Hokkaido Mathematical Journal Vol. 37 (2008) p. 331–348

C^ℓ-G-triviality of map germs and Newton polyhedra, $G = \mathcal{R}$, $\mathcal{C}$ and $\mathcal{K}$

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(Received November 8, 2006; Revised May 18, 2007)

Abstract. We provide estimates for the C^ℓ-G-triviality, for $0 \le \ell < \infty$ and G is one of Mather's groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, of deformations of analytic map germs $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ of type $f_t(x) = f(x) + \theta(x, t)$ which satisfy a non-degeneracy condition with respect to some Newton polyhedron. We apply the method of construction of controlled vector fields and, for each group G, the control function is determined from the choice of a convenient *Newton filtration* in the ring of real analytic germs. The results are given in terms of the filtration of the coordinate function germs $f_1, \ldots, f_p$ of f.

Key words: C^ℓ-determinacy, Newton filtration, controlled vector fields.

1. Introduction

A classical problem in singularity theory is the classification of families of analytic map germs which are equivalent with respect to some equivalence relation. In general, this classification is done in terms of the orbits of some group action in such a way that, the elements of one orbit preserve the desired properties.

Following the original ideas of Thom and Mather, several authors studied the action of Mather's groups $\mathcal{R}$, $\mathcal{C}$ and $\mathcal{K}$, see section 3 for these definitions. However, in several situations these group actions are too strong, giving rise to a very large quantity of orbits, therefore the action by weaker groups which could give a nice "stratification by its orbits" for the set of map germs becomes interesting.

In real case there exists an special interest in the C^ℓ-G-actions, where G is one of the groups above, and the diffeomorphisms are of *class* C^ℓ with $0 \le \ell < \infty$. Many works are devoted to the characterization of topological classification, given by diffeomorphisms of class C^0, with respect to various equivalence relations, see [4], [5] or [8], for example.

Concerning the C^ℓ-classification for $0 < \ell < \infty$, we see the work of

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2000 Mathematics Subject Classification : 58C27.

The first named author is partially supported by CNPq-Grant 300556/92-6.

Bromberg and Lopes de Medrano in [3]. They consider families of germs of real analytic functions and gave estimates for the C^ℓ-$\mathcal{R}$-triviality in the semi-weighted homogeneous case. Kuiper in [9] gave estimates for the C^1-$\mathcal{R}$-equivalence of isolated singularities. In these articles this estimation was made using the Lojasiewicz exponent of the Jacobian ideal of the germ.

More recently, Abderrahmane in [1] studies the C^0-$\mathcal{R}$-triviality in *function germs* which satisfy a non-degeneracy condition which depends of a convenient *Newton filtration*, extending the results of Kuo in [8]. The extension made by Abderrahmane refers to an estimation of the degree of determinacy, using a suitable Lojasiewicz exponent, with respect to a given Newton polyhedron of a given function germ. Moreover, in this paper it is also proved a version for Newton filtrations, of the results given by J. Bochnak and S. Lojasiewicz in [2]. We remark that the results of Abderrahmane also extend the results given by L. Paunescu in [10] for the weighted homogeneous case.

Considering families of real analytic *semi-weighted homogeneous map germs*, in [11] there are given estimates for the C^ℓ-G-triviality, where $0 \leq \ell < \infty$ and G is one of Mather's groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$.

In this paper we investigate the C^ℓ-G-triviality of families of real analytic *map germs* which satisfy a non-degeneracy condition with respect to some fixed Newton polyhedron, for $0 \leq \ell < \infty$ and G is one of Mather's groups $\mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$. We give explicit orders such that the C^ℓ geometrical structure of a non-degenerate map germ is preserved after higher order perturbations. Our method consists of constructing controlled vector fields, where the control functions are determined from an appropriate choice of a Newton polyhedron.

2. Newton polyhedra and functions of class C^ℓ

The main tool to provide estimates for the C^ℓ-triviality of analytic map germs $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is the determinacy of the class of differentiability of functions of type $h(x)/g(x)$, with $g(x)$ satisfying some non-degeneracy condition with respect to a convenient filtration in the ring of analytic germs $f\colon (\mathbb{R}^n, 0) \to \mathbb{R}$. Here we consider a *Newton filtration*, which is determined from a Newton polyhedron.

To construct a Newton polyhedron we fix an $n \times m$ matrix $A = (a_i^j)$, with $i = 1, \ldots, n$, $j = 1, \ldots, m$, $a^j = (a_1^j, \ldots, a_n^j) \in \mathbb{Q}_+^n$ and $m \geq n$, such that the first n columns of A are $(0, \ldots, 0, a_j^j, 0, \ldots, 0)$ with $a_j^j > 0$, for all $j = 1, \ldots, n$.

Definition 2.1 The Newton polyhedron $\Gamma_+(A)$ is the convex hull in $\mathbb{R}_+^n$ of $\mathrm{Supp}(A)+\mathbb{R}_+^n$, where $\mathrm{Supp}(A) = \{a^j,\, j = 1, \ldots, m\}$. The Newton diagram of A, denoted $\Gamma(A)$, is the union of the compact faces of $\Gamma_+(A)$.

Associated to a Newton polyhedron $\Gamma_+(A)$ we define a control function, which is fundamental to describe the class of differentiability of the controlled vector fields which guarantee the C^ℓ-triviality.

For any vector a^j of the matrix A and $k \in \mathbb{R}_+$, we denote $ka^j = (ka_1^j, ka_2^j, \ldots, ka_n^j)$. We fix the smallest integer p such that pa_i^j is integer for all i, j and for any non-negative rational number d define the function $\rho^d \colon \mathbb{R}^n \to \mathbb{R}$ as:

$$\rho^d(x) = \left(\sum_{j=1}^{m} x_1^{2pa_1^j} x_2^{2pa_2^j} \cdots x_n^{2pa_n^j}\right)^{d/(2p)}. \tag{1}$$

We remark that the function $\rho^{2p}(x) = \sum_{j=1}^m x_1^{2pa_1^j} x_2^{2pa_2^j} \cdots x_n^{2pa_n^j}$ is a polynomial.

We call ρ^d *control* of $\Gamma_+(dA)$, where dA denotes the matrix $dA = (da_i^j)$ and denote by $\Gamma_+(\rho^d)$, the Newton polyhedron of the matrix $dA = (da_i^j)$, i.e. $\Gamma_+(\rho^d) = \Gamma_+(dA)$.

Example 2.2 For $n = 2$, such an A is written as $A = \begin{pmatrix} a_1^1 & 0 & a_1^3 & \cdots & a_1^m \\ 0 & a_2^2 & a_2^3 & \cdots & a_2^m \end{pmatrix}$.

