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Global estimates of maximal operators generated by dispersive equations

Yonggeun Cho and Yongsun Shim

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Abstract. Let $Tf(x, t) = e^{it\phi(D)}f$ be the solution of a general dispersive equation with phase function ϕ and initial data f in a Sobolev space. When the phase ϕ has a suitable growth condition and the initial data f has an angular regularity, we prove global and local L^p estimates for maximal operators generated by T. Here we do not assume the radial symmetry for the initial data. These results reveal some sufficient conditions on initial data for the boundedness of maximal operators in contrast to the negative results of [28]. We also prove a weighted L^2 maximal estimate, which is an extension of [19] to nonradial initial data.

Key words: dispersive equation, maximal operator, phase function, angular regularity.

1. Introduction

The general dispersive equation is defined by

$$\begin{aligned} &iu_t(x) = -\phi(D)u(x), \quad \text{on } \mathbb{R}^n \times \mathbb{R}, \\ &u(x, 0) = f(x), \quad (x, t) \in \mathbb{R}^{n+1} (n \geq 2), \end{aligned}$$

where $D = -i\nabla$ and ϕ is a smooth phase function. The formal solution of this equation is

$$u(x, t) = Tf(x, t) = \frac{1}{(2\pi)^n} \int e^{i(x\cdot\xi + t\phi(\xi))} \widehat{f}(\xi) d\xi, \tag{1.1}$$

where $\widehat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)dx$. Now let us define maximal operators:

$$T^* f(x) \equiv \sup_{-1\leq t\leq 1} |Tf(x, t)|, \quad T^{**} f(x) \equiv \sup_{t\in\mathbb{R}} |Tf(x, t)|.$$

In this paper, we try to find mapping properties of T^{**} or T^* associated to various phase functions. A typical phase is $|\xi|^a$ $(a \neq 0)$. Mappings are from a mixed Sobolev space $H^s(H^m_\omega)$ to a weighted $L^p(wdx)$. The estimates

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are of the form:

$$\|T^*f\|_{L^p(wdx)} \quad \text{or} \quad \|T^{**}f\|_{L^p(wdx)} \le C\|f\|_{H^s(H^m_\omega)}. \tag{1.2}$$

Here, $\|g\|^p_{L^p(wdx)} = \int_{\mathbb{R}^n} |g(x)|^p w(x)dx$ for some nonnegative measurable function w, $L^p = L^p(\mathbb{R}^n)$. The mixed Sobolev space norm is

$$\|f\|_{H^s(H^m_\omega)} = \|(1-\Delta)^{s/2}(1-\Delta_\omega)^{m/2}f\|_{L^2}$$

where Δ_ω is the Laplace-Beltrami operator defined on the unit sphere S^{n-1}. Let us now impose an assumption on the phase ϕ as follows:

A. Let ϕ be a radial function such that for some $a \in \mathbb{R}$ $(\neq 0,\, 1)$, $\phi \in C^2(\mathbb{R}^n \setminus \{0\})$ and there exist positive constants c_1, c_2 such that

$$c_1|\xi|^{a-k} \le |\phi^{(k)}(\xi)| \le c_2|\xi|^{a-k} \ (k = 0,\, 1,\, 2), \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.$$

The maximal inequality (1.2) is motivated from the well-known pointwise convergence problem: $\lim_{t\to 0} u(x,\, t) = f(x)$ a.e. x, for $f \in H^{1/4}(\mathbb{R}^n)$. The local and global L^p estimates of the maximal operators have been studied by many authors [3, 4, 8, 10, 12, 15, 25] and [26]. P. Sjölin [20] and L. Vega [26] obtained the strong necessary condition ($s \ge 1/4$) on the pointwise convergence problem. In particular, P. Sjölin showed that the maximal operator T^{**} cannot have the global L^2 boundedness (see [17]). Thus we consider global L^p or local (or weighted) L^2 estimates. For a global weighted and a local L^2 estimates see [6] and [9].

In this paper we show a nonweighted global L^p estimate under an angular regularity condition on f. See (1) of the main theorem below. In view of the negative result of [28], where S. Wang showed that if $p > 2$, then there exist f_0 and Y_k such that if $f(x) = |x|^k f_0(|x|)Y_k(x')$ for some spherical harmonic function Y_k of order k (for instance see [23]), then $\lim_{k\to\infty} \|T^*f\|_{L^p(B)}/\|f\|_{H^{1/4}} = \infty$ for any ball B, we see that our result suggests a sufficient condition on f for L^p maximal inequality (1.2). The same method leads us to an extension of [27], which is stated in part (2) of the Theorem 1.1.

P. Sjölin also showed the global estimate of T^*. Namely, $\|T^*f\|_{L^2} \le C\|f\|_{H^s}$ holds for $s > a/4$ and radial f, and fails for $s < a/4$ $(a > 1)$ (see [19, 20]). We still do not know about the critical case $s = a/4$. As related topics, we refer the readers to [1] and [16] in which some global smoothing properties of the critical case ($s = 1/2$, $a = 2$) are considered. Our last result ((4) below) is an extension of the sufficiency condition of Sjölin. We consider a weighted L^2 inequality for a general f with angular regularity even in the case $0 < a < 1$.

Now we state our main results.

Theorem 1.1 *Let f be a function in $H^s(H^m_\omega)$. Then*

(1) $\|T^{**}f\|_{L^p} \le C\|f\|_{H^s(H^m_\omega)}$ *holds for* $s \in (1/4, 1/2)$, $m > (3n-5)/6 - 2/p$, $4n/(2n-1) \le p < 2n/(n-2s)$ *and* $0 < a \ne 1$

(2) $\|T^{**}f\|_{L^p(w\,dx)} \le C\|f\|_{H^s(H^m_\omega)}$ *holds for* $s \in [1/4, 1/2)$, $m > (3n+1)/6 - 2s/n$, $p = 2n/(n-2s)$, $0 < a \ne 1$ *and* $w = |x|^b/(1+|x|)^b$ *with* $b > 0$

(3) $\|T^{*}f\|_{L^p(wdx)} \le C\|f\|_{H^s(H^m_\omega)}$ *holds for* $0 < a < 1$, $m > (3n+1)/6 - 2s/n$, $a/4 < s < 1/4$, $p = 4n(1-a)/\{2n(1-a)+a-4s\}$ *and* $w = |x|^{b_1}/(1+|x|)^{b_1+b_2}$ *with* $b_1 > 0$ *and* $b_2 > 1-(n-1)(p/2-1)$

(4) $\|T^{*}f\|_{L^2(w\,dx)} \le C\|f\|_{H^s(H^m_\omega)}$ *holds for* $s > a/4$, $m > n/2$, $0 < a \ne 1$ *and* $w = (1+|x|)^{-b}$ *with* $b > 0$.

The parts (1), (2) and (3) are obtained by using a boundedness property of one dimensional oscillatory integral of the form $\int_{\mathbb{R}} e^{i(t\phi(\xi)+x\xi)}|\xi|^{-s}d\xi$. Many authors referred in this paper have tried to handle such an integral and obtained various bounds for the corresponding growth rate a of ϕ (i.e. $a > 1$ or $a < 1$). Here, we show that the integral is bounded by a constant multiple of $|x|^{-(1-s)}$ for any $0 < a \ne 1$ where the constant is independent of t. See Lemma 2.3 below. Recently P. Sjölin showed this lemma for $a > 1$ in [21]. An earlier version of the proof of Lemma 2.3 can be also found in [5].

For the proof of theorem, the uniform bound of Bessel function is important. We use the following asymptotic behavior of Bessel function: $J_\nu(r) = r^{-1/2}(b_+re^{ir}+b_-e^{-ir})+\Psi(r)$, $|\Psi| \le Cr^{-1}$ for $r > \nu$, where C is independent of ν. Owing to the slow decay of the tail Ψ, we cannot attack the end point case $p = 2n/(n-2s)$ in (1). To avoid this difficulty, we have used weights in (2) and (3).

