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# CMO estimates for higher-order commutators of integral operators with rough kernels

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**Abstract.** In this paper, we establish the CMO estimate for the higher-order commutator of Carlderón-Zygmund singular integral with rough kernel. This estimate enables us to get the CMO estimates for the higher-order commutators of a class of Marcinkiewicz integral operators $\mu_\Omega$, $\mu^*_{\Omega,\lambda}$ and $\mu_{\Omega,S}$ with rough kernels, which are corresponding to the Littlewood-Paley $g$ function, Littlewood-Paley $g^*_\lambda$ function and the Lusin area integral, respectively.

*Key words*: Commutator, Carlderón-Zygmund singular integral, Marcinkiewicz integral, Littlewood–Paley operator, CMO function, Herz space.

## 1. Introduction

Suppose that $S^{n-1}$ is the unit sphere of $\mathbb{R}^n (n \geq 2)$ equipped with normalized Lebesgue measure. Let $\Omega \in L^1(S^{n-1})$ be homogeneous of degree zero and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$. In 1958, Stein [15] introduced the Marcinkiewicz integral as follows.

$$\mu_\Omega(f)(x) = \left( \int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\, dy,$$

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and proved that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$, then $\mu_\Omega$ is of type $(p,p)$ for $1 < p \le 2$ and of weak type $(1,1)$. In [1], Benedek et al. proved that if $\Omega \in C^1(S^{n-1})$, then $\mu_\Omega$ is of type $(p,p)$ for $1 < p < \infty$. Ding et al. [3] improved the above results to the case $\Omega \in L^r(S^{n-1})$.

Let $m$ be a non-negative integer, $\vec{b} = (b_1, b_2, \ldots, b_m)$, $b_i (i = 1, 2, \ldots, m)$ be a locally integrable function. The higher-order commutators $T_\Omega^{\vec{b}}$, $\mu_\Omega^{\vec{b}}$, $\mu_{\Omega,\lambda}^{*,\vec{b}}$ and $\mu_{\Omega,S}^{\vec{b}}$ are defined respectively as follows:

$$T_\Omega^{\vec{b}}(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \prod_{i=1}^m (b_i(x) - b_i(y)) f(y)\, dy, \tag{1.2}$$

$$\mu_\Omega^{\vec{b}}(f)(x) = \left( \int_0^\infty \left| F_{\Omega,t}^{\vec{b}}(x) \right|^2 \frac{dt}{t^3} \right)^{1/2}, \tag{1.3}$$

$$\mu_{\Omega,\lambda}^{*,\vec{b}}(f)(x) = \left( \iint_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t + |x-y|} \right)^{n\lambda} \left| F_{\Omega,t}^{\vec{b}}(y) \right|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}, \quad \lambda > 1 \tag{1.4}$$

and

$$\mu_{\Omega,S}^{\vec{b}}(f)(x) = \left( \iint_{\Gamma(x)} \left| F_{\Omega,t}^{\vec{b}}(y) \right|^2 \frac{dydt}{t^{n+3}} \right)^{1/2}, \tag{1.5}$$

where

$$F_{\Omega,t}^{\vec{b}}(x) = \int_{|x-y| \le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \prod_{i=1}^m (b_i(x) - b_i(y)) f(y)\, dy$$

and $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}$.

Obviously, when $m = 0$, $T_\Omega^{\vec{b}} = T_\Omega$, which is the Calderón-Zygmund singular integral operator, and if $m = 1$, $T_\Omega^{\vec{b}} = [b, T_\Omega]$, which is the well known commutator generated by $b$ and $T_\Omega$.

**Definition 1.1** ([6], [10]) Let $1 \le q < \infty$. The space $\dot{\mathrm{CMO}}^q(\mathbb{R}^n)$ is defined by

$$\dot{\mathrm{CMO}}^q(\mathbb{R}^n) = \left\{ f \in L_{\mathrm{loc}}^q(\mathbb{R}^n) : \|f\|_{\dot{\mathrm{CMO}}^q} < \infty \right\},$$

where

$$\|f\|_{\dot{\mathrm{CMO}}^q} = \sup_{\mathrm{R}>0}\left(\frac{1}{|B(0,\mathrm{R})|}\int_{B(0,\mathrm{R})}|f(x)-f_{\mathrm{B}(0,\mathrm{R})}|^q dx\right)^{1/q}$$

and $f_{B(0,\mathrm{R})} = \frac{1}{|B(0,\mathrm{R})|}\int_{B(0,\mathrm{R})} f(x)dx$.

**Remark 1.1** It is easy to see that $\mathrm{BMO}(\mathbb{R}^n) \subsetneqq \dot{\mathrm{CMO}}^p(\mathbb{R}^n)$ and $\dot{\mathrm{CMO}}^q(\mathbb{R}^n) \subsetneqq \dot{\mathrm{CMO}}^p(\mathbb{R}^n)$ for any $1 \le p < q < \infty$.

