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Linear differential relations satisfied by Wirtinger integrals

(Dedicated to Professor Masaaki Yoshida on his sixtieth birthday)

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Abstract. We will derive linear differential relations satisfied by Wirtinger integrals by exploiting classical formulas of Jacobi's theta functions. Although Wirtinger integrals are related to Gauss' hypergeometric functions, we will do that without referring to Gauss' hypergeometric differential equation. We believe that our method to derive them will be applicable to another definite integral defined on the torus which has as integrand a power product of not necessarily four theta functions, and which is not a lift of any definite integral defined on the complex projective line.

Key words: Wirtinger integral, theta function.

0. Introduction

In our recent paper [3] we studied twisted homology and cohomology groups with coefficients in local systems associated to a power product of Jacobi's four theta functions (see also Theorems 1.1 and 1.2 below). As is well-known ([5]), the pairing of non-vanishing homology and cohomology groups is expressed as a definite integral defined in our case on the torus by the de Rham theory. We propose to call such an integral *Wirtinger integral*, because the integral arising from a certain special choice of a homology class and a cohomology class is identical with the one considered by Wirtinger [4] (see also Remark 3 in Section 1). The purpose of this paper is to derive linear differential relations satisfied by Wirtinger integrals (Theorem 1.3). As is seen in [4] (see also [2]), Wirtinger integrals are the lifts of Gauss' hypergeometric functions to the upper half plane. So one can obtain the linear differential relations in Theorem 1.3 also by taking the universal covering space of the space of the independent variable of Gauss' hypergeometric differential equation. Nevertheless in this paper, without referring to Gauss' hypergeometric differential equation, we will derive them directly by exploit-

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ing classical formulas of Jacobi's theta functions. Thus we can say that the results of the present paper and our previous ones [2] and [3] constitute the theory of Gauss' hypergeometric functions based on theta functions and developed on the upper half plane such that the analytic continuation of Gauss' hypergeometric functions is translated in the language of modular transformations, and that they therefore suggest various new generalizations of Gauss' hypergeometric functions from the viewpoint of modular groups and theta functions. In fact, we believe that our method in this paper (Section 2) will be applicable to deriving differential relations of another definite integral defined on the torus which has as integrand a power product of not necessarily four theta functions, and which is not a lift of any definite integral defined on the complex projective line.

1. Preliminaries

In this paper we follow Chandrasekharan's notation for theta functions ([1]). Namely, the four symbols $\theta(u,\tau)$, $\theta_i(u,\tau)$ $(i=1,2,3)$ are defined by

$$\theta(u,\tau) = \frac{1}{i}\sum_{n=-\infty}^{+\infty}(-1)^n e^{(n+\frac{1}{2})^2\pi i\tau}e^{(2n+1)\pi iu},$$

$$\theta_1(u,\tau) = \sum_{n=-\infty}^{+\infty} e^{(n+\frac{1}{2})^2\pi i\tau}e^{(2n+1)\pi iu},$$

$$\theta_2(u,\tau) = \sum_{n=-\infty}^{+\infty}(-1)^n e^{n^2\pi i\tau}e^{2n\pi iu},$$

$$\theta_3(u,\tau) = \sum_{n=-\infty}^{+\infty} e^{n^2\pi i\tau}e^{2n\pi iu},$$

where $u,\tau\in\mathbf{C}$ and $\mathrm{Im}(\tau)>0$. We set $\Gamma=\mathbf{Z}+\mathbf{Z}\tau$, $D=\{0,\frac{1}{2},\frac{\tau}{2},\frac{1+\tau}{2}\}$, and $M=\mathbf{C}/\Gamma-D$. Let α,β,γ be complex parameters. Throughout this paper we assume the following conditions for α,β,γ:

$$\alpha,\ \beta,\ \gamma,\ \gamma-\alpha,\ \gamma-\beta,\ \gamma-\alpha-\beta,\ \text{and}\ \alpha-\beta\notin\frac{1}{2}\mathbf{Z},$$