Let $A = \begin{pmatrix} 1/b & 0 & (b-1)/\{(b+1)b\} \\ 0 & 1/(b+1) & 1/\{(b+1)b\} \end{pmatrix}$ where b is a positive integer.

Then, we obtain $p = b(b+1)$ and the control function ρ of $\Gamma_+(A)$ is given as

$$\rho(x, y) = (x^{2b+2} + y^{2b} + x^{2b-2}y^2)^{1/\{2b(b+1)\}}.$$

The corresponding Newton polyhedron $\Gamma_+(\rho^{2b(b+1)})$ has two faces with vertices $\{(2b+2, 0), (2b-2, 1), (0, 2b)\}$.

Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function germ written as $f(x) = \sum a_\gamma x^\gamma$, where x^γ denotes the monomial $x^\gamma = x_1^{\gamma_1} \ldots x_n^{\gamma_n}$. For any fixed germ f, call d the biggest rational integer such that $\gamma = (\gamma_1, \ldots, \gamma_n) \in$

$\Gamma_+(\rho^d)$ for all $a_\gamma \neq 0$.

Abderrahmane shows that the control function ρ^d satisfies a Lojasiewicz condition with respect to such germs.

Lemma 2.3 ([1], p. 524) *There exists a constant $c_1 > 0$ and a neighborhood V of the origin such that, for all $x \in V$, $\|f(x)\| \leq c_1\rho^d(x)$.*

The main condition to guarantee the class of differentiability of function germs of type $h(x)/g(x)$ is given in terms of a non-degeneracy condition, called A-isolated, which we describe here.

For any germ g with Taylor series $g(x) = \sum a_\alpha x^\alpha$, if Δ is a subset of a Newton polyhedron $\Gamma_+(A)$, call g_Δ the germ $g_\Delta(x) = \sum_{\alpha\in\Delta} a_\alpha x^\alpha$.

Definition 2.4 ([1], p. 525) The origin is an A-isolated point of a germ f if for each compact face Δ of $\Gamma(\rho^d)$, the equation $f_\Delta(x) = 0$ does not have solution in $(\mathbb{R} - \{0\})^n$.

In this case we say that the germ f is *A-isolated*. For any germ satisfying this condition, it is possible to obtain a Lojasiewicz condition with respect to the control function ρ^d.

Lemma 2.5 ([1], p. 525) *Suppose that f is A-isolated for some matrix A, then there exists a real $c > 0$ such that $c\rho^d(x) \leq \|f(x)\|$ for all x in a neighborhood of the origin.*

For a Newton polyhedron $\Gamma_+(A)$ and for each $w = (w_1, \ldots, w_n) \in \mathbb{R}^n_+$ define:

Definition 2.6
(a) $\ell(w) = \min\{\langle w, k\rangle \colon k \in \Gamma_+(g)\}$, $\langle w, k\rangle = \sum_{i=1}^n w_i k_i$.
(b) $\Delta(w) = \{k \in \Gamma_+(g) \colon \langle w, k\rangle = \ell(w)\}$.
(c) Two vectors $a, a' \in \mathbb{R}^{n*}_+$ are equivalent if $\Delta(a) = \Delta(a')$.

The next lemmas form the key tools that determine the class of differentiability of the controlled vector fields.

Lemma 2.7 *For any germ $h(x) = \sum a_\gamma x^\gamma$, with γ in the interior of $\Gamma_+(\rho^d)$ for all $a_\gamma \neq 0$, then $\lim_{x\to 0} h(x)/\rho^d(x) = 0$.*

Proof. It is sufficient to prove that for all $a = (a_1, \ldots, a_n)$ in the interior of $\Gamma_+(\rho^d)$, $\lim_{x\to 0} x^a/\rho^d(x) = 0$.

If we suppose by contradiction that there exists a monomial x^a with

$a = (a_1, \ldots, a_n)$ in the interior of $\Gamma_+(\rho^d)$ and $\lim_{x\to 0} x^a/\rho^d(x) \neq 0$, then for each neighborhood V of the origin there exists a constant $c > 0$ such that $\|x^a/\rho^d(x)\| \geq c$, for some x in V.

Therefore the origin is in the closure of the set $X := \{x \in \mathbb{R}^n \mid \|x\| \geq c\rho^d(x)\}$.

Since X is semi-analytic, from the Curve Selection Lemma we conclude that there exists an analytic curve $\lambda\colon (0, \epsilon] \to X$, $\lambda := (\lambda_1, \ldots, \lambda_n)$, with $\lambda(0) = 0$ and

$$\lambda_1(t) = t^{\alpha_1} + o(\alpha_1), \ldots, \lambda_n(t) = t^{\alpha_n} + o(\alpha_n),$$

where $o(\alpha_i)$ denotes the terms of order higher than α_i in the Taylor series of $\lambda_i(t)$.

Call $\alpha = (\alpha_1, \ldots, \alpha_n)$ and as $\rho^d(\lambda(t)) \leq (1/c)\|\lambda(t)^a\|$, we obtain $\inf_j\{\langle da^j, \alpha\rangle\} \geq \langle a, \alpha\rangle$.

But $\Delta(\alpha) := \{b \in \Gamma_+(\rho^d) \mid \langle b, \alpha\rangle = \ell(\alpha)\}$ is a face of $\Gamma_+(\rho^d)$ with

$$\ell(\alpha) := \min\{\langle c, \alpha\rangle \mid c \in \Gamma_+(\rho^d)\}.$$

Since each da^j is one of the vertices of $\Delta(\alpha)$, we have $\langle da^j, \alpha\rangle = \ell(\alpha)$, however, $\langle a, \alpha\rangle \leq \langle da^j, \alpha\rangle = \ell(\alpha)$, hence $a \in \Gamma(\rho^d)$ and we obtain a contradiction to the hypothesis. $\square$

Now we fix a Newton polyhedron $\Gamma_+(A)$ and for each $(n-1)$-dimensional compact face Δ_k of $\Gamma(A)$ we denote by $v^k = (v_1^k, \ldots, v_n^k)$, the vector in $\mathbb{Z}_+^n - \{0\}$ with minimum length which is associated to Δ_k i.e. $\Delta_k = \Delta(v^k)$.