The angular regularity is due to the estimate of Bessel function which appears in the Fourier transform of spherical harmonics. The condition on m is far from being optimal. We do not know even the necessary condition on this angular regularity, which might be interesting if found.

For the last result (4), we use an oscillatory integral to show that a scaling property is related to the growth rate of the phase ϕ for a local time. See Lemma 2.4 below. If $a > 1$, the dispersiveness is quite strong, hence an L^2 weighted global maximal inequality is possible which covers the part (4) when $a > 1$ (see [6]).

For the more general initial data f, by using a bilinear estimate, T. Tao in [25] showed that the global estimate for T^{**} holds for $p > 2(n+3)/(n+1)$ and $s > n(1/2-1/p)$, if the phase ϕ is of an elliptic type. Recently, in [13] S. Lee improved Tao's results in 2 dimensional case up to $s > 3/8$. As another global estimates, there are several results about the weighted estimates for $s > 1/2$. For these results, one may refer [7, 10, 26, 27]. However, it remains still an open problem whether even a local estimate of T^* holds or not for $s = 1/4$.

If not specified, throughout this paper, C denotes a generic positive constant that depends on c_1, c_2, a, s, n.

2. Preliminary lemmas

We begin with the weighted inequality for the Fourier transform.

Lemma 2.1 (see [14]) *If $1 \le q \le 2$, $0 \le \alpha < 1/2$, $0 \le \alpha_1 < 1/q'$ and $\alpha_1 = \alpha + 1/2 - 1/q$, then the following inequality holds*

$$\Big(\int_{\mathbb{R}} |\xi|^{-2\alpha} |\widehat{f}(\xi)|^2 d\xi\Big)^{1/2} \le C\Big(\int_{\mathbb{R}} |f(x)|^q |x|^{\alpha_1 q} dx\Big)^{1/q}.$$

Now we introduce some estimates of oscillatory integrals. Let us first state a stationary phase lemma which can be found in [12] etc..

Lemma 2.2 *Let ψ be a monotone function and $I = \int_\alpha^\beta e^{i\varphi(\xi)}\psi(\xi)d\xi$. Then if $|d\varphi/d\xi| \ge \lambda > 0$ in $[\alpha, \beta]$ and $d\varphi/d\xi$ is monotone, $|I| \le C\lambda^{-1}\sup_{[\alpha,\beta]}|\psi(\xi)|$, and if $|d^2\varphi/d\xi^2| \ge \lambda > 0$, then $|I| \le C\lambda^{-1/2}\sup_{[\alpha,\beta]}|\psi(\xi)|$. The constant C doesn't depend on α, β, λ, φ and ψ.*

Utilizing the lemma above, we get the following lemma.

Lemma 2.3 *Suppose ϕ satisfies the assumption* **A** *for $0 < a \ne 1$. Let A, B, s be the real numbers such that A, $B \ne 0$, $1/2 \le s < 1$. Consider the following integral:*

$$I = \int_{\xi\in\mathbb{R}} e^{i(A\phi(\xi)+B\xi)}|\xi|^{-s}d\xi.$$

Then $|I| \le C(a, s, c_1, c_2)|B|^{-(1-s)}$.

If $0 < a < 1$, $a/2 < s < 1/2$ *and* $|A| \le 2$, *then*

$$I \le C(|B|^{-(1-s)} + |B|^{-(1-s)-a(1-2s)/2(1-a)}).$$

Proof of Lemma 2.3.

Case $a > 1$: Without loss of generality, we may assume that $A > 0$ and $B > 0$. Let $D = B/A^{1/a}$. Then by the change of variable, we have

$$I = A^{-(1-s)/a} \int e^{i(A\phi(A^{-1/a}\xi)+D\xi)}|\xi|^{-s}d\xi = \int_{\xi<0} + \int_{\xi>0} = I_- + I_+.$$

We may only have to consider I_+. Let us denote it by I again.

Now we first consider the case when $\phi' > 0$. Observe that

$$E \equiv (A\phi(A^{-1/a}\xi) + D\xi)' \ge c_1\xi^{a-1} + D.$$

Let M be a large positive number depending only on a, s, c_1, c_2. If $D \le M$, then

$$I = A^{-(1-s)/a}\Big(\int_0^1 + \int_1^\infty\Big) = I_1 + I_2$$

For I_1, by a direct integration, we have $|I_1| \le CA^{-(1-s)/a} \le CB^{-(1-s)}$. For I_2, since $E \ge C^{-1}$, by the first part of Lemma 2.2, we have $|I_2| \le CA^{-(1-s)/a} \le CB^{-(1-s)}$. If $D > M$, then since $E \ge D$, by the first part of Lemma 2.2, we have $|I_2| \le CA^{-(1-s)/a}D^{-1} \le A^{s/a}B^{-1} \le B^{-(1-s)}$. For I_1, using the change of variable, we have

$$I_1 = A^{-(1-s)/a}D^{-(1-s)} \int_0^D e^{i(A\phi(D^{-1}A^{-1/a}\xi)+\xi)}\xi^{-s}d\xi.$$

Thus $I_1 = \int_0^1 + \int_1^D = I_{1,1} + I_{1,2}$. By the integration, $|I_{1,1}| \le CB^{-(1-s)}$. For $I_{1,2}$, since $(A\phi(D^{-1}A^{-1/a}\xi) + \xi)' \ge 1$, from the first part of Lemma 2.2, we have $|I_{1,2}| \le CB^{-(1-s)}$ and hence $|I_1| \le CB^{-(1-s)}$.

Now we consider the case when $\phi' < 0$. We observe that

$$-c_2\xi^{a-1} + D \le E = (A\phi(A^{-1/a}\xi) + D\xi)' \le -c_1\xi^{a-1} + D.$$

If $D \le M$, then we split I into two parts as follows:

$$I = A^{-(1-s)/a}\Big(\int_0^{(2M/c_2)^{1/(a-1)}} + \int_{(2M/c_2)^{1/(a-1)}}^\infty\Big) = I_3 + I_4.$$

For I_3, we have by direct integration $|I_3| \le CA^{-(1-s)/a} \le CB^{-(1-s)}$. For I_4, since $E \le -c_1\xi^{a-1} + D \le -1$, by the first part of Lemma 2.2, we get $|I_4| \le CA^{-(1-s)/a} \le CB^{-(1-s)}$. If $D > M$, then we split I into four parts as follows:

$$\begin{aligned} I &= A^{-(1-s)/a}\Big(\int_0^1 + \int_1^{(D/2c_2)^{1/(a-1)}} + \int_{(D/2c_2)^{1/(a-1)}}^{(2D/c_1)^{1/(a-1)}} + \int_{(2D/c_1)^{1/(a-1)}}^{\infty}\Big) \\ &\equiv I_5 + I_6 + I_7 + I_8. \end{aligned} \tag{2.1}$$

For I_5, we use the change of variable so that

$$I_5 = A^{-(1-s)/a}D^{-(1-s)}\int_0^D e^{i(A\phi(D^{-1}A^{-1/a}\xi)+\xi)}\xi^{-s}d\xi.$$

We split I_5 into two part: $I_5 = A^{-(1-s)/a}D^{-(1-s)}(\int_0^1 + \int_1^D) = I_{5,1} + I_{5,2}$. For $I_{5,1}$ and $I_{5,2}$, using the direct integration and the first part of Lemma 2.2 respectively, we have $|I_{5,1}| + |I_{5,2}| \le CA^{-(1-s)/a}D^{-(1-s)} = CB^{-(1-s)}$. For I_6, since $E \ge C^{-1}D \ge C^{-1}D^{1-s}$, using the first part of Lemma 2.2, we have $|I_6| \le CA^{-(1-s)/a}D^{-(1-s)} = CB^{-(1-s)}$.