**Remark 1.2** $\dot{\mathrm{CMO}}^q(\mathbb{R}^n)$ is a Banach space and the dual of $H\dot{K}_{q'}^{n(1-1/q),1}(\mathbb{R}^n)$ with $1/q+1/q' = 1, 1 < q < \infty$, where $H\dot{K}_{q'}^{n(1-1/q),1}(\mathbb{R}^n)$ is the Herz–type Hardy space [12].

**Definition 1.2** ([11]) Let $B_k = \{x \in \mathbb{R}^n : |x| < 2^k\}$, $C_k = B_k \backslash B_{k-1}$ and $\chi_k = \chi_{C_k}$ be the characteristic function of the set $C_k$ for $k \in \mathbb{Z}$. For $\alpha \in \mathbb{R}$, $0 < p < \infty, 0 < q \le \infty$, the homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \left\{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty\right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f\chi_k\|_q^p\right)^{1/p},$$

with usual modification made when $p = \infty$.

**Remark 1.3** $\dot{K}_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $\dot{K}_p^{\alpha/p,p}(\mathbb{R}^n) = L^p(|x|^\alpha dx)$ for $0 < p \le \infty$ and $\alpha \in \mathbb{R}$.

A famous result of Coifman, Rochderg and Weiss[2] is that when $\Omega$ is smooth, the commutator $[b, T_\Omega]$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Because of $\mathrm{BMO}(\mathbb{R}^n) \subsetneqq \dot{\mathrm{CMO}}^q(\mathbb{R}^n)$, the commutator $[b, T_\Omega]$ may not be a bounded operator on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ if $b \in \dot{\mathrm{CMO}}^q(\mathbb{R}^n)$. But Grafakos et al. [7] found the fact that $[b, T_\Omega]$ is bounded on homogenous Herz spaces when $\Omega$ is smooth and $b \in \dot{\mathrm{CMO}}^q(\mathbb{R}^n)$. Lu and Wu [13] improved this result to the case of $\Omega \in L^r(S^{n-1})$ for $r > 1$.

Recently, Ding et al. [4] obtained the weighted $L^p$ boundedness for the higher-order commutators of Marcinkiewicz integrals with $\Omega \in L^r(S^{n-1})$. Inspired by [4], [7] and [13], we establish the $\dot{\mathrm{CMO}}(\mathbb{R}^n)$ estimates for the higher-order commutators $T_{\Omega}^{\vec{b}}$, $\mu_{\Omega}^{\vec{b}}$, $\mu_{\Omega,\lambda}^{*,\vec{b}}$ and $\mu_{\Omega,S}^{\vec{b}}$ with $\Omega \in L^r(S^{n-1})$. Throughout this paper, $C$ denotes a constant independent by the essential variables.

## 2. Auxiliary and main results

To prove our main results, we introduce the following lemmas.

**Lemma 2.1** ([5]) *$b \in \dot{\mathrm{CMO}}^1(\mathbb{R}^n)$ and $i, k \in \mathbb{Z}$. Then*

$$|b(t) - b_{B_k}| \leq |b(t) - b_{B_i}| + C|i-k|\|b\|_{\dot{\mathrm{CMO}}^1}. \tag{2.1}$$

**Lemma 2.2** ([14]) *Suppose $\Omega \in L^r(S^{n-1})$, $1 < r \leq \infty$. If $a > 0$, $0 < d \leq r$ and $-n + (n-1)d/r < \beta < \infty$, then*

$$\left(\int_{|x|<a|y|} |\Omega(x-y)|^d |x|^\beta \, dx\right)^{1/d} \leq C|y|^{(\beta+n)/d}\|\Omega\|_{L^r}.$$

**Lemma 2.3** ([8]) *The dual space of $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is $\dot{K}_{q'}^{-\alpha,p'}(\mathbb{R}^n)$, where $1/p + 1/p' = 1$, $1/q + 1/q' = 1$ with $1 < p, q < \infty$.*

**Lemma 2.4** ([9]) *If $-n/q < \alpha < n(1-1/q)$, $0 < p \leq \infty$, $1 < q < \infty$, then the Hardy-Littlewood maximal operator $M$ is bounded on homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$.*

Using the similar method of proving Lemma 2 in [3], we can easily get

**Lemma 2.5** *For any non-negative function $\phi$, there exists a constant $C_\lambda > 0$ such that*

$$\int_{\mathbb{R}^n} \left(\mu_{\Omega,\lambda}^{*,\vec{b}} f(x)\right)^2 \phi(x) dx \leq C_\lambda \int_{\mathbb{R}^n} \left(\mu_{\Omega}^{\vec{b}} f(x)\right)^2 M\phi(x) dx.$$

The main results in this paper are the following theorems.

