lambda_2)^2}{2}\langle W,E_3\rangle^2,\end{aligned}$$

where the principal Ricci curvatures are given by the equations

$$r(E_1,E_1)=\frac{\lambda_1^2-\lambda_2^2}{2},\quad r(E_2,E_2)=\frac{\lambda_2^2-\lambda_1^2}{2},\quad r(E_3,E_3)=-\frac{(\lambda_1-\lambda_2)^2}{2}.$$

Then $K(P(E_3))$ is a critical value of the curvature function $K(P(W))$ on $S^2(\mathfrak{g})$ which is neither the maximum nor the minimum.

3.2. Grassmann geometry for the case $\mathfrak{g}=\mathfrak{e}(1,1)$

Next let G be a simply connected Lie group whose Lie algebra $\mathfrak{g}$ is $\mathfrak{e}(1,1)$ and g a left invariant metric on G. We retain the same notations as in Section 2 and the previous Subsection 3.1, and in this case assume that $-\lambda_2\le\lambda_1<0<\lambda_2$ and $\lambda_3=0$. Then, since $\dim I_o(G,g)=3$, the Grassmann geometry on (G,g) is of trivial isotropy type and so the orbit space of $I_o(G,g)$-orbits is bijective to $\mathrm{Gr}^2(\mathfrak{g})$. Consider a G-orbit $\mathcal{O}(P(W))$ for $W\in S^2(\mathfrak{g})$. Then we have the following theorem.

Theorem 3.5 (Kuwabara [9]) *Let G be a simply connected Lie group with the Lie algebra $\mathfrak{e}(1,1)$ and g a left invariant metric on G. Take λ_i, E_i $(i=1,2,3)$ corresponding to g as in Section 2 and assume that $-\lambda_2\le\lambda_1<0<\lambda_2$ and $\lambda_3=0$. Then, for $W\in S^2(\mathfrak{g})$, the $\mathcal{O}(P(W))$-geometry is non-empty if and only if W satisfies the equation*

$$\lambda_1\langle W,E_1\rangle^2+\lambda_2\langle W,E_2\rangle^2=0. \tag{3.5}$$

Moreover, for such an $\mathcal{O}(W)$-geometry, an $\mathcal{O}(W)$-surface is a minimal surface in G of constant nonpositive curvature $\lambda_2(\lambda_1-\lambda_2)\langle W,E_2\rangle^2$, where $0\le\langle W,E_2\rangle^2\le\lambda_1/(\lambda_1-\lambda_2)$. In particular an $\mathcal{O}(W)$-geometry has a flat surface if and only if $P(W)=P(E_3)$. Also, an $\mathcal{O}(W)$-geometry has a totally geodesic surface if and only if g satisfies that $\lambda_1+\lambda_2=0$ and $P(W)$ is either of $P\big(\frac{1}{\sqrt{2}}(E_1\pm E_2)\big)$.

Remark 3.6 In this case the Grassmann geometry is of trivial isotropy type. Hence, by the same reason as in Remark 3.3 of the $\mathfrak{e}(2)$ case, we can see that for a non-empty $\mathcal{O}(P(W))$-geometry, the maximal $\mathcal{O}(P(W))$-surfaces are homogeneous surfaces in G which are congruent to each other.

Analogous to the case $\mathfrak{g} = \mathfrak{e}(2)$, the class of non-empty $\mathcal{O}(W)$-surfaces coincides with the class of parallel surfaces. See [6].

Remark 3.7 In this case, the sectional curvature $K(P(W))$ of (G, g) is given by the same form as for the case $\mathfrak{e}(2)$. Hence, if an $\mathcal{O}(P(W))$-geometry is not empty, it holds

$$\big(\lambda_2 + (3 + 2\sqrt{2})\lambda_1\big)\big(\lambda_2 + (3 - 2\sqrt{2})\lambda_1\big)/4 \leq K(P(W)) \leq (\lambda_1 - \lambda_2)^2/4.$$

Note that the set of $\mathcal{O}(P(W))$-orbits whose geometries are non-empty is identified with two projective lines in $\mathbb{R}P^2$. Because the equation (3.5) is decomposed into the product of two linear equations. Then, the common point of the projective lines corresponds to just the $\mathcal{O}(P(E_3))$-orbit whose geometry has a flat surface, and moreover the sectional curvature $K(P(E_3))$ attains the maximum value $(\lambda_1 - \lambda_2)^2/4$ in the above inequality, which also gives the maximum of the curvature function $K(P(W))$ on $S^2(\mathfrak{g})$. Also, in the case when $\lambda_1 + \lambda_2 = 0$, the sectional curvatures $K\big(P\big(\frac{1}{\sqrt{2}}(E_1 \pm E_2)\big)\big)$ for the $\mathcal{O}(P(W))$-orbits whose geometries have totally geodesic surfaces attain the minimum value $-\lambda_1^2$ in the above inequality.

3.3. Grassmann geometry for the case $\mathfrak{g} = \mathfrak{h}_3$

Next let G be a simply connected Lie group whose Lie algebra $\mathfrak{g}$ is $\mathfrak{h}_3$. Then G is isomorphic to the *Heisenberg group* H_3. Let g be a left invariant metric on G. We retain the same notations as in Section 2 and the Subsection 3.1, and assume that $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 > 0$. Then, since $\dim I_o(G, g) = 4$, the Grassmann geometry on (G, g) is of isotropy type $SO(2)$ and the action of the isotropy K_o on $\mathfrak{g}$ is given by the $SO(2)$-action on the (E_1E_2)-plane. Hence the orbit space of $I_o(G, g)$-orbits is bijective to the quotient space $SO(2)\backslash\mathrm{Gr}^2(\mathfrak{g})$, which is moreover identified with $SO(2)\backslash\mathbb{R}P^2(\mathfrak{g})$. Consider a G-orbit $\mathcal{O}(P(W))$ for $W \in S^2(\mathfrak{g})$. Then, for W, $W' \in S^2(\mathfrak{g})$, it holds that $\mathcal{O}(P(W)) = \mathcal{O}(P(W'))$ if and only if there exists $T \in SO(2)$ such that $T(W) = W'$. Hence we can parametrize the orbit space of $I_o(G, g)$-orbits by the *height* h of W from the (E_1E_2)-plane, which is defined by putting $h = \langle W, E_3 \rangle$ where $0 \leq h \leq 1$. We here remark that the orbit space can be also parametrized by the sectional curvature $K(P(W))$. In fact, by the formula (2.4), the sectional curvature $K(P(W))$ is given as follows:

$$\begin{aligned} K(P(W)) &= \frac{\lambda_3^2}{4}(\langle W, E_1\rangle^2 + \langle W, E_2\rangle^2 - 3\langle W, E_3\rangle^2) \\ &= \frac{\lambda_3^2}{4}(1 - 4h^2). \end{aligned} \tag{3.6}$$

In the previous paper [5], the parametrization of the orbit space is given by the sectional curvature, but we here use the height h as its parametrization. Denote by $\mathcal{O}(h)$ the $I_o(G, g)$-orbit which contains the planes $P(W)$ for the elements W in $S^2(\mathfrak{g})$ with height h. Then we have the following theorem.

Theorem 3.8 (Inoguchi, Kuwabara and Naitoh [5]) *Let G be a simply connected Lie group with Lie algebra $\mathfrak{h}_3$ and g a left invariant metric on G. Take λ_i, E_i $(i = 1, 2, 3)$ corresponding to g as in Section* 2 *and assume that $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 > 0$. Then, for $h \in [0, 1]$, the $\mathcal{O}(h)$-geometry is non-empty if and only if $h \neq 1$. Moreover the following* (i) *and* (ii) *hold*:

(i) *Let S be an $\mathcal{O}(0)$-surface in G. Then S is a flat surface with no geodesic point. Also, S is a minimal surface* (*resp. a surface of nonzero constant mean curvature*) *if and only if it is a part of a Hopf cylinder over a straight line* (*resp. a circle*) *in the u_1u_2-plane*;

(ii) *Let $0 < h < 1$. Then an $\mathcal{O}(h)$-surface S in G has constant negative curvature ${-\lambda_3}^2h^2$ and it has no geodesic point. Also, there exists no $\mathcal{O}(h)$-surface of constant mean curvature.*

Remark 3.9 In the statement (i) of Theorem 3.8, the notations u_i $(i = 1, 2, 3)$ denote the global coordinate functions on the Heisenberg group H_3 defined by the relation

$$x = \exp(u_1(x)E_1 + u_2(x)E_2 + u_3(x)E_3)$$

for $x \in H_3$, and a Hopf cylinder means a surface defined by the set $\{(u_1, u_2, u_3) : (u_1, u_2) \in \gamma\}$ for some curve γ in the (u_1u_2)-plane. We here remark that the $\mathcal{O}(0)$-surfaces are nothing but the Hopf cylinders and moreover the $\mathcal{O}(0)$-geometry is just the same as the geometry of Hopf cylinders in the Euclidean $(u_1u_2u_3)$-space. Hopf cylinders of constant mean curvature are characterized as the only surfaces in G with parallel second fundamental form. Note that Heisenberg group G does not contain totally geodesic surfaces. See [1].

Remark 3.10 In this case, by (3.6) the sectional curvature $K(P)$ of (G, g)

satisfies the inequality $-3{\lambda_3}^2/4 \leq K(P) \leq {\lambda_3}^2/4$. If $P \in \mathcal{O}(0)$, the sectional curvature $K(P)$ attains the maximum value ${\lambda_3}^2/4$ and if $P \in \mathcal{O}(1)$, it attains the minimum value $-3{\lambda_3}^2/4$.

Remark 3.11 There are simple mistakes of calculations in the argument of the paper [5]. They are related to the statement (ii) in Theorem 3.8 though the statement holds true. In the last of this paper we will give the corrections for them.

4. Grassmann geometry on the special unitary group $SU(2)$

Let G be a simply connected Lie group with the Lie algebra $\mathfrak{su}(2)$. Then G is isomorphic to $SU(2)$. In this section we take a left invariant metric g on G and consider the Grassmann geometry on $SU(2)$, first the case of trivial isotropy type and next the case of isotropy type $SO(2)$. We again retain the notations in Section 2 and Section 3.

4.1. Grassmann geometry of trivial isotropy type

In this subsection we assume that the triple $(\lambda_1, \lambda_2, \lambda_3)$ satisfies the condition $0 < \lambda_1 < \lambda_2 < \lambda_3$. Recall that the principal Ricci curvatures of (G, g) are generally given by the following formulas

$$r(E_1, E_1) = 2\mu_2\mu_3, \quad r(E_2, E_2) = 2\mu_3\mu_1, \quad r(E_3, E_3) = 2\mu_1\mu_2$$

where $\mu_1 = \frac{1}{2}(-\lambda_1+\lambda_2+\lambda_3)$, $\mu_2 = \frac{1}{2}(\lambda_1-\lambda_2+\lambda_3)$, and $\mu_3 = \frac{1}{2}(\lambda_1+\lambda_2-\lambda_3)$. In this case, by the condition $0 < \lambda_1 < \lambda_2 < \lambda_3$, the following two cases occur: the case (i) that the principal Ricci curvatures are all distinct, and the case (ii) that $\lambda_1 + \lambda_2 = \lambda_3$ and it holds $r(E_1, E_1) = r(E_2, E_2) = 0$ and $r(E_3, E_3) = 2\lambda_1\lambda_2$. The following lemma holds for both cases.

Lemma 4.1 *For both cases* (i) *and* (ii) *the Grassmann geometry on* (G, g) *is of trivial isotropy type.*

Proof. We note that K_o is connected since $I_o(G, g)$ is so. We first consider the case (i). Take any $\varphi \in K_o$. Then since φ preserves the Ricci quadratic form r, it follows that $\varphi(E_i) = \pm E_i$ for $i = 1, 2, 3$. Then φ is the identity since K_o is connected. This implies that K_o is trivial. We next consider the case (ii). Denote by R the curvature tensor of (G, g) and by ∇R its covariant derivative. Then, by (2.2), it follows

$$(\nabla_{E_1} R)(E_1, E_2)E_1 = 2\lambda_1\lambda_2^2 E_3, \quad (\nabla_{E_1} R)(E_3, E_1)E_1 = -2\lambda_1\lambda_2^2 E_2,$$
$$(\nabla_{E_2} R)(E_1, E_2)E_2 = 2\lambda_1^2\lambda_2 E_3, \quad (\nabla_{E_2} R)(E_2, E_3)E_1 = 2\lambda_1^2\lambda_2 E_2,$$
$$(\nabla_{E_i} R)(E_j, E_k)E_\ell = 0 \quad \text{for other } i, j, k, \ell.$$

Take any $\varphi \in K_o$. Then we may suppose that φ_{*e} has the following form:

$$\varphi_{*e}(E_1) = \cos\theta E_1 + \sin\theta E_2, \quad \varphi_{*e}(E_2) = -\sin\theta E_1 + \cos\theta E_2,$$
$$\varphi_{*e}(E_3) = E_3.$$

for some θ. (See the proof of Lemma 4.4 in the next Subsection 4.2.) Using the above results on ∇R, we calculate $(\nabla_{\varphi_*(E_1)} R)(\varphi_*(E_1), \varphi_*(E_2))\varphi_*(E_1)$ and $\varphi_*\{(\nabla_{E_1} R)(E_1, E_2)E_1\}$ as follows:

$$(\nabla_{\varphi_*(E_1)} R)(\varphi_*(E_1), \varphi_*(E_2))\varphi_*(E_1) = \left\{2\lambda_1\lambda_2^2\cos^2\theta + 2\lambda_1^2\lambda_2\sin^2\theta\right\}E_3,$$
$$\varphi_*\{(\nabla_{E_1} R)(E_1, E_2)E_1\} = 2\lambda_1\lambda_2^2 E_3.$$

Since φ is an isometry, these vectors coincide and it follows that $\lambda_2\cos^2\theta + \lambda_1\sin^2\theta = \lambda_2$, thus, $(\lambda_1 - \lambda_2)\sin^2\theta = 0$. Since $\lambda_1 \neq \lambda_2$, it holds $\sin\theta = 0$, and since K_o is connected, it follows $\varphi = 1$. This implies that K_o is also trivial. □

Now the following result holds for the Grassmann geometry of this type.

Theorem 4.2 *Let G be a simply connected Lie group with the Lie algebra $\mathfrak{su}(2)$ and g a left invariant metric on G which satisfies the condition $0 < \lambda_1 < \lambda_2 < \lambda_3$. Then for any $W \in S^2(\mathfrak{g})$, the $\mathcal{O}(P(W))$-geometry is empty.*

Proof. Let $\mathcal{D}(W)$ be the left invariant distribution on G defined by the $\mathcal{O}(P(W))$-orbit and take a basis $\{U, V\}$ of the distribution $\mathcal{D}(W)$ such that $U \times V = W$. Then, since it holds that $\langle [U, V], W\rangle = \langle L(W), W\rangle$, the distribution $\mathcal{D}(W)$ on G is involutive if and only if it holds that $\langle L(W), W\rangle = 0$. But, in this case, it does not occur since $\lambda_i > 0$ for all i. In fact, it follows that

$$\langle L(W), W\rangle = \lambda_1\langle W, E_1\rangle^2 + \lambda_2\langle W, E_2\rangle^2 + \lambda_3\langle W, E_3\rangle^2.$$

Hence, any $\mathcal{O}(P(W))$-geometry is empty. □

Remark 4.3 (1) For a general case of trivial isotropy type, it holds that an $\mathcal{O}(P(W))$-geometry is not empty if and only if $\langle L(W), W\rangle = 0$.

(2) The nonexistence of surfaces with parallel second fundamental form in G with $0 < \lambda_1 < \lambda_2 < \lambda_3$ is obtained in [7].

4.2. Grassmann geometry of isotropy type $SO(2)$

Next we consider the cases that $0 < \lambda_1 = \lambda_2 < \lambda_3$ or that $0 < \lambda_1 < \lambda_2 = \lambda_3$. In these cases the corresponding left invariant metrics g on $SU(2)$ are called the *Berger metrics* (*cf.* [3]). We first consider the case that $0 < \lambda_1 = \lambda_2 < \lambda_3$, where we put $\lambda_1 = \lambda_2 = \lambda$.

Set $G = SU(2)$ and let g be a left invariant metric on G of this case. We study the isotropy K_o in $I_o(G, g)$. In this case the principal Ricci curvatures of (G, g) are given in the following.

$$r(E_1, E_1) = \frac{1}{2}\lambda_3(2\lambda - \lambda_3), \quad r(E_2, E_2) = \frac{1}{2}\lambda_3(2\lambda - \lambda_3), \quad r(E_3, E_3) = \frac{1}{2}\lambda_3^2,$$

where, since $\lambda \neq \lambda_3$, it holds that $r(E_1, E_1) = r(E_2, E_2) \neq r(E_3, E_3)$. Then the following holds.

Lemma 4.4 *The Grassmann geometry of this case is of isotropy type $SO(2)$.*

Proof. Let $\varphi \in K_o$. Since φ is an isometry, its differential φ_{*e} preserves the eigenspaces of the Ricci quadratic form r. Hence $\varphi_{*e} \in O(2) \times \{\pm 1\}$ where $O(2)$ denotes the orthogonal group of the (E_1E_2)-plane and ± 1 the identity or minus the identity on the E_3-line, and so it follows that $K_o \subset O(2) \times \{\pm 1\}$. Note that K_o is connected since $I_o(G, g)$ is so. Then φ_{*e} has the following form:

$$\varphi_{*e}(E_1) = \cos\theta E_1 + \sin\theta E_2, \quad \varphi_{*e}(E_2) = -\sin\theta E_1 + \cos\theta E_2,$$
$$\varphi_{*e}(E_3) = E_3.$$

for some θ. Conversely suppose that a linear isometry ϕ of $\mathfrak{g}$ has this form. Then we can see that ϕ is an automorphism of the Lie algebra $\mathfrak{g}$. Since G is simply connected, ϕ induces a unique automorphism ρ of G such that $d\rho = \phi$, and moreover, since ϕ is an isometry, ρ preserves the left invariant metric g. Hence it follows that $\rho \in K_o$. These imply that the K_o-action on $\mathfrak{g}$ is the $SO(2)$-action on the (E_1E_2)-plane. □

By this lemma, our Grassmann geometry is the same type as the ones for the Heisenberg case in the Subsection 3.3, and so the orbit space of $I_o(G,g)$-orbits is the quotient space $SO(2)\backslash\mathbb{R}P^2(\mathfrak{g})$. In the following, similarly to the Heisenberg case, we use the height h from the (E_1E_2)-plane as a parametrization of the orbit space. Then $\mathcal{O}(h)$ where $0 \leq h \leq 1$ expresses the $I_o(G,g)$-orbit which contains the planes $P(W)$ for the elements W in $S^2(\mathfrak{g})$ with height h.

4.2.1 Existence of Grassmann geometry

We first consider the case of $\mathcal{O}(1)$-orbit. In this case, since $h = 1$, it follows that $\mathcal{O}(1) = \mathcal{O}(P(E_3))$ and $K_o(P(E_3)) = \{P(E_3)\}$. Hence the orbit $\mathcal{O}(1)$ induces a unique left invariant distribution on G. Similarly to the case of trivial isotropy type, we have the following proposition.