Call

$$M = \text{l.c.m.}\{\ell(v^k)\}, \quad R = \max_j \max_i \Big\{\frac{M}{\ell(v^k)} v_i^k\Big\}$$

and

$$r = \min_j \min_i \Big\{\frac{M}{\ell(v^k)} v_i^k\Big\}.$$

Definition 2.8 For an analytic real germ $f(x) = \sum a_\gamma x^\gamma$ call

$$\text{fil}(f) := \inf\{\text{fil}(\gamma) \mid a_\gamma \neq 0\}, \text{ where } \text{fil}(\gamma) = \min_k \Big\{\frac{M}{\ell(v^k)}\langle\gamma, v^k\rangle\Big\}.$$

Lemma 2.9 *Let $h\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function germ satisfying*

$$\operatorname{fil}(h) \geq \operatorname{fil}(\rho^d) + \ell R + 1.$$

Then h/ρ^d is differentiable of class C^ℓ.

Proof. The proof is done by induction on ℓ.

Then we first consider $\ell = 1$, and if $\operatorname{fil}(h) \geq \operatorname{fil}(\rho^d) + R + 1$, we show that h/ρ^d is of class C^1. To do it we show that the gradient map $\nabla(h/\rho^d)$ is continuous.

Since $\nabla(h/\rho^d) = (1/\rho^{2d})(\rho^d.\nabla h - h.\nabla\rho^d)$, in order to apply the Lemma 2.7 we need to show that $\operatorname{fil}(\rho^d(\partial h/\partial x_i) - h.(\partial\rho^d/\partial x_i)) > \operatorname{fil}(\rho^{2d})$, for all $i = 1, \ldots, n$.

But, for $\ell = 1$, $\operatorname{fil}(h) \geq \operatorname{fil}(\rho^d) + R + 1$, then

$$\begin{aligned}\operatorname{fil}\Big(\rho^d\frac{\partial h}{\partial x_i} - h.\frac{\partial\rho^d}{\partial x_i}\Big) &\geq \operatorname{fil}(h) + \operatorname{fil}(\rho^d) - R\\ &\geq 2\operatorname{fil}(\rho^d) + 1\\ &= \operatorname{fil}(\rho^{2d}) + 1.\end{aligned}$$

Therefore, from the Lemma 2.7 we see that $\lim_{x\to 0}\nabla(h/\rho^d)(x) = 0$. Then $\nabla(h/\rho^d)$ is continuous and h/ρ^d is of class C^1.

Suppose now that for h_1, with $\operatorname{fil}(h_1) \geq \operatorname{fil}(\rho^d) + (\ell-1)R + 1$, the function h_1/ρ^d is of class $C^{\ell-1}$.

For any h/ρ^d with $\operatorname{fil}(h) \geq \operatorname{fil}(\rho^d) + \ell R + 1$, we obtain that $\nabla(h/\rho^d) = H/\rho^d$, where $H = (1/\rho^d)(\rho^d.\nabla h - h.\nabla\rho^d)$, hence

$$\operatorname{fil}(H) \geq \operatorname{fil}(\rho^d) + (\ell - 1)R + 1$$

and by the induction hypothesis, we have that H/ρ^d is of class $C^{\ell-1}$. Therefore f is of class C^ℓ. □

3. C^ℓ-G-triviality

Two map germs $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ and g are $\mathcal{R}$-equivalent if there exists a germ of diffeomorphism $h\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $g = f \circ h^{-1}$. Two map germs f and g are $\mathcal{K}$-equivalent if there exists a pair of diffeomorphisms (h, H), with $h\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $H\colon (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ for which $H(\mathbb{R}^n \times 0) = \mathbb{R}^n \times 0$ and $H \circ (\mathrm{Id}, f) = (\mathrm{Id}, g) \circ h$, where Id denotes the germ of the identity in $\mathbb{R}^n$. We remark that the $\mathcal{K}$-equivalence can be seen as the decomposition of the $\mathcal{R}$-equivalence and the finer notion called $\mathcal{C}$-equivalence, defined as the $\mathcal{K}$-equivalence, but the germ h has the special property that it is the germ at 0 of the identity mapping on $\mathbb{R}^n$.

The groups C^ℓ-G, for $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, with $0 \leq \ell < \infty$ are defined as the groups $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, taking diffeomorphisms of class C^ℓ, if $\ell \geq 1$ or homeomorphisms when $\ell = 0$. These groups act on the space of map germs of class C^ℓ. Our interest however is rather in the induced equivalence relation, the C^ℓ-G-equivalence, in the space of analytic map germs $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$.

Ruas and Saia in [11] determined conditions for the C^ℓ-G-triviality, with $\ell \geq 0$, $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$ of families $f + t\theta$, where f is a weighted homogeneous map germ with isolated singularity, in terms of the weights and degrees. Here we generalize these results for the class of map germs that are A-homogenous for some fixed matrix A.

Definition 3.1 For a fixed matrix A, a germ f is called A-homogeneous of degree d if $f(x) = \sum_{\nu\in\Gamma(\rho^d)} c_\nu x^\nu$.

Example 3.2 The germ $f(x,y) = x^{2b+2} - y^{2b} + x^{2(b-1)}y^2$ is A-homogeneous of degree $2b(b+1)$ if we consider the matrix A of the Example 2.2.

We remark that when the matrix A is diagonal, we obtain that any A-homogeneous germ of degree d is in fact, a weighted homogenous germ of degree d, whose weights are determined by the elements of the matrix A.

Definition 3.3 An analytic map germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$; $f = (f_1, \ldots, f_p)$ is a A-homogenous of degree $d = (d_1, \ldots, d_p)$ if each f_i is A-homogenous of degree d_i.

The main idea is to choose, for each group $G = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$, a convenient A-isolated function germ which is equivalent to a control function $\rho(x)$ associated to $\Gamma_+(A)$. First we shall do it for the group $\mathcal{R}$.

3.1. The group $\mathcal{R}$

For a polynomial map germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ and each $I = \{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$ denote by M_I the corresponding minor of order p of the Jacobian matrix df.

For a fixed matrix A, we denote $s_I := \mathrm{fil}(M_I)$, call $\alpha := \mathrm{l.c.m.}\{s_I\}$

and define $N_{\mathcal{R}}f := \sum_I M_I^{2\alpha_I}$ where $\alpha_I = \alpha/s_I$. As we shall see in the main result of this section, the condition of the function germ $N_{\mathcal{R}}f$ to be A-isolated is the key tool to get the estimates for the C^ℓ-$\mathcal{R}$-triviality.

Write $N_{\mathcal{R}}f$ as the sum of its A-homogeneous parts H_i of degree i, that is

$$N_{\mathcal{R}}f = H_D + \cdots + H_{D+e}$$

with $e > 0$ and each function germ H_i is A-homogenous of degree i.