**Theorem 2.1** *Let $T_{\Omega}^{\vec{b}}$ be defined as (1.2), $\vec{b} = (b_1, b_2, \ldots, b_m)$, $b_i \in \dot{\mathrm{CMO}}^{s_i}(\mathbb{R}^n)$, $1 < s_i < \infty$, $i = 1, 2, \ldots, m$, $m \in \mathbb{N}$ and $\Omega \in L^r(S^{n-1}), 1 < r \leq \infty$. If $0 < p \leq \infty$, $1 < q_1, q_2 < \infty$, $1/q_2 = 1/q_1 + \sum_{i=1}^m (1/s_i)$, $\alpha_1$ satisfies either of the following two conditions:*

( i ) $-n/q_1 < \alpha_1 < n(1/r' - 1/q_1) + 1/r$ *when* $r' \leq q_2 < \infty$,

( ii ) $-n(1/q_1 - 1/r) - 1/r < \alpha_1 < n/q_1'$ *when* $1 < q_1 \le r$,

*and* $\alpha_2 = \alpha_1 - n\sum_{i=1}^{m}(1/s_i)$, *then*

$$\|T_\Omega^{\vec{b}}(f)\|_{\dot{K}_{q_2}^{\alpha_2,p}} \le C\prod_{i=1}^{m}\|b_i\|_{\dot{\mathrm{CMO}}^{s_i}}\|f\|_{\dot{K}_{q_1}^{\alpha_1,p}}. \tag{2.2}$$

**Theorem 2.2** *Under the assumptions in Theorem* 2.1. *Let* $\mu_\Omega^{\vec{b}}$ *be defined as in* (1.3). *Then*

$$\|\mu_\Omega^{\vec{b}}(f)\|_{\dot{K}_{q_2}^{\alpha_2,p}} \le C\prod_{i=1}^{m}\|b_i\|_{\dot{\mathrm{CMO}}^{s_i}}\|f\|_{\dot{K}_{q_1}^{\alpha_1,p}}. \tag{2.3}$$

**Theorem 2.3** *Let* $\mu_{\Omega,\lambda}^{*,\vec{b}}$ *be defined as in* (1.4) *and* $\mu_{\Omega,S}^{\vec{b}}$ *in* (1.5), *respectively,* $\vec{b} = (b_1, b_2, \ldots, b_m)$, $b_i \in \dot{\mathrm{CMO}}^{s_i}(\mathbb{R}^n)$, $1 < s_i < \infty$, $i = 1, 2, \ldots, m$, $m \in \mathbb{N}$ *and* $\Omega \in L^r(S^{n-1})$, $1 < r \le \infty$. *If* $2 < p, q_1, q_2 < \infty$, $1/q_2 = 1/q_1 + \sum_{i=1}^{m}(1/s_i)$, $\alpha_1$ *satisfies either of the following two conditions*:

( i ) $-n/q_1 < \alpha_1 < n(1/2 - 1/q_1)$ *when* $\max\{2, r'\} \le q_2 < \infty$,
( ii ) $-n(1/q_1 - 1/r) - 1/r < \alpha_1 < n(1/2 - 1/q_1)$, *when* $2 < \max\{2(1 - 1/n), q_1\} \le r$, *and* $\alpha_2 = \alpha_1 - n\sum_{i=1}^{m}(1/s_i)$, *then*

$$\|\mu_{\Omega,\lambda}^{*,\vec{b}}(f)\|_{\dot{K}_{q_2}^{\alpha_2,p}} \le C\prod_{i=1}^{m}\|b_i\|_{\dot{\mathrm{CMO}}^{s_i}}\|f\|_{\dot{K}_{q_1}^{\alpha_1,p}}, \tag{2.4}$$

*and*

$$\|\mu_{\Omega,S}^{\vec{b}}(f)\|_{\dot{K}_{q_2}^{\alpha_2,p}} \le C\prod_{i=1}^{m}\|b_i\|_{\dot{\mathrm{CMO}}^{s_i}}\|f\|_{\dot{K}_{q_1}^{\alpha_1,p}}. \tag{2.5}$$

These results have corresponding inhomogeneous counterparts. More precisely, Theorem 2.1, Theorem 2.2 and Theorem 2.3 are still true if we replace $\dot{\mathrm{CMO}}(\mathbb{R}^n)$ and $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ by their inhomogeneous version $\mathrm{CMO}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$, respectively. We refer to [6] and [11] or the related references there for the definitions and properties of these inhomogeneous spaces.