To estimate I_7, we use the fact $|E'|$ is equivalent to ξ^{a-2} and hence to $D^{(a-2)/(a-1)}$. Then from the second part of Lemma 2.2, we obtain

$$\begin{aligned} |I_7| &\le CA^{-(1-s)/a}D^{-(a-2)/2(a-1)}D^{-s/(a-1)} \\ &= CA^{-(1-s)/a}D^{-(a-2+2s)/2(a-1)}. \end{aligned}$$

Since $a > 1$ and $s \ge 1/2$, we have $|I_7| \le CA^{-(1-s)/a}D^{-(1-s)} = CB^{-(1-s)}$. Finally, we estimate I_8. Since $E \ge C^{-1}D \ge C^{-1}D^{1-s}$, by the first part of Lemma 2.2, we have $|I_8| \le CA^{-(1-s)/a}D^{-(1-s)}D^{-s/(a-1)} \le CB^{-(1-s)}$.

Case $a < 1$: We first consider the case $1/2 \le s < 1$. We may assume A, $B > 0$. Let $\tilde{D} = A/B^a$. Then by the change of variable, we write

$$B^{1-s}I = \int e^{i(A\phi(\xi/B)+\xi)}|\xi|^{-s}d\xi = \int_0^\infty + \int_{-\infty}^0 = I_+ + I_-.$$

As in the previous case ($a > 1$), we only consider I_+ and denote it by I again.

In case that $\phi' > 0$, we have $E \equiv (A\phi(\xi/B) + \xi)' \ge c_1\tilde{D}\xi^{a-1} + 1 \ge 1$ for all $\xi > 0$. We divide I into two parts: $I = \int_0^1 + \int_1^\infty$. For the first integral, we just integrate and for the second one, we use the first part of Lemma 2.2. Then we can see $|I| \lesssim 1$.

Now we consider the case when $\phi' < 0$. Then we obtain

$$-c_2\tilde{D}\xi^{a-1} + 1 \le E \le -c_1\tilde{D}\xi^{a-1} + 1.$$

If $c_2\tilde{D} < 2$, then we divide I into two parts: $I = \int_0^{(1/4)^{1/(a-1)}} + \int_{(1/4)^{1/(a-1)}}^{\infty} = I_1 + I_2$. By the integration, we get $|I_1| \le C$. And since $c_2\tilde{D} < 2$ and hence $E \ge C^{-1}$, by the first part of Lemma 2.2, we have $|I_2| \le C$.

If $c_1\tilde{D} > 2$, then we divide I into four parts:

$$\begin{aligned} I &= \int_0^1 + \int_1^{(2/c_1\tilde{D})^{1/(a-1)}} + \int_{(2/c_1\tilde{D})^{1/(a-1)}}^{(1/2c_2\tilde{D})^{1/(a-1)}} + \int_{(1/2c_2\tilde{D})^{1/(a-1)}}^{\infty} \\ &= I_3 + I_4 + I_5 + I_6. \end{aligned}$$

For I_3, by the integration, $|I_3| \le C$. For $|I_5|$, since $|E'|$ is equivalent to $\tilde{D}\tilde{D}^{-(a-2)/(a-1)} = \tilde{D}^{1/(a-1)}$ and $s \ge 1/2$, by the second part of Lemma 2.2, we have $|I_5| \le C\tilde{D}^{(2s-1)/2(a-1)} \le C$. And since $E \le -C$ on $[1, (2/c_1\tilde{D})^{1/(a-1)}]$ and $E \ge C^{-1}$ on $[(1/2c_2\tilde{D})^{1/(a-1)}, \infty)$, we also have $|I_4|$, $|I_6| \le C$.

If $2/c_2 \le \tilde{D} \le 2/c_1$, choose a large number M depending only on c_1, c_2, and divide I as follows: $I = \int_0^M + \int_M^\infty$. Then as the estimate of I_1 and I_2, we can obtain $|I| \le C$.

If $0 < a < 1$ and $a/2 < s < 1/2$, then except for the integral I_5, we can treat every integral by the same method as above. For I_5, since $|E'|$ is equivalent to $\tilde{D}\tilde{D}^{-(a-2)/(a-1)} = \tilde{D}^{1/(a-1)}$, $|A| \le 2$ and $s < 1/2$, by the second part of Lemma 2.2, we have $|I_5| \le CB^{-a(1-2s)/2(1-a)}$. This completes the proof of the lemma. □

Lemma 2.4 *Let N be a positive number and α, β be real numbers satisfying $0 < |\alpha| \le 1$, $\beta \ne 0$. Let φ be a $C_0^\infty(\mathbb{R})$ function with the support away from the orgin. Consider the oscillatory integral*

$$I_N(\alpha, \beta) = N \int e^{i(\alpha\phi(N\xi) + N\beta\xi)}\varphi(\xi)d\xi,$$

where ϕ satisfies the assumption **A**. *If $N \ge 1$ and $0 < a \ne 1$, then $\int |I_N(\alpha, \beta)|d\beta \le CN^{a/2}$. The constant C_a does not depend on α and N.*

Proof of Lemma 2.4. If $N|\beta| > CN^a|\alpha|$ and $N|\beta| \ge 1$ for some C depending only on ϕ, then by the integration by part, we have

$$|I_N| \le C_\mu N(1 + N|\beta|)^{-\mu}, \tag{2.2}$$

where C_μ is a constant depending on the parameters of C to be denoted by

C again and μ is any positive number less than equal to 2. By the second part of Lemma 2.2, we have

$$|I_N| \leq CN(N^\alpha|\alpha|)^{-1/2} \quad \text{and} \quad |I_N| \leq CN. \tag{2.3}$$

We divide the integral $\int |I_N| d\beta$ into four part as follows.

$$\begin{aligned}\int |I_N| d\beta &= \int_{N|\beta|\leq 1} + \int_{\substack{1\leq N|\beta|\leq CN^a,\\ N|\beta|\leq CN^a|\alpha|}} + \int_{\substack{N|\beta|\geq 1,\\ CN^a|\alpha|\leq|\beta|\leq CN^a}} + \int_{N|\beta|>CN^a}\\ &\equiv \sum_{i=1}^{4} II_i.\end{aligned}$$

Now we estimate each term. At first, by the second part of (2.3), $II_1 \leq CNN^{-1} = C$. For II_2, using the first part of (2.3),

$$\begin{aligned}II_2 &\leq CN \int_{|\beta|\leq CN^{a-1}|\alpha|} (N^a|\alpha|)^{-1/2} d\beta\\ &\leq CN^{1/2} \int_{|\beta|\leq CN^{a-1}} |\beta|^{-1/2} d\beta \leq CN^{a/2}.\end{aligned}$$

Using (2.2) with $\mu = 1 - a/(2+2a)$, for II_3, we have

$$II_3 \leq CN^{1-\mu} \int_{\substack{N|\beta|\geq 1,\\ CN^a|\alpha|\leq|\beta|\leq CN^a}} |\beta|^{-\mu} d\beta \leq CN^{(1-\mu)(1+a)} = CN^{a/2}.$$

Finally, for II_4, using (2.2) with $1 < \mu \leq 2$, we have

$$II_4 \leq C(a, \mu)N^{1-\mu} \int_{|\beta|>CN^{a-1}} |\beta|^{-\mu} d\beta \leq CN^{a(1-\mu)} \leq CN^{a/2}.$$

This completes the proof. □

3. Proof of Theorem 1.1

3.1. Proof of the part (1)

We first consider the simple case that $f(r\omega) = f_0(r)Y_k(\omega)$, where $\{Y_k\}$ is the orthonormal basis of the space of spherical harmonic functions on the unit sphere of order k. Using Fourier transform of the radial function and spherical harmonic function (see [23]), we have

$$\widehat{f}(\rho\omega) = g_0(\rho)Y_k(\omega), \quad \|f_0\|_{L^2} = \|g_0\|_{L^2},$$

where

$$g_0(\rho) = c_{n,k} \int_0^\infty f_0(r) J_{\nu(k)}(r\rho) r^{n/2} \rho^{-(n-2)/2} dr, \tag{3.1}$$

with $\nu(k) = (n+2k-2)/2$ and $|c_{n,k}| \le C$. We define an auxiliary operator T_B by

$$T_B f(x,\, t) = \frac{1}{(2\pi)^n} \int e^{i(x\cdot\xi + t\phi(\xi))} \widehat{f}(\xi) \frac{d\xi}{(1+|\xi|^2)^{s/2}}.$$