From the Lemma 2.3, there exist constants $c_D, \ldots, c_{D+e}$ and a neighborhood V_1 of the origin such that for all $x \in V_1$:

$$\begin{aligned} N_{\mathcal{R}}f(x) &\leq \|H_D(x)\| + \cdots + \|H_{D+e}(x)\| \\ &\leq c_D\rho^D(x) + \cdots + c_{D+e}\rho^{D+e}(x) \leq k\rho^D(x), \end{aligned}$$

where $c_D + \cdots + c_{D+e} = k$.

If we suppose that $N_{\mathcal{R}}f$ is A-isolated, from the Lemma 2.5 there exist constants k_1 and $k_2 > 0$ and a neighborhood V of the origin such for all $x \in V$:

$$k_1\rho^D(x) \leq N_{\mathcal{R}}f(x) \leq k_2\rho^D(x).$$

From now on we shall consider $f_t(x) = f(x) + \theta(x, t)$ be a deformation of a map germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, where $\theta = (\theta_1, \ldots, \theta_p)$, and $\theta_i(x, t) = \sum_{s=1}^{\ell_s} \delta_s^i(t)\theta_s^i(x)$, where $\delta_s^i\colon (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and $\theta_s^i\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ are polynomial germs of functions with $\delta_s^i \neq 0$.

Define $N_{\mathcal{R}}f_t := \sum_I M_{t_I}^{2\alpha_I}$, where M_{t_I} denotes the minor of order p of the Jacobian matrix df_t and α_I is as above.

Lemma 3.4 *Suppose that for some matrix A, $N_{\mathcal{R}}f = \sum_I M_I^{2\alpha_I} = H_D + \cdots + H_{D+e}$ is A-isolated. If $\mathrm{fil}(\theta_s^i) > \mathrm{fil}(f_i)$ for all $i = 1, \ldots, p$, then there exist constants k_1 and $k_2 > 0$ and a neighborhood V of 0 such that for all $x \in V$*

$$k_1\rho^D(x) \leq N_{\mathcal{R}}f_t(x) \leq k_2\rho^D(x).$$

Proof. Since $N_{\mathcal{R}}f_t = N_{\mathcal{R}}f + t\Theta$ where Θ satisfies $\mathrm{fil}(\Theta) > \mathrm{fil}(N_{\mathcal{R}}f)$, we can write

$$N_{\mathcal{R}}f \leq N_{\mathcal{R}}f_t + \|\Theta\|, \quad \text{for all } 0 \leq t \leq 1.$$

From the Lemma 2.5 there exist a constant $k_1 > 0$ and a neighborhood

V_1 of 0 such that for all $x \in V_1$,

$$k_1\rho^D(x) \leq N_{\mathcal{R}}f(x) \leq N_{\mathcal{R}}f_t(x) + \|\Theta(x,t)\|.$$

Since $\mathrm{fil}(\Theta) > \mathrm{fil}(N_{\mathcal{R}}f)$, $\lim_{x\to 0}\Theta(x,\, t)/\rho^D(x) = 0$, therefore $k_1\rho^D(x) \leq N_{\mathcal{R}}f_t(x)$.

On the other hand, since $\mathrm{fil}(\Theta) > \mathrm{fil}(N_{\mathcal{R}}f)$ it follows from the Lemma 2.3 that there exists a constant c_3 and a neighborhood V_2 of 0 such that $|\Theta(x,\, t)\| \leq c_3\rho^D(x)$ for all $x \in V_2$, hence if $k_2 = c_2 + c_3$, for all $x \in V_2 \cap V_1$:

$$N_{\mathcal{R}}f_t(x) \leq N_{\mathcal{R}}f(x) + \|\Theta(x,\, t)\| \leq c_2\rho^D(x) + \|\Theta(x,\, t)\| \leq k_2\rho^D(x).$$

□

We show now the main result of this section:

Theorem 3.5 *Suppose that* $N_{\mathcal{R}}f := \sum_i M_I^{2\alpha_I}$ *is A-isolated for some matrix A.*

(a) *If* $\mathrm{fil}(\theta_s^i) \geq \mathrm{fil}(f_i) + \ell R - r + 1$, *then for* $\ell \geq 1$ *and* $t \in [0,\, 1]$, f_t *is* C^ℓ-$\mathcal{R}$*-trivial;*

(b) *If* $\mathrm{fil}(\theta_s^i) \geq \mathrm{fil}(f_i)$, *then* f_t *is* C^0-$\mathcal{R}$*-trivial for small values of* t.

In order to better discuss the hypothesis given here, we show some examples.

In the first example we show that the A-isolated condition of the function $N_{\mathcal{R}}f$ with respect to some matrix A is essential for the estimates.

Example 3.6 Let $f_t(x,\, y) = x^8 + y^6 + y^2x^4 - y^4x^2 + tx^ay^b$.

Here $N_{\mathcal{R}}f = \big(8x^7 + 4xy^2(x^2-y^2)\big)^{2\alpha} + 4y^4(x^2-3y^2)^2(x^2-y^2)^{2\beta}$ is *not A-isolated, for any matrix A.* From the calculation of the Milnor numbers $\mu(f_t)$ for small values of t, we obtain that a necessary condition for the C^0-triviality of such family is $a + b \geq 8$. If we fix, for example $f_t(x,\, y) = f(x,\, y) + tx^4y^3$ and consider the matrix $A = \begin{pmatrix} 7 & 0 & 1 \\ 0 & 5 & 1 \end{pmatrix}$, we have $\mathrm{fil}(x^4y^3) > \mathrm{fil}(f)$, but f_t is not C^0-$\mathcal{R}$-trivial, since for $t \neq 0$ the Milnor number of f_t is smaller than the Milnor number of f.

We remark that in this example we are using the fact that, even in the real case, which is our subject here, the constancy of the Milnor number is a necessary condition for the C^0-$\mathcal{R}$-triviality.

Now we show that these estimates can not be improved.

Example 3.7 Let $f_t(x, y) = (x^2 + y^2)^2 + tx^a y^b$, we consider in this case $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which gives the usual filtration by the degree, then $\mathrm{fil}(f_0) = 4$ and $R = r = 1$. From the Theorem 3.5 we see that the family is C^ℓ-$\mathcal{R}$-trivial if

$$\mathrm{fil}(x^a y^b) = a + b \geq \mathrm{fil}(f_0) + \ell = 4 + \ell.$$

If we consider the monomial x^{p+5}, for all $p \geq 0$, Kuiper showed in the Theorem 5B of [9] that the family $f_t(x, y) = (x^2+y^2)^2+tx^{p+5}$ is C^1-trivial. As a consequence of our Theorem 3.5 we improve this result to get that the family is C^{p+1}-trivial, since $\mathrm{fil}(x^{5+p}) = p + 5$.