## 3. Proof of Theorems

### 3.1. Proof of Theorem 2.1

Without loss of generality, we may assume $m = 2$ and $\prod_{i=1}^{2} \|b_i\|_{\dot{\mathrm{CMO}}^{s_i}} = 1$. Set $f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x)$, we get

$$
\begin{aligned}
\|T_\Omega^{\vec{b}}(f)\|_{\dot{K}_{q_2}^{\alpha_2,p}} \le\ & C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\Big(\sum_{j=-\infty}^{k-2} \|\chi_k T_\Omega^{\vec{b}}(f_j)\|_{q_2}\Big)^p\Big\}^{1/p} \\
& + C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\Big(\sum_{j=k-1}^{k+1} \|\chi_k T_\Omega^{\vec{b}}(f_j)\|_{q_2}\Big)^p\Big\}^{1/p} \\
& + C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\Big(\sum_{j=k+2}^{\infty} \|\chi_k T_\Omega^{\vec{b}}(f_j)\|_{q_2}\Big)^p\Big\}^{1/p} \\
:=\ & C(I_1 + I_2 + I_3).
\end{aligned}
$$

For $I_2$, set $\lambda_i = \frac{1}{|C_j|}\int_{C_j} b_i(x)\,dx$ with $i = 1, 2$, applying Hölder's inequality and the facts that the operator $T_\Omega$ is bounded on $L^q$ $(1 < q < \infty)$ and $|C_j| \sim |C_k|$ with $j = k-1, k, k+1$, we have

$$
\begin{aligned}
\|T_\Omega^{\vec{b}}(f_j)\chi_k\|_{q_2} &= \Big\|\chi_k(\cdot)\int_{\mathbb{R}^n} \frac{\Omega(\cdot - y)}{|\cdot - y|^n}\prod_{i=1}^{2}(b_i(\cdot) - b_i(y)) f_j(y)\,dy\Big\|_{q_2} \\
&\le C\Big\|\chi_k \prod_{i=1}^{2}(b_i - \lambda_i) T_\Omega(f_j)\Big\|_{q_2} + C\Big\|\chi_k T_\Omega\Big(\prod_{i=1}^{2}(b_i - \lambda_i) f_j\Big)\Big\|_{q_2} \\
&\quad + C\|\chi_k (b_1 - \lambda_1) T_\Omega((b_2 - \lambda_2) f_j)\|_{q_2} \\
&\quad + C\|\chi_k (b_2 - \lambda_2) T_\Omega((b_1 - \lambda_1) f_j)\|_{q_2} \\
&\le C\|T_\Omega(f_j)\|_{q_1}\Big\|\chi_k \prod_{i=1}^{2}(b_i - \lambda_i)\Big\|_{\frac{s_1 s_2}{s_1 + s_2}} + C\Big\|\prod_{i=1}^{2}(b_i - \lambda_i) f_j\Big\|_{q_2} \\
&\quad + C\|\chi_k(b_1 - \lambda_1)\|_{s_1}\|(b_2 - \lambda_2) f_j\|_{\frac{q_1 s_2}{q_1 + s_2}} \\
&\quad + C\|\chi_k(b_2 - \lambda_2)\|_{s_2}\|(b_1 - \lambda_1) f_j\|_{\frac{q_1 s_1}{q_1 + s_1}} \\
&\le C 2^{nk(1/s_1 + 1/s_2)}\|f_j\|_{q_1}.
\end{aligned}
$$

Thus we arrive at

$$
\begin{aligned}
I_2 &\le C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\Big(\sum_{j=k-1}^{k+1} 2^{nk(1/s_1+1/s_2)}\|f_j\|_{q_1}\prod_{i=1}^{2}\|b_i\|_{\dot{\mathrm{CMO}}^{s_i}}\Big)^p\Big\}^{1/p}\\
&\le C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p}\Big(\sum_{j=k-1}^{k+1}\|f_j\|_{q_1}\Big)^p\Big\}^{1/p}\\
&\le C\Big\{\sum_{k=-\infty}^{\infty} 2^{k\alpha_1 p}\|f_k\|_{q_1}^p\Big\}^{1/p} = C\|f\|_{\dot{K}_{q_1}^{\alpha_1,p}}.
\end{aligned}
$$

For $I_1$, we assume (i) holds and note that if $x\in C_k, y\in C_j$ and $j\le k-2$, then $|x-y|\sim|x|$ and $\Big|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\Big|\le C\frac{|y|}{|x-y|^3}$, we have