where $\frac{1}{2}\mathbf{Z}$ denotes the group consisting of all integers and half integers. We set $T(u) = \theta(u)^{2\alpha}\theta_1(u)^{2\gamma-2\alpha-2}\theta_2(u)^{2\beta-2\gamma+2}\theta_3(u)^{-2\beta}$, where $\theta_i(u)$ means $\theta_i(u,\tau)$. We define a connection ∇ by $\nabla\varphi = d\varphi + \omega \wedge \varphi$ for a differential form φ, where d denotes the exterior differential with respect to u, and $\omega = d(\log T(u))$. Then we have $\nabla\nabla = 0$ and $\nabla(1) = \omega$. Let $\mathcal{L}$ and $\check{\mathcal{L}}$ be the local systems on M defined by $T(u)^{-1}$ and $T(u)$, respectively: $\mathcal{L} = \mathbf{C}T(u)^{-1}$ and $\check{\mathcal{L}} = \mathbf{C}T(u)$. They are dual to each other. For a non-negative integer k, let $C_k(M,\check{\mathcal{L}})$ be the group of twisted k-chains with coefficients in $\check{\mathcal{L}}$, and let ∂ be the boundary operator of the complex $C_\bullet(M,\check{\mathcal{L}})$. Let $\Omega^k(M)$ be the vector space of single-valued holomorphic k-forms on M. Then the connection ∇ induces a natural homomorphism $\nabla : \Omega^k(M) \to \Omega^{k+1}(M)$. The following theorem is fundamental:

Stokes' Theorem For $\sigma \in C_k(M,\check{\mathcal{L}})$ and $\varphi \in \Omega^{k-1}(M)$, we have

$$\int_\sigma T(u)\cdot\nabla\varphi = \int_{\partial\sigma} T(u)\cdot\varphi.$$

Let $Z_k(M,\check{\mathcal{L}})$ be the group of twisted k-cycles with coefficients in $\check{\mathcal{L}}$, and let $B_k(M,\check{\mathcal{L}})$ be the group of twisted k-boundaries with coefficients in $\check{\mathcal{L}}$. We set $H_k(M,\check{\mathcal{L}}) = Z_k(M,\check{\mathcal{L}})/B_k(M,\check{\mathcal{L}})$: the k-th twisted homology group with coefficients in $\check{\mathcal{L}}$. Concerning the twisted homology groups, we have obtained in our paper [3] the following:

Theorem 1.1 *We have $H_2(M,\check{\mathcal{L}}) = H_0(M,\check{\mathcal{L}}) = 0$, $H_1(M,\check{\mathcal{L}}) \cong \mathbf{C}c_1 \oplus \mathbf{C}c_2 \oplus \mathbf{C}c_3 \oplus \mathbf{C}c_4$. Here c_1, c_2, c_3, c_4 denote the homology classes of the cycles $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, respectively, which are given by*

$$\sigma_1 = \frac{1}{(1-e^{4\pi i\alpha})(1-e^{4\pi i(\gamma-\alpha)})}\left(\frac{1}{2}+, 0+, \frac{1}{2}-, 0-\right),$$

$$\sigma_2 = \frac{1}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\gamma-\alpha)})}\left(\frac{1+\tau}{2}+, \frac{1}{2}+, \frac{1+\tau}{2}-, \frac{1}{2}-\right),$$

$$\sigma_3 = \frac{1}{(1-e^{-4\pi i\beta})(1-e^{4\pi i(\beta-\gamma)})}\left(\frac{\tau}{2}+, \frac{1+\tau}{2}+, \frac{\tau}{2}-, \frac{1+\tau}{2}-\right),$$

$$\sigma_4 = \frac{1}{(1-e^{-2\pi i\gamma})(1-e^{4\pi i\alpha})}(l, 0+, -l, 0-),$$

where the right-hand sides of the expressions for σ_1, σ_2, σ_3 are Pochham-

mer cycles multiplied by complex constants, and the right-hand side of the expression for σ_4 *is the cycle multiplied by a constant, obtained by connecting the four curves* l, $(0+)$, $-l$, $(0-)$ *in this order, where* l *and* $-l$ *are the global cycles of* $\mathbf{C}/\Gamma$ *defined by the equations* $l(s) = -\frac{\tau}{4} + s$ *and* $(-l)(s) = -\frac{\tau}{4} - s$ $(0 \le s \le 1)$*, respectively, and* $(0+)$ *and* $(0-)$ *are small circles turning around the center* 0 *once with anti-clockwise and clockwise directions, respectively.*

