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A formula for the Łojasiewicz exponent at infinity in the real plane via real approximations

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Abstract. We compute the Łojasiewicz exponent of $f = (f_1, \ldots, f_n)\colon \mathbb{R}^2 \to \mathbb{R}^n$ via the real approximation of Puiseux's expansions at infinity of the curve $f_1 \ldots f_n = 0$. As a consequence we construct a collection of real meromorphic curves which provide a testing set for properness of f as well as a condition, which is very easy to check, for a local diffeomorphism to be a global one.

Key words: Łojasiewicz exponent at infinity, Puiseux expansion at infinity, Testing sets for properness of polynomial mappings.

1. Introduction

Let M, N be finite dimensional real vector spaces and let $f\colon M \to N$ be semi-algebraic mapping. For $X \subset M$, put

$$\mathcal{L}_\infty(f|_X) := \sup\big\{\nu \in \mathbb{R}\colon \exists C,\ R > 0,\ \forall x \in X(\|x\| \geq R \Rightarrow \|f(x)\| \geq C\|x\|^\nu)\big\}$$

and

$$\widetilde{\mathcal{L}}_\infty(f|_X) = \inf_\Phi \frac{\deg f \circ \Phi}{\deg \Phi},$$

where Φ runs over the set of meromorphic functions at infinity such that $\deg \Phi > 0$ and $\Phi(\tau) \in X$, for all τ enough large.

According to [Sk, Theorem 2.1], we know that

$$\widetilde{\mathcal{L}}_\infty(f|_X) = \mathcal{L}_\infty(f|_X).$$

The number $\mathcal{L}_\infty(f) := \mathcal{L}_\infty(f|_M)$ is called *the Łojasiewicz exponent at infinity of the mapping f*.

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We refer the reader to the recent survey [K] for more information on the Łojasiewicz exponent at infinity of mappings.

Remark 1.1 It is clear that the Łojasiewicz exponent does not change by a linear transformation.

Following Jelonek [Je], $X \subset M$ is called *a testing set for properness of the map* f, if $f|_X \colon X \to N$ is proper, then f is proper, too. It is clear that if $\mathcal{L}_\infty(f|_X) = \mathcal{L}_\infty(f)$ then $X \subset M$ is a testing set for properness of the map f.

In this note we restrict our investigation to a very restrictive setting, namely we consider polynomial mappings in two real variables. We give a formula for the Łojasiewicz exponent in terms of real approximations of Puiseux's expansions at infinity. As a consequence we construct a collection of real meromorphic curves which provide a testing set for properness of polynomial maps as well as a condition, which is very easy to check, for a local diffeomorphism to be a global one.

In [Je], Z. Jelonek has given various conditions for a given set to be a testing set for properness of a polynomial mapping from a complex affine variety to $\mathbb{C}^n$. In particular, if $f = (f_1, \ldots, f_n) \colon \mathbb{C}^m \to \mathbb{C}^n$ is polynomial mapping then the set $\{f_1 f_2 \ldots f_n = 0\}$ is a testing set for properness of f. The same result was also proven in [C-K2]. Moreover if $m = n = 2$, the authors of [C-K2] have given a formula expressing the Łojasiewicz exponent via Puiseux's expansions at infinity of the curve $f_1 f_2 = 0$ ([C-K1]). It is not difficult to see that these results are not longer true for the case of real variables (see Remark 2.5 bellow).

2. Main result

If $\varphi(\tau)$ is a series of the form

$$\varphi(\tau) = a_0 \tau^\alpha + \text{ terms of lower degree,}$$

where $\tau \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$), $a_0 \in \mathbb{K}^n$, $n \in \mathbb{N}$, $a_0 \neq 0$, then the number α is denoted by $\deg \varphi$.

Let us consider a series $x = \lambda(y)$ in the form:

$$x = \lambda(y) = a_1 y^{\alpha_1} + a_2 y^{\alpha_2} + \cdots + a_{s-1} y^{\alpha_{s-1}} + a_s y^{\alpha_s} + \cdots$$

where $\alpha_1 > \alpha_2 > \cdots$, $a_i \in \mathbb{C}$.

If $a_1, a_2, \ldots, a_{s-1} \in \mathbb{R}$ and $a_s \notin \mathbb{R}$, we put

$$\lambda^{\mathbb{R}}(y) := a_1 y^{\alpha_1} + a_2 y^{\alpha_2} + \cdots + a_{s-1} y^{\alpha_{s-1}} + c y^{\alpha_s},$$

where c is a generic real number. We call $\lambda^{\mathbb{R}}(y)$ the *real approximation of* $\lambda(y)$.