Then using Fourier transform of the spherical harmonic function again, it can be written as:

$$\begin{aligned}
T_B f(r\omega,\, t) &= \frac{1}{(2\pi)^n} \int e^{i(x\cdot\xi + t\phi(\xi))} Y_k\Big(\frac{\xi}{|\xi|}\Big) g_0(|\xi|) \frac{d\xi}{(1+|\xi|^2)^{s/2}} \\
&= \frac{1}{(2\pi)^n} \int_0^\infty e^{it\phi(\rho)} \rho^{n-1} g_0(\rho) \Big(\int_{S^{n-1}} e^{ir\rho\omega\cdot\omega'} Y_k(\omega') d\omega' \Big) \frac{d\rho}{(1+\rho^2)^{s/2}} \\
&= \frac{1}{(2\pi)^n} \int_0^\infty e^{it\phi(\rho)} \rho^{n-1} g_0(\rho) (r\rho)^{-(n-2)/2} J_{\nu(k)}(r\rho) \frac{d\rho}{(1+\rho^2)^{s/2} Y_k(-\omega)} \\
&= \frac{1}{(2\pi)^n} r^{-(n-2)/2} \int_0^\infty e^{it\phi(\rho)} \rho^{1/2} G_0(\rho) J_{\nu(k)}(r\rho) \frac{d\rho}{(1+\rho^2)^{s/2} Y_k(-\omega)} \\
&\equiv T_{0,k}(G_0)(r,\, t) Y_k(-\omega),
\end{aligned}$$

where $G_0(\rho) = \rho^{n-1/2} g_0(\rho)$. Then we have $\|T_B^{**} f\|_{L^p} \le C \|T_{0,k}^{**}(G_0)\|_{L^p} \times \|Y_k\|_{L^p(S^{n-1})}$. For the proof of (1) we have only to show that for $p = 2n/(n-2s')$ and $s' \in [1/4,\, s)$ (hence $4n/(2n-1) \le p < 2n/(n-2s)$)

$$\|T_{0,k}^{**} G_0(\cdot)\|_{L^p} \le C \nu(k)^{3/2-s'} \|G_0\|_{L^2(\mathbb{R}_+)}. \tag{3.2}$$

Here the L^p norm LHS of (3.2) is the usual $L^p = L^p(\mathbb{R}^n)$. For simplicity, we denote $\|g\|_{L^p(\mathbb{R}_+)}$ by $\|g\|_{L^p_+}$. Since $\|G_0\|_{L^2(\mathbb{R}_+)} = C\|g_0\|_{L^2(\mathbb{R}^n)}$, once (3.2) is proven, then from the observation[1] that $\|Y_k\|_{L^\infty(S^{n-1})} \le Ck^{(n-2)/2}$ and the condition $p = 2n/(n-2s')$, we have for $k \ge 1$,

$$\begin{aligned}
\|T_B^{**} f\|_{L^p} &\le Ck^{3/2-s'} \|f_0\|_{L^2} \|Y_k\|_{L^p(S^{n-1})} \\
&\le Ck^{3/2-s'+(p-2)(n-2)/2p} \|f\|_{L^2} \\
&\le Ck^{3/2-2s'/n} \|f\|_{L^2}.
\end{aligned}$$

[1]This follows from the facts that $Y_k(\omega) = \int_{S^{n-1}} Y_k(\sigma) Z_\omega^k(\sigma) d\sigma$ for some zonal harmonic $Z_\omega^k(\sigma)$ and $\|Z_\omega^k\|_{L^2(S^{n-1})} \lesssim k^{(n-2)/2}$. For instance see p. 143–144 of [23].

Hence, for general f with expansion $f(r\omega) = \sum_{k\ge 0, 1\le l\le d(k)} f_k^l(r)Y_k^l(\omega)$, where $\{Y_k^l\}$ is the orthonormal basis of spherical harmonics of order k and $d(k)$ is its dimension, we get

$$\begin{aligned}
\|T_B^{**}f\|_{L^p} &\le C \sum_{k\ge 0, 1\le l\le d(k)} (1+k)^{3/2-2s'/n}\|f_k^lY_k^l\|_{L^2}\\
&\le C\sum_{k\ge 0}(1+k)^{3/2-2s'/n}d(k)^{1/2}\Big(\sum_{1\le l\le d(k)}\|f_k^lY_k^l\|_{L^2}^2\Big)^{1/2}\\
&\le C\Big(\sum_{k\ge 0}(1+k)^{2(3/2-2s'/n+(n-2)/2-m)}\Big)^{1/2}\\
&\quad\times\Big(\sum_{k,l}(1+k)^{2m}\|f_k^lY_k^l\|_{L^2}^2\Big)^{1/2}\\
&\le C\|f\|_{L^2(H_\omega^m)}.
\end{aligned}$$

We have used the bound $d(k) \le C(1+k)^{n-2}$, the condition $m > (3n+1)/6 - 2s'/n = (3n-5)/6 - 2/p$ and the fact $-\Delta_\omega Y_k = k(k+n-2)Y_k$ for $k \ge 1$. This proves the first part (1) of the theorem.

From now on we prove (3.2). We may assume that k and $\nu(k)$ are sufficiently large. Let us denote $\nu(k)$ by ν for simplicity. Let $SG_0 = r^{n-1/p}T_{0,k}G_0$ and S^d be the dual operator of S. Let $\gamma = (n-1)(1/2 - 1/p)$. Then for any $F \in C_0^\infty(\mathbb{R}_+\times\mathbb{R})$, we may write S^dF as follows:

$$\begin{aligned}
S^dF(\rho) &= \frac{1}{(2\pi)^n}\rho^{1/2}(1+\rho^2)^{-s/2}\\
&\quad\times\int_{\mathbb{R}}\int_0^\infty e^{-it\phi(\rho)}J_\nu(r\rho)r^{1/2-\gamma}F(r,\,t)drdt\\
&\equiv \sum_{j=0,1,2} S_j^dF,
\end{aligned}$$

where

$$\begin{aligned}
S_j^dF &= \frac{1}{(2\pi)^n}\rho^{1/2}(1+\rho^2)^{-s/2}\\
&\quad\times\int_{\mathbb{R}}\int_0^\infty e^{-it\phi(\rho)}J_\nu(r\rho)\phi_j\Big(\frac{r\rho}{\nu}\Big)r^{1/2-\gamma}F(r,\,t)drdt
\end{aligned}$$

and ϕ_0, ϕ_1 and ϕ_3 are smooth cut-off functions such that $\phi_0 = 1$ on $\{|s| < 1/4\}$, $\phi_0 = 0$ on $\{|s| > 1/2\}$, $\phi_1 = 1$ on $\{|s| \sim 1\}$, $\phi_1 = 0$ otherwise, $\phi_2 = 0$

on $\{|s| < 2\}$, $\phi_2 = 1$ on $\{|s| > 3\}$, and $\phi_0 + \phi_1 + \phi_2 = 1$.

Before estimating each part, we list some asymptotic properties of Bessel function:

$$|J_\nu(r)| \le C\exp(-C\nu), \quad \text{if} \quad r \le \frac{\nu}{2}, \tag{3.3}$$

$$\Big|J_\nu(r)\phi_1\Big(\frac{r}{\nu}\Big)\Big| \le Cr^{-1/3} \quad \text{for all} \quad r \ge 1, \tag{3.4}$$

$$J_\nu(r)\phi_2\Big(\frac{r}{\nu}\Big) = r^{-1/2}(b_+e^{ir} + b_-e^{-ir})\phi_2\Big(\frac{r}{\nu}\Big) + \Phi_\nu(r)\phi_2\Big(\frac{r}{\nu}\Big), \tag{3.5}$$

where $|\Phi_\nu(r)| \le C/r$, $|b_\pm| \le C$ and the constant C is independent of ν. For the proof of (3.3), (3.4) and (3.5), see [24], 5.2 of [22] and [6], respectively.