$$
\begin{aligned}
&\big\|\chi_k T_\Omega^{\vec{b}}(f_j)\big\|_{q_2}\\
&\le C\Big\{\int_{C_k}\frac{1}{|x|^{nq_2}}\Big|\int_{C_j}\Omega(x-y)\prod_{i=1}^{2}(b_i(x)-b_i(y))f(y)\,dy\Big|^{q_2}dx\Big\}^{1/q_2}\\
&\le C2^{-nk}\Big\{\int_{C_k}\Big(\int_{C_j}|\Omega(x-y)||f(y)|\prod_{i=1}^{2}|b_i(x)-\lambda_i|\,dy\Big)^{q_2}dx\Big\}^{1/q_2}\\
&\quad + C2^{-nk}\Big\{\int_{C_k}\Big(\int_{C_j}|\Omega(x-y)||f(y)||b_1(x)-\lambda_1||b_2(y)-\lambda_2|\,dy\Big)^{q_2}dx\Big\}^{1/q_2}\\
&\quad + C2^{-nk}\Big\{\int_{C_k}\Big(\int_{C_j}|\Omega(x-y)||f(y)||b_2(x)-\lambda_2||b_1(y)-\lambda_1|\,dy\Big)^{q_2}dx\Big\}^{1/q_2}\\
&\quad + C2^{-nk}\Big\{\int_{C_k}\Big(\int_{C_j}|\Omega(x-y)||f(y)|\prod_{i=1}^{2}|b_i(y)-\lambda_i|\,dy\Big)^{q_2}dx\Big\}^{1/q_2}\\
&:= U_{11}+U_{12}+U_{13}+U_{14}.
\end{aligned}
$$

Observe that $r'<q_1<\infty$ and choose $\beta<0$ such that $\alpha_1<-\beta+n(1/r'-1/q_1)<1/r+n(1/r'-1/q_1)$. From Lemma 2.1 and Lemma 2.2, it follows that

$$
\begin{aligned}
U_{11} \le C2^{-nk}2^{-j\beta}\bigg\{ \int_{C_k} \prod_{i=1}^{2} |b_i(x)-\lambda_i|^{q_2} \bigg( \int_{C_j} |\Omega(x-y)|^r |y|^{\beta r}\, dy \bigg)^{q_2/r} \\
\times \bigg( \int_{C_j} |f(y)|^{r'}\, dy \bigg)^{q_2/r'} dx \bigg\}^{1/q_2} \\
\le C2^{-nk}2^{-j\beta+k\beta+kn/r}2^{nj(1/r'-1/q_1)} \bigg\{ \int_{C_k} \prod_{i=1}^{2} |b_i(x)-\lambda_i|^{q_2}\, dx \bigg\}^{1/q_2} \|f_j\|_{q_1} \\
\le C(k-j)^2 2^{(k-j)(\alpha_1+\beta-n/r'+n/q_1)}2^{-k\alpha_2}2^{j\alpha_1}\|f_j\|_{q_1}.
\end{aligned}
$$

By Hölder's inequality, we obtain

$$
\begin{aligned}
U_{12} \le C2^{-nk}2^{-j\beta}\bigg\{ \int_{C_k} |b_1(x)-\lambda_1|^{q_2} \bigg( \int_{C_j} |\Omega(x-y)|^r |y|^{\beta r}\, dy \bigg)^{q_2/r} \\
\times \bigg( \int_{C_j} |f(y)|^{r'}|b_2(y)-\lambda_2|^{r'}\, dy \bigg)^{q_2/r'} dx \bigg\}^{1/q_2} \\
\le C2^{k(\beta-n/r')+nj(1/r'-1/q_1-1/s_2)-j\beta}\|f_j\|_{q_1} \bigg( \int_{C_j} |b_2(y)-\lambda_2|^{s_2} dy \bigg)^{1/s_2} \\
\times \bigg( \int_{C_k} |b_1(x)-\lambda_1|^{q_2} dx \bigg)^{1/q_2} \\
\le C(k-j)2^{(k-j)(\alpha_1+\beta-n/r'+n/q_1)}2^{-k\alpha_2}2^{j\alpha_1}\|f_j\|_{q_1}.
\end{aligned}
$$

Using the similar method, we get

$$
U_{13} \le C(k-j)2^{(k-j)(\alpha_1+\beta-n/r'+n/q_1)}2^{-k\alpha_2}2^{j\alpha_1}\|f_j\|_{q_1}.
$$

By Hölder's inequality, we have

$$
\begin{aligned}
U_{14} \le C2^{-nk}\bigg\{ \int_{C_k} \bigg( \int_{C_j} |\Omega(x-y)|^r\, dy \bigg)^{q_2/r} \\
\times \bigg( \int_{C_j} |f(y)|^{r'} \prod_{i=1}^{2} |b_i(y)-\lambda_i|^{r'}\, dy \bigg)^{q_2/r'} dx \bigg\}^{1/q_2}
\end{aligned}
$$

$$\begin{aligned}&\leq C2^{-j\beta+k(\beta-n/r')}2^{nk/q_2}\|f_j\|_{q_1}\\&\quad\times\Big(\int_{C_j}\prod_{i=1}^{2}|b_i(y)-\lambda_i|^{q_1r'/(q_1-r')}\,dy\Big)^{1/r'-1/q_1}\\&\leq C2^{(k-j)(\alpha_1+\beta-n/r'+n/q_1)}2^{-k\alpha_2}2^{j\alpha_1}\|f_j\|_{q_1}.\end{aligned}$$