We remark that in this example, this estimate is the best possible, since Kuiper also showed that this family is *not* C^{p+2}-$\mathcal{R}$-*trivial*. Kuiper showed this by contradiction on the hypothesis of C^{p+2}-$\mathcal{R}$-triviality of the family, a direct analysis of the Taylor series of the composition of the function f with a C^{p+2}-diffeomorphism shows the contradiction on the term of degree $p + 7$.

Example 3.8 Let $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $f(x, y) = y^7 + x^4 y + x^9$. Then $df(x, y) = (4x^3 y + 9x^8, 7y^6 + x^4)$, with minors $M_1(x, y) = 4x^3 y + 9x^8$ and $M_2(x, y) = 7y^6 + x^4$.

Consider $A = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 6 & 1 \end{pmatrix}$, with the associate control $\rho(x, y) = y^{12} + x^6 y^2 + x^8$.

Call Δ_1 and Δ_2 the 1-dimensional compact faces of $\Gamma_+(\rho^2)$, then $N_{\mathcal{R}} f(x, y) = M_1^2(x, y) + M_2^2(x, y) = (4x^3 y + 9x^8)^2 + (7y^6 + x^4)^2$ is A-isolated, since $N_{\mathcal{R}} f|_{\Delta_1}(x, y) = (4x^3 y)^2 + (7y^6)^2$ and $N_{\mathcal{R}} f|_{\Delta_2}(x, y) = (4x^3 y)^2 + (x^4)^2$.

Now, for any monomial $x^a y^b$, we have $\varphi(x^a y^b) = \min\{2(5a + 3b), 9(a + b)\}$.

Then $\mathrm{fil}(f) = \min\{\varphi(y^7), \varphi(x^4 y), \varphi(x^9)\} = 42$, with $R = 10$ and $r = 6$.

Consider the family $f_t(x, y) = y^7 + x^4 y + x^9 + tx^2 y^5$, since $\mathrm{fil}(x^2 y^5) = 50$ from the Theorem 3.5 we conclude that f_t is C^1-$\mathcal{R}$-trivial.

We remark that if we consider the family $g_t(x, y) = y^7 + x^4 y + x^9 + tx^5 y^7$, as $\mathrm{fil}(x^5 y^7) = 92$ from the Theorem 3.5 we only can conclude that g_t is C^5-$\mathcal{R}$-trivial, however the monomial $x^5 y^7$ is in the $\mathcal{R}$-tangent space of the germ f, therefore this family is in fact C^ω-$\mathcal{R}$-trivial.

Example 3.9 Let $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$; $f(x, y) = (xy, x^{2b+2} - y^{2b} + x^{2b-2}y^2)$, with $b > 2$ and fix the matrix

$$A = \begin{pmatrix} 1/b & 0 & (b-1)/\{(b+1)b\} \\ 0 & 1/(b+1) & 1/\{(b+1)b\} \end{pmatrix}.$$

Since l.c.m.$\{\ell(v_1), \ell(v_2)\} = 2b(b+1)$, $\mathrm{fil}(xy) = 2b+2$, $\mathrm{fil}(y^{2b}) = 2b(b+1)$, $\mathrm{fil}(x^{2b-2}y^2) = 2b(b+1)$ and $\mathrm{fil}(x^{2b+2}) = 2b(b+1)$, then we obtain $R = 2b$ and $r = b$.

The 2×2 minor of df, $M(x, y) = -2\big((b+1)x^{2b+2} + by^{2b} + bx^{2b-2}y^2\big)$ is A-homogenous of degree $2b(b+1)$ and also A-isolated.

From the Theorem 3.5 we see that a family $f + t\theta$ with $\theta = (\theta_1, \theta_2)$ is C^ℓ-trivial if $\mathrm{fil}(\theta_1) \geq b + 2b\ell + 3$ and $\mathrm{fil}(\theta_2) \geq 2b^2 + b + 2b\ell + 1$.

Fixing $(\theta_1, \theta_2)(x, y) = (x^5y^9, y^{4(b+1)})$, $\mathrm{fil}(\theta_1) = 14b + 14$ and $\mathrm{fil}(\theta_2) = 4b^2 + 8b + 4$.

Therefore, for $b \geq 3$ the family $f_t(x, y) = (xy + tx^5y^9, x^{2b+2} - y^{2b} + x^{2b-2}y^2 + ty^{4(b+1)})$ is C^6-$\mathcal{R}$-trivial.

3.1.1. C^ℓ-$\mathcal{R}$ and bi-Lipschitz-$\mathcal{R}$-equivalence Recently in the Theorem 3.5 of [6], a similar result of the Theorem 3.5 is shown for the case of bi-Lipschitz $\mathcal{R}$-triviality of map germs, using the same method. In order to compare these two results, we recover this theorem here.

Theorem 3.10 ([6]) *Let $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a polynomial map-germ. Suppose that $N_{\mathcal{R}}f := \sum_i M_I^{2\alpha_I}$ is A-isolated for some matrix A. If $f_t = f + t\theta$ is a deformation of f with $\mathrm{fil}(\theta_i) \geq \mathrm{fil}(f_i) + R - r$, then f_t is bi-Lipschitz $\mathcal{R}$-trivial.*

We remark that the bi-Lipschitz equivalence is stronger than the C^0-equivalence and weaker than the C^1-equivalence, and these conditions clearly appear in these results, we can also see that the bi-Lipschitz equivalence is "closer" to the C^0-equivalence than the C^1 equivalence. It is interesting to remark that in the homogenous case, or when $R = r$, we obtain equal estimates for the C^0 and bi-Lipschitz equivalences.

Next, we give another example to show that the appropriate choice of the matrix A to get the A-isolated condition is also essential.

Example 3.11 Consider the family $f_t(x,y) = f(x,y) + tx^2y^2$, with $f(x, y) = x^3 + y^6$. Since f_t is weighted homogeneous, we can fix the matrix $A =$

$\begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$ to conclude that this family is C^0-$\mathcal{R}$-trivial.