We first estimate S_0^dF. Using (3.3) and the inequality $(1+\rho^2)^{-s/2} \le C\rho^{-s'}$, we have

$$\begin{aligned}|S_0^dF(\rho)| &\le Ce^{-C\nu}\rho^{(n-1)/2-s'}\int_0^{\nu/2\rho} r^{(n-1)/2-\gamma}\|F(r,\,\cdot\,)\|_{L_t^1}dr\\ &\le Ce^{-C\nu/2}\rho^{-s'}\int_0^{\nu/2\rho} r^{-\gamma}\|F(r,\,\cdot\,)\|_{L_t^1}dr.\end{aligned}$$

To use the identity $\|S_0^dF(\rho)\|_{L^2(\mathbb{R}_+)} = \|(1/\rho)S_0^dF(1/\rho)\|_{L^2(\mathbb{R}_+)}$, we estimate $(1/\rho)S_0^dF(1/\rho)$ as follows:

$$\begin{aligned}\frac{1}{\rho}\left|S_0^dF\left(\frac{1}{\rho}\right)\right| &\le Ce^{-C\nu/2}\nu^{1-s'}\int_0^{(\nu\rho/2)}\frac{r^{-\gamma}}{(\nu\rho/2)^{1-s'}}\|F(r,\,\cdot\,)\|_{L_t^1}dr\\ &\le Ce^{-C\nu/2}\nu^{1-s'}\mathcal{I}_{s'}(r^{-\gamma}\|F(r,\,\cdot\,)\|_{L_t^1})\Big(\frac{\nu\rho}{2}\Big),\end{aligned}$$

where $\mathcal{I}_{s'}$ is the one dimensional Riesz potential: $\mathcal{I}_{s'}(g)(r) = \int_{\mathbb{R}} g(r')/|r - r'|^{1-s'}dr'$. From Lemma 2.1 with $\alpha_1 = s' + 1/2 - 1/p'$ and $p = 2n/(n-2s')$, we get for large k,

$$\begin{aligned}\|S_0^dF\|_{L_+^2}^2 &\le Ce^{-C\nu}\nu^{1-2s'}\int|\xi|^{-2s'}|(r^{-\gamma}\|F(r,\,\cdot\,)\|_{L_t^1})^\wedge(\xi)|^2d\xi\\ &\le C\Big(\int_0^\infty(r^{-\gamma}\|F(r,\,\cdot\,)\|_{L_t^1})^{p'}r^{\alpha_1p'}dr\Big)^{2/p'}.\end{aligned}$$

Here the Fourier transform of the integrand of the second term is applied after extending the function $r^{-\gamma}\|F(r,\,\cdot\,)\|_{L_t^1}$ to 0 for $r \le 0$. Since $\alpha_1 = \gamma$, we get

$$\|S_0^dF\|_{L_+^2} \le C\|F\|_{L_r^{p'}L_t^1} \tag{3.6}$$

for $p = 2n/(n-2s')$, where $\|F\|_{L_r^{p'}L_t^1}^{p'} = \int_0^\infty (\int_{\mathbb{R}} |F(r,\, t)|dt)^{p'} dr$.

For $S_1^d F$, we have $|S_i^d F(\rho)| \le C\rho^{1/2-s'} \int_{\nu/(2\rho)}^{2\nu/\rho} |J_\nu(r\rho)| r^{1/2-\gamma} \|F(r,\,\cdot)\|_{L_t^1} dr$. Similarly as we did for the estimate of $S_0^d F$, with $p = 2n/(n-2s')$, we have from (3.4) that

$$\begin{aligned}\frac{1}{\rho}\Big|S_1^d F\Big(\frac{1}{\rho}\Big)\Big| &\le C\rho^{-7/6+s'} \int_{\nu/(2\rho)}^{2\nu/\rho} r^{1/6-\gamma} \|F(r,\,\cdot)\|_{L_t^1} dr \\ &\le C\nu^{7/6-s'} \int_0^{2\nu\rho} \frac{r^{-\gamma}}{(2\nu\rho)^{1-s'}} \|F\|_{L_t^1} dr \\ &\le C\nu^{7/6-s'} \mathcal{I}_{s'}(r^{-\gamma}\|F\|_{L_t^1})(2\nu\rho).\end{aligned}$$

Changing the variable $\rho \mapsto \nu\rho$, we have that

$$\|S_1^d F\|_{L_+^2} \le C\nu^{3/2-s'} \|F\|_{L_r^{p'}L_t^1}. \tag{3.7}$$

Now we estimate $S_2^d F$. Let us set $S_2^d F = S_+F + S_-F + S_3F$, where

$$\begin{aligned}S_\pm F(\rho) &= \frac{b_\pm}{(2\pi)^n}(1+\rho^2)^{-s/2} \\ &\quad\times \int_{\mathbb{R}} \int_0^\infty e^{i(\pm r\rho - t\phi(\rho))} \phi_2\Big(\frac{r\rho}{\nu}\Big) r^{-\gamma} F(r,\, t) drdt, \\ S_3 F(\rho) &= \frac{1}{(2\pi)^n}(1+\rho^2)^{-s/2} \\ &\quad\times \int_{\mathbb{R}} \int_0^\infty e^{-it\phi(\rho)} (r\rho)^{1/2} \Phi_\nu(r\rho) \phi_2\Big(\frac{r\rho}{\nu}\Big) r^{-\gamma} F(r,\, t) drdt.\end{aligned}$$

For the estimate $S_\pm F$, it suffices to consider S_+F. We decompose it into two parts as follows:

$$S_+F(\rho) = \mathcal{A}_1 + \mathcal{A}_2$$

where

$$\begin{aligned}\mathcal{A}_1 &= \frac{b_+}{(2\pi)^n}(1+\rho^2)^{-s/2} \int_{\mathbb{R}} \int_0^\infty e^{i(r\rho - t\phi(\rho))} r^{-\gamma} F(r,\, t) drdt, \\ \mathcal{A}_2 &= \frac{b_+}{(2\pi)^n}(1+\rho^2)^{-s/2} \\ &\quad\times \int_{\mathbb{R}} \int_0^\infty e^{i(r\rho - t\phi(\rho))} \Big(\phi_2\Big(\frac{r\rho}{\nu}\Big) - 1\Big) r^{-\gamma} F(r,\, t) drdt.\end{aligned}$$

For $\mathcal{A}_2$, we have

$$|\mathcal{A}_2(\rho)| \le C\rho^{-s'} \int_0^{3\nu/\rho} r^{-\gamma} \|F(r,\,\cdot\,)\|_{L^1} dr$$

and hence by the similar estimate to $S_0^d F$ except for the factor $e^{-C\nu}$

$$\|\mathcal{A}_2\|_{L^2_+} \le C\nu^{1/2-s'} \|F\|_{L^{p'}_r L^1_t}. \tag{3.8}$$

Now we estimate $\mathcal{A}_1$. Since F is compactly supported in $\mathbb{R}_+ \times \mathbb{R}$, obviously we may assume that

$$\mathcal{A}_1 = \frac{b_+}{(2\pi)^n}(1+\rho^2)^{-s/2} \int_{\mathbb{R}^2} e^{i(r\rho - t\phi(\rho))}|r|^{-\gamma} F(r,\,t) drdt.$$

Hence for $\mathcal{A}_1$, we have from the inequality $(1+\rho^2)^{-s/2} \le C\rho^{-s'}$

$$\|\mathcal{A}_1\|^2_{L^2_+} \le C \iiiint K(r,\,r',\,t,\,t')|r|^{-\gamma} F(r,\,t)|r'|^{-\gamma}\overline{F(r',\,t')} drdr'dtdt',$$

where

$$K(r,\,r',\,t,\,t') = \int e^{-i((t-t')\phi(\rho)+(r-r')\rho)}\rho^{-2s'} d\rho.$$