The following theorem is the main result of the article.

Theorem 2.1 *Let $f = (f_1, \ldots, f_n)\colon \mathbb{R}^2 \to \mathbb{R}^n$ be a real polynomial mapping, where* $\deg f_i = \deg_x f_i = d_i > 0$.

Let $x = x_j(y)$ be the Puiseux expansions at infinity of $f_1 \ldots f_n = 0$ and let $x_j^{\mathbb{R}}(y)$ be the real approximations of $x_j(y)$, for $j = 1, 2, \ldots, D$, where $D = d_1 \cdots d_n$. Then

$$\mathcal{L}_\infty(f) = \min_j \big\{ \deg f(x_j^{\mathbb{R}}(y), y) \big\}.$$

Let $f(x, y) \in \mathbb{C}[x, y]$ such that $\deg f = \deg_x f = d > 0$. Let Γ denote the zero set of f. Let $x = x_i(y)$, $i = 1, 2, \ldots, d$, be the Puiseux expansions at infinity of $f(x, y) = 0$ and let $x_i^{\mathbb{R}}(y)$ be the real approximations of $x_i(y)$. Put

$$\Gamma^{\mathbb{R}} := \cup_{i=1}^{d} \big\{ (x, y) \in \mathbb{R}^2 \colon |y| > R, x = x_i^{\mathbb{R}}(y) \big\}$$

and call it the real approximation of Γ.

Corollary 2.2 *Let $f = (f_1, \ldots, f_n)\colon \mathbb{R}^2 \to \mathbb{R}^n$ be a real polynomial mapping such that* $\deg f_i = \deg_x f_i = d_i$, *for all $i = 1, 2, \ldots, n$. Let $\Gamma := \{(x, y) \in \mathbb{C}^2 \colon f_1(x, y) \ldots f_n(x, y) = 0\}$ and let $\Gamma^{\mathbb{R}}$ denote the real approximation of Γ. Then $\Gamma^{\mathbb{R}}$ is a testing set for properness of the map f.*

Corollary 2.3 *With the notation as above, a local polynomial diffeomophism $f = (f_1, f_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a global diffeomorphism if and only if one of three equivalent conditions hold*

(i) *f is proper.*

(ii) *The restriction of f on $\Gamma^{\mathbb{R}}$ is proper.*

(iii) *The degree of $f(x_j^{\mathbb{R}}(y), y)$ is positive for every $j = 1, 2, \ldots, D$, where $D = d_1 \cdots d_n$.*

Remark 2.4

(i) It is well known that a local polynomial diffeomorphism might not be a global diffemorphism [P].

(ii) Some sufficient conditions for a local diffeomorphism to be a global diffeomorphism were given in [C-G], [R], [S].

Remark 2.5 It is easy to see that the restriction of the map $f = (f_1, f_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$, where $f_1(x,y) = (xy-1)^2+y^2$, $f_2(x,y) = [(xy-1)^2+y^2]x$, on the set $f_1 f_2 = 0$ is proper, nevertheless f is not proper. In fact $z_n = (n, \frac{1}{n}) \to \infty$ but $f(z_n) \to (0,0)$.

Example 2.6 (a) We will compute the Łojasiewicz exponent at infinity of the map in Remark 2.5. By the linear transformation $x := x$; $y := x + y$, we get $g = (g_1, g_2)$, where

$$g_1(x,y) = (x^2 + xy - 1)^2 + (x+y)^2,$$
$$g_2(x,y) = \big[(x^2 + xy - 1)^2 + (x+y)^2\big]x^2.$$

It follows from Remark 1.1 that $\mathcal{L}_\infty(f) = \mathcal{L}_\infty(g)$. Then

$$x_1(y) = i + y^{-1} + o(y^{-1}),$$
$$x_2(y) = -i + y^{-1} + o(y^{-1}),$$
$$x_3(y) = -y - y^{-1} + iy^{-2} + o(y^{-2}),$$
$$x_4(y) = -y - y^{-1} - iy^{-2} + o(y^{-2})$$

and $x_5(y) = 0$ are the Puiseux expansions at infinity of $g_1 g_2 = 0$. Therefore

$$x_1^{\mathbb{R}}(y) = x_2^{\mathbb{R}}(y) = c, \quad x_3^{\mathbb{R}}(y) = x_4^{\mathbb{R}}(y) = -y - y^{-1} + cy^{-2}$$