Thus we arrive at

$$\begin{aligned}I_1&\leq C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{j\alpha_1}(k-j)^2 2^{(k-j)(\alpha_1+\beta-n/r'+n/q_1)}\|f_j\|_{q_1}\Big)^p\Big\}^{1/p}\\&\leq C\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha_1 p}\|f_j\|_{q_1}^p\Big\}^{1/p}=C\|f\|_{\dot K_{q_1}^{\alpha_1,p}}.\end{aligned}$$

To estimate $I_3$, it is observed that $|x-y|\sim|y|$ and $\Big|\frac{1}{|x-y|^2}-\frac{1}{|x|^2}\Big|\leq C\frac{|x|}{|x-y|^3}$ for any $x\in C_k$ and $y\in C_j$ with $j\geq k+2$. Similar to the above estimates, we have

$$\begin{aligned}&\big\|\chi_k T_\Omega^{\vec b}(f_j)\big\|_{q_2}\\&\leq C2^{-nj}\Big\{\int_{C_k}\Big(\int_{C_j}|\Omega(x-y)||f(y)|\prod_{i=1}^{2}|b_i(x)-\lambda_i|\,dy\Big)^{q_2}dx\Big\}^{1/q_2}\\&\quad+C2^{-nj}\Big\{\int_{C_k}\Big(\int_{C_j}|\Omega(x-y)||f(y)||b_1(x)-\lambda_1||b_2(y)-\lambda_2|\,dy\Big)^{q_2}dx\Big\}^{1/q_2}\\&\quad+C2^{-nj}\Big\{\int_{C_k}\Big(\int_{C_j}|\Omega(x-y)||f(y)||b_2(x)-\lambda_2||b_1(y)-\lambda_1|\,dy\Big)^{q_2}dx\Big\}^{1/q_2}\\&\quad+C2^{-nj}\Big\{\int_{C_k}\Big(\int_{C_j}|\Omega(x-y)||f(y)|\prod_{i=1}^{2}|b_i(y)-\lambda_i|\,dy\Big)^{q_2}dx\Big\}^{1/q_2}\\&:=C(U_{31}+U_{32}+U_{33}+U_{34}).\end{aligned}$$

Similar to the estimates of $U_{11}$, $U_{12}$, $U_{13}$ and $U_{14}$, by Hölder's inequality and Lemma 2.1, we can get the estimates of $U_{31}$, $U_{32}$, $U_{33}$ and $U_{34}$, respectively. Thus we can finish the estimate for $I_3$ immediately. Here we omit the details.

Let's turn to the case (ii) now. By Lemma 2.1, Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}
U_{11} &\le C2^{-nk}\int_{C_j}\bigg(\int_{C_k}|\Omega(x-y)|^{q_2}|f(y)|^{q_2}\prod_{i=1}^{2}|b_i(x)-\lambda_i|^{q_2}\,dx\bigg)^{1/q_2}dy\\
&\le C2^{-nk}\int_{C_j}\bigg(\int_{C_k}|\Omega(x-y)|^{r}\,dx\bigg)^{1/r}\\
&\quad\times\bigg(\int_{C_k}\prod_{i=1}^{2}|b_i(x)-\lambda_i|^{rq_2/(r-q_2)}\,dx\bigg)^{1/q_2-1/r}|f(y)|\,dy\\
&\le C(k-j)^2 2^{(k-j)(\alpha_1-n/q_1{}')}2^{-k\alpha_2}2^{j\alpha_1}\|f_j\|_{q_1},
\end{aligned}$$

and

$$\begin{aligned}
U_{12} &\le C2^{-nk}\int_{C_j}\bigg(\int_{C_k}|\Omega(x-y)|^{r}\,dx\bigg)^{1/r}\\
&\quad\times\bigg(\int_{C_k}|b_1(x)-\lambda_1|^{rq_2/(r-q_2)}\,dx\bigg)^{1/q_2-1/r}|f(y)||b_2(y)-\lambda_2|\,dy\\
&\le C(k-j)2^{(k-j)(\alpha_1-n/q_1{}')}2^{j\alpha_1}2^{-k\alpha_2}\|f_j\|_{q_1}.
\end{aligned}$$

Similarly, we can obtain

$$U_{13}\le C(k-j)2^{(k-j)(\alpha_1-n/q_1{}')}2^{-k\alpha_2}2^{j\alpha_1}\|f_j\|_{q_1}$$

and

$$U_{14}\le C2^{(k-j)(\alpha_1-n/q_1{}')}2^{-k\alpha_2}2^{j\alpha_1}\|f_j\|_{q_1}.$$