We can ask now about the C^1-$\mathcal{R}$-triviality of this family. For this, consider the matrix $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 5 & 1 \end{pmatrix}$. We see that for this matrix, $N_{\mathcal{R}}f = 9x^4 + 36y^{10}$ is not A-isolated, $\mathrm{fil}(f) = 12$ and $\mathrm{fil}(x^2y^2) = 20$, hence for $\ell = 1$, $\mathrm{fil}(x^2y^2) > \mathrm{fil}(f) + 8\ell - 2 + 1$. Therefore if this hypothesis should be not necessary, we could obtain that the family f_t should be C^1-trivial. On the other side if we do the following exchange in the coordinate system $x = X - ty^2/3$ and $y = Y$, we see that f_t is C^ω-equivalent to $x^3 + y^6 + sxy^4$ and in [7] Henry and Parusiński showed that this family is not bi-Lipschitz trivial, hence not C^1-trivial.

3.1.2. Proof of the Theorem 3.5

(a) For each $p \times p$ minor M_{t_I} of df_t, we construct the vector field W_I defined by the co-factors of M_{t_I}:

$$W_I = \sum_{i=1}^n w_i \frac{\partial}{\partial x_i}, \quad \text{with} \begin{cases} w_i = 0, & \text{if } i \notin I \\ w_{i_m} = \displaystyle\sum_{j=1}^p N_{ji_m} \Big(\frac{\partial f_t}{\partial t}\Big)_j, & \text{if } i_m \in I, \end{cases}$$

where N_{ji_m} denotes the $(p-1) \times (p-1)$ minor cofactor of the element $\partial f_j / \partial x_{i_m}$ in the matrix df and $(\partial f_t/\partial t)_j$ is the j-coordinate of the map germ $\partial f_t / \partial t$.

Therefore $(\partial f_t/\partial t) M_{t_I} = df(W_I)$ and $(\partial f_t/\partial t) N_{\mathcal{R}} f_t = df_t(W_{\mathcal{R}})$, where $W_{\mathcal{R}}$ is the vector field $W_{\mathcal{R}} := \sum_I M_I^{2\alpha_I - 1} W_I$.

We remark here that for all $i = 1, \ldots, n$ we have

$$\begin{aligned}
&\mathrm{fil}\Big(w_i \sum_I M_I^{2\alpha_I-1}\Big) \\
&= \min_I \{\mathrm{fil}(M_I^{2\alpha_I-1}) + \mathrm{fil}(w_i)\} \\
&\geq \min\{2\alpha - \mathrm{fil}(M_I) + \mathrm{fil}(N_{ji_m}) + \mathrm{fil}(\theta_j)\} \\
&\geq \min\Big\{2\alpha - \mathrm{fil}(M_I) + \mathrm{fil}(M_I) - \mathrm{fil}\Big(\frac{\partial f_j}{\partial x_{i_m}}\Big) + \mathrm{fil}(\theta_j)\Big\} \\
&\geq \min\{2\alpha - (\mathrm{fil}(f_j) - r) + \mathrm{fil}(\theta_j)\} \\
&\geq 2\alpha + \ell R + 1.
\end{aligned}$$

Now consider the vector field $V\colon (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0)$, $V :=$

$W_{\mathcal{R}}/N_{\mathcal{R}}f_t$.

Since $W_{\mathcal{R}} = \sum_{i=1}^n (w_i \sum_I M_I^{2\alpha_I - 1})\partial/\partial x_i$, it follows from the equivalence between $N_{\mathcal{R}}f_t$ and ρ^d given in the Lemma 3.4 that we can apply the Lemma 2.9 to conclude that the vector field V is of class C^ℓ.

Then the C^ℓ-$\mathcal{R}$-triviality, for small values of t, follows from the equation

$$\frac{\partial f_t}{\partial t}(x,\, t) = (df_t)_x(V(x,\, t)).$$

A similar argument shows that the result follows for all $t_0 \in [0,\, 1]$.

(b) Since for all $i = 1,\, \dots,\, n$,

$$\begin{aligned}
&\operatorname{fil}\Big(w_i \sum_I M_I^{2\alpha_I - 1}\Big)\\
&\geq \min_I\Big\{2\alpha - \operatorname{fil}(M_I) + \operatorname{fil}(M_I) - \operatorname{fil}\Big(\frac{\partial f_j}{\partial x_{i_m}}\Big) + \operatorname{fil}(\theta_j)\Big\}\\
&\geq \min\{2\alpha - (\operatorname{fil}(f_j) - r) + \operatorname{fil}(f_j) + R - r\}\\
&= 2\alpha + r = \operatorname{fil}(\rho^D) + r
\end{aligned}$$

then $\operatorname{fil}(w_i \sum_I M_I^{2\alpha_I - 1}) \geq \operatorname{fil}(\rho^D) + r = \operatorname{fil}(\rho^D \|x\|)$ and we conclude that $W_{\mathcal{R}}/(\rho^D\|x\|)$ is limited. Therefore, from the Lemma 3.4 we obtain the inequalities

$$\Big\|\frac{W_{\mathcal{R}}}{N_{\mathcal{R}}f_t}\Big\| \leq c\Big\|\frac{W_{\mathcal{R}}}{\rho^D}\Big\| \leq c'\|x\|$$

to get that the vector field $W_{\mathcal{R}}/N_{\mathcal{R}}f_t$ is integrable and the C^0-$\mathcal{R}$-triviality follows. $\square$

3.2. The group $\mathcal{C}$

For a map germ $f\colon (\mathbb{R}^n,\, 0) \to (\mathbb{R}^p,\, 0)$; with $f = (f_1,\, \dots,\, f_p)$ and a fixed matrix A, we call $\beta_i := \text{l.c.m.}\{\operatorname{fil}(f_j),\, j = 1,\, \dots,\, p\}/\operatorname{fil}(f_i)$.

In this case we consider the coordinate functions $\{f_1,\, \dots,\, f_p\}$ and define the function germ $N_{\mathcal{C}}f := \sum_{i=1}^p (f_i)^{2\beta_i}$. Here the condition of the function $N_{\mathcal{C}}f$ to be A-isolated is the key tool to get the estimates for the C^ℓ-$\mathcal{C}$-triviality.

Then we write $N_{\mathcal{C}}f = H_D + \cdots + H_{D+e}$ with $e > 0$ and from the Lemma 2.3 we conclude that there exists constants $c_D,\, \dots,\, c_{D+e}$ and a neighborhood V of the origin such that

$$N_{\mathcal{C}}f(x) \leq c_D\rho^D(x) + \cdots + c_{D+e}\rho^{D+e}(x) \leq (c_D + \cdots + c_{D+e})\rho^D(x).$$

Therefore if $N_{\mathcal{C}}f$ is A-isolated, from the Lemma 2.5 there exist constants k_1 and $k_2 > 0$ such that $k_1\rho^D \leq N_{\mathcal{C}}f \leq k_2\rho^D$ in a neighborhood of the origin.

