INROADS

Real Analysis Exchange
Vol. 43(2), 2018, pp. 445–450

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UNIQUENESS PROPERTIES OF HARMONIC FUNCTIONS

Abstract

We study the zero set of a harmonic function of several real variables. Using the theory of real analytic functions, we analyze such sets. We generalize these results to solutions of elliptic partial differential equations with constant coefficients.

1 Introduction

The zero set of a holomorphic function of one complex variable is easy to characterize: the set must be discrete. But, even in this context, the zero set of a harmonic function is more complicated. For instance, the set $\{z \in \mathbb{C} : \Im z = 0\}$ is the zero set of the harmonic function $u(z) = z - \overline{z}$. Matters for harmonic functions of more than two real variables are even more subtle.

In this paper we prove a basic result about these zero sets. While mostly well known to experts, this result is not well documented in the literature. It is useful to have a crisp, clean proof of the result recorded and documented.

There are other proofs of this result using the mean value property for harmonic functions. Those proofs are needlessly complicated and obscure. The point here is to give the most elegant possible presentation.

2 Definitions and Basic Results

Of course $\mathbb{R}^N$ is the usual Euclidean space. We let $B(P, R)$ denote the open ball with center P and radius R in $\mathbb{R}^N$. Also $\overline{B}(P, R)$ is the corresponding closed ball.

Mathematical Reviews subject classification: Primary: 32T35; Secondary: 32T05, 32T27
Key words: harmonic function, zero set, real analytic, uniqueness
Received by the editors February 3, 2018
Communicated by: Andrei K. Lerner

A *domain* $U \subseteq \mathbb{R}^N$ is defined to be a connected, open set. A function $u : U \to \mathbb{C}$ is said to be *harmonic* if $\triangle u \equiv 0$ on U, where $\triangle$ is the classical Laplace operator. Our first main result is this:

Theorem 1. *Let $U \subseteq \mathbb{R}^N$ be a domain. Let $u : U \to \mathbb{C}$ be harmonic. If u vanishes on a set $E \subseteq U$ of positive measure, then $u \equiv 0$.*

We shall prove this result in Section 4.

3 Motivation

In this section we treat a toy version of the main result.

Proposition 2. *Let p be a polynomial in N real variables. Then the zero set of p is a set of zero measure in $\mathbb{R}^N$.*

PROOF. We induct on dimension.

In dimension 1, the zero set is a finite set so certainly has measure 0.

Assume now that we have proved the result in dimension N. Consider now a polynomial p in $(N+1)$ real variables. For each fixed value of x_{N+1}, we have a polynomial of N real variables. By the inductive hypothesis, this polynomial has a zero set of 0 N-dimensional measure. Since that is true for each fixed value of x_{N+1}, we conclude by Fubini's theorem that the zero set of p has $(N+1)$-dimensional measure 0. □

4 Proof of Theorem 1

A function u on a domain $U \subseteq \mathbb{R}^N$ is said to be *real analytic* if u has a convergent power series expansion about each point of U. The reference [3] contains the chapter and verse about real analytic functions.

One useful characterization of real analytic functions is this

Proposition 3. *A function u on a domain U is real analytic if and only if, for each point $P \in U$, there are constants $C, R > 0$ so that*

$$\left|\frac{\partial^\alpha}{\partial x^\alpha} u(x)\right| \leq C \cdot \frac{|\alpha!|}{R^{|\alpha|}}$$

for $x \in B(P, R) \subseteq U$ and α a multi-index.

PROOF. We refer the reader to [3] for the details of the proof. □

The Poisson kernel for the unit ball $B \subseteq \mathbb{R}^N$ is given by

$$P_B(x,t) = \frac{\Gamma(N/2)}{2\pi^{N/2}} \cdot \frac{1-|x|^2}{|x-t|^N} \,.$$

Here Γ is the classical gamma function. We see by inspection that P_B is real analytic in both the x and t variables.

Proposition 4. *If U is a domain in $\mathbb{R}^N$ and u is harmonic on U then u is real analytic on U.*

Remark 5. In real dimension $N = 2$ ($\approx \mathbb{C}$), the matter is trivial. For, in that context, a harmonic function is the real part of a holomorphic function.