Since $1/4 \le s' < 1/2$, by Lemma 2.3, we have $|K(r,\,r',\,t,\,t')| \le C|r - r'|^{-(1-2s')}$. Thus we have from the Fourier transform of Riesz potential and the fact $\int g(x)h(x)dx = (2\pi)^{-n}\int \widehat{g}(\xi)\widehat{h}(\xi)d\xi$ that

$$\begin{aligned}\|\mathcal{A}_1\|^2_{L^2_+} &\le C \iint |r-r'|^{-(1-2s')}|r|^{-\gamma}\|F(r,\,\cdot\,)\|_{L^1_t} \\ &\qquad\qquad \times |r'|^{-\gamma}\|F(r',\,\cdot\,)\|_{L^1_{t'}} drdr' \\ &= C\int |\xi|^{-2s'} \big|(|r|^{-\gamma}\|F(r,\,\cdot\,)\|_{L^1_t})^\wedge(\xi)\big|^2 d\xi.\end{aligned}$$

Invoking Lemma 2.1, we can get

$$\|\mathcal{A}_1\|^2_{L^2_+} \le C\Big(\int_0^\infty (r^{-\gamma}\|F\|_{L^1_t})^{p'} r^{\alpha p'} dr\Big)^{2/p'},$$

provided $\alpha = s' + 1/2 - 1/p'$. Since $\alpha = \gamma$, we have

$$\|\mathcal{A}_1\|_{L^2_+} \le C\|F\|_{L^{p'}_r L^1_t}. \tag{3.9}$$

Now it remains to estimate S_3F. The uniform decay of the function Φ_ν

on ν shows that

$$|S_3F(\rho)| \le C\rho^{-1/2}(1+\rho^2)^{-s/2}(\chi_{(0,\nu)}(\rho)+\chi_{[\nu,\infty]}(\rho)) \times \int_{2\nu/\rho}^{\infty} r^{-1/2-\gamma}\|F(r,\,\cdot\,)\|_{L_t^1}dr.$$

By Hölder's inequality, we get

$$\begin{aligned}|S_3F(\rho)| &\le C\rho^{-1/2}(1+\rho^2)^{-s/2}(\chi_{(0,\nu)}(\rho)+\chi_{[\nu,\infty]}(\rho))\Big(\frac{\nu}{\rho}\Big)^{-s'}\|F\|_{L_r^{p'}L_t^1}\\ &\le C\nu^{-s'}\big(\rho^{-1/2+s'}\chi_{(0,\nu)}(\rho)+\rho^{-1/2-s+s'}\chi_{[\nu,\infty)}\big)\|F\|_{L_r^{p'}L_t^1}\end{aligned}$$

and hence

$$\|S_3F\|_{L_+^2} \le C\|F\|_{L_r^{p'}L_t^1}. \tag{3.10}$$

Therefore the claim (3.2) follows from the estimates (3.6), (3.7), (3.8), (3.9) and (3.10). This completes the proof of the part (1).

3.2. Proof of the parts (2) and (3)

For the part (2), we follow almost the same way as in the proof of (1). One can prove (3.6), (3.7), (3.8) and (3.9) by replacing s' with s and $r^{-\gamma}$ factor with $r^{-\gamma}r^{b/p}(1+r)^{-b/p} = r^{-\gamma}w(r)^{1/p}$. We leave the details to the readers.

We have to check the inequality (3.10) with $r^{-\gamma}$ replaced by $r^{-\gamma}w(r)^{1/p}$. Let us observe from the estimate

$$\int_L^{\infty} r^{-(1/2+\gamma)p}w(r)dr \le CL^{-(sp-b)} \quad \text{or} \quad CL^{-sp}$$

for $p = 2n/(n-2s)$ and $\gamma = (n-1)(1/2-1/p)$ that

$$\begin{aligned}&|S_3F(\rho)|\\ &\le C\rho^{-1/2}(1+\rho^2)^{-s/2}(\chi_{(0,\nu)}(\rho)+\chi_{[\nu,\infty]}(\rho))\\ &\qquad\times \int_{2\nu/\rho}^{\infty} r^{-1/2-\gamma}w\|F(r,\,\cdot\,)\|_{L_t^1}dr\\ &\le C\rho^{-1/2}(1+\rho^2)^{-s/2}\bigg(\chi_{(0,\nu)}(\rho)\Big(\frac{\nu}{\rho}\Big)^{-s}+\chi_{[\nu,\infty]}(\rho)\Big(\frac{\nu}{\rho}\Big)^{-(s-b/p)}\bigg)\\ &\qquad\times\|F\|_{L_r^{p'}L_t^1}\\ &\le C\nu^{-s}\big(\rho^{-1/2+s}\chi_{(0,\nu)}(\rho)+\rho^{-1/2-b/p}\chi_{[\nu,\infty)}\big)\|F\|_{L_r^{p'}L_t^1}.\end{aligned}$$

Taking L^2_+ norm to both side, we get the desired result.

Similarly, one treats the part (3). Replacing $r^{-\gamma}$ with $r^{-\gamma}r^{b_1/p}(1+r)^{-(b_1+b_2)/p}$, one can readily get (3.6), (3.7), (3.8) and (3.10). For the inequality (3.9), let us consider the modified functional

$$\begin{aligned}\mathcal{A}_1 = &\frac{b_+}{(2\pi)^n}(1+\rho^2)^{-s/2}\\ &\times \int_{\mathbb{R}^2} e^{i(r\rho - t\phi(\rho))}|r|^{-\gamma}|r|^{b_1/p}(1+|r|)^{-(b_1+b_2)/p}F(r,\, t)drdt.\end{aligned}$$

Then we have from the second part of Lemma 2.3 that

$$\begin{aligned}&\|\mathcal{A}_1\|^2_{L^2_+}\\ &\le C\iiiint |K(r,\, r',\, t,\, t')||r|^{-\gamma}(1+|r'|)^{-b_2/p}\\ &\quad\times|\widetilde{F}(r,\, t)||r'|^{-\gamma}(1+|r'|)^{-b_2/p}|\widetilde{F}(r',\, t')|drdr'dtdt'\\ &\le C\iint |r-r'|^{-(1-2s)}(1+|r-r'|^{-a(1-4s)/2(1-a)})|r|^{-\gamma}(1+|r'|)^{-b_2/p}\\ &\quad\times\|\widetilde{F}(r,\,\cdot\,)\|_{L^1_t}|r'|^{-\gamma}(1+|r'|)^{-b_2/p}\|\widetilde{F}(r',\,\cdot\,)\|_{L^1_t}drdr'\\ &\le C\iint_{|r-r'|\le 1} + \iint_{|r-r'|>1},\end{aligned}$$

where $\widetilde{F}(r,\, t) = |r|^{b_1/p}(1+|r|)^{-b_1/p}$ and

$$K(r,\, r',\, t,\, t') = \int e^{-i((t-t')\phi(\rho)+(r-r')\rho)}\rho^{-2s}d\rho.$$

It follows from Lemma 2.1 with $\alpha = s - a(1-4s)/4(1-a)$, $\alpha_1 = \gamma$ and $q = p'$ that the first integral is bounded by

$$\begin{aligned}&\iint |r-r'|^{-(1-2(s-a(1-4s)/4(1-a))}|r|^{-\gamma}\|\widetilde{F}(r,\,\cdot\,)\|_{L^1_t}\\ &\qquad\qquad\qquad\qquad\times|r'|^{-\gamma}\|\widetilde{F}(r',\,\cdot\,)\|_{L^1_t}drdr'\\ &\le C\|\widetilde{F}\|^2_{L^{p'}_r L^1_t} \le C\|F\|^2_{L^{p'}_r L^1_t}.\end{aligned}$$

The second integral is bounded by

$$C\Big(\int_0^\infty r^{-\gamma}(1+r)^{-b_2/p}\|F\|_{L^1_t}dr\Big)^2 \le C\Big(\Big(\int_0^1 + \int_1^\infty\Big)(\cdots)\Big)^2$$

Since $\gamma p < 1$ and $\gamma p + b_2 > 1$ by the choice of p and b_2, $(\int_0^1 + \int_1^\infty)(\cdots)+ \le C\|F\|_{L_r^{p'} L_t^1}$ and hence $\|\mathcal{A}_1\|_{L_+^2} \le C\|F\|_{L_r^{p'} L_t^1}$.