Proposition 4.5 *The $\mathcal{O}(1)$-geometry is empty.*

Proof. By Remark 4.3, the $\mathcal{O}(1)$-geometry is empty if and only if $\langle L(E_3), E_3\rangle \neq 0$. In this case $\langle L(E_3), E_3\rangle = \lambda_3 \neq 0$. □

Before we consider the $\mathcal{O}(h)$-geometries where $h \neq 1$, we recall a local coordinate system of $SU(2)$ diffeomorphic to S^3. Let S^3 regard the unit sphere in Euclidena 4-space $\mathbb{R}^4$ centered at the origin of $\mathbb{R}^4$ and consider the correspondence:

$$S^3 \ni (x,y,z,w) \to \begin{pmatrix} x+iy & z+iw \\ -z+iw & x-iy \end{pmatrix} \in SU(2).$$

Set $D = \{(y,z,w) \in \mathbb{R}^3 : y^2+z^2+w^2 < 1\}$ and put $x = \sqrt{1-y^2-z^2-w^2}$ on D. The mapping $D \ni (y,z,w) \longmapsto (x,y,z,w) \in S^3$ gives a local coordinate system of S^3 on D. We regard this as a local coordinate system of $SU(2)$ through the above correspondence. Then the origin $(0,0,0)$ in D corresponds to the unit e in $SU(2)$. Let e_i $(i = 1,2,3)$ be the elements in $\mathfrak{g} = \mathfrak{su}(2)$ defined by the equations

$$e_1 = \frac{\sqrt{\lambda\lambda_3}}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \frac{\sqrt{\lambda\lambda_3}}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \frac{\lambda}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and $\hat{E}_i$ $(i = 1,2,3)$ the vector fields of $SU(2)$ defined by putting $\hat{E}_i(q) = q\cdot e_i$ for $q \in SU(2)$. Then the vector fields $\hat{E}_i$ are left invariant and they satisfy the following relations

$$\left[\hat{E}_1, \hat{E}_2\right] = \lambda_3 \hat{E}_3, \quad \left[\hat{E}_2, \hat{E}_3\right] = \lambda \hat{E}_1, \quad \left[\hat{E}_3, \hat{E}_1\right] = \lambda \hat{E}_2.$$

In the following we identify the vector fields $\hat{E}_i$ with the elements $E_i \in \mathfrak{g}$. Then, the following holds on D.

$$(E_1, E_2, E_3) = \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial w}\right) \begin{pmatrix} \frac{\sqrt{\lambda\lambda_3}}{2} x & -\frac{\sqrt{\lambda\lambda_3}}{2} w & \frac{\lambda}{2} z \\ \frac{\sqrt{\lambda\lambda_3}}{2} w & \frac{\sqrt{\lambda\lambda_3}}{2} x & -\frac{\lambda}{2} y \\ -\frac{\sqrt{\lambda\lambda_3}}{2} z & \frac{\sqrt{\lambda\lambda_3}}{2} y & \frac{\lambda}{2} x \end{pmatrix}. \tag{4.7}$$

Now we consider the $\mathcal{O}(h)$-geometry on $SU(2)$ where $h \neq 1$. In this case the set $\mathcal{O}(h) \cap \mathrm{Gr}^2(\mathfrak{g})$ is identified with the circle in $S^2(\mathfrak{g})$ which consists of the unit vectors with height h. For a local function θ on an open set O in $G = SU(2)$), set

$$\begin{aligned} X(q) &= h \sin(\theta(q)) E_1(q) - h \cos(\theta(q)) E_2(q) + \sqrt{1-h^2} E_3(q), \\ Y(q) &= \cos(\theta(q)) E_1(q) + \sin(\theta(q)) E_2(q), \\ N(q) &= \sqrt{1-h^2} \sin(\theta(q)) E_1(q) - \sqrt{1-h^2} \cos(\theta(q)) E_2(q) - h E_3(q) \end{aligned} \tag{4.8}$$

where $q \in O$, and moreover put $P_q = \mathbb{R}X(q) \oplus \mathbb{R}Y(q)$ where $q \in O$ and $\mathcal{D}^\theta = \{P_q : q \in O\}$. Then the set $\{X, Y, N\}$ is an orthonormal frame of vector fields on O, the planes P_q belong to $\mathcal{O}(h)$, and $\mathcal{D}^\theta$ gives the distribution on O generated by the planes P_q. By Lemma 3.1, the $\mathcal{O}(h)$-geometry is non-empty if and only if there exist an open set O in G and a function θ on O such that $\mathcal{D}^\theta$ is involutive. We may here suppose that O is a neighbourhood of the unity e which is contained in the domain D, since (G, g) is a Riemannian homogeneous space.

We now study a necessary and sufficient condition that $\mathcal{D}^\theta$ is involutive. By the definition of involutivity it is the condition that $g([X, Y], N) = 0$, thus,

$$\begin{aligned} &h\sqrt{1-h^2} \sin\theta(E_1\theta) - h\sqrt{1-h^2} \cos\theta(E_2\theta) \\ &\qquad + (1-h^2)(E_3\theta) + \lambda(1-h^2) + \lambda_3 h^2 = 0. \end{aligned} \tag{4.9}$$

Rewriting this in terms of x, y, z, w by (4.7), we have the following quasi-linear partial differential equation (shortly PDE) of the first order with unknown function θ:

$$\begin{aligned}
&\left\{\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}x\sin\theta+\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}w\cos\theta+(1-h^2)z\right\}\left(\frac{\partial\theta}{\partial y}\right)\\
&\quad+\left\{\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}w\sin\theta-\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}x\cos\theta-(1-h^2)y\right\}\left(\frac{\partial\theta}{\partial z}\right)\\
&\quad+\left\{-\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}z\sin\theta-\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}y\cos\theta+(1-h^2)x\right\}\left(\frac{\partial\theta}{\partial w}\right)\\
&\quad+2(1-h^2)+2\frac{\lambda_3}{\lambda}h^2=0.
\end{aligned}\tag{4.10}$$

Let us consider the existence of solutions for this equation. The characteristic ordinary differential equations (shortly ODE's) associated with this equation are given in the following.

$$\frac{dy}{dt}=\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}x\sin\theta+\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}w\cos\theta+(1-h^2)z,\tag{4.11}$$

$$\frac{dz}{dt}=\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}w\sin\theta-\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}x\cos\theta-(1-h^2)y,$$

$$\frac{dw}{dt}=-\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}z\sin\theta-\sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}y\cos\theta+(1-h^2)x,$$

$$\frac{d\theta}{dt}=-2(1-h^2)-2\frac{\lambda_3}{\lambda}h^2.\tag{4.12}$$

Take an initial plane Q, initial values of solutions x, y, z, w and an initial function of θ when $t=0$, as follows: $Q=\{(y,z,w)\in D : y=0\}$, and for a, b such that $(0,a,b)\in Q$

$$y(t,a,b)|_{t=0}=0,\quad z(t,a,b)|_{t=0}=a,\quad w(t,a,b)|_{t=0}=b,$$

$$x(t,a,b)|_{t=0}=\sqrt{1-a^2-b^2},\quad \theta(t,a,b)|_{t=0}=\varphi(a,b)$$

where $\varphi(a,b)$ is an arbitrary function. Then, from (4.12), it follows that

$$\theta(t,a,b) = -2\left\{(1-h^2) + \frac{\lambda_3}{\lambda}h^2\right\}t + \varphi(a,b), \tag{4.13}$$

and the Jacobian $\frac{\partial(y,z,w)}{\partial(t,a,b)}\Big|_{t=0}$ when $t=0$ is given by

$$\frac{\partial(y,z,w)}{\partial(t,a,b)}\bigg|_{t=0} = \sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}\big(\sqrt{1-a^2-b^2}\sin\varphi + b\cos\varphi\big) + (1-h^2)a. \tag{4.14}$$

Hence we have the following proposition.

Proposition 4.6 *For height h such that $0 \leq h < 1$, the $\mathcal{O}(h)$-geometry is not empty.*

Proof. Let $\varphi = 0$. Then, by (4.14), it follows

$$\frac{\partial(y,z,w)}{\partial(t,a,b)}\bigg|_{t=0} = \sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}b + (1-h^2)a.$$

Since in this case $1-h^2 \neq 0$, it holds that $\frac{\partial(y,z,w)}{\partial(t,a,b)}\Big|_{t=0} \neq 0$ for sufficiently small b and nozero a. Then, by the inverse mapping theorem, the variables t, a, b can be solved as functions of the variables y, z, w for a, b as above and sufficiently small t. This implies that the equation (4.10) has a local solution. Hence the $\mathcal{O}(h)$-geometry is non-empty. $\square$

4.2.2 Geometry of $\mathcal{O}$-surfaces

We next consider the geometry of $\mathcal{O}(h)$-surfaces where $0 \leq h < 1$. Let θ be a local solution of the equation (4.9) and $\mathcal{D}^\theta$ the involutive distribution assosiated with θ. The integral manifolds S of $\mathcal{D}^\theta$ are $\mathcal{O}(h)$-surfaces. Then the vector fields X and Y generate their tangent spaces and the vector field N gives a unit normal vector field on the surfaces. By the equations (2.2) and (4.8) we can calculate the covariant derivatives $\nabla_X X$, $\nabla_X Y$, $\nabla_Y X$, and $\nabla_Y Y$ as follows.

$$\begin{aligned} \nabla_X X &= -\lambda_3\frac{h}{\sqrt{1-h^2}}Y, & \nabla_Y Y &= -F_\theta\big(hX + \sqrt{1-h^2}N\big), \\ \nabla_X Y &= \lambda_3\frac{h}{\sqrt{1-h^2}}X + \frac{\lambda_3}{2}N, & \nabla_Y X &= F_\theta hY + \frac{\lambda_3}{2}N \end{aligned} \tag{4.15}$$

where $F_\theta = \cos\theta(E_1\theta) + \sin\theta(E_2\theta)$. Here, to calculate $\nabla_X X$, we use the equation (4.9) which is the existence condition of the $\mathcal{O}(h)$-geometry. Taking the tangent and the normal parts of these, we have the following.

$$\begin{aligned}
&\nabla^\theta_X X = -\lambda_3 \frac{h}{\sqrt{1-h^2}} Y, && \Pi^\theta(X,X) = 0,\\
&\nabla^\theta_Y Y = -F_\theta h X, && \Pi^\theta(Y,Y) = -F_\theta\sqrt{1-h^2}N,\\
&\nabla^\theta_X Y = \lambda_3 \frac{h}{\sqrt{1-h^2}} X, && \Pi^\theta(X,Y) = \Pi^\theta(Y,X) = \frac{\lambda_3}{2}N,\\
&\nabla^\theta_Y X = F_\theta h Y
\end{aligned} \tag{4.16}$$

where ∇^θ and Π^θ give the Riemanian connection and the second fundamental form of $\mathcal{O}(h)$-surfaces S. Also, the Gauss curvature K^θ and the mean curvature H^θ of S are given by the equations

$$K^\theta = -h(XF_\theta) - \left(\frac{\lambda_3 h}{\sqrt{1-h^2}}\right)^2 - (F_\theta h)^2 \tag{4.17}$$

$$H^\theta = -\frac{F_\theta\sqrt{1-h^2}}{2}. \tag{4.18}$$

Hence, by (2.4), (4.16) and (4.17), we have the following.

Proposition 4.7 *Let $h \neq 1$. Then, for a plane $P \in \mathcal{O}(h)$ the sectional curvature $K(P)$ of (G,g) is given by $\{\lambda_3^2 - 4\lambda_3(\lambda_3-\lambda)h^2\}/4$. Also, any $\mathcal{O}(h)$-surface has no geodesic point and any $\mathcal{O}(0)$-surface is moreover flat.*

4.2.3 Constant Mean Curvature surface equations

Next we consider whether in the $\mathcal{O}(h)$-geometry there exists an $\mathcal{O}(h)$-surface of constant mean curvature, or not. By (4.18) the condition is given by the following equation

$$F_\theta = \cos\theta(E_1\theta) + \sin\theta(E_2\theta) = -k/2 \tag{4.19}$$

where k is constant and H^θ is given by $k\sqrt{1-h^2}/4$. This equation is also rewritten in terms of the local coordinates y, z, w and the local function x as follows.

$$(x\cos\theta - w\sin\theta)\left(\frac{\partial\theta}{\partial y}\right) + (w\cos\theta + x\sin\theta)\left(\frac{\partial\theta}{\partial z}\right)$$

$$+ (-z\cos\theta + y\sin\theta)\left(\frac{\partial\theta}{\partial w}\right) + \frac{k}{\sqrt{\lambda\lambda_3}} = 0. \tag{4.20}$$

In this paper we call these (4.19) or (4.20) the *CMC surface equations*, in particular, the *minimal surface equations* when $k = 0$.

To analyse the CMC surface equation we first solve the system (4.11) of the characteristic ODE's associated with the existence PDE of $\mathcal{O}(h)$-surfaces. The system (4.11) is completed by adding to it an ODE with respect to x which is obtained by differentiating the local function $x = \sqrt{1 - y^2 - z^2 - w^2}$ as follows:

$$\begin{aligned}\frac{dx}{dt} &= -\frac{1}{x}\left(y\frac{dy}{dt} + z\frac{dz}{dt} + w\frac{dw}{dt}\right) \\ &= \sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2}(-y\sin\theta + z\cos\theta) - (1-h^2)w.\end{aligned}$$

The completed system is now represented by the following form

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \\ \frac{dw}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -\mu\sin\theta & \mu\cos\theta & -\nu \\ \mu\sin\theta & 0 & \nu & \mu\cos\theta \\ -\mu\cos\theta & -\nu & 0 & \mu\sin\theta \\ \nu & -\mu\cos\theta & -\mu\sin\theta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \tag{4.21}$$

where $\theta = \theta(t) = -2((1-h^2) + \frac{\lambda_3}{\lambda}h^2)t + \varphi(a,b)$ and

$$\mu = \sqrt{\frac{\lambda_3}{\lambda}}h\sqrt{1-h^2} > 0, \quad \nu = 1 - h^2 > 0. \tag{4.22}$$

We write the system (4.21) as $d\mathbb{X}/dt = A(t)\mathbb{X}$. Since the system is linear and $A(t)$ is a skew symmetric matrix with a finite period, we can transform it to a linear system with constant coefficients and solve it concretely. We practice this process. Take an orthogonal matrix $O(t)$ as follows:

$$O(t) = \begin{pmatrix} -\frac{\nu}{\ell}\cos\theta & \sin\theta & \frac{\mu}{\ell}\cos\theta & 0 \\ \frac{\mu}{\ell} & 0 & \frac{\nu}{\ell} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\nu}{\ell}\sin\theta & \cos\theta & -\frac{\mu}{\ell}\sin\theta & 0 \end{pmatrix}$$

where $\ell = \sqrt{\mu^2+\nu^2}$. Then it follows

$$ {}^tO(t)A(t)O(t) = \ell \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and}$$

$$ {}^tO(t)\frac{d}{dt}O(t) = 2\ell \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \frac{\mu}{\nu} & 0 \\ 0 & -\frac{\mu}{\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Put the right hands of these into A and B, respectively, and change $\mathbb{X}(t)$ into $\mathbb{Y}(t)$ by the equation $\mathbb{X}(t) = O(t)\mathbb{Y}(t)$. Then it follows

$$\frac{d\mathbb{X}}{dt} = \left(\frac{d}{dt}O(t)\right)\mathbb{Y} + O(t)\frac{d\mathbb{Y}}{dt} \quad \text{and} \quad A(t)\mathbb{X} = O(t)A\,{}^tO(t)O(t)\mathbb{Y} = O(t)A\mathbb{Y},$$

and since $d\mathbb{X}/dt = A(t)\mathbb{X}$, it follows

$$\frac{d\mathbb{Y}}{dt} = (A-B)\mathbb{Y} = \ell \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\frac{2\mu}{\nu} & 0 \\ 0 & \frac{2\mu}{\nu} & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \mathbb{Y}.$$

We next put $A - B = C$ and solve the linear system $d\mathbb{Y}/dt = C\mathbb{Y}$ with constant coefficients. Set

$$\alpha = \frac{\mu+\ell}{\nu} > 0 \quad \text{and} \quad \beta = \frac{-\mu+\ell}{\nu} > 0 \tag{4.23}$$

where $\alpha\beta = 1$ and $\alpha \geq \beta$ and take an orthogonal matrix O as follows:

$$O = \begin{pmatrix} \frac{1}{\sqrt{1+\alpha^2}} & 0 & \frac{1}{\sqrt{1+\beta^2}} & 0 \\ 0 & -\frac{\alpha}{\sqrt{1+\alpha^2}} & 0 & -\frac{\beta}{\sqrt{1+\beta^2}} \\ -\frac{\alpha}{\sqrt{1+\alpha^2}} & 0 & \frac{\beta}{\sqrt{1+\beta^2}} & 0 \\ 0 & -\frac{1}{\sqrt{1+\alpha^2}} & 0 & \frac{1}{\sqrt{1+\beta^2}} \end{pmatrix}.$$

Then it follows

$${}^tOCO = \ell \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix}.$$

Put the right hand of this into D and change $\mathbb{Y}(t)$ into $\mathbb{Z}(t)$ by the relation $\mathbb{Y}(t) = O\mathbb{Z}(t)$. Then it follows

$$\frac{d\mathbb{Y}}{dt} = O\frac{d\mathbb{Z}}{dt} \quad \text{and} \quad C\mathbb{Y} = OD\mathbb{Z},$$

and since $d\mathbb{Y}/dt = C\mathbb{Y}$, it follows

$$\frac{d\mathbb{Z}}{dt} = D\mathbb{Z} = \ell \begin{pmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix} \mathbb{Z}.$$

Set

$$S(t) = \begin{pmatrix} \cos(\ell\alpha t) & \sin(\ell\alpha t) & 0 & 0 \\ -\sin(\ell\alpha t) & \cos(\ell\alpha t) & 0 & 0 \\ 0 & 0 & \cos(\ell\beta t) & \sin(\ell\beta t) \\ 0 & 0 & -\sin(\ell\beta t) & \cos(\ell\beta t) \end{pmatrix}.$$

Then it follows $\mathbb{Z}(t) = S(t)\mathbb{Z}(0)$, and since $\mathbb{X}(t) = O(t)O\mathbb{Z}(t)$, it follows

$$\mathbb{X}(t) = O(t)OS(t)^t O^t O(0)\mathbb{X}(0)$$

where $\mathbb{X}(0) = {}^t(\sqrt{1-a^2-b^2}, 0, a, b)$. Note that we can rewrite $\theta(t)$ by using α, β as follows:

$$\theta(t) = -\ell(\alpha+\beta)t + \varphi(a,b). \tag{4.24}$$

Then, by explicit calculations, we have the following solution $\mathbb{X}(t) = {}^t(x(t), y(t), z(t), w(t))$:

$$\begin{aligned}
x(t) &= \left\{\frac{\alpha^2}{1+\alpha^2}\cos(\ell\beta t) + \frac{\beta^2}{1+\beta^2}\cos(\ell\alpha t)\right\}\sqrt{1-a^2-b^2} \\
&\quad + \left\{-\frac{\alpha}{1+\alpha^2}\sin(\ell\beta t-\varphi) + \frac{\beta}{1+\beta^2}\sin(\ell\alpha t-\varphi)\right\}a \\
&\quad + \left\{-\frac{\alpha^2}{1+\alpha^2}\sin(\ell\beta t) - \frac{\beta^2}{1+\beta^2}\sin(\ell\alpha t)\right\}b, \\
y(t) &= \left\{\frac{\alpha}{1+\alpha^2}\cos(\ell\alpha t-\varphi) - \frac{\beta}{1+\beta^2}\cos(\ell\beta t-\varphi)\right\}\sqrt{1-a^2-b^2} \\
&\quad + \left\{\frac{1}{1+\alpha^2}\sin(\ell\alpha t) + \frac{1}{1+\beta^2}\sin(\ell\beta t)\right\}a \\
&\quad + \left\{\frac{\alpha}{1+\alpha^2}\sin(\ell\alpha t-\varphi) - \frac{\beta}{1+\beta^2}\sin(\ell\beta t-\varphi)\right\}b, \\
z(t) &= \left\{-\frac{\alpha}{1+\alpha^2}\sin(\ell\alpha t-\varphi) + \frac{\beta}{1+\beta^2}\sin(\ell\beta t-\varphi)\right\}\sqrt{1-a^2-b^2} \\
&\quad + \left\{\frac{1}{1+\alpha^2}\cos(\ell\alpha t) + \frac{1}{1+\beta^2}\cos(\ell\beta t)\right\}a \\
&\quad + \left\{\frac{\alpha}{1+\alpha^2}\cos(\ell\alpha t-\varphi) - \frac{\beta}{1+\beta^2}\cos(\ell\beta t-\varphi)\right\}b, \\
w(t) &= \left\{\frac{\alpha^2}{1+\alpha^2}\sin(\ell\beta t) + \frac{\beta^2}{1+\beta^2}\sin(\ell\alpha t)\right\}\sqrt{1-a^2-b^2} \\
&\quad + \left\{\frac{\alpha}{1+\alpha^2}\cos(\ell\beta t-\varphi) - \frac{\beta}{1+\beta^2}\cos(\ell\alpha t-\varphi)\right\}a \\
&\quad + \left\{\frac{\alpha^2}{1+\alpha^2}\cos(\ell\beta t) + \frac{\beta^2}{1+\beta^2}\cos(\ell\alpha t)\right\}b.
\end{aligned} \tag{4.25}$$

Moreover we can calculate the derivatives of coordinate functions y, z, w with respect to the variables t, a, b. We need these derivatives to represent the CMC surface equation (4.20) in terms of the coordinates t, a, b. They are given in the following.