Let D be the effective divisor on $\mathbf{C}/\Gamma$ given by $D = 2[0]+[\frac{1}{2}]+[\frac{\tau}{2}]+[\frac{1+\tau}{2}]$. Let Ω_D be the sheaf of meromorphic 1-forms on $\mathbf{C}/\Gamma$ which are multiples of the divisor $-D$. Let $H^k(M, \mathcal{L})$ be the k-th twisted cohomology group with coefficients in $\mathcal{L}$. Our result concerning the twisted cohomology groups is as follows ([3]):

Theorem 1.2 *We have* $H^0(M,\mathcal{L}) = H^2(M,\mathcal{L}) = 0$*,* $H^1(M,\mathcal{L}) \cong H^0(\mathbf{C}/\Gamma, \Omega_D)/\nabla(\mathbf{C}) = \mathbf{C}[\varphi_1] \oplus \mathbf{C}[\varphi_2] \oplus \mathbf{C}[\varphi_3] \oplus \mathbf{C}[\varphi_4]$*. Here* $[\varphi]$ *denotes the image of an element* φ *of* $H^0(\mathbf{C}/\Gamma, \Omega_D)$ *by the natural map* $H^0(\mathbf{C}/\Gamma, \Omega_D) \to H^0(\mathbf{C}/\Gamma, \Omega_D)/\nabla(\mathbf{C})$*, and* $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ *are elements of* $H^0(\mathbf{C}/\Gamma, \Omega_D)$ *given by*

$$\varphi_1 = \pi\theta_3^2 du, \quad \varphi_2 = \pi\theta_1^2 \frac{\theta_2(u)^2}{\theta(u)^2} du,$$

$$\varphi_3 = \pi\theta_3^2 \frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)} du, \quad \varphi_4 = \pi\theta_3^2 \frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)} du,$$

where θ_i *denotes the theta constant* $\theta_i(0)$*.*

Since $H_1(M, \check{\mathcal{L}})$ and $H^1(M,\mathcal{L})$ are dual to each other, we have a natural nondegenerate bilinear form $H_1(M, \check{\mathcal{L}}) \times H^1(M, \mathcal{L}) \to \mathbf{C}$. For $[\sigma] \in H_1(M, \check{\mathcal{L}})$ and $[\varphi] \in H^1(M,\mathcal{L})$ with $\sigma \in Z_1(M, \check{\mathcal{L}})$ and $\varphi \in H^0(\mathbf{C}/\Gamma, \Omega_D)$, let $\langle [\sigma], [\varphi] \rangle$ be the image by this bilinear form. By the standard procedure for regarding twisted cycles and cocycles as currents ([5]), we obtain the expression $\langle [\sigma], [\varphi] \rangle = \int_\sigma T(u)\varphi$, which we call *Wirtinger integral* (see Remark 3 below for the reason why we call it so). Every Wirtinger integral is a single-valued and holomorphic function of τ defined on the upper half plane H. We set $\int_{\sigma_j} T(u)\varphi_i = I_{ij}$ $(i, j = 1, 2, 3, 4)$. It is easy to see that, for a fixed j, the four integrals $I_{1j}, I_{2j}, I_{3j}, I_{4j}$ are linearly independent over $\mathbf{C}$, and that, for a fixed i, $I_{i1}, I_{i2}, I_{i3}, I_{i4}$ are linearly independent over $\mathbf{C}$. The main theorem in this paper is as follows:

Theorem 1.3 *The* 16 *functions* I_{ij} $(i,j = 1,2,3,4)$ *of the variable* τ *satisfy the following system of linear differential equations with coefficients invariant under the modular action of the principal congruence subgroup* $\Gamma(2)$ *of level* 2 :

$$\frac{i}{\pi\theta_3^4}\frac{d}{d\tau}\begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix}, \tag{1.1}$$

where a_{ij} *'s and* b_{ij} *'s are given by*

$$a_{11} = \frac{\alpha-\beta-1}{2}\frac{\theta_1^4}{\theta_3^4} + \frac{\gamma-1}{2}, \quad a_{12} = \frac{1-2\alpha}{2}, \quad a_{21} = \frac{2\gamma-2\beta-3}{2}\frac{\theta_1^4}{\theta_3^4},$$