Thus we arrive at

$$I_1\le C\bigg\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\bigg(\sum_{j=-\infty}^{k-2}(k-j)^2 2^{(k-j)(\alpha_1-n/q_1{}')-k\alpha_2+j\alpha_1}\|f_j\|_{q_1}\bigg)^p\bigg\}^{1/p}$$

$$\le C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k-2}2^{j\alpha_1}(k-j)^2 2^{(k-j)(\alpha_1-n/q_1{}')}\|f_j\|_{q_1}\Big)^p\Big\}^{1/p}$$

$$\le C\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha_1 p}\|f_j\|_{q_1}^p\Big\}^{1/p} = C\|f\|_{\dot{K}_{q_1}^{\alpha_1,p}}.$$

To estimate $I_3$, we choose $\gamma < 0$ such that $\alpha_1 > \gamma + n/r - n/q_1 > -1/r + n/r - n/q_1$, we have

$$\begin{aligned} U_{31} &\le C2^{-nj}2^{-k\gamma}\int_{C_j}\Big(\int_{C_k}|\Omega(x-y)|^r|x|^{r\gamma}\,dx\Big)^{1/r}|f(y)|\,dy \\ &\quad\times\Big(\int_{C_k}|b_1(x)-\lambda_1|^{s_1}\,dx\Big)^{1/s_1} \\ &\quad\times\Big(\int_{C_k}|b_2(x)-\lambda_2|^{s_1q_2r/(s_1r-s_1q_2-q_2r)}\,dx\Big)^{1/q_2-1/r-1/s_1} \\ &\le C(j-k)^2 2^{(k-j)(\alpha_1-\gamma+n/q_1-n/r)}2^{-k\alpha_2}2^{j\alpha_1}\|f_j\|_{q_1}. \end{aligned}$$

Similarly, we can get the estimates of $U_{32}$, $U_{33}$ and $U_{34}$, respectively. Thus we arrive at

$$\begin{aligned} I_3 &\le C\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p} \\ &\qquad\times\Big(\sum_{j=k+2}^{\infty}(j-k)^2 2^{(k-j)(\alpha_1-\gamma+n/q_1-n/r)-k\alpha_2+j\alpha_1}\|f_j\|_{q_1}\Big)^p\Big\}^{1/p} \\ &\le C\Big\{\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k+2}^{\infty}2^{j\alpha_1}(j-k)^2 2^{(k-j)(\alpha_1-\gamma+n/q_1-n/r)}\|f_j\|_{q_1}\Big)^p\Big\}^{1/p} \\ &\le C\Big\{\sum_{j=-\infty}^{\infty}2^{j\alpha_1 p}\|f_j\|_{q_1}^p\Big\}^{1/p} = C\|f\|_{\dot{K}_{q_1}^{\alpha_1,p}}. \end{aligned}$$

Combining the estimates of $I_1$, $I_2$ and $I_3$, the proof of Theorem 2.1 is completed. $\square$

### 3.2. Proof of Theorem 2.2

It is easy to see that

$$\begin{aligned}\|\mu_\Omega^{\vec{b}}(f)\|_{\dot{K}_{q_2}^{\alpha_2,p}} &\le C\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\Big(\sum_{j=-\infty}^{k-2}\|\mu_\Omega^{\vec{b}}(f_j)\chi_k\|_{q_2}\Big)^p\Big\}^{1/p}\\ &\quad + C\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\Big(\sum_{j=k-1}^{k+1}\|\mu_\Omega^{\vec{b}}(f_j)\chi_k\|_{q_2}\Big)^p\Big\}^{1/p}\\ &\quad + C\Big\{\sum_{k=-\infty}^{\infty}2^{k\alpha_2 p}\Big(\sum_{j=k+2}^{\infty}\|\mu_\Omega^{\vec{b}}(f_j)\chi_k\|_{q_2}\Big)^p\Big\}^{1/p}\\ &:= C(J_1+J_2+J_3).\end{aligned}$$

Let us estimate $J_2$ now. Similar to the estimate for $I_2$ in Theorem 2.1, we get