$$
\begin{aligned}
y_t &= -\ell\rho\sqrt{1-a^2-b^2}\{\alpha\sin(\ell\alpha t-\varphi)-\beta\sin(\ell\beta t-\varphi)\}\\
&\quad+\ell a\{\eta\alpha\cos(\ell\alpha t)+\xi\beta\cos(\ell\beta t)\}\\
&\quad+\ell\rho b\{\alpha\cos(\ell\alpha t-\varphi)-\beta\cos(\ell\beta t-\varphi)\},\\
z_t &= -\ell\rho\sqrt{1-a^2-b^2}\{\alpha\cos(\ell\alpha t-\varphi)-\beta\cos(\ell\beta t-\varphi)\}\\
&\quad-\ell a\{\eta\alpha\sin(\ell\alpha t)+\xi\beta\sin(\ell\beta t)\}\\
&\quad-\ell\rho b\{\alpha\sin(\ell\alpha t-\varphi)-\beta\sin(\ell\beta t-\varphi)\},\\
w_t &= \ell\sqrt{1-a^2-b^2}\{\eta\alpha\cos(\ell\alpha t)+\xi\beta\cos(\ell\beta t)\}\\
&\quad+\ell\rho a\{-\alpha\sin(\ell\alpha t-\varphi)+\beta\sin(\ell\beta t-\varphi)\}\\
&\quad-\ell b\{\eta\alpha\sin(\ell\alpha t)+\xi\beta\sin(\ell\beta t)\},\\
y_a &= \varphi_a\rho\sqrt{1-a^2-b^2}\{\sin(\ell\alpha t-\varphi)-\sin(\ell\beta t-\varphi)\}\\
&\quad+\rho\{\sin(\ell\alpha t)+\sin(\ell\beta t)\}\\
&\quad-\left(\frac{\rho a}{\sqrt{1-a^2-b^2}}+\varphi_a\rho b\right)\{\cos(\ell\alpha t-\varphi)-\cos(\ell\beta t-\varphi)\},\\
z_a &= \varphi_a\rho\sqrt{1-a^2-b^2}\{\cos(\ell\alpha t-\varphi)-\cos(\ell\beta t-\varphi)\}\\
&\quad+\{\eta\cos(\ell\alpha t)+\xi\cos(\ell\beta t)\}\\
&\quad+\left(\frac{\rho a}{\sqrt{1-a^2-b^2}}+\varphi_a\rho b\right)\{\sin(\ell\alpha t-\varphi)-\sin(\ell\beta t-\varphi)\},\\
w_a &= -\frac{a}{\sqrt{1-a^2-b^2}}\{\eta\sin(\ell\alpha t)+\xi\sin(\ell\beta t)\}\\
&\quad+\varphi_a\rho a\{-\sin(\ell\alpha t-\varphi)+\sin(\ell\beta t-\varphi)\}\\
&\quad+\rho\{-\cos(\ell\alpha t-\varphi)+\cos(\ell\beta t-\varphi)\},\\
y_b &= \left(\varphi_b\rho\sqrt{1-a^2-b^2}+\rho\right)\{\sin(\ell\alpha t-\varphi)-\sin(\ell\beta t-\varphi)\}\\
&\quad-\left(\frac{\rho b}{\sqrt{1-a^2-b^2}}+\varphi_b\rho b\right)\{\cos(\ell\alpha t-\varphi)-\cos(\ell\beta t-\varphi)\},
\end{aligned}
$$

$$
\begin{aligned}
z_b &= \big(\varphi_b\rho\sqrt{1-a^2-b^2}+\rho\big)\{\cos(\ell\alpha t-\varphi)-\cos(\ell\beta t-\varphi)\}\\
&\quad+\left(\frac{\rho b}{\sqrt{1-a^2-b^2}}+\varphi_b\rho b\right)\{\sin(\ell\alpha t-\varphi)-\sin(\ell\beta t-\varphi)\},\\
w_b &= -\frac{b}{\sqrt{1-a^2-b^2}}\{\eta\sin(\ell\alpha t)+\xi\sin(\ell\beta t)\}\\
&\quad+\varphi_b\rho a\{-\sin(\ell\alpha t-\varphi)+\sin(\ell\beta t-\varphi)\}+\{\eta\cos(\ell\alpha t)+\xi\cos(\ell\beta t)\}
\end{aligned}
\tag{4.26}
$$

where

$$
\begin{gathered}
\rho=\frac{\alpha}{1+\alpha^2}=\frac{\beta}{1+\beta^2}>0,\\
\xi=\frac{\alpha^2}{1+\alpha^2}=\frac{1}{1+\beta^2}>0,\quad \eta=\frac{\beta^2}{1+\beta^2}=\frac{1}{1+\alpha^2}>0.
\end{gathered}
\tag{4.27}
$$

These constants ρ, ξ and η satisfy the following relations.

$$
\xi+\eta=1,\quad \rho\alpha=\xi,\quad \rho\beta=\eta,\quad \eta\alpha=\xi\beta=\rho,\quad \eta\xi=\rho^2. \tag{4.28}
$$

We now rewrite the CMC surface equation (4.20) in terms of coordinates t, a and b. Since $\partial\theta/\partial t=-\ell(\alpha+\beta)$, $\partial\theta/\partial a=\varphi_a$ and $\partial\theta/\partial b=\varphi_b$, we have the following CMC surface equation:

$$
\begin{aligned}
&(x\cos\theta-w\sin\theta)\bigg(-\ell(\alpha+\beta)\frac{\partial(z,w)}{\partial(a,b)}+\varphi_a\frac{\partial(z,w)}{\partial(b,t)}+\varphi_b\frac{\partial(z,w)}{\partial(t,a)}\bigg)\\
&\quad+(w\cos\theta+x\sin\theta)\bigg(-\ell(\alpha+\beta)\frac{\partial(w,y)}{\partial(a,b)}+\varphi_a\frac{\partial(w,y)}{\partial(b,t)}+\varphi_b\frac{\partial(w,y)}{\partial(t,a)}\bigg)\\
&\quad+(-z\cos\theta+y\sin\theta)\bigg(-\ell(\alpha+\beta)\frac{\partial(y,z)}{\partial(a,b)}+\varphi_a\frac{\partial(y,z)}{\partial(b,t)}+\varphi_b\frac{\partial(y,z)}{\partial(t,a)}\bigg)\\
&\quad+\frac{k}{\sqrt{\lambda\lambda_3}}\bigg(y_t\frac{\partial(z,w)}{\partial(a,b)}+z_t\frac{\partial(w,y)}{\partial(a,b)}+w_t\frac{\partial(y,z)}{\partial(a,b)}\bigg)=0.
\end{aligned}
\tag{4.29}
$$

Here it should be assumed that $\frac{\partial(y,z,w)}{\partial(t,a,b)}\neq 0$. But, by (4.14), this assumption is possible if we take a suitable domain in the (t,a,b)-space.

Now we consider the existence problem of $\mathcal{O}(h)$-surfaces of constant mean curvature. To do this we explicitly calculate the factors $(x\cos\theta - w\sin\theta)$, $(w\cos\theta + x\sin\theta)$, $(-z\cos\theta + y\sin\theta)$ and the Jacobians such as $\frac{\partial(z,w)}{\partial(a,b)}$, $\frac{\partial(z,w)}{\partial(b,t)}$, $\frac{\partial(z,w)}{\partial(t,a)}$ and so on. By (4.24) and (4.25) three factors in the above are calculated as follows:

$$
\begin{aligned}
x\cos\theta - w\sin\theta &= \xi\sqrt{1-a^2}\sin(\ell\alpha t - \varphi + \tau) + a\rho\sin(\ell\alpha t) \\
&\quad + \eta\sqrt{1-a^2}\sin(\ell\beta t - \varphi + \tau) - a\rho\sin(\ell\beta t), \\
w\cos\theta + x\sin\theta &= \xi\sqrt{1-a^2}\cos(\ell\alpha t - \varphi + \tau) + a\rho\cos(\ell\alpha t) \\
&\quad + \eta\sqrt{1-a^2}\cos(\ell\beta t - \varphi + \tau) - a\rho\cos(\ell\beta t), \\
-z\cos\theta + y\sin\theta &= \rho\sqrt{1-a^2}\cos(\ell\alpha t - \tau) - a\xi\cos(\ell\alpha t - \varphi) \\
&\quad - \rho\sqrt{1-a^2}\cos(\ell\beta t - \tau) - a\eta\cos(\ell\beta t - \varphi),
\end{aligned}
\tag{4.30}
$$

where τ is defined by

$$
\sin\tau = \frac{\sqrt{1-a^2-b^2}}{\sqrt{1-a^2}} \quad \text{and} \quad \cos\tau = \frac{b}{\sqrt{1-a^2}}. \tag{4.31}
$$

Also, by using (4.26) we can explicitly caluculate the terms which contain Jacobians, y_t, z_t and w_t. After these explicit calculations, by using the additive laws of Cosine and Sine functions, we rearrange the left hand of the equation (4.29) in the following form

$$
\sum_{\omega\in\Omega}\sum_i \left\{ f_{\omega,i}\sin(\ell\omega t + r_{\omega,i}) + g_{\omega,i}\cos(\ell\omega t + s_{\omega,i}) \right\} = 0 \tag{4.32}
$$

where $\Omega = \{3\alpha, 3\beta, 2\alpha \pm \beta, 2\beta \pm \alpha, \alpha, \beta\}$, and $f_{\omega,i}$, $g_{\omega,i}$, $r_{\omega,i}$, $s_{\omega,i}$ are some functions dependent only on a, b and φ. We here note the following (i), (ii), (iii): (i) If $\alpha \neq \beta$, the elements in Ω are all distinct since $\alpha > \beta$ and $\alpha, \beta > 0$, and if $\alpha = \beta$, it holds $\Omega = \{3\alpha, \alpha\}$. The first case occurs when $0 < h < 1$ and the second case occurs when $h = 0$. (ii) Moreover, if the equation (4.32) holds on a small open interval of variable t, it also holds for all real numbers t since the Cosine and Sine functions are analytic. (iii) Also, by the property about the period of the Cosine and Sine functions, it follows that for each distinct period ω, it holds that $\sum_i\{f_{\omega,i}\sin(\ell\omega t + r_{\omega,i}) + g_{\omega,i}\cos(\ell\omega t + s_{\omega,i})\} = 0$.

We now divide the case into the case that $0 < h < 1$ and the case that $h = 0$, and first consider the case that $0 < h < 1$. In this case we take notice of the periods $3\ell\alpha$ and $3\ell\beta$. These periods are different.

Let $\omega = 3\alpha$ and extract all the terms with period $3\ell\alpha$ in the equation (4.29). Then, from the term

$$-\ell(\alpha+\beta)\Big\{(x\cos\theta - w\sin\theta)\frac{\partial(z,w)}{\partial(a,b)} + (w\cos\theta + x\sin\theta)\frac{\partial(w,y)}{\partial(a,b)} + (-z\cos\theta + y\sin\theta)\frac{\partial(y,z)}{\partial(a,b)}\Big\},$$

we can extract the following ones

$$\begin{aligned}
&-(1/4)\ell(\alpha+\beta)(\rho^3-\rho^2\xi)a\sqrt{1-a^2}\varphi_a\cos(3\ell\alpha t-2\varphi+\tau),\\
&-(1/4)\ell(\alpha+\beta)(\rho^2\eta-\rho^3)\frac{(1-a^2)}{\sqrt{1-a^2-b^2}}\cos(3\ell\alpha t-\varphi),\\
&-(1/4)\ell(\alpha+\beta)(\rho^2\eta-\rho^3)a^2\varphi_a\cos(3\ell\alpha t-\varphi),\\
&-(1/4)\ell(\alpha+\beta)(\rho\eta^2-\rho^2\eta)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\cos(3\ell\alpha t-\tau),\\
&-(1/4)\ell(\alpha+\beta)(\rho^2\eta-\rho^3)\frac{(1-a^2)}{\sqrt{1-a^2-b^2}}\\
&\quad\times\big(\varphi_b\sqrt{1-a^2-b^2}+1\big)\cos(3\ell\alpha t-\varphi),\\
&-(1/4)\ell(\alpha+\beta)(\rho^2\xi-\rho^3)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\\
&\quad\times\big(\varphi_b\sqrt{1-a^2-b^2}+1\big)\cos(3\ell\alpha t-2\varphi+\tau),
\end{aligned}$$

and from the term

$$\varphi_b\Big\{(x\cos\theta - w\sin\theta)\frac{\partial(z,w)}{\partial(t,a)} + (w\cos\theta + x\sin\theta)\frac{\partial(w,y)}{\partial(t,a)} + (-z\cos\theta + y\sin\theta)\frac{\partial(y,z)}{\partial(t,a)}\Big\},$$

the following ones

$$\begin{aligned}
&(1/4)\ell\varphi_b(2\rho^2\xi - 2\rho^3)a^2\cos(3\ell\alpha t - \varphi),\\
&(1/4)\ell\varphi_b(\eta\rho^2 - \rho^3 - \xi\rho^2 + \rho\xi^2)a\sqrt{1-a^2}\cos(3\ell\alpha t - \tau),\\
&(1/4)\ell\varphi_b(-\rho\xi^2 + \rho^3 - \rho^2\xi + \xi^3)(1-a^2)\cos(3\ell\alpha t - \varphi),\\
&(1/4)\ell\varphi_b(2\rho\xi^2)a\sqrt{1-a^2}\cos(3\ell\alpha t - 2\varphi + \tau),
\end{aligned}$$

and from the term

$$\frac{k}{\sqrt{\lambda\lambda_3}}\left\{y_t\frac{\partial(z,w)}{\partial(a,b)} + z_t\frac{\partial(w,y)}{\partial(a,b)} + w_t\frac{\partial(y,z)}{\partial(a,b)}\right\},$$

the following ones

$$\begin{aligned}
&(1/4)\big(k/\sqrt{\lambda\lambda_3}\big)\ell(-\rho^3 - \rho^2\xi)a\sqrt{1-a^2}\varphi_a\sin(3\ell\alpha t - 2\varphi + \tau),\\
&(1/4)\big(k/\sqrt{\lambda\lambda_3}\big)\ell(-\rho^2\eta + \rho^3)\frac{(1-a^2)}{\sqrt{1-a^2-b^2}}\sin(3\ell\alpha t - \varphi),\\
&(1/4)\big(k/\sqrt{\lambda\lambda_3}\big)\ell(\rho\xi^2 - \rho^2\xi)\big(\varphi_b\sqrt{1-a^2-b^2} + 1\big)\sin(3\ell\alpha t - \varphi),\\
&(1/4)\big(k/\sqrt{\lambda\lambda_3}\big)\ell(-\rho^2\eta + \rho^3)a^2\varphi_a\sin(3\ell\alpha t - \varphi),\\
&(1/4)\big(k/\sqrt{\lambda\lambda_3}\big)\ell(-\rho\eta^2 + \rho^2\eta)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\sin(3\ell\alpha t - \tau),\\
&(1/4)\big(k/\sqrt{\lambda\lambda_3}\big)\ell(\rho^2\xi - \rho^3)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\\
&\quad\times\big(\varphi_b\sqrt{1-a^2-b^2} + 1\big)\sin(3\ell\alpha t - 2\varphi + \tau).
\end{aligned}$$

Also, in the term

$$\begin{aligned}
\varphi_a\bigg\{&(x\cos\theta - w\sin\theta)\frac{\partial(z,w)}{\partial(b,t)} + (w\cos\theta + x\sin\theta)\frac{\partial(w,y)}{\partial(b,t)}\\
&+ (-z\cos\theta + y\sin\theta)\frac{\partial(y,z)}{\partial(b,t)}\bigg\},
\end{aligned}$$

there is no term with period $3\ell\alpha$. The sum of these terms with period $3\ell\alpha$ gives the left hand of the equality $\sum_i\{f_{\omega,i}\sin(\ell\omega t + r_{\omega,i}) + g_{\omega,i}\cos(\ell\omega t + s_{\omega,i})\} = 0$ when $\omega = 3\alpha$. Substitute the value $\varphi/(3\ell\alpha)$ for t in the obtained explicit equality and simplify it by using (4.27) and (4.28). Then we can obtain the following equality which does not contain the parameter t.

$$\begin{aligned}
&\Big\{ -\rho(\rho-\xi)a\sqrt{1-a^2}\varphi_a + (\rho-\eta)(\eta-\xi)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}} \\
&\quad + 2(\rho\xi^2 - \rho^3 + \eta\rho^2)a\sqrt{1-a^2}\varphi_b\Big\}\cos(\varphi-\tau) \\
&\quad + \Big\{2\rho(\rho-\eta)\frac{(1-a^2)}{\sqrt{1-a^2-b^2}} + \rho(\rho-\eta)a^2\varphi_a + (\eta^2+\xi^2)(\xi-\rho)\varphi_b \\
&\qquad + (2\rho^2\xi - 2\rho^3 + \rho\eta^2 - \eta\rho^2 + \rho\xi^2 - \xi^3)a^2\varphi_b\Big\} \\
&\quad + \frac{k}{\sqrt{\lambda\lambda_3}}\Big\{(\rho^3+\rho^2\xi)a\sqrt{1-a^2}\varphi_a + (\rho^3-\rho^2\xi)a\sqrt{1-a^2}\varphi_b \\
&\qquad + (\rho^3-\rho\eta^2)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\Big\}\sin(\varphi-\tau) = 0.
\end{aligned} \tag{4.33}$$

Next, after differentiating the obtained explicit equality when $\omega = 3\alpha$ with respect to the parameter t, substitute the value $\varphi/(3\ell\alpha)$ for t. Then we can also obtain the following equality.

$$\begin{aligned}
&\Big\{\rho(\rho-\xi)a\sqrt{1-a^2}\varphi_a - 2\rho^2\xi a\sqrt{1-a^2}\varphi_b + (\rho-\eta)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\Big\}\sin(\varphi-\tau) \\
&\quad - \frac{k}{\sqrt{\lambda\lambda_3}}\rho\Big\{ -\rho(\rho+\xi)a\sqrt{1-a^2}\varphi_a + \xi(\rho-\eta)a\sqrt{1-a^2}\varphi_b \\
&\qquad + (\rho-\eta)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\Big\}\cos(\varphi-\tau) \\
&\quad - \frac{k}{\sqrt{\lambda\lambda_3}}\rho\Big\{\rho(\rho-\eta)a^2\varphi_a + \xi(\xi-\rho)\sqrt{1-a^2-b^2}\varphi_b \\
&\qquad + \rho(\rho-\eta)\frac{(1-a^2)}{\sqrt{1-a^2-b^2}} + \xi(\xi-\rho)\Big\} = 0.
\end{aligned} \tag{4.34}$$

Next let $\omega = 3\beta$. Similarly to the case $\omega = 3\alpha$, we extract the following ones with period $3\ell\beta$ in the equation (4.29): From the terms which contain the factor $\ell(\alpha+\beta)$,

$$
\begin{aligned}
&-(1/4)\ell(\alpha+\beta)(-\rho^3+\eta\rho^2)a\sqrt{1-a^2}\varphi_a\cos(3\ell\beta t-2\varphi+\tau),\\
&-(1/4)\ell(\alpha+\beta)(\rho^2\xi-\rho^3)\frac{(1-a^2)}{\sqrt{1-a^2-b^2}}\cos(3\ell\beta t-\varphi),\\
&-(1/4)\ell(\alpha+\beta)(\rho^2\xi-\rho^3)a^2\varphi_a\cos(3\ell\beta t-\varphi),\\
&-(1/4)\ell(\alpha+\beta)(\rho^2\xi-\rho^3)\frac{(1-a^2)}{\sqrt{1-a^2-b^2}}\\
&\quad\times\left(\varphi_b\sqrt{1-a^2-b^2}+1\right)\cos(3\ell\beta t-\varphi),\\
&-(1/4)\ell(\alpha+\beta)(-\rho^2\eta+\rho^3)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\\
&\quad\times\left(\varphi_b\sqrt{1-a^2-b^2}+1\right)\cos(3\ell\beta t-2\varphi+\tau),
\end{aligned}
$$

and from the terms which contain the factor φ_b,

$$
\begin{aligned}
&(1/4)\ell\varphi_b(-\rho\eta^2+\rho^3-\rho^2\eta+\eta^3)(1-a^2)\cos(3\ell\beta t-\varphi),\\
&(1/4)\ell\varphi_b(2\rho^2\eta-2\rho^3)a^2\cos(3\ell\beta t-\varphi),\\
&(1/4)\ell\varphi_b(-2\rho\eta^2)a\sqrt{1-a^2}\cos(3\ell\beta t-2\varphi+\tau),\\
&(1/4)\ell\varphi_b(\rho^3-\rho\eta^2)a\sqrt{1-a^2}\cos(3\ell\beta t-\tau),
\end{aligned}
$$

and from the terms which contain the factor $k/\sqrt{\lambda\lambda_3}$,

$$
\begin{aligned}
&(1/4)\left(k/\sqrt{\lambda\lambda_3}\right)\ell(-\rho^3+\rho^2\eta)a\varphi_a\sin(3\ell\beta t-2\varphi+\tau),\\
&(1/4)\left(k/\sqrt{\lambda\lambda_3}\right)\ell(\rho^2\xi-\rho^3)\frac{(1-a^2)}{\sqrt{1-a^2-b^2}}\sin(3\ell\beta t-\varphi),\\
&(1/4)\left(k/\sqrt{\lambda\lambda_3}\right)\ell(\rho^2\xi-\rho^3)a^2\varphi_a\sin(3\ell\beta t-\varphi),
\end{aligned}
$$

$$(1/4)\big(k/\sqrt{\lambda\lambda_3}\big)\ell(-\rho\eta^2+\rho^2\eta)\frac{(1-a^2)}{\sqrt{1-a^2-b^2}}$$
$$\times\big(\varphi_b\sqrt{1-a^2-b^2}+1\big)\sin(3\ell\beta t-\varphi),$$
$$(1/4)\big(k/\sqrt{\lambda\lambda_3}\big)\ell(\rho^2\eta-\rho^3)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}$$
$$\times\big(\varphi_b\sqrt{1-a^2-b^2}+1\big)\sin(3\ell\beta t-2\varphi+\tau).$$

Also, in the terms which contain the factor φ_a, there is no term with period $3\ell\beta$. The sum of these terms with period $3\ell\beta$ gives the left hand of the equality $\sum_i\{f_{\omega,i}\sin(\ell\omega t+r_{\omega,i})+g_{\omega,i}\cos(\ell\omega t+s_{\omega,i})\}=0$ when $\omega=3\beta$. After differentiating the obtained explicit equality with respect to the parameter t, substitute the value $\varphi/(3\ell\beta)$ for t and simplify it by using (4.27) and (4.28). Then we can obtain the following equality:

$$\begin{aligned}
&\Big\{(-\rho^2+\eta\rho)a\sqrt{1-a^2}\varphi_a+(\rho\eta^2+\rho^3)a\sqrt{1-a^2}\varphi_b\\
&\quad+(-\rho\eta+\rho^2)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\big(\varphi_b\sqrt{1-a^2-b^2}+1\big)\Big\}\sin(\varphi-\tau)\\
&-\frac{k}{\sqrt{\lambda\lambda_3}}\Big\{(-\rho^3+\rho^2\eta)a\varphi_a\\
&\qquad+(\rho^2\eta-\rho^3)\frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\big(\varphi_b\sqrt{1-a^2-b^2}+1\big)\Big\}\cos(\varphi-\tau)\\
&-\frac{k}{\sqrt{\lambda\lambda_3}}\Big\{(\rho^2\xi-\rho^3)\frac{(1-a^2)}{\sqrt{1-a^2-b^2}}+(\rho^2\xi-\rho^3)a^2\varphi_a\\
&\qquad+(-\rho\eta^2+\rho^2\eta)\frac{(1-a^2)}{\sqrt{1-a^2-b^2}}\big(\varphi_b\sqrt{1-a^2-b^2}+1\big)\Big\}=0.
\end{aligned}\tag{4.35}$$

Under the above preparation, we have the following two propositions for the case that $0<h<1$.