$$a_{22} = \frac{\beta-\alpha+1}{2}\frac{\theta_1^4}{\theta_3^4} + \frac{1-\gamma}{2}, \quad b_{11} = \frac{(1+\alpha-\gamma)(\alpha+\beta)}{2(\beta-\alpha)} + \frac{\beta}{2}\frac{\theta_1^4}{\theta_3^4} + \frac{\alpha}{2}\frac{\theta_2^4}{\theta_3^4},$$

$$b_{12} = \frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{\beta-\alpha}, \quad b_{21} = \frac{\alpha\beta}{\alpha-\beta},$$

$$b_{22} = \frac{\alpha(\alpha+\beta-2\gamma+2)}{2(\alpha-\beta)} + \frac{\gamma-\alpha-1}{2}\frac{\theta_2^4}{\theta_3^4} + \frac{\gamma-\beta-1}{2}\frac{\theta_1^4}{\theta_3^4}.$$

The proof is given in the next section.

Remark 1 Note that the differential operator $\frac{i}{\pi\theta_3^4}\frac{d}{d\tau}$ is invariant under the action of the group $\Gamma(2)$. In fact we have $\frac{i}{\pi\theta_3^4}\frac{d}{d\tau} = \lambda(\lambda-1)\frac{d}{d\lambda}$, where λ denotes the lambda function: $\lambda = \frac{\theta_1^4}{\theta_3^4}$.

Remark 2 The system (1.1) is splitted into the following two systems of differential equations:

$$\frac{i}{\pi\theta_3^4}\frac{d}{d\tau}\begin{bmatrix} I_{1j} \\ I_{2j} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} I_{1j} \\ I_{2j} \end{bmatrix}$$

and

$$\frac{i}{\pi\theta_3^4}\frac{d}{d\tau}\begin{bmatrix} I_{3j} \\ I_{4j} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} I_{3j} \\ I_{4j} \end{bmatrix},$$

each of which is equivalent to Gauss' hypergeometric differential equation.

Remark 3 As was shown by Wirtinger [4] (see also [2]), the integrals I_{ij}'s are related to Gauss' hypergeometric function $F(\alpha, \beta, \gamma, z)$. In fact we have for example

$$I_{11} = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(\gamma-\alpha-\frac{1}{2})}{2\Gamma(\gamma)}\lambda^{\frac{\gamma-1}{2}}(1-\lambda)^{\frac{\alpha+\beta-\gamma+2}{2}}F\left(\alpha+\frac{1}{2}, \beta+\frac{3}{2}, \gamma, \lambda\right),$$

$$I_{21} = \frac{e^{\frac{\pi i}{2}(\gamma-\alpha-\beta-1)}\Gamma(\alpha-\frac{1}{2})\Gamma(\gamma-\alpha-\frac{1}{2})}{2\Gamma(\gamma+1)}\lambda^{\frac{\alpha+\beta-\gamma+1}{2}}$$
$$\times (1-\lambda)^{-\frac{\alpha+\beta+1}{2}}F\left(\alpha-\frac{1}{2}, \beta+\frac{1}{2}, \gamma-1, \frac{1}{1-\lambda}\right),$$

$$I_{31} = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{2\Gamma(\gamma)}\lambda^{\frac{\gamma-1}{2}}(1-\lambda)^{\frac{\alpha+\beta-\gamma+1}{2}}F(\alpha, \beta+1, \gamma, \lambda),$$

$$I_{41} = \frac{\Gamma(\alpha+1)\Gamma(\gamma-\alpha-1)}{2\Gamma(\gamma)}\lambda^{\frac{\gamma-1}{2}}(1-\lambda)^{\frac{\alpha+\beta-\gamma+1}{2}}F(\alpha+1, \beta, \gamma, \lambda),$$

where λ denotes the lambda function: $\lambda = \frac{\theta_1^4}{\theta_3^4}$. The other 12 integrals I_{ij}'s have analogous expressions, too.