3.3. Proof of the part (4)

Let us first define an auxiliary operator $\widetilde{T}_B$ by

$$\widetilde{T}_B f(x,\,t) = \frac{w(|x|)^{1/2}}{(2\pi)^n}\int e^{i(t\phi(\xi)+x\cdot\xi)}\widehat{f}(\xi)\frac{d\xi}{(1+|\xi|)^s},$$

where $w(|x|) = (1+|x|)^{-b}$ with $0 < b < s$. If $f(r\omega) = f_0(r)Y_k(\omega)$ for some smooth radial function f_0 and a spherical harmonic Y_k of order k, by the Fourier transform of the spherical harmonic function, the operator $\widetilde{T}_B$ can be written as:

$$\begin{aligned}\widetilde{T}_B f(r\omega,\,t) &= \frac{w(r)^{1/2}}{(2\pi)^n} r^{-(n-2)/2}\\ &\quad\times\int_0^\infty e^{it\phi(\rho)}\rho^{1/2}G_0(\rho)J_\nu(r\rho)\frac{d\rho}{(1+\rho^2)^{s/2}}Y_k(-\omega)\\ &\equiv \tilde{T}_{0,k}G_0(r,\,t)Y_k(-\omega),\end{aligned}$$

where $\nu = \nu(k) = (n+2k-2)/2$ and G_0 is the same function as in (3.1). From the proof of the part (1), we have only to prove that

$$\|\tilde{T}_{0,k}^* G_0(\,\cdot\,)\|_{L^2} \le C\nu^{1/2}\|G_0\|_{L_+^2}. \tag{3.11}$$

Let $\tilde{S}G_0 = r^{(n-1)/2}\tilde{T}_{0,k}G_0$. Then for the proof of (3.11) it suffices to show that

$$\|\tilde{S}^* G_0(\,\cdot\,)\|_{L_+^2} \le C\nu^{1/2}\|G_0\|_{L_+^2}. \tag{3.12}$$

Now we divide the integral of $\tilde{S}G_0$ by the cut-off functions ϕ_i introduced in the previous section:

$$\begin{aligned}\tilde{S}G_0(r,\,t) &= \frac{w(r)^{1/2}}{(2\pi)^n} r^{1/2}\int_0^\infty e^{it\phi(\rho)}\rho^{1/2}G_0(\rho)J_\nu(r\rho)\frac{d\rho}{(1+\rho^2)^{s/2}}\\ &\equiv \tilde{S}_0G_0 + \tilde{S}_1G_0 + \tilde{S}_2G_0,\end{aligned}$$

where

$$\tilde{S}_jG_0 = \frac{w(r)^{1/2}}{(2\pi)^n} r^{1/2}\int_0^\infty e^{it\phi(\rho)}\rho^{1/2}G_0(\rho)\phi_j\Big(\frac{r\rho}{\nu}\Big)J_\nu(r\rho)\frac{d\rho}{(1+\rho^2)^{s/2}}.$$

Let us define the maximal functions $\tilde{S}_0^*(r) = \sup_{0<t<1} |\tilde{S}_0 G_0(r, t)|$ and $\tilde{S}_1^*(r) = \sup_{0<t<1} |\tilde{S}_1 G_0(r, t)|$. Then from (3.3) we have that

$$\frac{1}{r}\tilde{S}_0^*\Big(\frac{1}{r}\Big) \le C\nu^{1/2}e^{-C\nu}r^{-1}\int_0^{2\nu r}|G_0(\rho)|d\rho \le C\nu^{3/2}e^{-C\nu}\mathcal{M}G_0(\nu r),$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function. We did not use the weight w here. Hence for large ν,

$$\|\tilde{S}_0^*\|_{L_+^2} \le C\nu^{3/2}e^{-C\nu}\|G_0\|_{L_+^2}. \tag{3.13}$$

Using the inequality $1/r\int_0^r |J_\nu(t)|^2 tdt \le C$ for any $r > 0$ (see [27] for instance), for $\tilde{S}_1^*$, we have

$$\begin{aligned}
\tilde{S}_1^*(r) \le{}& C\nu^{-s}\chi_{(0,1)}(r)r^{1/2+s}\int_{\nu/2r}^{2\nu/r}\rho^{1/2}|G_0(\rho)J_\nu(r\rho)|d\rho \\
&\qquad + C\chi_{[0,\infty)}(r)r^{1/2-\varepsilon}\int_{\nu/2r}^{2\nu/r}\rho^{1/2}|G_0(\rho)J_\nu(r\rho)|d\rho \\
\le{}& C\Big(\int_{\nu/2r}^{2\nu/r}|J_\nu(r\rho)|^2\rho d\rho\Big)^{1/2} \\
&\times\big(\nu^{-s}\chi_{(0,1)}(r)r^{1/2+s} + \chi_{[1,\infty)}(r)r^{1/2-\varepsilon}\big)\|G_0\|_{L_+^2} \\
\le{}& C\nu^{1/2}\big(\nu^{-s}\chi_{(0,1)}(r)r^{-1/2+s} + \chi_{[1,\infty)}(r)r^{-1/2-\varepsilon}\big)\|G_0\|_{L_+^2}
\end{aligned}$$

and hence that

$$\|\tilde{S}_1^*\|_{L_+^2} \le C\nu^{1/2}\|G_0\|_{L_+^2}. \tag{3.14}$$

Now we treat $\tilde{S}_2$ similarly as we did in (1). Using the asymptotic behavior (3.5), we divide the integral of $\tilde{S}_2 G_0$ into three parts:

$$\tilde{S}_2 G_0 = \tilde{S}_+ G_0 + \tilde{S}_- G_0 + \tilde{S}_3 G_0,$$

where

$$\begin{aligned}
\tilde{S}_\pm G_0 &= \frac{b_\pm w(r)^{1/2}}{(2\pi)^n}\int_0^\infty e^{i(\pm r\rho + t\phi(\rho))}G_0(\rho)\phi_2\Big(\frac{r\rho}{\nu}\Big)\frac{d\rho}{(1+\rho^2)^{s/2}} \\
\tilde{S}_3 G_0 &= \frac{w(r)^{1/2}}{(2\pi)^n}r^{1/2}\int_0^\infty e^{it\phi(\rho)}\rho^{1/2}G_0(\rho)\Phi_\nu(r\rho)\phi_2\Big(\frac{r\rho}{\nu}\Big)\frac{d\rho}{(1+\rho^2)^{s/2}}.
\end{aligned}$$