Consider now a deformation $f_t = f + \theta(x, t)$ of f with $\operatorname{fil}(\theta_s^i) > \operatorname{fil}(f_i)$ and define $N_{\mathcal{C}}f_t := \sum_{i=1}^{p}(f_{ti})^{2\beta_i}$.

Lemma 3.12 *Suppose that $N_{\mathcal{C}}f$ is A-isolated for some matrix A. If f_t is a deformation of f with $\operatorname{fil}(\theta_s^i) > \operatorname{fil}(f_i)$, there exist constants k_1 and $k_2 > 0$ and a neighborhood V of 0 such that for all $x \in V$,*

$$k_1\rho^{D_1}(x) \leq N_{\mathcal{C}}f_t(x) \leq k_2\rho^{D_1}(x).$$

Proof. We can write $N_{\mathcal{C}}f_t = N_{\mathcal{C}}f + t\Theta(x, t)$, with $\operatorname{fil}(\Theta) > \operatorname{fil}(N_{\mathcal{C}}f)$, then $N_{\mathcal{C}}f \leq N_{\mathcal{C}}f_t + \|\Theta\|$ for all t with $0 \leq t \leq 1$.

As $N_{\mathcal{C}}f = H_D + \cdots + H_{D+e}$ is A-isolated there exist constants k_1 and $k_2 > 0$ and a neighborhood V of 0 such that for all $x \in V$: $k_1\rho^D(x) \leq N_{\mathcal{C}}f(x) \leq k_2\rho^D(x)$.

Therefore

$$k_1\rho^D(x) \leq N_{\mathcal{C}}f(x) \leq N_{\mathcal{C}}f_t(x) + \|\Theta(x, t)\|,$$

since $\operatorname{fil}(\Theta) > \operatorname{fil}(N_{\mathcal{C}}f)$ and $\lim_{x\to 0}\Theta(x, t)/\rho^{D_1}(x) = 0$ this implies that $k_1\rho^{D_1}(x) \leq N_{\mathcal{C}}f_t(x)$.

On the other hand, $N_{\mathcal{C}}f_t(x) \leq N_{\mathcal{C}}f(x) \leq k_2\rho^D(x)$ and the result follows. $\square$

We show now the main result of this section.

Theorem 3.13 *Let $f_t(x) = f(x) + \theta(x, t)$, be a deformation of a polynomial map germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. Suppose that $N_{\mathcal{C}}f$ is A-isolated for some matrix A. Then, if $\operatorname{fil}(\theta_s^i) \geq d + \ell R + 1$ for all s, all $i = 1, \ldots, n$ and $\ell \geq 1$ with $d := \max\{\operatorname{fil}(f_i)\}$, the family f_t is C^ℓ-$\mathcal{C}$-trivial for all $t \in [0, 1]$.*

Example 3.14 Let $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$; $f(x, y) = (xy + x^2y^2, x^{2(c+1)} + xy - y^{2c})$ with $c \geq 2$.

Fix $A = \begin{pmatrix} 2(c+1) & 0 & 1 \\ 0 & 2c & 1 \end{pmatrix}$, then l.c.m.$\{\ell(v^1), \ell(v^2)\} = 2c(c+1)$, $R = 2c^2 + c$ and $\varphi(a, b) = \min\{(c+1)\langle(a, b), (2c-1, 1)\rangle, c\langle(a, b), (1, 2c+1)\rangle\}$.

Since $N_{\mathcal{C}}f$ is A-isolated, we apply the Theorem 3.13 to obtain that

$$f_t(x, y) = (xy + x^2y^2 + txy^{2c+1}, x^{2(c+1)} + xy - y^{2c})$$

is C^1-$\mathcal{C}$-trivial, for $c \geq 3$.

Example 3.15 Let $f\colon (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$; $f(x, y) = (x^4y^4, x^8 + x^3y^5 + y^{11})$.

Fix $A = \begin{pmatrix} 8 & 0 & 3 \\ 0 & 11 & 2 \end{pmatrix}$, then l.c.m.$\{\ell(v^1), \ell(v^2)\} = 88$, $R = 16$ and for any monomial x^ay^b, $\varphi(a, b) = \min\{16a + 8b, 11a + 11b\}$, hence $\mathrm{fil}(f_1) = \mathrm{fil}(f_2) = 88$.

Since $N_{\mathcal{C}}f$ is A-isolated, we can apply the Theorem 3.13. If we consider monomials of type x^ky^2 with $8 \leq k \leq 11$ we obtain that $f_t(x, y) = (x^4y^4, x^8 + x^3y^5 + y^{11} + tx^ky^2)$ is C^1-$\mathcal{C}$-trivial for $k = 8$, C^2-$\mathcal{C}$-trivial for $k = 9$ or $k = 10$ and C^3-$\mathcal{C}$-trivial for $k = 11$.

Proof of the Theorem 3.13. To show the C^ℓ-$\mathcal{C}$-triviality of f_t we consider the germs of vector fields $V_i\colon (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^p \times \mathbb{R}, 0)$; $V_i = (V_{i1}, \ldots, V_{ip})$ of class C^ℓ, where $V_{ij}(x, 0) = \delta_{ij}(x)$, in such a way that

$$\frac{\partial f_t}{\partial t} = \sum_{i=1}^{p} V_i(x, t)(f_{ti}).$$

Write $\partial f_t/\partial t = \partial f_t/\partial t \cdot (\sum_{i=1}^{p} f_{ti}^{2\beta_i - 1} f_{ti}/N_{\mathcal{C}}f_t)$ and define $W_i = (\partial f_t/\partial t)\cdot f_{ti}^{2\beta_i-1}$, therefore

$$\frac{\partial f_t}{\partial t}(x, t) = \sum_{i=1}^{p} \frac{W_i}{N_{\mathcal{C}}f_t}(f_{ti})(x, t).$$

Since $B := \text{l.c.m.}\{\mathrm{fil}(f_j),\ j = 1, \ldots, p\}$ we obtain

$$\begin{aligned} \mathrm{fil}(W_i) &= \min_j\{\mathrm{fil}(f_i^{2\beta_i-1}) + \mathrm{fil}(\theta_j)\} \\ &\geq 2B - d + d + \ell R + 1 \\ &= 2B + \ell R + 1, \quad \forall i. \end{aligned}$$

Let $V\colon (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}, 0)$ the germ of vector field defined as $(0, V_p, 0)$, where

$$V_p(x, y, t) := \sum_{i=1}^{p} \frac{W_i(x, t)}{N_{\mathcal{C}}f_t} y_i.$$

Therefore, from the inequalities given in the Lemma 3.12 we see that we can apply the Lemma 2.9 to conclude that V is of class C^ℓ and the result follows by integrating the vector field V. $\square$

3.3. The group $\mathcal{K}$

To define the germ of function $N_{\mathcal{K}}f$ with respect to the group $\mathcal{K}$ we fix a matrix A and use the functions $N_{\mathcal{R}}f$ and $N_{\mathcal{C}}f$ to consider the smallest integer numbers a and b such that $\mathrm{fil}([N_{\mathcal{R}}f]^a) = \mathrm{fil}([N_{\mathcal{C}}f]^b)$.