and $x_5^{\mathbb{R}}(y) = 0$, where c is a generic real number. Hence by Theorem 2.1 we have

$$\mathcal{L}_\infty(f) = \mathcal{L}_\infty(g) = -2.$$

(b) We consider the map $f = (f_1, f_2)\colon \mathbb{K}^2 \to \mathbb{K}^2$, where

$$f_1(x,y) = (x^2 + xy - 1)^2 + (x+y)^2 \text{ and } f_2(x,y) = x^2 + 1.$$

Then

$$
\begin{aligned}
x_1(y) &= i + y^{-1} + o(y^{-1}),\\
x_2(y) &= -i + y^{-1} + o(y^{-1}),\\
x_3(y) &= -y - y^{-1} + iy^{-2} + o(y^{-2}),\\
x_4(y) &= -y - y^{-1} - iy^{-2} + o(y^{-2}),
\end{aligned}
$$

$x_5(y) = i$ and $x_6(y) = -i$ are the Puiseux expansions at infinity of $f_1 f_2 = 0$. Therefore

$$
\begin{aligned}
x_1^{\mathbb{R}}(y) &= x_2^{\mathbb{R}}(y) = x_5^{\mathbb{R}}(y) = x_6^{\mathbb{R}}(y) = c,\\
x_3^{\mathbb{R}}(y) &= x_4^{\mathbb{R}}(y) = -y - y^{-1} + cy^{-2},
\end{aligned}
$$

where c is a generic real number. Thus, by the result of [C-K2]

$$\mathcal{L}_\infty(f) = -1, \text{ if } \mathbb{K} = \mathbb{C}$$

while by Theorem 2.1, we have

$$\mathcal{L}_\infty(f) = 2, \text{ if } \mathbb{K} = \mathbb{R}.$$

3. Proof of the main result

Let $f\colon \mathbb{K}^2 \to \mathbb{K}$ be a polynomial. For a series

$$x = \varphi(y) = c_1 y^{n_1/N} + c_2 y^{n_2/N} + \cdots, \quad c_i \in \mathbb{K}, c_1 \neq 0$$

we put

$$M(X,Y) = f\left(X + \varphi\left(\frac{1}{Y}\right), \frac{1}{Y}\right) = \sum_{i,j} c_{ij} X^i Y^{j/N}.$$

For each $c_{ij} \neq 0$, let us plot a dot at $(i, j/N)$, called a Newton dot. The set of Newton dots is called the Newton diagram. They generate a convex hull, whose boundary is called *the Newton polygon of f relative to* φ, to be denoted by $\mathbb{P}(f,\varphi)$ or $\mathbb{P}(M)$.

Assume that $x = \varphi(y)$ is not a Puiseux root at infinity of $f = 0$. Then the Y-axis contains at least one dot of M. Let $(0, h_M)$ be the lowest Newton dot. We see that $h_M = -\deg f(\varphi(y), y)$.

By "*the highest Newton edge*" H_M of M we mean the edge of $\mathbb{P}(M)$, one of its extremities is $(0, h_M)$ and all of Newton dots of M are lying on or above the line containing H_M. Let $\theta_M = \tan \varphi$, here φ is the angle between H_M and the X-axis. Note that if $(i, j/N)$ is a Newton dot of M then $\theta_M i + j/N \geq h_M$ and $(i, j/N) \in H_M$ if and only if $\theta_M i + j/N = h_M$. If $x = \varphi(y)$ is a Puiseux root at infinity of $f = 0$, we set $h_M = +\infty$ and $\theta_M = +\infty$.

We associate H_M with the polynomial $\varepsilon_M(x) := \varepsilon(x, 1)$, where

$$\varepsilon(X, Y) = \sum c_{ij} X^i Y^{j/N}, \text{ with } (i, j/N) \in H_M.$$

Lemma 3.1 ([H-D, Lemma 2.1]) *Let $\widetilde{M}(X, Y) = M(X + cY^\theta, Y)$, where θ is a real number. We have*

(a) *If $\theta > \theta_M$, then $h_{\widetilde{M}} = h_M$ and $\theta_{\widetilde{M}} = \theta_M$.*
(b) *If $\theta = \theta_M$ and c is a non-zero root of $\varepsilon_M(x)$, then $h_{\widetilde{M}} > h_M$ and $\theta_{\widetilde{M}} > \theta_M$.*
(c) *If $\theta = \theta_M$ and $\varepsilon_M(c) \neq 0$, then $h_{\widetilde{M}} = h_M$ and $\theta_{\widetilde{M}} = \theta_M$.*