For $\tilde{S}_3 G_0$, we have from the uniform bound of Φ_ν,

$$\begin{aligned}
|\tilde{S}_3 G_0(r,\, t)| &\le C\chi_{(0,1)}(r) r^{-1/2} \int_{2\nu/r}^{\infty} \rho^{-1/2-s} |G_0(\rho)| d\rho \\
&\qquad + C\chi_{[1,\infty)}(r) r^{-1/2-b/2} \int_{2\nu/r}^{\infty} \rho^{-1/2} |G_0(\rho)| \frac{d\rho}{(1+\rho^2)^{s/2}} \\
&\le C\nu^{-\delta} \chi_{(0,1)}(r) r^{-1/2+\delta} \int_{2\nu/r}^{\infty} \rho^{-1/2-s+\delta} |G_0(\rho)| d\rho \\
&\qquad + C\chi_{[1,\infty)}(r) r^{-1/2-b/2} \int_{2\nu/r}^{\infty} \rho^{-1/2-\varepsilon_0} |G_0(\rho)| d\rho,
\end{aligned}$$

where δ is any number smaller than s. For the last integrand we have used $(1+\rho^2)^{-s/2} \le C\rho^{-\varepsilon_0}$ for $\varepsilon_0 < \min(b/2,\, s)$. Then by Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
\tilde{S}_3^*(r) &\le C\big(\nu^{-s}\chi_{(0,1)}(r) r^{-1/2+s} + \nu^{-\varepsilon_0}\chi_{[1,\infty)}(r) r^{-1/2-(b/2-\varepsilon_0)}\big) \\
&\quad \times \|G_0\|_{L^2_+},
\end{aligned}$$

where $\tilde{S}_3^*(r) = \sup_{0<t<1} |\tilde{S}_3 G_0(r,\, t)|$. Hence

$$\|\tilde{S}_3^*\|_{L^2_+} \le C\nu^{-\varepsilon_0} \|G_0\|_{L^2_+}. \tag{3.15}$$

Now we estimate $\tilde{S}_\pm G_0$. We use the extended operator $\mathcal{B}$ defining G_0 by $G_0(-\rho)$ for $\rho \le 0$ as follows:

$$\mathcal{B}G_0(r,\, t) = \frac{b_\pm w(r)^{1/2}}{(2\pi)^n} \int_{\mathbb{R}} e^{i(t\phi(|\rho|) \pm r\rho)} \phi_2\Big(\frac{r\rho}{\nu}\Big) G_0(\rho) \frac{d\rho}{(1+|\rho|)^s}.$$

We can rewrite this by

$$\begin{aligned}
\mathcal{B}G_0(r,\, t) &= \frac{b_\pm w(r)^{1/2}}{(2\pi)^n} \int_{\mathbb{R}} (\cdots) + \frac{b_\pm w(r)^{1/2}}{(2\pi)^n} \int_{\mathbb{R}} \Big(1 - \phi_2\Big(\frac{r\rho}{\nu}\Big)\Big)(\cdots) \\
&\equiv \mathcal{B}_1 G_0 + \mathcal{B}_2 G_0
\end{aligned}$$

as $\mathcal{A}_1$ and $\mathcal{A}_2$ in Section 3.1. Let $\mathcal{B}_1^* G_0(r) = \sup_{|t|<1} |\mathcal{B}_1 G_0|$ and $\mathcal{B}_2^* G_0(r) = \sup_{|t|<1} |\mathcal{B}_2 G_0|$. Then we first have $\mathcal{B}_2^* G(r) \le C \int_0^{3\nu/r} |G_0(\rho)| d\rho$ and hence

$$\frac{1}{r}\mathcal{B}_2^* G_0\Big(\frac{1}{r}\Big) \le C\frac{1}{r}\int_0^{3\nu r} |G_0(\rho)| d\rho \le C\nu \mathcal{M}(|G_0|)\Big(\frac{3\nu r}{2}\Big),$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal function. Thus

$$\|\mathcal{B}_2^* G_0\|_{L^2_+} \leq C\nu^{1/2}\|G_0\|_{L^2(\mathbb{R})} \leq C\nu^{1/2}\|G_0\|_{L^2_+}. \tag{3.16}$$

Next, we consider a global L^2 estimate of the local maximal operator $\mathcal{B}_1^*$. To do this, we employ the Kolmogorv-Seliverstov-Plessner method (see [8]). Let us define an operator $\mathcal{T}$ as

$$\mathcal{T}G_0(r) = \int e^{i(\pm r\rho + t(r)\phi(|\rho|))} G_0(\rho) \frac{d\rho}{(1+\rho^2)^{s/2}},$$

where $t(r)$ is any measurable function with $|t(r)| < 1$ on $\mathbb{R}_+$. Then we may write the operator $\mathcal{T}$ by $\mathcal{T}_j$ as $\mathcal{T}G_0(r) = \sum_{j\geq 0} \mathcal{T}_j G_0(r)$, where

$$\mathcal{T}_j G_0(r) = \int e^{i(\pm r\cdot\rho + t(r)\phi(|\rho|))} G_0(\rho)\varphi_j(\rho) \frac{d\rho}{(1+\rho^2)^{s/2}} \quad \text{for} \quad j \in \mathbb{Z},$$

where $\{\varphi_j\}$ are Littlewood-Paley functions such that $\varphi_0 \in C_0^\infty(B(0,\,1))$, $\varphi_j(\,\cdot\,) = \varphi(\,\cdot\,/2^j))$, $\varphi \in C_0^\infty(B(0,\,2)\backslash B(0,\,2^{-1}))$ and $\sum_{j\geq 0}\varphi_j = 1$. We claim that $\|\mathcal{T}_j G_0\|_{L^2_+} \lesssim 2^{aj/4}\|\Delta_j \check{G}_0\|_{L^2(\mathbb{R})}$, where $\widehat{\Delta_j g} = \varphi_j \widehat{g}$ is the n-dimensional Fourier transform. The $L^2(\mathbb{R})$ was taken for $\Delta_j\check{G}_0$ as a one-dimensional function of ρ. To show that, let $\mathcal{T}_j^d$ be the dual operator of $\mathcal{T}_j$. Then for any $F(r) \in C_0^\infty(\mathbb{R}_+)$ and $j \geq 1$,

$$\|\mathcal{T}_j^d F\|_{L^2_+}^2 = \iint K_j(r,\,r')F(r)\overline{F(r')}drdr'$$

where

$$K_j(r,\,r') = 2^{(1-2s)j}\int e^{-i((t(r)-t(r'))\phi(2^j|\rho|)+2^j(r-r')\rho)} \times 2^{2sj}\varphi^2(\rho)\frac{d\rho}{(1+2^{2j}\rho^2)^s}.$$

Since $2^{2sj}\varphi^2(\rho)/(1+2^{2j}\rho^2)^s$ and its derivatives are uniformly bounded on j, from Lemma 2.4 replacing δ by 2^j, we have

$$\sup_{r'\in\mathbb{R}}\int |K_j(r,\,r')|dr + \sup_{r\in\mathbb{R}}\int |K_j(r,\,r')|dr' \leq C2^{(a/2-2s)j}.$$

It follows from the Schur's lemma (see the lemma in p. 284 of [22]) that

$$\|\mathcal{T}_j^d F\|_{L^2_+} \leq C2^{(a/4-s)j}\|F\|_{L^2_+}. \tag{3.17}$$

If $j = 0$, then

$$\|\mathcal{T}_0^d F\|_{L_+^2}^2 = \iint K_0(r, r')F(r)\overline{F(r')}drdr'$$

where

$$K_0(r, r') = \int e^{-i((t(r)-t(r'))\phi(|\rho|)\pm(r-r')\rho)}\varphi_0^2(\rho)\frac{d\rho}{(1+\rho^2)^s}.$$

Since $|t(r) - t(r')| \le 2$, using the integration by parts twice, we have

$$|K_0(r, r')| \le C(1 + |r - r'|)^{-2}.$$

Thus using Schur's lemma again, we have $\|\mathcal{T}_0^d F\|_{L^2} \le C\|F\|_{L^2}$. Combining this and (3.17), we get

$$\|\mathcal{B}_1^* G_0\|_{L_+^2} \le C\|G_0\|_{L_+^2}. \tag{3.18}$$

Therefore the part (3) of Theorem 1 follows from the estimates (3.13)–(3.18).

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Y. Cho
Department of Mathematics
Institute of Pure and Applied Mathematics
Chonbuk National University
Jeonju 561-756, Republic of Korea
E-mail: changocho@chonbuk.ac.kr

Y. Shim
Department of Mathematics
Pohang University of Science and Technology
Pohang 790-784, Korea
E-mail: shim@postech.ac.kr