2. Proof of Theorem 1.3: Derivation of differential relations

Since the integrals I_{ij} introduced in Section 1 are related to Gauss' hypergeometric function (Remark 3 in Section 1), one can of course obtain the equation (1.1) by lifting the variable λ of Gauss' hypergeometric differential equation

$$\lambda(1-\lambda)\frac{d^2F}{d\lambda^2} + \{\gamma-(\alpha+\beta+1)\lambda\}\frac{dF}{d\lambda} - \alpha\beta F = 0$$

to the variable τ via the equality $\lambda = \frac{\theta_1^4}{\theta_3^4}$. Nevertheless, we give here another proof of Theorem 1.3 based on the theory of theta functions. To derive the 16 relations in (1.1), it suffices to prove the following

Proposition 2.1 *The following formulas hold*:

$$\frac{i}{\pi\theta_3^4}\frac{\partial}{\partial\tau}(T(u)\varphi_1) \equiv a_{11}T(u)\varphi_1 + a_{12}T(u)\varphi_2 \mod B^1(M,\mathcal{L}); \tag{2.1}$$

$$\frac{i}{\pi\theta_3^4}\frac{\partial}{\partial\tau}(T(u)\varphi_2) \equiv a_{21}T(u)\varphi_1 + a_{22}T(u)\varphi_2 \mod B^1(M,\mathcal{L}); \tag{2.2}$$

$$\frac{i}{\pi\theta_3^4}\frac{\partial}{\partial\tau}(T(u)\varphi_3) \equiv b_{11}T(u)\varphi_3 + b_{12}T(u)\varphi_4 \mod B^1(M,\mathcal{L}); \tag{2.3}$$

$$\frac{i}{\pi\theta_3^4}\frac{\partial}{\partial\tau}(T(u)\varphi_4) \equiv b_{21}T(u)\varphi_3 + b_{22}T(u)\varphi_4 \mod B^1(M,\mathcal{L}), \tag{2.4}$$

where $B^1(M,\mathcal{L})$ *denotes the group of twisted* 1*-coboundaries with coefficients in* $\mathcal{L}$.

In fact, the 16 relations in (1.1) follows immediately if one integrates each of the four relations in the proposition along suitable cycles.

Proof of Proposition 2.1. We prove the formula (2.3) only, since the other formulas are proved similarly. We have

$$\begin{aligned}\frac{\partial}{\partial\tau}(T(u)\varphi_3) = \Bigg[2\frac{\theta_{3\tau}}{\theta_3} &+ (2\gamma-2\alpha-1)\left(\frac{\theta_{1\tau}(u)}{\theta_1(u)} - \frac{\theta_{2\tau}(u)}{\theta_2(u)}\right)\\ &+ (2\beta-2\alpha+2)\left(\frac{\theta_{2\tau}(u)}{\theta_2(u)} - \frac{\theta_{3\tau}(u)}{\theta_3(u)}\right)\\ &+ (1-2\alpha)\left(\frac{\theta_{3\tau}(u)}{\theta_3(u)} - \frac{\theta_{\tau}(u)}{\theta(u)}\right)\Bigg]T(u)\varphi_3,\end{aligned} \tag{2.5}$$

where $\theta_{3\tau}$ denotes $\frac{\partial\theta_3}{\partial\tau}(0,\tau)$, and $\theta_{1\tau}(u)$ denotes $\frac{\partial\theta_1}{\partial\tau}(u,\tau)$, etc. Since the four theta functions $\theta(u)$ and $\theta_i(u)$ satisfy the common partial differential equation $4\pi i\frac{\partial\Theta}{\partial\tau} = \frac{\partial^2\Theta}{\partial u^2}$, (2.5) is turned to

$$\frac{\partial}{\partial \tau}(T(u)\varphi_3) = \Bigg[2\frac{\theta_{3\tau}}{\theta_3} + \frac{2\gamma - 2\alpha - 1}{4\pi i}\left(\frac{\theta_1''(u)}{\theta_1(u)} - \frac{\theta_2''(u)}{\theta_2(u)}\right) + \frac{2\beta - 2\alpha + 2}{4\pi i}\left(\frac{\theta_2''(u)}{\theta_2(u)} - \frac{\theta_3''(u)}{\theta_3(u)}\right) + \frac{1 - 2\alpha}{4\pi i}\left(\frac{\theta_3''(u)}{\theta_3(u)} - \frac{\theta''(u)}{\theta(u)}\right)\Bigg] T(u)\varphi_3, \tag{2.6}$$