Proposition 4.8 *Let $0<h<1$. Then there exists no minimal $\mathcal{O}(h)$-surface.*

Proof. We first note that if there exists a minimal $\mathcal{O}(h)$-surface, the CMC surface equation (4.29) when $k=0$ must have a local solution. Because we can put the minimal surface in a desired position in the ambient space G by the transitivity of $I_o(G,g)$-action on G. Hence, from (4.34) and (4.35), they hold

$$\left\{\rho(\rho-\xi)\varphi_a - 2\rho^2\xi\varphi_b + (\rho-\eta)\frac{1}{\sqrt{1-a^2-b^2}}\right\}\sin(\varphi-\tau)=0, \tag{4.36}$$

$$\begin{aligned}&\Big\{(-\rho^2+\eta\rho)\varphi_a + (\rho^3+\rho^2+\rho\eta^2-\rho\eta)\varphi_b\\ &\quad + (-\rho\eta+\rho^2)\frac{1}{\sqrt{1-a^2-b^2}}\Big\}\sin(\varphi-\tau)=0.\end{aligned} \tag{4.37}$$

We here consider the following system of equations and we prove that this system has no local solution.

$$\begin{aligned}\rho(\rho-\xi)\varphi_a \qquad\qquad\quad - 2\rho^2\xi\varphi_b + \quad (\rho-\eta)\frac{1}{\sqrt{1-a^2-b^2}} &= 0,\\ (-\rho+\eta)\varphi_a + (\rho^2+\rho+\eta^2-\eta)\varphi_b + (-\eta+\rho)\frac{1}{\sqrt{1-a^2-b^2}} &= 0.\end{aligned} \tag{4.38}$$

We regard this system as a linear system with variables φ_a and φ_b. Then the (2×3) matrix of coefficients has rank two. In fact, it follows by (4.28) that

$$\begin{aligned}&\rho(\rho-\xi)\times(-\eta+\rho) - (\rho-\eta)\times(-\rho+\eta)\\ &\quad = (\rho-\eta)(\rho^2-\rho\xi+\rho-\eta) = (\rho-\eta)(\xi\eta+\rho(1-\xi)-\eta)\\ &\quad = (\rho-\eta)(\xi\eta+\rho\eta-\eta) = (\rho-\eta)\eta(\xi+\rho-1) = \eta(\rho-\eta)^2 \neq 0.\end{aligned}$$

Because, if $\rho=\eta$, by (4.27) it follows $\alpha=\beta=1$. This is not the case. Now, if the system (4.38) has a solution, φ_a and φ_b must have the following form:

$$\varphi_a = \frac{c}{\sqrt{1-a^2-b^2}} \quad \text{and} \quad \varphi_b = \frac{d}{\sqrt{1-a^2-b^2}}$$

for some constants c and d. The integrability condition of φ implies that $c=d=0$, thus, $\varphi_a=\varphi_b=0$. This contradicts (4.38). Hence (4.38) has no solution. So, by (4.36) or (4.37), it follows that $\sin(\varphi - \tau) = 0$, thus, $\varphi = \tau + c$ for some constant c. In particular, it follows by (4.31) that

$$\varphi_a = -\frac{ab}{(1-a^2)\sqrt{1-a^2-b^2}}, \quad \varphi_b = -\frac{1}{\sqrt{1-a^2-b^2}}.$$

Substitute these for φ_a and φ_b in the equation (4.33) when $k = 0$. Then we have an identical equality with respect to variables a and b. Noting that $\rho \neq \xi$ and observing the order of the variables a and b in the obtained equality, we can easily induce a contradiction. Hence our statement has been proved. The fact that $\rho \neq \xi$ follows by (4.27), since $\alpha \neq \beta$. □

Proposition 4.9 *Let $0 < h < 1$. Then there exists no $\mathcal{O}(h)$-surface of nonzero constant mean curvature.*

Proof. If there exists an $\mathcal{O}(h)$-surface of nonzero constant mean curvature, the equations (4.34) and (4.35) when $k \neq 0$ have a local solution. Since we can put the $\mathcal{O}(h)$-surface in a desired position of G, we may assume that $a = 0$. Substitute 0 for a in the equations (4.34) and (4.35). Then, since $k \neq 0$, the equations are simplified as follows:

$$\xi(\xi-\rho)\sqrt{1-b^2}\varphi_b|_{a=0} + \rho(\rho-\eta)\frac{1}{\sqrt{1-b^2}} + \xi(\xi-\rho) = 0, \tag{4.39}$$

$$(\rho^2\xi - \rho^3 - \rho\eta^2 + \rho^2\eta)\frac{1}{\sqrt{1-b^2}} + (-\rho\eta^2 + \rho^2\eta)\varphi_b|_{a=0} = 0. \tag{4.40}$$

Since in this case $\rho \neq \xi, \eta$, it follows by (4.39) that

$$\varphi_b|_{a=0} = -\frac{\rho(\rho-\eta)}{\xi(\xi-\rho)}\frac{1}{(1-b^2)} - \frac{1}{\sqrt{1-b^2}} \tag{4.41}$$

where $\rho(\rho-\eta)/\{\xi(\xi-\rho)\} \neq 0$. Also, the coefficient of $1/\sqrt{1-b^2}$ in (4.40) is simplified by (4.28) as follows:

$$\begin{aligned} \rho^2\xi - \rho^3 - \rho\eta^2 + \rho^2\eta &= \rho^2\xi - \xi\eta\rho - \rho\eta(\eta-\rho) \\ &= \rho(\rho-\eta)(\xi+\eta) = \rho(\rho-\eta). \end{aligned}$$

Hence, it follows by (4.40) that

$$\rho(\rho-\eta)\frac{1}{\sqrt{1-b^2}}+\rho(\rho-\eta)\eta\varphi_b|_{a=0}=0, \quad \text{thus,} \quad \varphi_b|_{a=0}=-\frac{1}{\eta\sqrt{1-b^2}}. \tag{4.42}$$

These results for $\varphi_b|_{a=0}$ are contrary to each other, since they have different orders with respect to the variable b. Hence our statement has been proved. □

Next we consider the case that $h=0$. In this case it holds that

$$\mu=0, \quad \nu=1, \quad \alpha=\beta=1, \quad \rho=\xi=\eta=1/2, \tag{4.43}$$

and moreover

$$\theta=-2t+\varphi \tag{4.44}$$

$$\begin{aligned}
&x=\sqrt{1-a^2-b^2}\cos t-b\sin t, \quad y=a\sin t,\\
&z=a\cos t, \quad w=\sqrt{1-a^2-b^2}\sin t+b\cos t \qquad\qquad (4.45)\\
&y_t=a\cos t, \quad z_t=-a\sin t, \quad w_t=\sqrt{1-a^2-b^2}\cos t-b\sin t\\
&y_a=\sin t, \quad z_a=\cos t, \quad w_a=-\frac{a}{\sqrt{1-a^2-b^2}}\sin t,\\
&y_b=0, \quad z_b=0, \quad w_b=-\frac{b}{\sqrt{1-a^2-b^2}}\sin t+\cos t.
\end{aligned}$$

Then the CMC surface equation (4.29) is explicitly given as follows:

$$\begin{aligned}
&\left\{-\frac{a(1-a^2)}{\sqrt{1-a^2-b^2}}\cos(\varphi-\tau)\sin(t-\tau)\right\}\varphi_a\\
&\quad+\left\{\frac{a^2\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\sin(t-\varphi+\tau)-a^2\cos(t-\varphi)\right.\\
&\qquad\left.+\sqrt{1-a^2}\sin(\varphi-\tau)\left(\frac{1-b^2}{\sqrt{1-a^2-b^2}}\cos t-b\sin t\right)\right\}\varphi_b\\
&\quad+\left\{-\frac{\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}}\left\{2\sqrt{1-a^2}\sin(\varphi-\tau)+\frac{ka}{\sqrt{\lambda\lambda_3}}\right\}\sin(t-\tau)\right\}=0
\end{aligned} \tag{4.46}$$

We note that the left hand of the above equation is constructed by Cosine and Sine functions with the same period 1. In the case that $0 < h < 1$, there appear terms with periods $3\ell\alpha$ and $3\ell\beta$ in the equation (4.29) because $\alpha > \beta$, but, in this case these terms are canceled each other. Hence the equalities (4.33), (4.34) and (4.35) when $\alpha < \beta$ are invalid in this case.

Denote by $f(t, a, b)$ the function defined by the left hand of the equation (4.46). Then, the equation $f(t, a, b) = 0$ holds for any value t if and only if at any fixed value t_0, both equations $f(t_0, a, b) = 0$ and $f'(t_0, a, b) = 0$ hold, where f' denotes the first differential with respect to t. Let $t_0 = \tau$. Then it always holds that $f(t_0, a, b) = 0$. Moreover we can see that $f'(t_0, a, b) = 0$ if and only if the following equation holds:

$$\left[-\frac{a(1-a^2)}{\sqrt{1-a^2-b^2}}\varphi_a + \frac{a^2 b}{\sqrt{1-a^2-b^2}}\varphi_b \right] \cos(\varphi - \tau) - \left[\varphi_b + \frac{2(1-a^2)}{\sqrt{1-a^2-b^2}} \right] \sin(\varphi - \tau) - \frac{a\sqrt{1-a^2}}{\sqrt{1-a^2-b^2}} \frac{k}{\sqrt{\lambda\lambda_3}} = 0. \tag{4.47}$$

Now we have the following propositions.

Proposition 4.10 *Let $h = 0$. Then there exists a minimal $\mathcal{O}(0)$-surface.*

Proof. We consider the equation (4.47) when $k = 0$. Let $\varphi = \tau$ where τ is given by the equation $\tau = \arctan \frac{\sqrt{1-a^2-b^2}}{b}$. Then, since

$$\varphi_a = -\frac{ab}{(1-a^2)\sqrt{1-a^2-b^2}} \quad \text{and} \quad \varphi_b = -\frac{1}{\sqrt{1-a^2-b^2}},$$

the coefficient of $\cos(\varphi - \tau)$ in the equation (4.47) is zero. Moreover the second term in (4.47) is also zero since $\sin(\varphi - \tau) = 0$. Hence our φ satisfies the equation (4.47) when $k = 0$. □

Proposition 4.11 *Let $h = 0$. Then for any nonzero number H there exists an $\mathcal{O}(0)$-surface of constant mean curvature H.*

Proof. We show that for any nonzero k, the equation (4.47) has a local solution. Then $k = 4H$. We put

$$u = \sin(\varphi - \tau) \quad \text{and} \quad v = \cos(\varphi - \tau).$$

The equation (4.47) is rewritten as follows:

$$\begin{cases} u^2+v^2=1, \\ -a(1-a^2)u_a + a^2 b u_b + \sqrt{1-a^2-b^2}v_b + 2(a^2-1)u - a\sqrt{1-a^2}\frac{k}{\sqrt{\lambda\lambda_3}} = 0. \end{cases}$$

Using the equality $uu_b+vv_b=0$, we have the following quasi-linear equation of the first order with respect to the variable u.

$$\begin{aligned} &-a(1-a^2)u_a + \left(a^2 b - \sqrt{1-a^2-b^2}\frac{u}{\sqrt{1-u^2}}\right)u_b \\ &\quad + 2(a^2-1)u - a\sqrt{1-a^2}\frac{k}{\sqrt{\lambda\lambda_3}} = 0. \end{aligned} \tag{4.48}$$

We show that this equation has a local solution u near the zero function. The characteristic ODE of the PDE (4.48) is given by the following:

$$\begin{aligned} \frac{da}{ds} &= -a(1-a^2), \\ \frac{db}{ds} &= a^2 b - \sqrt{1-a^2-b^2}\frac{u}{\sqrt{1-u^2}}, \\ \frac{du}{ds} &= -2(a^2-1)u + a\sqrt{1-a^2}\frac{k}{\sqrt{\lambda\lambda_3}}. \end{aligned}$$

Take an initial line L and initia values of solutions a, b and u when $s=0$, as follows: $L=\{(a,b) : a^2<1,\ b=0\}$ and for r such that $(r,0)\in L$

$$a(s,r)|_{s=0}=r, \quad b(s,r)|_{s=0}=0, \quad u(s,r)|_{s=0}=\epsilon(r)$$

where $\epsilon(r)$ is an arbitrary nonzero function near the zero function. Then the Jacobian $\frac{\partial(a,b)}{\partial(s,r)}\Big|_{s=0}$ when $s=0$ is given by the equation

$$\left.\frac{\partial(a,b)}{\partial(s,r)}\right|_{s=0} = \sqrt{1-r^2}\frac{\epsilon(r)}{\sqrt{1-\epsilon(r)^2}} \neq 0.$$

Hence s, r can be solved by a and b, and so $u(s,r)$ is a function of a and b. This $u(a,b)$ gives a local solution of (4.48). □

4.2.4 Summary

Now we can summarize our results in this section as follows.

Theorem 4.12 *Let $G = SU(2)$ and let g be a left invariant metric on G such that $0 < \lambda_1 = \lambda_2 < \lambda_3$. Then the Grassmann geometry on (G, g) is of isotropy type $SO(2)$ and the orbit space of $I_o(G, g)$-orbits is parametrized by the height h where $0 \leq h \leq 1$, which is defined for the unit sphere in the tangent space T_eG at the unity e. Denote by $\mathcal{O}(h)$ the orbit with height h. Then the $\mathcal{O}(h)$-geometry is nonempty if and only if $h \neq 1$. Moreover each $\mathcal{O}(h)$-geomery where $h \neq 1$ has the following properties* (i) *and* (ii):

(i) *Let $0 < h < 1$. Then any $\mathcal{O}(h)$-surface has no geogesic point. Also, there exists no $\mathcal{O}(h)$-surface of constant mean curvature*;

(ii) *Let $h = 0$. Then any $\mathcal{O}(0)$-surface is a flat surface without geodesic points. Also, for any real number H there exists an $\mathcal{O}(0)$-surface of constant mean curvature H, in particular, a minimal $\mathcal{O}(0)$-surface.*

Another proof[1]. Though this theorem is summarized as the result of our argument done until now, we here give another proof for the latter part in the statement (i). Assume that $0 < h < 1$. If there exists an $\mathcal{O}(h)$-surface of constant mean curvature, then for a local solution θ of the equation (4.9) the Gauss curvature K^θ and the mean curvature H^θ satisfy the equations (4.17) and (4.18). On the other hand, by the Gauss equation together with (2.4) and (4.16), it follows

$$\begin{aligned} K^\theta &= K(P) - \{\langle \Pi(X,Y), \Pi(X,Y)\rangle - \langle \Pi(X,X), \Pi(Y,Y)\rangle\} \\ &= \frac{\left(\lambda_3^2 - 4\lambda_3(\lambda_3 - \lambda)h^2\right)}{4} - \frac{\lambda_3^2}{4} = -\lambda_3(\lambda_3 - \lambda)h^2 \end{aligned}$$

where P, X, Y mean the notations in the subsection 4.2.1. Note that the function F_θ, which appears in (4.17) and (4.18), is constant since H^θ is constant by the assumption. Hence, the above equation of K^θ, together with (4.17), induces the following equality

$$-\left(\frac{\lambda_3 h}{\sqrt{1-h^2}}\right)^2 - (F_\theta h)^2 = -\lambda_3(\lambda_3 - \lambda)h^2,$$

[1] This proof is suggested by the referee.

thus, noting that $h \neq 0$,

$$F_\theta^2 = -\lambda\lambda_3 - \frac{h^2}{1-h^2}\lambda_3^2.$$

The right hand of this equality is strictly negative since λ and λ_3 are positive in this case. This is a contradiction. □

Remark 4.13 In this case, from (2.4), the sectional curvature $K(P)$ of (G, g) satisfies the inequality $\lambda_3(\lambda-(3/4)\lambda_3) \leq K(P) \leq {\lambda_3}^2/4$. If $P \in \mathcal{O}(0)$, the sectional curvature $K(P)$ attains the maximum value ${\lambda_3}^2/4$. Then the Grassmann geometry has a flat surface. Also, if $P \in \mathcal{O}(1)$, it attains the minimum value $\lambda_3(\lambda - (3/4)\lambda_3)$. Then the Grassmann geomery is empty.

Remark 4.14 If a left invariant metric g on $SU(2)$ satisfies that $\lambda_1 = \lambda_2 = \lambda_3(>0)$, it is the metric of symmetric space, thus, $(SU(2), g)$ is the standard sphere S^3. In our case, if $h = 0$, the existence equation (4.10) for the Grassmann geometry and the CMC surface equation (4.47) are essentially independent on the constant λ and λ_3, by a suitable change of the constant k. This implies that the $\mathcal{O}(0)$-geometry of this case is just the same as the geometry of surfaces of cylindrical type in the standard sphere S^3. This phenomenon occurs for the $\mathcal{O}(0)$-geometry on the Heisenberg group. (See Remark 3.9.)

Next we consider the case that $0 < \lambda_1 < \lambda_2 = \lambda_3$. We put $\lambda = \lambda_2 = \lambda_3$. In this case the principal curvatures are given by

$$r(E_1, E_1) = \frac{1}{2}{\lambda_1}^2, \quad r(E_2, E_2) = \frac{1}{2}\lambda_1(2\lambda - \lambda_1),$$
$$r(E_3, E_3) = \frac{1}{2}\lambda_1(2\lambda - \lambda_1),$$

and by the same way as the case that $0 < \lambda_1 = \lambda_2 < \lambda_3$, the Grassmann geometry on (G, g) is of isotropy type $SO(2)$, where the $SO(2)$ action on $\mathfrak{g}$ is the standard action on the (E_2E_3)-plane. Moreover, the state of geometry is the same as the one of the case that $0 < \lambda_1 = \lambda_2 < \lambda_3$ where $\lambda = \lambda_1 = \lambda_2$, under the following changes (i), (ii) and (iii) of notations: (i) The left invariant vector fields E_1 (or e_1), E_2 (or e_2) and E_3 (or e_3) for the case that $0 < \lambda_1 = \lambda_2 < \lambda_3$ are respectively changed into the left invariant vector fields

E_2 (or e_2), E_3 (or e_3) and E_1 (or e_1) for the case that $0 < \lambda_1 < \lambda_2 = \lambda_3$; (ii) The local coordinates x, y, z and w for the case that $0 < \lambda_1 = \lambda_2 < \lambda_3$ are respectively changed into the local coordinates x, w, y and z for the case that $0 < \lambda_1 < \lambda_2 = \lambda_3$; (iii) The constants λ_3 and λ for the case that $0 < \lambda_1 = \lambda_2 < \lambda_3$ are respectively changed into the constants λ_1 and λ for the case that $0 < \lambda_1 < \lambda_2 = \lambda_3$.