If c is a non-zero root of $\varepsilon_M(x)$, the series $\varphi_1(y) = \varphi(y) + cy^{-\theta_M}$ will be called *the sliding of $\varphi(y)$ along f*. A recursive sliding $\varphi \to \varphi_1 \to \cdots$ produces a limit, φ_∞, where $\varphi_\infty(y) = \varphi_i(y)$ if $f(\varphi_i(y), y) = 0$. The series φ_∞ is a Puiseux expansion at infinity of $f = 0$ (see [H-P] for more information about Puiseux expansions at infinity) and will be called *a final result of sliding φ along f*.

Lemma 3.2 ([H-D, Lemma 2.3]) *Let $f, g \colon \mathbb{R}^2 \to \mathbb{R}$ be polynomials. For a series $x = \varphi(y)$, we put*

$$M(X, Y) = f\left(X + \varphi\left(\frac{1}{Y}\right), \frac{1}{Y}\right)$$

and

$$N(X, Y) = g\left(X + \varphi\left(\frac{1}{Y}\right), \frac{1}{Y}\right).$$

Let $x = \varphi_\infty(y)$ be a final result of sliding φ along f and $\varphi_\infty^{\mathbb{R}}(y)$ be the real approximation of $\varphi_\infty(y)$. We have

(a) *If* $\theta_M > \theta_N$*, then* $\deg g(\varphi^{\mathbb{R}}_\infty(y), y) = \deg g(\varphi(y), y)$;

(b) *If* $\theta_M = \theta_N$*, then* $\deg g(\varphi^{\mathbb{R}}_\infty(y), y) \leq \deg g(\varphi(y), y)$,

in particular with $g = f$*, we have* $\deg f(\varphi^{\mathbb{R}}_\infty(y), y) \leq \deg f(\varphi(y), y)$.

Proof of Theorem 2.1. We know that $\mathcal{L}_\infty(f) \leq \max\{\deg f_i\}$. Assume that $\mathcal{L}_\infty(f) = \max\{\deg f_i\}$. From the hypothesis $\deg f_{i_0} = \deg_x f_{i_0}$, we have $\deg x_j(y) \leq 1$ and therefore $\deg x^{\mathbb{R}}_j(y) \leq 1$. It follows that $\deg f(x^{\mathbb{R}}_j(y), y) \leq \deg f$. Thus

$$\mathcal{L}_\infty(f) = \max\{\deg f_i\} \geq \min_j \big\{ \deg f(x^{\mathbb{R}}_j(y), y)\big\}.$$

Hence

$$\mathcal{L}_\infty(f) = \min_j \big\{ \deg f(x^{\mathbb{R}}_j(y), y)\big\},$$

since

$$\mathcal{L}_\infty(f) \leq \min_j \big\{ \deg f(x^{\mathbb{R}}_j(y), y)\big\}.$$

Assume now that $\mathcal{L}_\infty(f) < \max\{\deg f_i\}$. Let $x = \varphi(y)$ be a any series satisfying the condition

$$\frac{\deg f(\varphi(y), y)}{\deg(\varphi(y), y)} < \max\{\deg f_i\} = \deg f_{i_0}.$$

Then $\deg \varphi \leq 1$, since $\deg f_{i_0} = \deg_x f_{i_0}$. Put

$$M_i(X, Y) = f_i\left(X + \varphi\left(\frac{1}{Y}\right), \frac{1}{Y}\right).$$

Let $\theta_{M_{i_0}} = \max\{\theta_{M_i}\}$, Lemma 3.2 yields that

$$\deg f_i\big(\varphi^{\mathbb{R}}_\infty(y), y\big) \leq \deg f_i(\varphi(y), y), \quad \forall i = 1, \ldots, n$$

where $x = \varphi_\infty(y)$ is the final result of the sliding φ along f_{i_0}. Thus

$$\mathcal{L}_\infty(f) \geq \min_j \big\{ \deg f(x^{\mathbb{R}}_j(y), y)\big\}.$$

On the other hand, the inequality

$$\mathcal{L}_\infty(f) \leq \min_j \left\{ \deg f(x_j^{\mathbb{R}}(y), y) \right\}$$

is always satisfied. Hence

$$\mathcal{L}_\infty(f) = \min_j \{\deg f(x_j^{\mathbb{R}}(y), y)\}. \qquad \square$$

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