where $\theta_i''(u)$ denotes $\frac{\partial^2 \theta_i}{\partial u^2}(u,\tau)$. Now we note the following formulas: $\theta_2''(u)\theta_1(u) - \theta_1''(u)\theta_2(u) = \pi\theta_3^2(\theta'(u)\theta_3(u) + \theta_3'(u)\theta(u))$, $\theta_2''(u)\theta_3(u) - \theta_3''(u)\theta_2(u) = \pi\theta_1^2(\theta'(u)\theta_1(u) + \theta_1'(u)\theta(u))$, $\theta''(u)\theta_3(u) - \theta_3''(u)\theta(u) = \pi\theta_3^2(\theta_1'(u)\theta_2(u) + \theta_2'(u)\theta_1(u))$, where $\theta_i'(u)$ denotes $\frac{\partial \theta_i}{\partial u}(u,\tau)$. Applying these formulas to the right-hand side of (2.6), we have

$$\frac{\partial}{\partial \tau}(T(u)\varphi_3) = \Bigg[2\frac{\theta_{3\tau}}{\theta_3} - \frac{2\gamma - 2\alpha - 1}{4i}\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)}\left(\frac{\theta'(u)}{\theta(u)} + \frac{\theta_3'(u)}{\theta_3(u)}\right) + \frac{2\beta - 2\alpha + 2}{4i}\theta_1^2\frac{\theta(u)\theta_1(u)}{\theta_2(u)\theta_3(u)}\left(\frac{\theta'(u)}{\theta(u)} + \frac{\theta_1'(u)}{\theta_1(u)}\right) - \frac{1 - 2\alpha}{4i}\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}\left(\frac{\theta_1'(u)}{\theta_1(u)} + \frac{\theta_2'(u)}{\theta_2(u)}\right)\Bigg] T(u)\varphi_3. \tag{2.7}$$

Substituting the formulas:

$$\frac{\theta_3'(u)}{\theta_3(u)} = \frac{\theta_1'(u)}{\theta_1(u)} + \pi\theta_2^2\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)} \quad \text{and} \quad \frac{\theta_2'(u)}{\theta_2(u)} = \frac{\theta'(u)}{\theta(u)} - \pi\theta_2^2\frac{\theta_1(u)\theta_3(u)}{\theta(u)\theta_2(u)}$$

into the right-hand side of (2.7) and making some calculation, we have

$$\begin{aligned}
&\frac{\partial}{\partial \tau}(T(u)\varphi_3) \\
&= 2\frac{\theta_{3\tau}}{\theta_3}T(u)\varphi_3 + \frac{\theta_3^2}{4i}\Bigg[-(2\gamma - 2\alpha - 1)\pi\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)} \\
&\qquad + (2\beta - 2\alpha + 2)\pi\theta_1^2\frac{\theta(u)\theta_1(u)}{\theta_2(u)\theta_3(u)} - (1 - 2\alpha)\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}\Bigg] \\
&\quad \times \left(\frac{\theta'(u)}{\theta(u)} + \frac{\theta_1'(u)}{\theta_1(u)}\right)\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}du
\end{aligned}$$

$$+\frac{\pi^2\theta_2^2\theta_3^4}{4i}\Big[-(2\gamma-2\alpha-1)\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}+(1-2\alpha)\frac{\theta_1(u)^3\theta_2(u)}{\theta(u)^3\theta_3(u)}\Big]T(u)du. \tag{2.8}$$

Here we note the equality

$$\Big[-(2\gamma-2\alpha-1)\pi\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)}+(2\beta-2\alpha+2)\pi\theta_1^2\frac{\theta(u)\theta_1(u)}{\theta_2(u)\theta_3(u)}$$
$$-(1-2\alpha)\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}\Big]\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}$$
$$=\frac{d}{du}\big\{\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}\big\}. \tag{2.9}$$

Applying (2.9) to the right-hand side of (2.8), we have

$$\frac{\partial}{\partial\tau}(T(u)\varphi_3)$$
$$=2\frac{\theta_{3\tau}}{\theta_3}T(u)\varphi_3$$
$$+\frac{\theta_3^2}{4i}d\Big[\Big(\frac{\theta'(u)}{\theta(u)}+\frac{\theta_1'(u)}{\theta_1(u)}\Big)\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}\Big]$$
$$+\frac{\theta_3^2}{4i}\Big(\frac{\theta'(u)^2}{\theta(u)^2}-\frac{\theta''(u)}{\theta(u)}+\frac{\theta_1'(u)^2}{\theta_1(u)^2}-\frac{\theta_1''(u)}{\theta_1(u)}\Big)$$
$$\times\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}du$$
$$+\frac{\pi^2\theta_2^2\theta_3^4}{4i}\Big[-(2\gamma-2\alpha-1)\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}+(1-2\alpha)\frac{\theta_1(u)^3\theta_2(u)}{\theta(u)^3\theta_3(u)}\Big]T(u)du. \tag{2.10}$$