5. Grassmann geometry on the special linear group $SL(2, \mathbb{R})$

Throughout this section, let G be a simply connected Lie group with Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. We take a left invariant metric g on G and consider the Grassmann geometry on (G, g), firstly the case of trivial isotropy type and next the case of isotropy type $SO(2)$. We again retain the notations in Section 2 and Section 3. The process of arguments is the same as in the previous section.

5.1. Grassmann geometry of trivial type

We first assume that the triple $(\lambda_1, \lambda_2, \lambda_3)$ satisfies $\lambda_1 < 0 < \lambda_2 < \lambda_3$. In this case, similarly to the case of the Subsection 4.1, we can see that the following two cases occur: the case (i) that the principal Ricci curvatures are all distinct; the case (ii) that $\lambda_1 + \lambda_3 = \lambda_2$ and it holds $r(E_1, E_1) = r(E_3, E_3) = 0$ and $r(E_2, E_2) = 2\lambda_1\lambda_3$. But, the following lemma holds for both cases.

Lemma 5.1 *For both cases* (i) *and* (ii) *the Grassmann geometry on* (G, g) *is of trivial isotropy type.*

Proof. We can show our statement by the same way as the proof of Lemma 4.1 for the $SU(2)$ case. For the case (i), the statement is obvious. We consider the case (ii). Denote by R the curvature tensor of (G, g) and by ∇R its covariant derivative. Then, by (2.2), it holds

$$
\begin{aligned}
&(\nabla_{E_1} R)(E_1, E_2)E_1 = -2\lambda_1{\lambda_3}^2 E_3, \quad (\nabla_{E_1} R)(E_3, E_1)E_1 = 2\lambda_1{\lambda_3}^2 E_2, \\
&(\nabla_{E_3} R)(E_2, E_3)E_1 = 2{\lambda_1}^2\lambda_3 E_3, \qquad (\nabla_{E_3} R)(E_3, E_1)E_2 = -2{\lambda_1}^2\lambda_3 E_3, \\
&(\nabla_{E_i} R)(E_j, E_k)E_\ell = 0 \quad \text{for other } i, j, k, \ell.
\end{aligned}
$$

Take any $\varphi \in K_o$. Then we may suppose that φ_{*e} has the following form:

$$\varphi_{*e}(E_1) = \cos\theta E_1 + \sin\theta E_3, \quad \varphi_{*e}(E_3) = -\sin\theta E_1 + \cos\theta E_3,$$
$$\varphi_{*e}(E_2) = E_2.$$

for some θ. Using the above results on ∇R, we calculate $(\nabla_{\varphi_*(E_1)}R)\cdot (\varphi_*(E_1), \varphi_*(E_2))\varphi_*(E_1)$ and $\varphi_*\{(\nabla_{E_1}R)(E_1, E_2)E_1\}$ as follows:

$$\begin{aligned}
&(\nabla_{\varphi_*(E_1)}R)(\varphi_*(E_1), \varphi_*(E_2))\varphi_*(E_1)\\
&\quad = \big\{2\lambda_1{\lambda_3}^2 \sin\theta\cos^2\theta + 2{\lambda_1}^2\lambda_3\sin^3\theta\big\}E_1\\
&\qquad + \big\{-2\lambda_1{\lambda_3}^2\cos^3\theta - 2{\lambda_1}^2\lambda_3\sin^2\theta\cos\theta\big\}E_3,\\
&\varphi_*\{(\nabla_{E_1}R)(E_1, E_2)E_1\} = 2\lambda_1{\lambda_3}^2\sin\theta E_1 - 2\lambda_1{\lambda_3}^2\cos\theta E_3.
\end{aligned}$$

Since these vectors coincide, it follows

$$\lambda_3\sin\theta\cos^2\theta + \lambda_1\sin^3\theta = \lambda_3\sin\theta, \quad \lambda_3\cos^3\theta + \lambda_1\sin^2\theta\cos\theta = \lambda_3\cos\theta,$$

and thus

$$(\lambda_1 - \lambda_3)\sin^3\theta = 0, \quad (\lambda_1 - \lambda_3)\cos\theta\sin^2\theta = 0.$$

Since $\lambda_1 - \lambda_3 < 0$, it follows $\sin\theta = 0$. This implies that $\varphi = 1$, since K_o is connected. Hence K_o is trivial, which implies that the Grassmann geometry of this case is of trivial isotropy type. □

Now, noting Remark 4.3 in the previous section, we have the following proposition for the existence of Grassmann geometry.

Proposition 5.2 *Let $W \in S^2(\mathfrak{g})$ and let $\mathcal{O}(P(W))$ be the $I_o(G, g)$-orbit which contains the plane $P(W)$ in $\mathfrak{g}$. Then the $\mathcal{O}(P(W))$-geometry is not empty if and only if W satisfies that*

$$\lambda_1\langle W, E_1\rangle^2 + \lambda_2\langle W, E_2\rangle^2 + \lambda_3\langle W, E_3\rangle^2 = 0. \tag{5.49}$$

Since in this case $\lambda_1 < 0$ and $\lambda_2, \lambda_3 > 0$, the equation (5.49) implies that the set of $I_o(G, g)$-orbits whose geometries are not empty corresponds to an ellipse in $\mathbb{R}P^2$.

We next see the geometric properties of $\mathcal{O}(P(W))$-surfaces for $W \in S^2(\mathfrak{g})$ which satisfies the condition (5.49). We prepare the following lemmas.

Lemma 5.3 *If* $W \in S^2(\mathfrak{g})$ *satisfies the condition* (5.49), *it holds that* $\langle W, E_1 \rangle \neq 0$.

Proof. Put $w_i = \langle W, E_i \rangle$ where $i = 1, 2, 3$. Then it holds that ${w_1}^2 + {w_2}^2 + {w_3}^2 = 1$ and $\lambda_1 {w_1}^2 + \lambda_2 {w_2}^2 + \lambda_3 {w_3}^2 = 0$. Assume that $w_1 = 0$. Then, since $0 < \lambda_2 < \lambda_3$, it follows that ${w_2}^2 = \frac{\lambda_3}{\lambda_3 - \lambda_2} > 1$. This is a contradiction. Hence $w_1 \neq 0$. $\square$

Next, for $W \in S^2(\mathfrak{g})$ such that $w_1 \neq 0$, put $X = w_2 E_1 - w_1 E_2$ and $Y = w_3 E_1 - w_1 E_3$. Then X and Y are linear independent vectors in the plane $P(W)$. Let $\mathcal{D}(W)$ the left invariant distribution on G generated by the plane $P(W)$, and denote by ∇ the Riemannian connection on (G, g) and by Π the normal component of ∇ with respect to the orthogonal decomposition $TG = \mathcal{D}(W) \oplus \mathcal{D}(W)^{\perp}$.

Lemma 5.4 *Then it holds that*

$$\begin{aligned}
\nabla_X X &= w_1 w_2 (\mu_2 - \mu_1) E_3, \\
\nabla_Y Y &= w_1 w_3 (\mu_1 - \mu_3) E_2, \\
\nabla_X Y &= {w_1}^2 \mu_2 E_1 + w_1 w_2 \mu_1 E_2 + w_1 w_3 \mu_2 E_3, \\
\nabla_Y X &= -{w_1}^2 \mu_3 E_1 - w_1 w_2 \mu_3 E_2 - w_1 w_3 \mu_1 E_3,
\end{aligned} \tag{5.50}$$

$$\begin{aligned}
\Pi(X, X) &= w_1 w_2 w_3 (\mu_2 - \mu_1) W, \\
\Pi(Y, Y) &= w_1 w_2 w_3 (\mu_1 - \mu_3) W, \\
\Pi(X, Y) &= w_1 \big(\mu_1 {w_2}^2 + \mu_2 {w_3}^2 + \mu_2 {w_1}^2\big) W, \\
\Pi(Y, X) &= -w_1 \big(\mu_1 {w_3}^2 + \mu_3 {w_2}^2 + \mu_3 {w_1}^2\big) W,
\end{aligned} \tag{5.51}$$

where μ_i $(i = 1, 2, 3)$ *are the constants defined in* (2.3).

This lemma can be directly calculated by (2.2). Also, if W satisfies the condition (5.49), it holds that $\Pi(X, Y) = \Pi(Y, X)$, and then Π gives the second fundamental forms of $\mathcal{O}(P(W))$-surfaces, which are obtained as the integral surfaces of $\mathcal{D}(W)$.

We now consider the geometry of $\mathcal{O}(P(W))$-surfaces.

Proposition 5.5 *Let* $W \in S^2(\mathfrak{g})$ *satisfy the condition* (5.49). *Then any* $\mathcal{O}(P(W))$*-surface is minimal.*

Proof. Let S be an $\mathcal{O}(P(W))$-surface, i.e., an integral surface of $\mathcal{D}(W)$. The mean curvature H of S is given by the following general formula

$$2H = \frac{\langle X, X\rangle\langle \Pi(Y,Y), W\rangle + \langle Y, Y\rangle\langle \Pi(X,X), W\rangle - 2\langle X, Y\rangle\langle \Pi(X,Y), W\rangle}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}.$$

Then, the numerator T of the above fractional equation is calculated by (5.51) and (5.49) as follows:

$$\begin{aligned} T &= \left(w_1^2 + {w_3}^2\right) w_1 w_2 w_3 (\mu_2 - \mu_1) + \left(w_1^2 + {w_2}^2\right) w_1 w_2 w_3 (\mu_1 - \mu_3) \\ &\quad - 2 w_2 w_3 \left(w_1 {w_2}^2 \mu_1 + w_1 {w_3}^2 \mu_2 + {w_1}^3 \mu_2\right) \\ &= -w_1 w_2 w_3 \left\{(\mu_2 + \mu_3){w_1}^2 + (\mu_3 + \mu_1){w_2}^2 + (\mu_1 + \mu_2){w_3}^2\right\} \\ &= -w_1 w_2 w_3 \left(\lambda_1 {w_1}^2 + \lambda_2 {w_2}^2 + \lambda_3 {w_3}^2\right) \\ &= 0. \end{aligned}$$

Hence it follows $H = 0$. □

Proposition 5.6 *Let $W \in S^2(\mathfrak{g})$ satisfy the condition* (5.49). *Then an $\mathcal{O}(P(W))$-surface is totally geodesic if and only if it is the case $\lambda_1 + \lambda_3 = \lambda_2$ and W satisfies that*

$$\langle W, E_1\rangle^2 = \frac{\lambda_3}{\lambda_3 - \lambda_1}, \quad \langle W, E_2\rangle^2 = 0, \quad \langle W, E_3\rangle^2 = -\frac{\lambda_1}{\lambda_3 - \lambda_1}. \tag{5.52}$$

Proof. Let S be an integral surface of $\mathcal{D}(W)$. Then S is totally geodesic if and only if it holds that $\Pi(X,X) = \Pi(X,Y) = \Pi(Y,Y) = 0$. By (5.51) and Lemma 5.3, this condition is moreover equivalent to the following equations:

$$\begin{aligned} &w_2 w_3 (\mu_2 - \mu_1) = 0, \quad w_2 w_3 (\mu_1 - \mu_3) = 0, \\ &\mu_1 {w_2}^2 + \mu_2 \left(w_1^2 + w_3^2\right) = \mu_1 {w_3}^2 + \mu_3 \left({w_1}^2 + {w_2}^2\right) = 0. \end{aligned} \tag{5.53}$$

We first show that $w_3 \neq 0$. If $w_3 = 0$, it follows by (5.53) that $\mu_3 = 0$, thus, $\lambda_1 + \lambda_2 - \lambda_3 = 0$. Since $\lambda_1 < 0$ and $\lambda_2 - \lambda_3 < 0$, this does not occur. Hence it holds $w_3 \neq 0$. We next show that $w_2 = 0$. If $w_2 \neq 0$, together with the condition $w_3 \neq 0$, it follows from (5.53) that $\mu_1 = \mu_2 = \mu_3$, thus, it holds that $\lambda_1 = \lambda_2 = \lambda_3$. This is not the case. Hence it holds $w_2 = 0$.

Under these conditions, the condition (5.53) is equivalent to the conditions $\mu_2 = 0$ and $\mu_1 {w_3}^2 + \mu_3 {w_1}^2 = 0$. Since $\mu_2 = (\lambda_1 - \lambda_2 + \lambda_3)/2$, the condition $\mu_2 = 0$ implies that $\lambda_1 + \lambda_3 = \lambda_2$. Also, when $w_2 = 0$, it follows together with (5.49) that ${w_1}^2 = \lambda_3/(\lambda_3 - \lambda_1)$ and ${w_3}^2 = -\lambda_1/(\lambda_3 - \lambda_1)$. □

Proposition 5.7 *Let $W \in S^2(\mathfrak{g})$ satisfy the condition (5.49). Then an $\mathcal{O}(P(W))$-surface has the positive constant Gauss curvature $\lambda_2\lambda_3 + (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)\langle W, E_1\rangle^2$.*

Proof. Let S be an integral surface of $\mathcal{D}(W)$. The Gauss curvature K of S is generally given by the following Gauss equation

$$K = K(P(W)) - \frac{\langle \Pi(X,Y), \Pi(X,Y)\rangle - \langle \Pi(X,X), \Pi(Y,Y)\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

Using this formula together with (2.4) and (5.51), we can calculate K as follows:

$$\begin{aligned} K &= \big\{(\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1) - 2\big(\mu_2\mu_3 {w_1}^2 + \mu_3\mu_1 {w_2}^2 + \mu_1\mu_2 {w_3}^2\big)\big\} \\ &\quad - \frac{1}{{w_1}^2}\big\{{w_1}^2\big(\mu_1 {w_2}^2 + \mu_2({w_1}^2 + {w_3}^2)\big)^2 \\ &\qquad\qquad - {w_1}^2{w_2}^2{w_3}^2(\mu_2 - \mu_1)(\mu_1 - \mu_3)\big\} \\ &= \lambda_2\lambda_3 + (\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3){w_1}^2 > 0, \end{aligned}$$

where we cancel ${w_2}^2$ and ${w_3}^2$ by using the equalities

$${w_1}^2 + {w_2}^2 + {w_3}^2 = 1 \quad \text{and} \quad \lambda_1 {w_1}^2 + \lambda_2 {w_2}^2 + \lambda_3 {w_3}^2 = 0,$$

and moreover replace μ_i's by λ_j's under the following relations

$$\mu_1 = \frac{1}{2}(-\lambda_1 + \lambda_2 + \lambda_3), \quad \mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2 + \lambda_3),$$
$$\mu_3 = \frac{1}{2}(\lambda_1 + \lambda_2 - \lambda_3).$$

□

Now we can summarize our results in this subsection as follows.

Theorem 5.8 *Let G be the simply connected Lie group with the Lie algebra*

$\mathfrak{sl}(2,\mathbb{R})$ *and* g *a left invariant metric on* G *such that* $\lambda_1 < 0 < \lambda_2 < \lambda_3$. *Then the Grassmann geometry on* (G,g) *is of trivial isotropy type and the orbit space of* $I_o(G,g)$*-orbits is bijective to the projective plane* $\mathbb{R}P^2$ *over* $\mathfrak{g}$. *Moreover, for a plane* $P(W)$ *associated with* $W \in S^2(\mathfrak{g})$, *the* $\mathcal{O}(P(W))$*-geometry is nonempty if and only if* W *satisfies the condition*

$$\lambda_1 \langle W, E_1 \rangle^2 + \lambda_2 \langle W, E_2 \rangle^2 + \lambda_3 \langle W, E_3 \rangle^2 = 0.$$

Also, when the $\mathcal{O}(P(W))$*-geometry is not empty, any* $\mathcal{O}(P(W))$*-surface is a minimal surface of constant positive Gauss curvature* $\lambda_2\lambda_3 + (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)\langle W, E_1\rangle^2$, *and particularly it is totally geodesic if and only if the left invariant metric* g *is the case* $\lambda_1 + \lambda_3 = \lambda_2$ *and the plane* $P(W)$ *is either of* $P\left(\sqrt{\frac{\lambda_3}{\lambda_3-\lambda_1}}E_1 \pm \sqrt{\frac{-\lambda_1}{\lambda_3-\lambda_1}}E_3\right)$.
Also, the maximal $\mathcal{O}(P(W))$*-surfaces are homogeneous Riemannian surfaces of* (G,g) *which are congruent to each other.*

Remark 5.9 When an $\mathcal{O}(P(W))$-geometry is not empty, the sectional curvature $K(P(W))$ of (G,g) is given by

$$K(P(W)) = \left\{\frac{1}{4}(\lambda_1 - \lambda_2 - \lambda_3)^2 + \lambda_2\lambda_3\right\} - 2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)w_1^2,$$

and thus it satisfies the following inequality

$$\begin{aligned}\left\{\frac{1}{4}(\lambda_1 - \lambda_2 - \lambda_3)^2 + \lambda_2\lambda_3\right\} &- 2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \\ \leq K(P(W)) &< \left\{\frac{1}{4}(\lambda_1 - \lambda_2 - \lambda_3)^2 + \lambda_2\lambda_3\right\}.\end{aligned}$$

When it is the case $\lambda_1 + \lambda_3 = \lambda_2$, the planes $P\left(\sqrt{\frac{\lambda_3}{\lambda_3-\lambda_1}}E_1 \pm \sqrt{\frac{-\lambda_1}{\lambda_3-\lambda_1}}E_3\right)$ does not attain the minimum of this inequality. This state is different from the other cases of Grassmann geometry.

5.2. Grassmann geometry of isotropy type $SO(2)$

Next we consider the case that $\lambda_1 < 0 < \lambda_2 = \lambda_3$ where we put $\lambda = \lambda_2 = \lambda_3$. Let G be a simply connected Lie group with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ and let g be a left invariant metric on G of this case. We first consider the isotropy K_o in the isometry group $I_o(G, g)$ at the unity e. In this case the principal Ricci curvatures of (G, g) are given by

$$r(E_1, E_1) = \frac{1}{2}{\lambda_1}^2, \quad r(E_2, E_2) = r(E_3, E_3) = \frac{1}{2}\lambda_1(2\lambda - \lambda_1),$$

where since $\lambda \neq \lambda_1$, it holds that $r(E_2, E_2) = r(E_3, E_3) \neq r(E_1, E_1)$. Then, by the same way as Lemma 4.4 for the $SU(2)$ case, we have the following lemma.

Lemma 5.10 *The Grassmann geometry of this type is of isotropy type $SO(2)$.*

We here note that the $SO(2)$-action on $\mathfrak{g}$ is the standard action on the (E_2E_3)-plane. Also, by this lemma, the orbit space of $I_o(G, g)$-orbits is the quotient space $SO(2)\backslash\mathbb{R}P^2(\mathfrak{g})$. In the following, similarly to the $SU(2)$ case, we use the height h $(0 \leq h \leq 1)$ from the (E_2E_3)-plane as a parametrization of the orbit space. Then $\mathcal{O}(h)$ where $0 \leq h \leq 1$ expresses the $I_o(G, g)$-orbit which contains the planes $P(W)$ for the elements W in $S^2(\mathfrak{g})$ with height h.

5.2.1 Existence and geometry of $\mathcal{O}$-surfaces

We divide the case into the case that $h \neq 1$ and the case that $h = 1$. In next subsection 5.2.2, the first case will be moreover devided into 4 cases.

We first consider the case that $h = 1$. Since $h = 1$, it follows that $\mathcal{O}(1) = \mathcal{O}(P(E_1))$ and $K_o(P(E_1)) = \{P(E_1)\}$. Hence the orbit $\mathcal{O}(1)$ induces a unique left invariant distribution on G. By the same way as Proposition 4.5. we have the following proposition.