Then we define $N_{\mathcal{K}}f = [N_{\mathcal{R}}f]^a + [N_{\mathcal{C}}f]^b$ and show the following:

Theorem 3.16 *Suppose that $N_{\mathcal{K}}f$ is A-isolated for some matrix A. Then deformations $f_t(x) = f(x) + \theta(x, t)$, with $\mathrm{fil}(\theta_s^i) \geq d + \ell R + 1$ for all i and $\ell \geq 1$, are C^ℓ-$\mathcal{K}$-trivial for all $t \in [0, 1]$.*

We remember that if for some map germ f the function $N_{\mathcal{R}}(f)$ is A-isolated, it is better to apply the Theorem 3.5, since it gives a better estimate and as the group $\mathcal{R}$ is a subgroup of the group $\mathcal{K}$, the C^ℓ-$\mathcal{R}$-triviality implies the C^ℓ-$\mathcal{K}$-triviality. On the other side, if $N_{\mathcal{C}}(f)$ is A-isolated we obtain the same estimates for the C^ℓ-$\mathcal{K}$-triviality. However there exist map germs which $N_{\mathcal{R}}(f)$ and $N_{\mathcal{C}}(f)$ are not A-isolated for any matrix A and $N_{\mathcal{K}}(f)$ is A-isolated for some matrix A, as we can see in the example below, in these cases we are only able to compute estimates for the C^ℓ-$\mathcal{K}$-triviality of the family.

Example 3.17 Let $f\colon (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$, $f(x, y, z) = (3x^6 + 2y^6 + xz^4, x^6 + y^6 + yz^3)$.

Then $df = \begin{bmatrix} 18x^5 + z^4 & 12y^5 & 4xz^3 \\ 6x^5 & 6y^5 + z^3 & 3yz^2 \end{bmatrix}$ with minors $M_{12} = 36x^5y^5 + 18x^5z^3 + 6y^5z^4 + z^7$, $M_{13} = 54x^5yz^2 + 3yz^6 - 24x^6z^3$, and $M_{23} = 36y^6z^2 - 24xy^5z^3 - 4xz^6$.

Here $N_{\mathcal{R}}(f)$ and $N_{\mathcal{C}}(f)$ are not A-isolated for any matrix A and $N_{\mathcal{K}}(f)$ is A-isolated for the matrix $A = \begin{pmatrix} 6 & 0 & 0 & 1 & 0 \\ 0 & 6 & 0 & 0 & 1 \\ 0 & 0 & 7 & 4 & 3 \end{pmatrix}$, then we obtain $R = 288$ and $\mathrm{fil}(f_1) = \mathrm{fil}(f_2) = 504$.

Now, if we consider the monomial y^5z^3 we get $\mathrm{fil}(y^5z^3) = 840$, therefore $f_t(x, y, z) = (3x^6 + 2y^6 + xz^4, x^6 + y^6 + yz^3 + ty^5z^3)$ is C^1-$\mathcal{K}$-trivial.

We remark here that the vector $(0, y^5z^3)$ is not in the $\mathcal{K}$-tangent space of the germ f, hence f_t is not C^ω-$\mathcal{K}$-trivial.

Proof of the Theorem 3.16. Define $N_{\mathcal{K}} f_t := [N_{\mathcal{R}} f_t]^a + [N_{\mathcal{C}} f_t]^b$, then

$$\begin{aligned} N_{\mathcal{K}} f_t \cdot \frac{\partial f_t}{\partial t} &= [N_{\mathcal{R}} f_t]^a \cdot \frac{\partial f_t}{\partial t} + [N_{\mathcal{C}} f_t]^b \cdot \frac{\partial f_t}{\partial t} \\ &= [N_{\mathcal{R}} f_t]^{a-1} \cdot [df_t]_x(W_{\mathcal{R}}) + [N_{\mathcal{C}} f_t]^{b-1} \cdot \sum W_i(f_{ti}) \\ &= [df_t]_x([N_{\mathcal{R}} f_t]^{a-1} W_{\mathcal{R}}) + \sum([N_{\mathcal{C}} f_t]^{b-1} W_i)(f_{ti}). \end{aligned}$$

Hence $\partial f_t/\partial t = [df_t]_x(\xi) + \sum(\eta_i)(f_{ti})$ where ξ is the vector field defined as $\xi := ([N_{\mathcal{R}} f_t]^{a-1}/N_{\mathcal{K}} f_t) W_{\mathcal{R}}$, where $W_{\mathcal{R}}$ is the vector field defined in the case of the group $\mathcal{R}$, and $\eta_i := ([N_{\mathcal{C}} f_t]^{b-1}/N_{\mathcal{K}} f_t) W_i$, where W_i is the vector field defined in the case of the group $\mathcal{C}$.

Since

$$\begin{aligned} \mathrm{fil}([N_{\mathcal{R}} f_t]^{a-1} W_{\mathcal{R}}) &\geq (a-1) \cdot 2\alpha + 2\alpha + \ell R + 1 \\ &= 2\alpha a + \ell R + 1, \quad \text{with } \alpha := \text{l.c.m.}\{\mathrm{fil}(M_I)\}; \\ \mathrm{fil}([N_{\mathcal{C}} f_t]^{b-1} W_i) &\geq (b-1) \cdot 2B + 2B + \ell R + 1 \\ &= 2Bb + \ell R + 1, \quad \text{with } B := \text{l.c.m.}\{\mathrm{fil}(f_i)\}; \end{aligned}$$

and

$$\begin{aligned} \mathrm{fil}(N_{\mathcal{K}} f_t) &= \mathrm{fil}([N_{\mathcal{R}} f_t]^a) = \mathrm{fil}([N_{\mathcal{C}} f_t]^b) \\ &= 2\alpha a = 2Bb, \end{aligned}$$

we obtain from the Lemma 2.9 that the vector fields ξ and $\eta = (\eta_1, \ldots, \eta_p)$ are of class C^ℓ and the result follows. □

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