We note that

$$\frac{\theta'(u)^2}{\theta(u)^2}-\frac{\theta''(u)}{\theta(u)}=-4\pi i\frac{\theta_{1\tau}}{\theta_1}+\pi^2\theta_2^2\theta_3^2\frac{\theta_1(u)^2}{\theta(u)^2},$$
$$\frac{\theta_1'(u)^2}{\theta_1(u)^2}-\frac{\theta_1''(u)}{\theta_1(u)}=-4\pi i\frac{\theta_{1\tau}}{\theta_1}+\pi^2\theta_2^2\theta_3^2\frac{\theta(u)^2}{\theta_1(u)^2},$$

and

$$d\left[\left(\frac{\theta'(u)}{\theta(u)}+\frac{\theta_1'(u)}{\theta_1(u)}\right)\theta(u)^{2\alpha-1}\theta_1(u)^{2\gamma-2\alpha-1}\theta_2(u)^{2\beta-2\gamma+3}\theta_3(u)^{-2\beta-1}\right]$$
$$\in B^1(M,\mathcal{L}).$$

Then the equality (2.10) is turned to

$$\begin{aligned}\frac{\partial}{\partial\tau}&(T(u)\varphi_3)\\ &\equiv 2\left(\frac{\theta_{3\tau}}{\theta_3}-\frac{\theta_{1\tau}}{\theta_1}\right)T(u)\varphi_3\\ &\quad+\frac{\pi^2\theta_2^2\theta_3^4}{2i}\left[(1+\alpha-\gamma)\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}+(1-\alpha)\frac{\theta_1(u)^3\theta_2(u)}{\theta(u)^3\theta_3(u)}\right]T(u)du\\ &\qquad\qquad \mod B^1(M,\mathcal{L}).\qquad (2.11)\end{aligned}$$

Substituting the equalities

$$\frac{\theta_1(u)^3\theta_2(u)}{\theta(u)^3\theta_3(u)}=\frac{\theta_1^2\theta_1(u)\theta_2(u)\theta_3(u)}{\theta_3^2\theta(u)^3}-\frac{\theta_2^2\theta_1(u)\theta_2(u)}{\theta_3^2\theta(u)\theta_3(u)}$$

and

$$\frac{\theta_{3\tau}}{\theta_3}-\frac{\theta_{1\tau}}{\theta_1}=\frac{\pi}{4i}\theta_2^4$$

into the right-hand side of (2.11), we have

$$\begin{aligned}\frac{\partial}{\partial\tau}&(T(u)\varphi_3)\\ &\equiv\frac{\pi}{2i}\theta_2^4T(u)\varphi_3+\frac{\pi^2\theta_2^2\theta_3^4}{2i}\left[(1+\alpha-\gamma)\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}\right.\\ &\qquad\left.+(1-\alpha)\frac{\theta_1^2}{\theta_3^2}\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}-(1-\alpha)\frac{\theta_2^2}{\theta_3^2}\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}\right]T(u)du\\ &\qquad\qquad \mod B^1(M,\mathcal{L}).\qquad (2.12)\end{aligned}$$

We need the following

Lemma 2.2 *The following formulas hold*:

$$\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}du \equiv \frac{\alpha}{\beta-\alpha}\frac{1}{\pi\theta_2^2}\varphi_3 + \frac{\beta-\gamma+1}{\beta-\alpha}\frac{1}{\pi\theta_2^2}\varphi_4 \mod B^1(M,\mathcal{L}); \tag{2.13}$$

$$\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du \equiv \left[\frac{\beta}{1-\alpha}\frac{\theta_1^2}{\pi\theta_2^2\theta_3^2} + \frac{(\alpha-\gamma+1)\beta}{(1-\alpha)(\beta-\alpha)}\frac{\theta_3^2}{\pi\theta_1^2\theta_2^2}\right]\varphi_3 + \frac{(\alpha-\gamma+1)(\beta-\gamma+1)}{(1-\alpha)(\beta-\alpha)}\frac{\theta_3^2}{\pi\theta_1^2\theta_2^2}\varphi_4 \mod B^1(M,\mathcal{L}). \tag{2.14}$$