Proposition 5.11 *The $\mathcal{O}(1)$-geometry is empty.*

Next we give a local coordinate system of $SL(2, \mathbb{R})$. Since in the following subsections we consider the local geometry of $\mathcal{O}(h)$-surfaces, we may regard a local coordinate system of $SL(2, \mathbb{R})$ as a local coordinate system of G. Set $D = \{(y, z, w) \in \mathbb{R}^3 : -y^2 + z^2 - w^2 < 1\}$ and put $x = \sqrt{1 + y^2 - z^2 + w^2}$ on D. Then the correspondence

$$D \ni (y, z, w) \longmapsto \begin{pmatrix} x + y & z + w \\ -z + w & x - y \end{pmatrix} \in SL(2, \mathbb{R})$$

gives a local coordinate system of $SL(2, \mathbb{R})$, by which the point $(0, 0, 0) \in D$

corresponds to the unit e of $SL(2,\mathbb{R})$. Let e_i $(i=1,2,3)$ be the elements in $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{R})$ defined by putting

$$e_1=\frac{\lambda}{2}\begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad e_2=\frac{\sqrt{\lambda|\lambda_1|}}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad e_3=\frac{\sqrt{\lambda|\lambda_1|}}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

and $\hat{E}_i$ $(i=1,2,3)$ the vector fields of $SL(2,\mathbb{R})$ defined by putting $\hat{E}_i(q)=q\cdot e_i$ for $q\in SL(2,\mathbb{R})$. Then $\hat{E}_i$ are left invariant and they satisfy the following relations

$$\big[\hat{E}_1,\hat{E}_2\big]=\lambda\hat{E}_3,\quad \big[\hat{E}_2,\hat{E}_3\big]=\lambda_1\hat{E}_1,\quad \big[\hat{E}_3,\hat{E}_1\big]=\lambda\hat{E}_2.$$

In the following we identify the vector fields $\hat{E}_i$ with the elements $E_i\in\mathfrak{g}$. Then, the following holds on D.

$$(E_1,E_2,E_3)=\left(\frac{\partial}{\partial y},\frac{\partial}{\partial z},\frac{\partial}{\partial w}\right)\begin{pmatrix}-\frac{\lambda}{2}w & \frac{\sqrt{\lambda|\lambda_1|}}{2}z & \frac{\sqrt{\lambda|\lambda_1|}}{2}x\\ \frac{\lambda}{2}x & \frac{\sqrt{\lambda|\lambda_1|}}{2}y & -\frac{\sqrt{\lambda|\lambda_1|}}{2}w\\ \frac{\lambda}{2}y & \frac{\sqrt{\lambda|\lambda_1|}}{2}x & -\frac{\sqrt{\lambda|\lambda_1|}}{2}z\end{pmatrix}. \tag{5.54}$$

Now we consider the $\mathcal{O}(h)$-geometry on G where $h\neq 1$. In this case the orbit $\mathcal{O}(h)\cap\mathrm{Gr}^2(\mathfrak{g})$ is identified with the circle in $S^2(\mathfrak{g})$ of unit vectors with height h. For a local function θ on an open set O in G, set

$$\begin{aligned}X(q)&=h\sin(\theta(q))E_2(q)-h\cos(\theta(q))E_3(q)+\sqrt{1-h^2}E_1(q),\\ Y(q)&=\cos(\theta(q))E_2(q)+\sin(\theta(q))E_3(q),\\ N(q)&=\sqrt{1-h^2}\sin(\theta(q))E_2(q)-\sqrt{1-h^2}\cos(\theta(q))E_3(q)-hE_1(q)\end{aligned}\tag{5.55}$$

where $q\in O$, and moreover set $P_q=\mathbb{R}\cdot X(q)\oplus\mathbb{R}Y(q)$ and $\mathcal{D}^\theta=\{P_q : q\in O\}$. Then the set $\{X,Y,N\}$ is an orthonormal frame of vector fields on O, the planes P_q belong to $\mathcal{O}(h)$, and $\mathcal{D}^\theta$ defines the distribution on O generated by the planes P_q. By Lemma 3.1, the $\mathcal{O}(h)$-geometry is non-empty if and only if there exist an open set O in G and a function θ on it such that $\mathcal{D}^\theta$ is involutive. We may here suppose that O is a neighbourhood of the unity e which is contained in the domain D, since (G,g) is a Riemannian homogeneous space.

By the same way as the $SU(2)$ case, $\mathcal{D}^\theta$ is involutive if and only if it holds

$$\begin{aligned} &h\sqrt{1-h^2}\sin\theta(E_2\theta) - h\sqrt{1-h^2}\cos\theta(E_3\theta) \\ &\quad + (1-h^2)(E_1\theta) + \lambda(1-h^2) + \lambda_1 h^2 = 0. \end{aligned} \tag{5.56}$$

Rewriting this in terms of x, y, z, w by (5.54), we have the following quasi-linear PDE of the first order with unknown function θ:

$$\begin{aligned} &\left\{\sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}z\sin\theta - \sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}x\cos\theta - (1-h^2)w\right\}\left(\frac{\partial\theta}{\partial y}\right) \\ &\quad + \left\{\sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}y\sin\theta + \sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}w\cos\theta + (1-h^2)x\right\}\left(\frac{\partial\theta}{\partial z}\right) \\ &\quad + \left\{\sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}x\sin\theta + \sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}z\cos\theta + (1-h^2)y\right\}\left(\frac{\partial\theta}{\partial w}\right) \\ &\quad + 2(1-h^2) + 2\frac{\lambda_1}{\lambda}h^2 = 0. \end{aligned} \tag{5.57}$$

We call (5.56) or (5.57) the *existence equations of Grassmann geometry.*

Let us consider the existence problem of solutions for the equation (5.57). The associated characteristic ODE's are given in the following.

$$\begin{aligned} \frac{dy}{dt} &= \sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}z\sin\theta - \sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}x\cos\theta - (1-h^2)w, \\ \frac{dz}{dt} &= \sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}y\sin\theta + \sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}w\cos\theta + (1-h^2)x, \\ \frac{dw}{dt} &= \sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}x\sin\theta + \sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}z\cos\theta + (1-h^2)y, \end{aligned} \tag{5.58}$$

$$\frac{d\theta}{dt} = -2(1-h^2) - 2\frac{\lambda_1}{\lambda}h^2. \tag{5.59}$$

Take an initial plane Q and initial values of solutions x, y, z, w and an initial function of θ when $t = 0$, as follows: $Q = \{(y, z, w) \in D : y = 0\}$, and for a,

b such that $(0, a, b) \in Q$

$$y(t, a, b)|_{t=0} = 0, \quad z(t, a, b)|_{t=0} = a, \quad w(t, a, b)|_{t=0} = b,$$
$$x(t, a, b)|_{t=0} = \sqrt{1 - a^2 + b^2}, \quad \theta(t, a, b)|_{t=0} = \varphi(a, b)$$

where $\varphi(a, b)$ is an arbitrary function. Then, from (5.59), it follows that

$$\theta(t, a, b) = -2\left\{(1 - h^2) + \frac{\lambda_1}{\lambda} h^2\right\} t + \varphi(a, b), \tag{5.60}$$

and the Jacobian $\frac{\partial(y,z,w)}{\partial(t,a,b)}\Big|_{t=0}$ when $t = 0$ is given by the equation

$$\frac{\partial(y, z, w)}{\partial(t, a, b)}\bigg|_{t=0} = \sqrt{\frac{|\lambda_1|}{\lambda}} h\sqrt{1 - h^2}\left(a \sin\varphi - \sqrt{1 - a^2 + b^2} \cos\varphi\right) - (1 - h^2)b. \tag{5.61}$$

Hence, similarly to the $SU(2)$ case, we have the following proposition.

Proposition 5.12 *For a height h such that $0 \leq h < 1$, the $\mathcal{O}(h)$-geometry is non-empty.*

We next consider the geometry of $\mathcal{O}(h)$-surfaces. The state of geometry is the same as the $SU(2)$ case if we replace λ_3, E_1, E_2 for the $SU(2)$ case by λ_1, E_2, E_3, respectively. Let θ be a local solution of the equation (5.57) and $\mathcal{D}^\theta$ the involutive distribution assosiated with θ. The integral surfaces S of $\mathcal{D}^\theta$ are $\mathcal{O}(h)$-surfaces, and their tangent spaces are generated by the vector fields X and Y, and the vector field N gives a unit normal vector field on the surfaces. By (2.2) and (5.55) we can calculate the covariant derivatives $\nabla_X X$, $\nabla_X Y$, $\nabla_Y X$, and $\nabla_Y Y$ as follows.

$$\begin{aligned}
\nabla_X X &= -\lambda_1 \frac{h}{\sqrt{1 - h^2}} Y, & \nabla_Y Y &= -F_\theta\left(hX + \sqrt{1 - h^2} N\right),\\
\nabla_X Y &= \lambda_1 \frac{h}{\sqrt{1 - h^2}} X + \frac{\lambda_1}{2} N, & \nabla_Y X &= F_\theta h Y + \frac{\lambda_1}{2} N
\end{aligned} \tag{5.62}$$

where $F_\theta = \cos\theta(E_2\theta) + \sin\theta(E_3\theta)$. Here, to calculate $\nabla_X X$, we use the existence equation (5.56) for the $\mathcal{O}(h)$-geometry. Taking the tangent and the normal parts of these covariant derivatives, we have the following.

$$
\begin{aligned}
&\nabla^\theta_X X = -\lambda_1 \frac{h}{\sqrt{1-h^2}} Y, && \Pi^\theta(X,X) = 0,\\
&\nabla^\theta_Y Y = -F_\theta h X, && \Pi^\theta(Y,Y) = -F_\theta\sqrt{1-h^2}N,\\
&\nabla^\theta_X Y = \lambda_1 \frac{h}{\sqrt{1-h^2}} X, && \Pi^\theta(X,Y) = \Pi^\theta(Y,X) = \frac{\lambda_1}{2}N,\\
&\nabla^\theta_Y X = F_\theta h Y
\end{aligned} \tag{5.63}
$$

where ∇^θ and Π^θ give the Riemanian connection and the second fundamental form of $\mathcal{O}(h)$-surfaces S. Also, the Gauss curvature K^θ and the mean curvature H^θ of $\mathcal{O}(h)$-surfaces S are given by the following equations:

$$K^\theta = -h(XF_\theta) - \left(\frac{\lambda_1 h}{\sqrt{1-h^2}}\right)^2 - (F_\theta h)^2 \tag{5.64}$$

$$H^\theta = -\frac{F_\theta\sqrt{1-h^2}}{2}. \tag{5.65}$$

Hence, by (2.4), (5.63) and (5.64), we have the following proposition.

Proposition 5.13 *Let $h \neq 1$. Then, for a plane $P \in \mathcal{O}(h)$ the sectional curvature $K(P)$ of (G,g) is given by $\{{\lambda_1}^2 - 4\lambda_1(\lambda_1-\lambda)h^2\}/4$. Also, any $\mathcal{O}(h)$-surface has no geodesic point and in particular any $\mathcal{O}(0)$-surface is moreover flat.*

5.2.2 Existence equations of Grassmann geometry

In this subsection we divide the case that $0 \leq h < 1$ into 4 cases according to the types of the characteristic ODE's (5.58). To complete the system (5.58) we differentiate the local function $x = \sqrt{1+y^2-z^2+w^2}$ as follows.

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{1}{x}\left(y\frac{dy}{dt} - z\frac{dz}{dt} + w\frac{dw}{dt}\right)\\
&= \sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2}(-y\cos\theta + w\sin\theta) - (1-h^2)z.
\end{aligned}
$$

The completed system is now represented as follows.

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \\ \frac{dw}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -\mu\cos\theta & -\nu & \mu\sin\theta \\ -\mu\cos\theta & 0 & \mu\sin\theta & -\nu \\ \nu & \mu\sin\theta & 0 & \mu\cos\theta \\ \mu\sin\theta & \nu & \mu\cos\theta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \tag{5.66}$$

where $\theta = \theta(t) = -2\big((1-h^2) + \frac{\lambda_1}{\lambda}h^2\big)t + \varphi(a,b)$ and

$$\mu = \sqrt{\frac{|\lambda_1|}{\lambda}}h\sqrt{1-h^2} > 0, \qquad \nu = 1-h^2 > 0. \tag{5.67}$$

We write the system (5.66) as $d\mathbb{X}/dt = A(t)\mathbb{X}$ and moreover put $A(t) = \nu A_1 + \mu A_2(t)$ where

$$A_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2(t) = \begin{pmatrix} 0 & -\cos\theta & 0 & \sin\theta \\ -\cos\theta & 0 & \sin\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \\ \sin\theta & 0 & \cos\theta & 0 \end{pmatrix}. \tag{5.68}$$

The matrix A_1 is skew symmetric and orthogonal, the matrix $A_2(t)$ is symmetric and orthogonal, and they satisfy that $A_1A_2(t) + A_2(t)A_1 = 0$. We note that the matrix $A(t)$ is periodic similarly to the $SU(2)$ case. Hence the ODE $d\mathbb{X}/dt = A(t)\mathbb{X}$ can be transformed into a linear ODE of constant efficients by a suitable change of variables. We practice this process and after that, solve the linear ODE of constant efficients concretely.

Take an orthogonal matrix O as follows:

$$O = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Then it follows

$${}^tOA(t)O = \begin{pmatrix} \nu J & \mu R(t) \\ \mu R(t) & \nu J \end{pmatrix},$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad R(t) = \begin{pmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}.$$

Changing $\mathbb{X}(t)$ into $\mathbb{Y}(t)$ by the relation $\mathbb{X}(t) = O\mathbb{Y}(t)$, we have the following ODE

$$\frac{d\mathbb{Y}}{dt} = \begin{pmatrix} \nu J & \mu R(t) \\ \mu R(t) & \nu J \end{pmatrix} \mathbb{Y}.$$

Put $\mathbb{Y} = \left(\begin{smallmatrix} \mathbb{Y}_1 \\ \mathbb{Y}_2 \end{smallmatrix}\right)$ where $\mathbb{Y}_i$ are 2×1 matrices. Then the above ODE is composed by the following two linear systems

$$\frac{d\mathbb{Y}_1}{dt} = \nu J\mathbb{Y}_1 + \mu R(t)\mathbb{Y}_2 \quad \text{and} \quad \frac{d\mathbb{Y}_2}{dt} = \mu R(t)\mathbb{Y}_1 + \nu J\mathbb{Y}_2.$$

Take an orthogonal 2×2 matrix $T(t)$ as follows:

$$T(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{-\cos\theta}{\sqrt{1-\sin\theta}} & \frac{-\cos\theta}{\sqrt{1+\sin\theta}} \\ \frac{\sin\theta-1}{\sqrt{1-\sin\theta}} & \frac{\sin\theta+1}{\sqrt{1+\sin\theta}} \end{pmatrix}$$

where it is assumed that $|\sin\theta| \neq 1$. Then it follows that

$${}^tT(t)JT(t) = \begin{pmatrix} 0 & \mathrm{sgn}(\cos\theta) \\ -\mathrm{sgn}(\cos\theta) & 0 \end{pmatrix}, \quad {}^tT(t)R(t)T(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\left(\frac{d\,{}^tT(t)}{dt}\right)T(t) = \frac{\gamma}{2} \begin{pmatrix} 0 & \mathrm{sgn}(\cos\theta) \\ -\mathrm{sgn}(\cos\theta) & 0 \end{pmatrix}$$

where sign$(\cos\theta)$ denotes the sign of $\cos\theta$ and γ denotes the constant such that $\theta = \gamma t + \varphi$, thus, $\gamma = -2\{(1-h^2) + \frac{\lambda_1}{\lambda}h^2\}$. We here note that the assumption $|\sin\theta| \neq 1$ is no problem and we may moreover assume that sign$(\cos\theta) = 1$, since we can select a suitable initial deta of φ in the subsequent arguments of this paper. Under these preparations, transform the variables $\mathbb{Y}_i$ into new variables $\mathbb{Z}_i$ where $i = 1, 2$ by the equations $\mathbb{Y}_i = T(t)\mathbb{Z}_i$. Then it follows that

$$\begin{aligned}
\frac{d\mathbb{Z}_1}{dt} &= \left\{\frac{d}{dt}\,{}^tT(t)\right\}\mathbb{Y}_1 + {}^tT(t)\frac{d\mathbb{Y}_1}{dt} \\
&= \frac{d}{dt}\{{}^tT(t)\}T(t)\mathbb{Z}_1 + \nu\,{}^tT(t)JT(t)\mathbb{Z}_1 + \mu\,{}^tT(t)R(t)T(t)\mathbb{Z}_2 \\
&= \left(\nu + \frac{\gamma}{2}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbb{Z}_1 + \mu\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbb{Z}_2, \\
\frac{d\mathbb{Z}_2}{dt} &= \left\{\frac{d}{dt}\,{}^tT(t)\right\}\mathbb{Y}_2 + {}^tT(t)\frac{d\mathbb{Y}_2}{dt} \\
&= \frac{d}{dt}\{{}^tT(t)\}T(t)\mathbb{Z}_2 + \nu\,{}^tT(t)JT(t)\mathbb{Z}_2 + \mu\,{}^tT(t)R(t)T(t)\mathbb{Z}_1 \\
&= \left(\nu + \frac{\gamma}{2}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\mathbb{Z}_2 + \mu\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mathbb{Z}_1.
\end{aligned}$$

Putting $\mathbb{Z} = \binom{\mathbb{Z}_1}{\mathbb{Z}_2}$ and $\upsilon = \nu + \gamma/2 = -\frac{\lambda_1}{\lambda}h^2$, we obtain the following linear system of ODE's of constant coefficients:

$$\frac{d\mathbb{Z}}{dt} = \begin{pmatrix} 0 & \upsilon & \mu & 0 \\ -\upsilon & 0 & 0 & -\mu \\ \mu & 0 & 0 & \upsilon \\ 0 & -\mu & -\upsilon & 0 \end{pmatrix}\mathbb{Z}. \tag{5.69}$$

The change of the variable $\mathbb{Z}$ into the variable $\mathbb{X}$ is given by the following relation.

$$\mathbb{X} = O\begin{pmatrix} T(t) & 0 \\ 0 & T(t) \end{pmatrix}\mathbb{Z} = \frac{1}{2}\begin{pmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ q & -p & 0 & 0 \\ 0 & 0 & q & -p \end{pmatrix}\mathbb{Z} \tag{5.70}$$

where p and q are the functions of variable θ which are defined when $\cos\theta > 0$ as follows:

$$\begin{aligned}
p(\theta) &= \frac{1 - \cos\theta - \sin\theta}{\sqrt{1 - \sin\theta}} = -\frac{1 - \cos\theta + \sin\theta}{\sqrt{1 + \sin\theta}} = 2\sin\frac{\theta}{2}, \\
q(\theta) &= -\frac{1 + \cos\theta - \sin\theta}{\sqrt{1 - \sin\theta}} = -\frac{1 + \cos\theta + \sin\theta}{\sqrt{1 + \sin\theta}} = 2\cos\frac{\theta}{2}.
\end{aligned}$$

We here note that

$$p^2 + q^2 = 4, \quad \frac{dp}{d\theta} = \frac{1}{2}q, \quad \frac{dq}{d\theta} = -\frac{1}{2}p.$$

Denote by B the 4×4 matrix in the equation (5.69). Then the characteristic polynomial of B is given by $\{x^2 - (\mu^2 - v^2)\}^2$ where $\mu^2 - v^2 = -\frac{\lambda_1}{\lambda} h^2 \big(1 - \frac{\lambda - \lambda_1}{\lambda} h^2\big)$. We divide the case into the following 4 cases (I) through (IV) according to each Jordan type of the matrix B: Case (I) is the case that $h = 0$, thus, $\mu = v = 0$; Case (II) is the case that $h = \sqrt{\frac{\lambda}{\lambda - \lambda_1}} < 1$, in this case, $\mu^2 = v^2$ but $\mu, v \neq 0$; Case (III) is the case that $0 < h < \sqrt{\frac{\lambda}{\lambda - \lambda_1}}$, in this case, $\mu^2 > v^2$ and the eigenvalues of B are two distinct real numbers; Case (IV) is the case that $\sqrt{\frac{\lambda}{\lambda - \lambda_1}} < h < 1$, in this case, $\mu^2 < v^2$ and the eigenvalues of B are two distinct purely imaginary complex numbers. This division will give a different state of Grassmann geometry for each case.