$$\begin{aligned}&\|\mu_\Omega^{\vec{b}}(f_j)\chi_k\|_{q_2}\\ &\le \Big\|\chi_k(\cdot)\Big(\int_0^\infty\Big|\int_{|\cdot-y|<t}\frac{\Omega(\cdot-y)}{|\cdot-y|^{n-1}}\prod_{i=1}^2(b_i(\cdot)-b_i(y))f_j(y)\,dy\Big|^2\frac{dt}{t^3}\Big)^{1/2}\Big\|_{q_2}\\ &\le \Big\|\chi_k(\cdot)\prod_{i=1}^2(b_i(\cdot)-\lambda_i)\mu_\Omega(f_j)(\cdot)\Big\|_{q_2}\\ &\quad+\|\chi_k(\cdot)|b_1(\cdot)-\lambda_1|\mu_\Omega((b_2(y)-\lambda_2)f_j)(\cdot)\|_{q_2}\\ &\quad+\|\chi_k(\cdot)|b_2(\cdot)-\lambda_2|\mu_\Omega((b_1(y)-\lambda_1)f_j)(\cdot)\|_{q_2}\\ &\quad+\Big\|\chi_k(\cdot)\mu_\Omega\Big(\prod_{i=1}^2(b_i(y)-\lambda_i)f_j\Big)(\cdot)\Big\|_{q_2}\\ &\le C\|\mu_\Omega(f_j)\|_{q_1}\Big\|\chi_k\prod_{i=1}^2(b_i-\lambda_i)\Big\|_{\frac{s_1s_2}{s_1+s_2}}+C\|\chi_k|b_1-\lambda_1|\|_{s_1}\|(b_2-\lambda_2)f_j\|_{\frac{q_1s_2}{q_1+s_2}}\\ &\quad+C\|\chi_k|b_2-\lambda_2|\|_{s_2}\|(b_1-\lambda_1)f_j\|_{\frac{q_1s_1}{q_1+s_1}}+C\Big\|\prod_{i=1}^2((b_i-\lambda_i)f_j)\Big\|_{q_2}\\ &\le C2^{nk(1/s_1+1/s_2)}\|f_j\|_{q_1}.\end{aligned}$$

Thus

$$J_2 \le C\Big\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha_2 p}\Big\{ \sum_{j=k-1}^{k+1} 2^{nk(1/s_1+1/s_2)}\|f_j\|_{q_1}^p \Big\}\Big\}^{1/p}$$

$$= C\Big\{ \sum_{k=-\infty}^{\infty} \Big( \sum_{j=k-1}^{k+1} 2^{k\alpha_1 p}\|f_j\|_{q_1}^p \Big)\Big\}^{1/p} = C\|f\|_{\dot{K}_{q_1}^{\alpha_1,p}}.$$

For $J_1$ and $J_3$, we only need to note the following fact

$$\mu_\Omega^{\vec{b}}(f_j)(x) = \left( \int_0^\infty \left| \frac{1}{t} \int_{|x-y|<t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \prod_{i=1}^2 (b_i(x)-b_i(y))f_j(y)dy \right|^2 \frac{dt}{t} \right)^{1/2}$$

$$\le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1}} \prod_{i=1}^2 |b_i(x)-b_i(y)||f_j(y)| \left( \int_{|x-y|}^\infty \frac{1}{t^3} dt \right)^{1/2} dy$$

$$= \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n}} \prod_{i=1}^2 |b_i(x)-b_i(y)||f_j(y)|dy.$$

Repeating the process of the estimates for $I_1$ and $I_3$ in the proof of Theorem 2.1, we can obtain the desired result. □

### 3.3. Proof of Theorem 2.3

Since $\mu_{\Omega,S}^{\vec{b}}(f)(x) \le C_\lambda \mu_{\Omega,\lambda}^{*,\vec{b}}(f)(x)$, see [4], we only need to consider the operator $\mu_{\Omega,\lambda}^{*,\vec{b}}$. For $p$, $q_2 > 2$, we denote $u = p/2$ and $v = q_2/2$, it is easy to note that $u$, $v > 1$. Then by Lemma 2.3, we have

$$\big\|\mu_{\Omega,\lambda}^{*,\vec{b}}(f)\big\|_{\dot{K}_{q_2}^{\alpha_2,p}}^2 = \sup_{\|\phi\|_{\dot{K}_{v'}^{-2\alpha_2,u'}} \le 1} \left| \int_{\mathbb{R}^n} \Big(\mu_{\Omega,\lambda}^{*,\vec{b}}(f)(x)\Big)^2 \phi(x)dx \right|$$

$$\le C_\lambda \sup_{\|\phi\|_{\dot{K}_{v'}^{-2\alpha_2,u'}} \le 1} \left| \int_{\mathbb{R}^n} \Big(\mu_{\Omega}^{\vec{b}}(f)(x)\Big)^2 M\phi(x)dx \right|$$

$$\le C_\lambda \big\|\mu_{\Omega}^{\vec{b}}(f)\big\|_{\dot{K}_{q_2}^{\alpha_2,p}}^2 \sup_{\|\phi\|_{\dot{K}_{v'}^{-2\alpha_2,u'}} \le 1} \|M\phi\|_{\dot{K}_{v'}^{-2\alpha_2,u'}}.$$

This estimate, Lemma 2.4 and Theorem 2.2 give the proof of Theorem 2.3. □

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