Proof of Lemma 2.2. The formula (2.13) follows immediately from the equality

$$\omega = (2\beta-2\gamma+2)\pi\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)}du - (2\beta-2\alpha)\pi\theta_2^2\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}du + 2\alpha\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}du.$$

Now we see that

$$\begin{aligned}\nabla\left(\frac{\theta_2(u)^2}{\theta(u)^2}\right) &= -2\pi\theta_2^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du + \frac{\theta_2(u)^2}{\theta(u)^2}\Bigg[2\alpha\pi\theta_2^2\frac{\theta_1(u)\theta_3(u)}{\theta(u)\theta_2(u)} \\ &\quad + (2\alpha+2\beta-2\gamma+2)\pi\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)} - 2\beta\pi\theta_2^2\frac{\theta(u)\theta_2(u)}{\theta_1(u)\theta_3(u)}\Bigg]du \\ &= (\alpha-1)2\pi\theta_2^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du + (\alpha+\beta-\gamma+1)2\pi\theta_3^2\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}du \\ &\quad - 2\beta\pi\theta_2^2\frac{\theta_2(u)^3}{\theta(u)\theta_1(u)\theta_3(u)}du,\end{aligned}$$

from which it follows immediately that

$$\theta_2^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du \equiv \frac{\alpha+\beta-\gamma+1}{1-\alpha}\theta_3^2\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}du - \frac{\beta}{1-\alpha}\theta_2^2\frac{\theta_2(u)^3}{\theta(u)\theta_1(u)\theta_3(u)}du \mod B^1(M,\mathcal{L}). \tag{2.15}$$

Substituting the equality

$$\frac{\theta_2^2\theta_2(u)^3}{\theta(u)\theta_1(u)\theta_3(u)} = \frac{\theta_3^2\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)} - \frac{\theta_1^2\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}$$

into the right-hand side of (2.15), we have

$$\theta_2^2\frac{\theta_1(u)\theta_2(u)\theta_3(u)}{\theta(u)^3}du \equiv \frac{\alpha-\gamma+1}{1-\alpha}\theta_3^2\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}du + \frac{\beta}{1-\alpha}\theta_1^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}du \mod B^1(M,\mathcal{L}). \tag{2.16}$$

Now we have

$$\omega = (2\beta-2\gamma+2)\pi\theta_3^2\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)}du - (2\beta-2\alpha)\pi\theta_1^2\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}du + 2\beta\pi\theta_3^2\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}du,$$

from which it follows immediately that

$$\frac{\theta_2(u)\theta_3(u)}{\theta(u)\theta_1(u)}du \equiv \frac{\beta-\gamma+1}{\beta-\alpha}\frac{\theta_3^2}{\theta_1^2}\frac{\theta(u)\theta_3(u)}{\theta_1(u)\theta_2(u)}du + \frac{\beta}{\beta-\alpha}\frac{\theta_3^2}{\theta_1^2}\frac{\theta_1(u)\theta_2(u)}{\theta(u)\theta_3(u)}du \mod B^1(M,\mathcal{L}). \tag{2.17}$$

Substituting (2.17) into the right-hand side of (2.16), we have the desired equality (2.14), which proves Lemma 2.2.

Let us return to our proof of Proposition 2.1. Substituting the two equalities (2.13) and (2.14) into the right-hand side of (2.12), we have the desired equality (2.3). Proposition 2.1 is thereby proved.

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References

[1] Chandrasekharan, K., *Elliptic functions*, Springer, 1985.

[2] Watanabe, H., Transformation relations of matrix functions associated to the hypergeometric function of Gauss under modular transformations, J. Math. Soc. Japan **59** (2007), 113–126.

[3] Watanabe, H., Twisted homology and cohomology groups associated to the Wirtinger integral, J. Math. Soc. Japan **59** (2007), 1067–1080.

[4] Wirtinger, W., Zur Darstellung der hypergeometrischen Function durch bestimmte Integrale, Akad. Wiss. Wien. S.-B. IIa, **111** (1902), 894–900.

[5] Aomoto, K. and Kita, M., Hypergeometric functions, Springer-Verlag, Tokyo, 1994 (in Japanese).

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