Proposition 5.14 *For the Grassmann geometry of Case* (II) *an $\mathcal{O}(h)$-surface of constant mean curvature is always minimal, and for the Grassmann geometry of Case* (IV) *there exists no $\mathcal{O}(h)$-surface of constant mean curvature.*

Proof. The proof is done similarly to the "another proof" for Theorem 4.12 of the $SU(2)$ case. If there exists an $\mathcal{O}(h)$-surface of constant mean curvature for a Grassmann geometry of isotropy type $SO(2)$ in the $SL(2, \mathbb{R})$ case, by a similar way to the $SU(2)$ case together with (2.4), (5.63), (5.64), and (5.65), we have the equality

$$F_\theta^2 = -\lambda\lambda_1 - \frac{h^2}{1 - h^2}\lambda_1^2.$$

In this case note that $\lambda > 0$ and $\lambda_1 < 0$. Then, from the above equality, the non-negativity of F_θ^2 induces the inequlity

$$h \leq \sqrt{\frac{\lambda}{\lambda - \lambda_1}}$$

where $F_\theta = 0$ if and only if $h = \sqrt{\frac{\lambda}{\lambda - \lambda_1}}$. Hence if it is the case that $h > \sqrt{\frac{\lambda}{\lambda - \lambda_1}}$, we have a contradiction. Also, if it is the case that $h = \sqrt{\frac{\lambda}{\lambda - \lambda_1}}$, it follows that $F_\theta = 0$, thus $H^\theta = 0$ by (5.65). This implies that an $\mathcal{O}(h)$-surface of constant mean curvature is minimal. $\square$

5.2.3 Grassmann geometry of Case (I) that $h = 0$

In this subsection we solve the linear system (5.66) of ODE's for Case (I), and consider the existence problem of $\mathcal{O}(0)$-surfaces of constant mean curvature. In this case, since $B = 0$, it follows by (5.69) that $\mathbb{Z}$ is constant. Hence using the change of variables $\mathbb{Z}$ into variables $\mathbb{X}$, we can concretely solve (5.66). But we solve it directly since they are simple in this case. Note that $\mu = 0$, $\nu = 1$ and $\gamma = -2$. Then the system (5.66) are concretely given as follows:

$$\frac{dx}{dt} = -z, \quad \frac{dy}{dt} = -w, \quad \frac{dz}{dt} = x, \quad \frac{dw}{dt} = y \tag{5.71}$$

where $\theta(t) = -2t + \varphi(a, b)$. Solving these under the initial conditions

$$x(0, a, b) = \sqrt{1 - a^2 + b^2}, \quad y(0, a, b) = 0, \quad z(0, a, b) = a, \quad w(0, a, b) = b,$$

we have the following solutions

$$\begin{aligned} x(t, a, b) &= \sqrt{1 - a^2 + b^2}\cos t - a \sin t, & y(t, a, b) &= -b \sin t \\ z(t, a, b) &= a \cos t + \sqrt{1 - a^2 + b^2}\sin t, & w(t, a, b) &= b \cos t, \end{aligned} \tag{5.72}$$

and moreover the differntials of these are given in the following:

$$x_t = -\sqrt{1 - a^2 + b^2}\sin t - a \cos t, \quad x_a = -\frac{a}{\sqrt{1 - a^2 + b^2}}\cos t - \sin t,$$

$$x_b = \frac{b}{\sqrt{1 - a^2 + b^2}}\cos t,$$

$$y_t = -b \cos t, \quad y_a = 0, \quad y_b = -\sin t,$$

$$z_t = -a \sin t + \sqrt{1 - a^2 + b^2}\cos t, \quad z_a = \cos t - \frac{a}{\sqrt{1 - a^2 + b^2}}\sin t,$$

$$z_b = \frac{b}{\sqrt{1 - a^2 + b^2}}\sin t,$$

$$w_t = -b \sin t, \quad w_a = 0, \quad w_b = \cos t,$$

$$\theta_t = -2, \quad \theta_a = \varphi_a, \quad \theta_b = \varphi_b.$$

We here consider the existence problem of $\mathcal{O}(0)$-surfaces of constant mean curvature. By (5.65) the CMC surface equation is generally given by the following equation.

$$F_\theta = \cos\theta(E_2\theta) + \sin\theta(E_3\theta) = -k/2 \tag{5.73}$$

where k is constant and in Case (I), the mean curvature H^θ of an $\mathcal{O}(0)$-surface is given by $k/4$. By (5.54), this equation is rewritten in tems of the local coordinates y, z, w and the local function x, as follows.

$$\begin{aligned}(z\cos\theta + x\sin\theta)\left(\frac{\partial\theta}{\partial y}\right) + (y\cos\theta - w\sin\theta)\left(\frac{\partial\theta}{\partial z}\right)\\ + (x\cos\theta - z\sin\theta)\left(\frac{\partial\theta}{\partial w}\right) + K = 0\end{aligned} \tag{5.74}$$

where $K = \frac{k}{\sqrt{\lambda|\lambda_1|}}$.

Moreover, similarly to the $SU(2)$ case, rewriting this by using the variables t, a, b, we have the following CMC surface equation

$$\begin{aligned}&(z\cos\theta + x\sin\theta)\left(\theta_t\frac{\partial(z,w)}{\partial(a,b)} + \theta_a\frac{\partial(z,w)}{\partial(b,t)} + \theta_b\frac{\partial(z,w)}{\partial(t,a)}\right)\\ &+ (y\cos\theta - w\sin\theta)\left(\theta_t\frac{\partial(w,y)}{\partial(a,b)} + \theta_a\frac{\partial(w,y)}{\partial(b,t)} + \theta_b\frac{\partial(w,y)}{\partial(t,a)}\right)\\ &+ (x\cos\theta - z\sin\theta)\left(\theta_t\frac{\partial(y,z)}{\partial(a,b)} + \theta_a\frac{\partial(y,z)}{\partial(b,t)} + \theta_b\frac{\partial(y,z)}{\partial(t,a)}\right)\\ &+ K\left(y_t\frac{\partial(z,w)}{\partial(a,b)} + z_t\frac{\partial(w,y)}{\partial(a,b)}w_t\frac{\partial(y,z)}{\partial(a,b)}\right) = 0.\end{aligned} \tag{5.75}$$

We remark that the CMC surface equations for the other Cases (II), (III) and (IV) also have the same form as this equation. In Case (I) the factors in this equation, such as $z\cos\theta + x\sin\theta$, $\frac{\partial(z,w)}{\partial(a,b)}$ and so on, are explicitly calculated as follows:

$$\begin{aligned}&z\cos\theta + x\sin\theta = \sqrt{1+b^2}\cos(t - \varphi + \tau),\\ &y\cos\theta - w\sin\theta = b\sin(t-\varphi),\end{aligned}$$

$$
\begin{aligned}
& x\cos\theta - z\sin\theta = \sqrt{1+b^2}\sin(t-\varphi+\tau), \\
& \frac{\partial(z,w)}{\partial(a,b)} = -\frac{\sqrt{1+b^2}}{\sqrt{1-a^2+b^2}}\cos t\sin(t-\tau), \\
& \frac{\partial(z,w)}{\partial(t,a)} = -\frac{b\sqrt{1+b^2}}{\sqrt{1-a^2+b^2}}\sin t\sin(t-\tau), \\
& \frac{\partial(z,w)}{\partial(b,t)} = -\frac{b^2}{\sqrt{1-a^2+b^2}}\sin^2 t + \sqrt{1+b^2}\cos t\sin(t-\tau), \\
& \frac{\partial(w,y)}{\partial(a,b)} = 0, \quad \frac{\partial(w,y)}{\partial(b,t)} = -b, \quad \frac{\partial(w,y)}{\partial(t,a)} = 0, \\
& \frac{\partial(y,z)}{\partial(a,b)} = -\frac{\sqrt{1+b^2}}{\sqrt{1-a^2+b^2}}\sin t\sin(t-\tau), \\
& \frac{\partial(y,z)}{\partial(t,a)} = \frac{b\sqrt{1+b^2}}{\sqrt{1-a^2+b^2}}\cos t\sin(t-\tau), \\
& \frac{\partial(y,z)}{\partial(b,t)} = a\sin^2 t + \left(\frac{b^2}{\sqrt{1-a^2+b^2}} - \sqrt{1-a^2+b^2}\right)\sin t\cos t
\end{aligned}
$$

where τ is defined as the constant which satisfies that

$$
\cos\tau = \frac{a}{\sqrt{1+b^2}}, \quad \sin\tau = \frac{\sqrt{1-a^2+b^2}}{\sqrt{1+b^2}}. \tag{5.76}
$$

Substituting these equations into the CMC surface equation (5.75), we obtain the following equation.

$$
\begin{aligned}
& \Bigg[-\frac{b^2\sqrt{1+b^2}}{\sqrt{1-a^2+b^2}}\cos(t-\varphi+\tau) \\
& \qquad - b^2\sin(t-\varphi) + \sqrt{1+b^2}\cos(\varphi-\tau)\left(\frac{a^2-1}{\sqrt{1-a^2+b^2}}\cos t + a\sin t\right)\Bigg]\varphi_a \\
& \quad + \left[-\frac{b(1+b^2)}{\sqrt{1-a^2+b^2}}\sin(\varphi-\tau)\sin(t-\tau)\right]\varphi_b \\
& \quad + \Bigg[(-2)\left\{ -\frac{(1+b^2)}{\sqrt{1-a^2+b^2}}\cos(\varphi-\tau)\sin(t-\tau)\right\}
\end{aligned}
$$

$$+ K\left\{ \frac{b\sqrt{1+b^2}}{\sqrt{1-a^2+b^2}} \sin(t-\tau) \right\}\Bigg] = 0. \tag{5.77}$$

We note that the left hand of the above equation is constructed by Cosine and Sine functions with the same period 1 with respect to the variable t. Denote by $f(t,a,b)$ the function defined by the left hand of the equation. Then, the equation $f(t,a,b)=0$ holds for any value t if and only if at any fixed value t_0, both equations $f(t_0,a,b)=0$ and $f'(t_0,a,b)=0$ hold where f' denotes the first differential with respect to t. Let $t_0=\tau$. Then it always holds $f(t_0,a,b)=0$. Moreover we can see that $f'(t_0,a,b)=0$ if and only if the following equation holds:

$$\begin{aligned}&\left[-\frac{ab^2}{\sqrt{1-a^2-b^2}}\varphi_a - \frac{b(1+b^2)}{\sqrt{1-a^2+b^2}}\varphi_b \right] \sin(\varphi-\tau) \\ &\quad + \left[\varphi_a + \frac{2(1+b^2)}{\sqrt{1-a^2+b^2}} \right] \cos(\varphi-\tau) + \frac{Kb\sqrt{1+b^2}}{\sqrt{1-a^2+b^2}} = 0. \end{aligned} \tag{5.78}$$

Now we have the following propositions.

Proposition 5.15 *Let $h=0$. Then there exists a minimal $\mathcal{O}(0)$-surface.*

Proof. We consider the equation (5.78) when $K=0$. Let $\varphi = \tau + \frac{\pi}{2}$ where $\tau = \arctan \frac{\sqrt{1-a^2+b^2}}{a}$. Then, since

$$\varphi_a = -\frac{1}{\sqrt{1-a^2+b^2}} \quad \text{and} \quad \varphi_b = \frac{ab}{(1+b^2)\sqrt{1-a^2+b^2}},$$

the coefficient of $\sin(\varphi-\tau)$ in the equation (5.78) is zero. Moreover the second term of the equation is also zero since $\cos(\varphi-\tau)=0$. Hence our φ satisfies the CMC surface equation (5.75) when $K=0$. ☐

Proposition 5.16 *Let $h=0$. Then for any nonzero number H there exists an $\mathcal{O}(0)$-surface of constant mean curvature H.*

Proof. We show that for any nonzero K the equation (5.78) has a local solution where $K = 4H/\sqrt{\lambda|\lambda_1|}$. We put

$$u = \sin(\varphi-\tau) \quad \text{and} \quad v = \cos(\varphi-\tau).$$

Then the equation (5.78) is rewritten as follows:

$$\begin{cases} u^2+v^2=1, \\ ab^2v_a+b(1+b^2)v_b+\sqrt{1-a^2+b^2}u_a+2(1+b^2)v+Kb\sqrt{1+b^2}=0. \end{cases}$$

Using the equality $uu_a+vv_a=0$, we have the following quasi-linear equation of the first order with respect to the variable v.

$$\left(ab^2-\sqrt{1-a^2+b^2}\frac{v}{\sqrt{1-v^2}}\right)v_a+b(1+b^2)v_b+2(1+b^2)v-Kb\sqrt{1+b^2}=0. \tag{5.79}$$

We show that this equation has a local solution v near the zero function. The characteristic ODE of the PDE (5.79) is given in the following.

$$\begin{aligned} \frac{da}{ds}&=ab^2-\sqrt{1-a^2+b^2}\frac{v}{\sqrt{1-v^2}}, \\ \frac{db}{ds}&=b(1+b^2), \\ \frac{dv}{ds}&=-2(1+b^2)v+Kb\sqrt{1+b^2}. \end{aligned}$$

Take an initial line L and initia values of solutions a, b and v when $s=0$, as follows: $L=\{(a,b):\ a=0,\ b\in\mathbb{R}\}$ and for r such that $(0,r)\in L$

$$a(s,r)|_{s=0}=0, \quad b(s,r)|_{s=0}=r, \quad v(s,r)|_{s=0}=\epsilon(r)$$

where $\epsilon(r)$ is an arbitrary nonzero function near the zero function. Then the Jacobian $\frac{\partial(a,b)}{\partial(s,r)}\big|_{s=0}$ when $s=0$ is given by the following equation

$$\left.\frac{\partial(a,b)}{\partial(s,r)}\right|_{s=0}=\sqrt{1+r^2}\frac{\epsilon(r)}{\sqrt{1-\epsilon(r)^2}}\neq 0.$$

Hence s, r can be solved by a and b, and so $v(s,r)$ is a function of a and b. This $v(a,b)$ gives a local solution of (5.79). □

Remark 5.17 The proof for Proposition 5.15 and Proposition 5.16 is essentially the same as the one for Proposition 4.10 and Proposition 4.11 of the $SU(2)$ case.

5.2.4 Grassmann geometry of Case (II) that $\boldsymbol{h}=\sqrt{\frac{\lambda}{\lambda-\lambda_1}}$

In this subsection we solve the linear system (5.66) of ODE's for Case (II), and consider the existence problem of minimal $\mathcal{O}\Big(\sqrt{\frac{\lambda}{\lambda-\lambda_1}}\Big)$-surfaces. In this case it holds that $\mu=\upsilon=-\frac{\lambda_1}{\lambda-\lambda_1}>0$, thus,

$$B=-\frac{\lambda_1}{\lambda-\lambda_1}\begin{pmatrix}0&1&1&0\\-1&0&0&-1\\1&0&0&1\\0&-1&-1&0\end{pmatrix},$$

and $\gamma=0$. Solving the equation $\frac{d\mathbb{Z}}{dt}=B\mathbb{Z}$ by using the theory of ODE's of constant coefficients, we have the following general solutions:

$$\mathbb{Z}(t)={}^t\Big(c+d\,t,\ c'+d't,\ \frac{d}{\mu}-c'-d't,\ -\frac{d'}{\mu}-c-d\,t\Big)$$

where c,c',d,d' are arbitrary real constants. Change the variable $\mathbb{Z}$ into the variable $\mathbb{X}$ by the relation (5.70). Then it follows

$$\mathbb{X}(t)=\begin{pmatrix}x(t)\\y(t)\\z(t)\\w(t)\end{pmatrix}=\frac{1}{2}\begin{pmatrix}(cp+c'q)+(dp+d'q)t\\-\big\{\big(c'-\frac{d}{\mu}\big)p+\big(c+\frac{d'}{\mu}\big)q\big\}-(d'p+dq)t\\(cq-c'p)+(dq-d'p)t\\-\big\{\big(c'-\frac{d}{\mu}\big)q-\big(c+\frac{d'}{\mu}\big)p\big\}-(d'q-dp)t\end{pmatrix}.$$

We here note that $\theta=\varphi$ since $\gamma=0$. We can then determine the functions $x(t)$, $y(t)$, $z(t)$, $w(t)$ under the initial conditions $x(0)=\sqrt{1-a^2+b^2}$, $y(0)=0$, $z(0)=a$, $w(0)=b$ as follows:

$$\begin{aligned}x(t)&=(-a+b\sin\varphi)\mu t+\sqrt{1-a^2+b^2},\\y(t)&=\big(-\sqrt{1-a^2+b^2}\cos\varphi+a\sin\varphi-b\big)\mu t,\\z(t)&=\big(\sqrt{1-a^2+b^2}+b\cos\varphi\big)\mu t+a,\end{aligned}\tag{5.80}$$

$$w(t) = \big(\sqrt{1-a^2+b^2}\sin\varphi + a\cos\varphi\big)\mu t + b.$$

Now, since $\theta_t = 0$ in this case, the CMC surface equation (5.75) reduces to the following equation

$$\begin{aligned}
&(z\cos\varphi + x\sin\varphi)\left\{\varphi_a\frac{\partial(z,w)}{\partial(b,t)} + \varphi_b\frac{\partial(z,w)}{\partial(t,a)}\right\} \\
&\quad + (y\cos\varphi - w\sin\varphi)\left\{\varphi_a\frac{\partial(w,y)}{\partial(b,t)} + \varphi_b\frac{\partial(w,y)}{\partial(t,a)}\right\} \\
&\quad + (x\cos\varphi - z\sin\varphi)\left\{\varphi_a\frac{\partial(y,z)}{\partial(b,t)} + \varphi_b\frac{\partial(y,z)}{\partial(t,a)}\right\} \\
&\quad + K\left(y_t\frac{\partial(z,w)}{\partial(a,b)} + z_t\frac{\partial(w,y)}{\partial(a,b)}w_t\frac{\partial(y,z)}{\partial(a,b)}\right) = 0.
\end{aligned} \tag{5.81}$$

We directly have the following proposition.

Proposition 5.18 *In the case $h = \sqrt{\frac{\lambda}{\lambda-\lambda_1}}$, there exists a minimal $\mathcal{O}\Big(\sqrt{\frac{\lambda}{\lambda-\lambda_1}}\Big)$-surface, and any minimal $\mathcal{O}\Big(\sqrt{\frac{\lambda}{\lambda-\lambda_1}}\Big)$-surface has negative constant Gaussian curvature $\lambda_1\lambda$.*

Proof. Let φ be a constant. Then φ satisfies the equation (5.81) when $K = 0$. We here note that we can select a range of variables a, b by (5.61) so that $\frac{\partial(y,z,w)}{\partial(t,a,b)}|_{t=0} \neq 0$. Then variables t, a, b are local functions of variables y, z, w, thus, θ is so. Hence there exists a minimal $\mathcal{O}\Big(\sqrt{\frac{\lambda}{\lambda-\lambda_1}}\Big)$-surface. Also, by the formula (5.64), we can see that the Gaussian curvature of any $\mathcal{O}\Big(\sqrt{\frac{\lambda}{\lambda-\lambda_1}}\Big)$-surface of constant mean curvature is $\lambda_1\lambda$. □

5.2.5 Grassmann geometry of Case (III) that $0 < h < \sqrt{\frac{\lambda}{\lambda-\lambda_1}}$

In this subsection we solve the linear system (5.66) of ODE's for Case (III), and give the explicit data of solutions $x(t)$, $y(t)$, $z(t)$, $w(t)$. The existence problem of $\mathcal{O}(h)$-surfaces of constant mean curvature will be considered in the forthcoming paper II.