Proof of Proposition 4. Fix a point $P \in U$ and restrict attention to a closed ball $\overline{B}(P,r) \subseteq U$. After translating and dilating coordinates, we may as well assume that this ball is $\overline{B} = \overline{B(0,1)}$.

We write

$$u(x) = \int_{\partial B} u(t) P_B(x,t)\, d\sigma(t) \,,$$

where $d\sigma$ is rotationally invariant area measure on ∂B. It is a fact that the Poisson kernel is real analytic—see the discussion above preceding Proposition 4. Then, differentiating under the integral sign, and invoking Proposition 3, it is apparent that u is real analytic. $\square$

Proposition 6. *Let $U \subseteq \mathbb{R}^N$ be a domain. Let u be a real analytic function on U. If u vanishes on a set of positive measure in U then $u \equiv 0$.*

Proof. This proof follows classical lines, as may be found in [3]. A sketch of the idea is this.

It is most convenient to induct on dimension. In dimension 1, there is the well-known stronger result that, if the zero set has an interior accumulation point, then the function is identically 0.

Now suppose that the result has been proved in dimension N. Let f be a real analytic function of $N+1$ variables which vanishes on a set S of positive $(N+1)$-dimensional measure. Then a set of N-dimensional slices of S, parametrized over a 1-dimensional interval of positive measure, will each be of positive N-dimensional measure. The inductive hypothesis then implies that each of those N-dimensional slices is identically 0. But now we apply the 1-dimensional result in the orthogonal direction to conclude that u vanishes on an open set. And therefore $u \equiv 0$. $\square$

And now we may prove the main result (Theorem 1):

PROOF. If u is a harmonic function on a domain $U \subseteq \mathbb{R}^N$ that vanishes on a set of positive measure, then u is a real analytic function that vanishes on a set of positive measure. Hence $u \equiv 0$. $\square$

5 Strongly Elliptic Operators

Let

$$\mathcal{L} = \sum_{j,k=1}^{N} a_{jk} \frac{\partial^2}{\partial x_j \partial x_k}$$

be a strongly elliptic partial differential operator with constant coefficients (see [2]). This means, of course, that

$$(a_{jk})$$

is a positive definite matrix.

The Green's function $G(x,t)$ for $\mathcal{L}$ on a domain U is the fundamental solution Γ corrected with the solution of a certain Dirichlet problem (see [1] for the details). The fundamental solution will of course be real analytic off the diagonal. If we take the domain to be the unit ball in $\mathbb{R}^N$, then the Dirichlet data will be real analytic and the solution of the Dirichlet problem will be real analytic.

Next, the Poisson kernel is the normal derivative

$$\frac{\partial}{\partial \nu_t} G(x,t) \, .$$

Here

$$\frac{\partial}{\partial \nu_t} = \sum_j \frac{\partial \rho}{\partial t_j} \cdot \frac{\partial}{\partial t_j} \, .$$

Here ρ is the defining function for the domain—which in this instance is the unit ball.

It follows that the Poisson kernel is real analytic. Thus we have

Proposition 7. *Let u be a solution of the partial differential equation $\mathcal{L}u = f$, where f is real analytic. Then u is real analytic.*

PROOF. The proof is similar to that of Proposition 4, and we omit the details. $\square$

We conclude now with this result that generalizes Theorem 1.

Theorem 8. *Let u be a solution of the partial differential equation $\mathcal{L}u = 0$. If u vanishes on a set of positive measure, then u is identically 0.*

6 Concluding Remarks

The seminal theorem of Lojaciewicz (see [3] and of course [4]) gives a structure theorem for the zero set of a harmonic function. Our Theorem 1 follows immediately from that result.

Lojaciewicz's theorem is extremely complicated and difficult to prove. It is attractive to have a direct and accessible proof of Theorem 1. That is what we provide here.

References

[1] S. G. Krantz, *Function Theory of Several Complex Variables*, Second edition, Amer. Math. Soc., Providence, RI, 2001.

[2] S. G. Krantz, *Partial Differential Equations and Complex Analysis*, CRC Press, Boca Raton, FL, 1992.

[3] S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, Second edition, Birkhäuser Publishing, Boston, 2002.

[4] S. Lojasiewicz, *Introduction to Complex Analytic Geometry*, Translated from the Polish by Maciej Klimek, Birkhüser Verlag, Basel, 1991.