In this case it holds that $\mu > \upsilon$ and $\gamma < 0$. Solving the equation $\frac{d\mathbb{Z}}{dt} = B\mathbb{Z}$ by the theory of ODE's of constant coefficients, we have the following general solutions:

$$\mathbb{Z}(t) = \left\{ c \begin{pmatrix} 1 \\ \frac{\sigma}{\upsilon} \\ 0 \\ -\frac{\mu}{\upsilon} \end{pmatrix} + c' \begin{pmatrix} 0 \\ \frac{\mu}{\upsilon} \\ -1 \\ -\frac{\sigma}{\upsilon} \end{pmatrix} \right\} e^{\sigma t} + \left\{ d \begin{pmatrix} 1 \\ -\frac{\sigma}{\upsilon} \\ 0 \\ -\frac{\mu}{\upsilon} \end{pmatrix} + d' \begin{pmatrix} 0 \\ \frac{\mu}{\upsilon} \\ -1 \\ \frac{\sigma}{\upsilon} \end{pmatrix} \right\} e^{-\sigma t}$$

where $\sigma = \sqrt{\mu^2 - \upsilon^2}$ and c, c', d, d' are arbitrary real constants. Change the variable $\mathbb{Z}$ into the variable $\mathbb{X}$ by the relation (5.70). Then it follows

$$\mathbb{X}(t) =$$

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} p & q & 0 & 0 \\ 0 & 0 & p & q \\ q & -p & 0 & 0 \\ 0 & 0 & q & -p \end{pmatrix} \begin{pmatrix} ce^{\sigma t} + de^{-\sigma t} \\ \left(\frac{\sigma}{\upsilon} c + \frac{\mu}{\upsilon} c'\right) e^{\sigma t} +, \left(- \frac{\sigma}{\upsilon} d + \frac{\mu}{\upsilon} d' \right) e^{-\sigma t} \\ -c' e^{\sigma t} - d' e^{-\sigma t} \\ \left(- \frac{\sigma}{\upsilon} c' - \frac{\mu}{\upsilon} c \right) e^{\sigma t} +, \left(\frac{\sigma}{\upsilon} d' - \frac{\mu}{\upsilon} d \right) e^{-\sigma t} \end{pmatrix}.$$

We express these as follows:

$$\begin{aligned} x(t) &= A^1(t,a,b)e^{\sigma t} + A^2(t,a,b)e^{-\sigma t}, \\ y(t) &= B^1(t,a,b)e^{\sigma t} + B^2(t,a,b)e^{-\sigma t}, \\ z(t) &= C^1(t,a,b)e^{\sigma t} + C^2(t,a,b)e^{-\sigma t}, \\ w(t) &= D^1(t,a,b)e^{\sigma t} + D^2(t,a,b)e^{-\sigma t}. \end{aligned} \tag{5.82}$$

The functions A^i, B^i, C^i, D^i are linear combinations of some Cosine and Sine functions with respect to the variable t. After determinating the constants c, c', d, d' under the initial conditions $x(0) = \sqrt{1 - a^2 + b^2}$, $y(0) = 0$, $z(0) = a$, $w(0) = b$, we have the following explicit expression of the functions A^i, B^i, C^i, D^i.

$$A^1 = \frac{\mu}{2\sigma} \left\{ \cos\left(\frac{\gamma}{2}t - \tau\right) \sqrt{1 - a^2 + b^2} + \sin\left(\frac{\gamma}{2}t - \tau\right) a + \sin\left(\frac{\gamma}{2}t + \varphi\right) b \right\},$$

$$A^2 = \frac{\mu}{2\sigma} \left\{ \cos\left(\frac{\gamma}{2}t + \tau\right) \sqrt{1 - a^2 + b^2} + \sin\left(\frac{\gamma}{2}t + \tau\right) a - \sin\left(\frac{\gamma}{2}t + \varphi\right) b \right\},$$

$$B^1 = \frac{\mu}{2\sigma} \left\{ - \cos\left(\frac{\gamma}{2}t + \varphi\right) \sqrt{1 - a^2 + b^2} + \sin\left(\frac{\gamma}{2}t + \varphi\right) a + \sin\left(\frac{\gamma}{2}t - \tau\right) b \right\},$$

$$
\begin{aligned}
B^2 &= \frac{\mu}{2\sigma}\Big\{ \cos\Big(\frac{\gamma}{2}t+\varphi\Big)\sqrt{1-a^2+b^2} - \sin\Big(\frac{\gamma}{2}t+\varphi\Big)a + \sin\Big(\frac{\gamma}{2}t+\tau\Big)b \Big\},\\
C^1 &= \frac{\mu}{2\sigma}\Big\{ -\sin\Big(\frac{\gamma}{2}t-\tau\Big)\sqrt{1-a^2+b^2} + \cos\Big(\frac{\gamma}{2}t-\tau\Big)a + \cos\Big(\frac{\gamma}{2}t+\varphi\Big)b \Big\},\\
C^2 &= \frac{\mu}{2\sigma}\Big\{ -\sin\Big(\frac{\gamma}{2}t+\tau\Big)\sqrt{1-a^2+b^2} + \cos\Big(\frac{\gamma}{2}t+\tau\Big)a - \cos\Big(\frac{\gamma}{2}t+\varphi\Big)b \Big\},\\
D^1 &= \frac{\mu}{2\sigma}\Big\{ \sin\Big(\frac{\gamma}{2}t+\varphi\Big)\sqrt{1-a^2+b^2} + \cos\Big(\frac{\gamma}{2}t+\varphi\Big)a + \cos\Big(\frac{\gamma}{2}t-\tau\Big)b \Big\},\\
D^2 &= \frac{\mu}{2\sigma}\Big\{ -\sin\Big(\frac{\gamma}{2}t+\varphi\Big)\sqrt{1-a^2+b^2} - \cos\Big(\frac{\gamma}{2}t+\varphi\Big)a + \cos\Big(\frac{\gamma}{2}t+\tau\Big)b \Big\}
\end{aligned}
\tag{5.83}
$$

where τ is the constant defined by the following equations

$$
\cos\tau = \frac{\sigma}{\sqrt{\sigma^2+\upsilon^2}} = \frac{\sigma}{\mu}, \quad \sin\tau = \frac{\upsilon}{\sqrt{\sigma^2+\upsilon^2}} = \frac{\upsilon}{\mu}.
$$

5.2.6 Summary

Though we keep the unsolved problem of the existence of $\mathcal{O}$-surfaces of constant mean curvature for the Grassmann geometry of Case (III), we can summarize our argument in the Subsection 5.2 as follows.

Theorem 5.19 *Let G be the simply connected Lie group with the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ and g a left invariant metric on G such that $\lambda_1 < 0 < \lambda_2 = \lambda_3 = \lambda$. Then the Grassmann geometry on (G,g) is of isotropy type $SO(2)$ and the orbit space of $I_o(G,g)$-orbits is parametrized by the height h where $0 \le h \le 1$, which is defined for the unit sphere in the tangent space T_eG at the unity e. Denote by $\mathcal{O}(h)$ the orbit with height h. Then the $\mathcal{O}(h)$-geometry is nonempty if and only if $h \neq 1$. Moreover devide the nonempty case into the following 4 cases: Case* (I) *that $h=0$, Case* (II) *that $h=\sqrt{\frac{\lambda}{\lambda-\lambda_1}}$, Case* (III) *that $0<h<\sqrt{\frac{\lambda}{\lambda-\lambda_1}}$ and Case* (IV) *that $\sqrt{\frac{\lambda}{\lambda-\lambda_1}}<h<1$. Then the $\mathcal{O}(h)$-geometry of each case has the following properties*:

(i) *If it is Case* (I), *any $\mathcal{O}(0)$-surface is a flat surface without geodesic points. Also, in this case, for any number H there exists an $\mathcal{O}(0)$-surface of constant mean curvature H*;

(ii) *If it is Case* (II), *any* $\mathcal{O}\Big(\sqrt{\frac{\lambda}{\lambda-\lambda_1}}\Big)$*-surface has no geodesic point. Also, if an* $\mathcal{O}\Big(\sqrt{\frac{\lambda}{\lambda-\lambda_1}}\Big)$*-surface has constant mean curvature, it is a minimal surface with negative constant Gauss curvature* $\lambda_1\lambda$ *and there exists such an* $\mathcal{O}\Big(\sqrt{\frac{\lambda}{\lambda-\lambda_1}}\Big)$*-surface*;

(iii) *If it is either of Cases* (III) *or* (IV), *any* $\mathcal{O}(h)$*-surface has no geodesic point. Also, for the Grassmann geometry of Case* (IV) *there exists no* $\mathcal{O}(h)$*-surface of constant mean curvature.*

Remark 5.20 In this case, from (2.4), the sectional curvature $K(P)$ of (G, g) satisfies the inequality $\lambda_1(\lambda - (3/4)\lambda_1) \leq K(P) \leq \lambda_1^2/4$. If $P \in \mathcal{O}(0)$, the sectional curvature $K(P)$ attains the maximum value $\lambda_3^2/4$. Then the Grassmann geometry has a flat surface. Also, if $P \in \mathcal{O}(1)$, it attains the minimum value $\lambda_3(\lambda - (3/4)\lambda_3)$. Then the Grassmann geomery is empty.

Remark 5.21 In Case (I) that $h = 0$, the existence equation (5.57) for the Grassmann geometry and the CMC surface equation (5.77) are essentially independent on the constant λ and λ_3. This implies that the $\mathcal{O}(0)$-geometry is independent on the way of taking a left invariant metic g which satisfies $\lambda_1 < 0 < \lambda_2 = \lambda_3 = \lambda$.

6. Remarks and Problems

In this section we overview the Grassmann geometry on the 3-dimensional unimodular Lie groups, and summarize the state of some common phenomena all over the Grassmann geometry. Next we propose some problems which are unsolved in this paper. Lastly, as we described in Remark 3.11, we give the correction for our paper [5] on the Grassmann geometry of the Heisenberg group H_3.

6.1. Bianchi-Cartan-Vranceanu metrics

We first pay attention to the Grassmann geometry of isotropy type $SO(2)$ on the simply connected Lie groups G with Lie algebra $\mathfrak{h}_3$, $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$. Let (G, g) be such a Lie group with left invariant metric g and $(\lambda_1, \lambda_2, \lambda_3)$ the triple corresponding to the metric g, which is defined in Section 2. Then two of λ_1, λ_2 and λ_3 are equal to each other and the other is different from these. These metrics are called the *Bianchi-Cartan-Vranceanu metrics* [1]. We set $\lambda_i = \lambda_j \neq \lambda_k$. Then the principal Ricci curvatures $r(E_i, E_i)$, $r(E_j, E_j)$ are equal to each other and the other $r(E_k, E_k)$

is different from them, and the isotropy action $SO(2)$ rotationally acts on the (E_iE_j)-plane. Consider the integral curves of the left invariant vector field E_k. These are geodesics of (G,g) since $\nabla_{E_k}E_k = 0$ by (2.2). Denote by H the one parameter subgroup of G generated by the vector field E_k. Then we can see that $H = \mathbb{R}$ if G is of Heisenberg type and $H = SO(2)$ if G is either of $SU(2)$ and $SL(2,\mathbb{R})$ types. We here consider the set M^2 of H-orbits in G. This set M^2 has a homogeneous structure G/H, and moreover an H-invariant metric g^M induced from the left invariant metric g. Then the orbit space M^2 is a Riemannian symmetric space and it is isometric to one of the complex line $\mathbb{C}$, the complex projective line $\mathbb{C}P^1$ and the complex hyperbolic line $\mathbb{C}H^1$ according as G is of Heisenberg, $SU(2)$ and $SL(2,\mathbb{R})$ types. We now have a fibration $\pi : G \to M^2$, which can be regarded as a generalization of the Hopf fibration $\pi : S^3 \to \mathbb{C}P^1$ over the complex projective line $\mathbb{C}P^1$, and it induces a natural homogeneous contact Riemannian structure on G whose characteristic vector field (Reeb vector field) is given by E_k. Let us denote by η the dual 1-form of E_k. Then η is a left invariant contact form on G, *i.e.*, $d\eta \wedge \eta \neq 0$.

By definition the height h of an $\mathcal{O}(W)$-surface is $h = \eta(W)$. Denote by ϑ the angle between the plane $P(W)$ and the contact distribution $\eta = 0$. Then $h = \cos\vartheta$.

A surface S of G is called a *Hopf cylinder* if S is the inverse image by π of a curve in M^2. In our Grassmann geometry the $\mathcal{O}(0)$-surfaces with height 0 are nothing but the Hopf cylinders.

The nonexistence of $\mathcal{O}(1)$-surfaces in G with 4-dimensinal isometry group can be proved by contact geometry as follows.

The $\mathcal{O}(1)$-surfaces are integral surfaces of the contact distribution $\eta = 0$. However by the contact condition, the contact distribution is never integrable.

Remark 6.1 The simply connected naturally reductive homogeneous spaces of dimension 3 are classified by F. Tricerri and L. Vanhecke [17] as follows:

- space forms: Euclidean 3-space $\mathbb{R}^3$, 3-sphere S^3 and hyperbolic 3-space H^3;
- (reducible symmetric spaces) $S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$,
- The following Lie groups equipped with a left invariant metric with 4-dimensional isometry group: $SU(2)$, the universal covering $\widetilde{SL}(2,\mathbb{R})$

of $SL(2,\mathbb{R})$ and the Heisenberg group H_3.

Remark 6.2 Let (M^3, g) be a Riemannian 3-manifold. Then the Riemannian metric g and the canonical 1-form of the orthonormal frame bundle $\mathrm{O}(M)$ induce a Riemannian metric on the Grassmann bundle $\mathrm{Gr}^2(TM)$. With respect to this metric, the natural projection $\pi : \mathrm{Gr}^2(TM) \to M$ is a Riemannian submersion with totally geodesic fibers. Let $S \subset M$ be an immersed surface. Then the *Gauss map* of S is a smooth map of S into $\mathrm{Gr}^2(TM)$ defined by

$$\gamma(p) := T_pS \in \mathrm{Gr}^2(T_pM).$$

In case $M = \mathbb{R}^3$, harmonicity of the Gauss map is equivalent to the constancy of mean curvature of S. In [8], [13], [15], harmonicity of Gauss maps of constant mean curvature surfaces in 3-dimensional Riemannnian homogeneous manifolds of non-constant curvature was investigated.

6.2. Curvatures and unsolved problems

We next summarize the relationship between the sectional curvature of (G, g) and the behavior of Grassmann geometry. Since an orbit $\mathcal{O}$ of our Grassmann geometry is an $I_o(G, g)$-orbit, the sectional curvatures $K(P)$ of (G, g) are constant for all $V \in \mathcal{O}$. Let $K(\mathcal{O})$ denote the constant and consider the range of the values $K(\mathcal{O})$ when the $\mathcal{O}$-geometries are nonempty. As described in Theorem 3.2, Remark 3.7, Remark 3.10, Remark 4.13, Remark 5.9 and Remark 5.20, we observe that if an $\mathcal{O}$-geometry has a flat $\mathcal{O}$-surface, the sectional curvature $K(\mathcal{O})$ attains the maximum value, and that if an $\mathcal{O}$-geometry has a totally geodesic $\mathcal{O}$-surface, the sectional curvature $K(\mathcal{O})$ attains the minimum value, exclusively of the case of the Grassmann geometry on $SL(2,\mathbb{R})$ of trivial isotropy type. Also, the totally geodesic surfaces of the 3-dimensional unimodular Lie groups have been classified by K. Tsukada [18], and the parallel surfaces of them by M. Belkhelfa–F. Dillen–J. Inoguchi [1] and J. Inoguchi–J. Van der Veken [6], [7]. From their classifications we can see that all the parallel surfaces are $\mathcal{O}$-surfaces.

Next we propose two problems which are unsolved in this paper. As described in the beginning of the subsections 5.2.5, the one is the existence problem of $\mathcal{O}$-surfaces of constant mean curvature for the $\mathcal{O}$-geometries on $SL(2,\mathbb{R})$ of isotropy type $SO(2)$, Case (III). For this we will give an affirmative answer in the forthcoming paper II. The other is the existence problem of $\mathcal{O}$-surfaces of constant Gaussian curvature for the following cases: the $\mathcal{O}$-geometries on $SU(2)$ or $SL(2,\mathbb{R})$ of isotropy type $SO(2)$ and moreover with height h such that $0 < h < 1$. By (4.17), (5.64), in these cases there exists an $\mathcal{O}$-surface of constant Gaussian curvature if and only if the following equation has a local solution θ.

$$\sqrt{1-h^2}(XH^\theta) - (\lambda_k)^2 h - 4h(H^\theta h)^2 = c \tag{6.84}$$

where λ_k implies the one defined first in this section and c is a constant. These problems are closely related to the following problem.

Problem Classify all homogeneous surfaces of the 3-dimensional unimodular Lie groups with left invariant metrics.

6.3. Corrections for the paper [5]

Lastly we give the corrections of our paper [5]. In pages 385 through 388 of the paper [5] there are mistakes for calculation. These come from a wrong calculation for the derivative $\partial u_3/\partial a$, in line 16, page 386. We correct these mistakes in the following:

(1) In the equations (4.11) in page 385, for the equation "$u_3(t,a,b) = -\frac{1}{2}\{t + \frac{1}{(\rho^2 c)}\sin\rho^2 ct\} - \cdots$" read the following equation "$u_3(t,a,b) = -\frac{1-\rho^2}{2}\{t + \frac{1}{(\rho^2 c)}\sin\rho^2 ct\} - \cdots$".

(2) For the equation in line 16, page 386: "$\frac{\partial u_3}{\partial a} = -\frac{a\sqrt{1-\rho^2}}{2\rho}\{\cos(\rho^2 ct + \varphi(a,b)) - \cos\varphi(a,b)\}\frac{\partial\varphi}{\partial a}$", read the following equation "$\frac{\partial u_3}{\partial a} = -\frac{a\sqrt{1-\rho^2}}{2\rho}\cdot\{\cos(\rho^2 ct+\varphi(a,b)) - \cos\varphi(a,b)\}\frac{\partial\varphi}{\partial a} - \frac{\sqrt{1-\rho^2}}{2\rho}\{\sin(\rho^2 ct+\varphi(a,b)) - \sin\varphi(a,b)\}$".

(3) For the Jacobian Δ in lines 19 through 21, page 386, read the following Jacobian "$\Delta = \frac{1-\rho^2}{c}\sin\rho^2 ct\frac{\partial\varphi}{\partial a} - \frac{(1-\rho^2)^{3/2}}{c\rho}\sin\rho^2 ct\sin\varphi(a,b)\frac{\partial\varphi}{\partial b} + \rho\sqrt{1-\rho^2}\cos(\rho^2 ct + \varphi(a,b))$".

(4) For the derivatives $\partial t/\partial u_1$, $\partial t/\partial u_2$, $\partial b/\partial u_1$, and $\partial b/\partial u_2$ in page 387 read the following derivatives:

$$\begin{aligned}\frac{\partial t}{\partial u_1} = \frac{1}{\Delta}\bigg\{&\frac{\sqrt{1-\rho^2}}{\rho c}(\sin * - \sin\varphi)\varphi_a - \frac{a\sqrt{1-\rho^2}}{2\rho}(\cos * - \cos\varphi)\varphi_b \\ &+ \frac{1-\rho^2}{2\rho^2 c}(\sin * - \sin\varphi)^2\varphi_b + 1\bigg\},\end{aligned}$$

$$\begin{aligned}\frac{\partial t}{\partial u_2} &= \frac{1}{\Delta}\bigg\{-\frac{\sqrt{1-\rho^2}}{\rho c}(\cos * - \cos\varphi)\varphi_a \\ &\qquad - \frac{1-\rho^2}{2\rho^2 c}(\cos * - \cos\varphi)(\sin * - \sin\varphi)\varphi_b\bigg\},\end{aligned}$$

$$\begin{aligned}\frac{\partial b}{\partial u_1} &= \frac{1}{\Delta}\bigg\{\frac{a(1-\rho^2)}{2}\varphi_a \sin\rho^2 ct + \frac{(1-\rho^2)^{3/2}}{2\rho c}(\sin * - \sin\varphi)(1+\cos\rho^2 ct)\varphi_a \\ &\qquad + \frac{1-\rho^2}{2}(1+\cos\rho^2 ct) + \frac{ac\rho}{2}\sqrt{1-\rho^2}\cos * \\ &\qquad - \frac{1-\rho^2}{2}(\sin^2 * - \sin * \sin\varphi)\bigg\},\end{aligned}$$

$$\begin{aligned}\frac{\partial b}{\partial u_2} &= \frac{1}{\Delta}\bigg\{-\frac{(1-\rho^2)^{3/2}}{2\rho c}(\cos * - \cos\varphi)(1+\cos\rho^2 ct)\varphi_a \\ &\qquad + \frac{1-\rho^2}{2}(\cos * \sin * - \cos * \sin\varphi)\bigg\}.\end{aligned}$$

(5) For the equation (4.14), page 388, read the following equation

$$\begin{aligned}&\text{“}c\rho^2\sin(\rho^2 ct + \varphi(a,b)) - \rho\sqrt{1-\rho^2}\varphi_a\cos\rho^2 ct \\ &\quad + \frac{1-\rho^2}{2}\varphi_b\sin\varphi(\sin * \sin\varphi + \cos^2 * + \cos\rho^2 ct) \\ &\quad = k\bigg[\frac{1-\rho^2}{c}\varphi_a\sin\rho^2 ct - \frac{(1-\rho^2)^{3/2}}{c\rho}\varphi_b\sin\rho^2 ct\sin\varphi \\ &\qquad + \rho\sqrt{1-\rho^2}\cos(\rho^2 ct + \varphi(a,b))\bigg].\text{”}\end{aligned}$$

(6) For the arguments in the lines 10 through 24, page 388, read the following arguments: “Rewrite the equation (4.14) as follows: $Ar + B\sqrt{1-r^2} + Cr\sqrt{1-r^2} + Dr^2 + E = 0$ where $A = \rho^2 c\cos\varphi + \frac{1-\rho^2}{2}\varphi_b\cos\varphi\sin^2\varphi - \frac{k(1-\rho^2)}{c}\varphi_a + \frac{k(1-\rho^2)^{3/2}}{c\rho}\varphi_b\sin\varphi + \rho\sqrt{1-\rho^2}\sin\varphi$, $B = -\rho\sqrt{1-\rho^2}\varphi_a + \rho^2 c\sin\varphi + \frac{1-\rho^2}{2}\varphi_b\sin^3\varphi + \frac{1-\rho^2}{2}\varphi_b\sin\varphi - \rho\sqrt{1-\rho^2}k\cos\varphi$, $C = -(1-\rho^2)\varphi_b\cos\varphi\sin^2\varphi$, $D = \frac{1-\rho^2}{2}\varphi_b\sin\varphi(\sin^2\varphi - \cos^2\varphi)$, $E = \cos^2\varphi$. Then an asymptotic expansion as $r \to 0$ in the equation $Ar+B\sqrt{1-r^2}+Cr\sqrt{1-r^2}+$

$Dr^2 + E = 0$ induces the equality $E = 0$. This contradicts the assumption $\cos\varphi(a, b) \neq 0$."

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