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A NOTE ON LEVEL SETS OF DIFFERENTIABLE FUNCTIONS F(X,Y) WITH NON-VANISHING GRADIENT

Abstract

The purpose of this note is to give an alternate proof of a result of M.Elekes. We show that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function with everywhere non-zero gradient then for every point $x \in \mathbb{R}^2$ in the level set $\{x : f(x) = c\}$ there is a neighborhood V of x such that $\{f = c\} \cap V$ is homeomorphic to an open interval or the union of finitely many open segments passing through a point.

1 Introduction

The Inverse Function Theorem is frequently proved under the assumption that the mapping is continuously differentiable. In [4] continuity was removed from the assumptions and the theorem was generalized for differentiable mappings. The Implicit Function Theorem is commonly derived from the Inverse Function Theorem. Whether the Implicit Function Theorem may be proved under these more general assumptions was studied in [1] and [2].
More precisely, in [1] the following was proved.

Theorem 1. *Assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that $|\nabla f| > \eta > 0$ for all x in $\mathbb{R}^2$ then in a neighborhood of its points the level set, $\{x \in \mathbb{R}^2 : f(x) = c\}$ is homeomorphic to an open interval.*

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Moreover, in [1] a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, of non-vanishing gradient, such that in any neighborhood of the origin the level set is not homeomorphic to an open interval, was presented. Later, in [2] the following related result was proved.

Theorem 2. *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function with non-vanishing gradient then in a neighborhood of each of its points the level set $Z = \{x : f(x) = c\}$ is homeomorphic either to an open interval or a union of finitely many open segments passing through a point. Furthermore, the set of branch points (see below for definition) is discrete.*

In this note we will present an alternate account of Theorem 2.

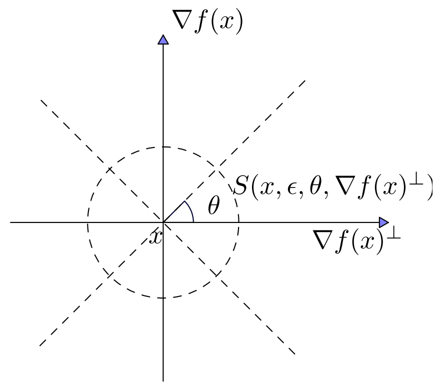
We first mention some terminology and facts that will frequently be used in the remainder of this note. An *arc* is the image of a continuous injective function from $[0, 1]$ into the plane. An *unbounded arc* will refer to a set which is the image of a continuous injective function from $[0, 1)$ to $\mathbb{R}^2$ that is not contained in any ball centered about the origin. A set $S \subset \mathbb{R}^2$ is said to be *locally arcwise connected* if each neighborhood in S contains an arcwise connected neighborhood. We will make extensive use of the fact that a complete locally connected space is locally arcwise connected, ([3] p. 254). If there are two distinct arcs between points x and y then there must exist a closed curve in the union of these arcs. As we are dealing with functions of non-vanishing gradient we observe that in this case a level set may not contain any closed paths as this will lead to a local extreme. By *branch point* we will refer to a point where the level set is not locally homeomorphic to an open interval. When $\{\Gamma_i\}$ is a collection of arcs with the terminal point of Γ_i coinciding with the starting point of Γ_{i+1} then $\sum \Gamma_i$ will refer to their concatenation. At times we will need to supply a specific parametrization of an arc, we will denote it by the lowercase of the respective letter that has been used to denote the arc. We will also use $A(x, a, b)$ to denote an annulus centered at x with inner radius a and outer radius b. $B(x, r)$ will denote a ball of radius r centered at x. By bdS we will denote the topoligical boundary of a set S.

As mentioned above we will use the fact that any closed, locally connected set is locally arcwise connected. The usefulness of this follows from the following fact which was proved in [2].

Theorem 3 ([2])**.** *Given a function f, as in Theorem 2, the level set $\{x : f(x) = c\}$ is a locally connected set.*

For any unit vector, v, let

$$S(x,\epsilon,\theta,v)=\{y\ :\ \left|\frac{(x-y)}{|x-y|}\cdot v\right|>|\cos(\theta)|,\ |x-y|<\epsilon\}\cup\{x\}.$$



It is an elementary fact that for every $x\in\mathbb{R}^2$ there exists an $\epsilon_0>0$ such that the level set Z containing a point x satisfies $Z\cap B(x,\epsilon)\subset S=S(x,\epsilon,\frac{\pi}{4},\nabla f(x)^{\perp})$ (note that any $\theta>0$ will do) whenever $0<\epsilon<\epsilon_0$ and $\nabla f(x)^{\perp}$ is the unit vector perpendicular to the gradient of f at x for which the basis $\{\nabla f(x),\nabla f(x)^{\perp}\}$ has positive orientation. As the directional derivative of f at x along $\nabla f(x)$ may not vanish it follows that the signs of $f-f(x)$ on the two connected components of $\left\{y:\frac{(x-y)}{|x-y|}\cdot\nabla f(x)^{\perp}=\frac{1}{\sqrt{2}}\right\}$ must be opposite. The same is true for the components of $\left\{y:\frac{(x-y)}{|x-y|}\cdot\nabla f(x)^{\perp}=\frac{-1}{\sqrt{2}}\right\}$. From this and the Intermediate Value Theorem it follows that if $f(x)=c$ then elements of $\{f=c\}$ will accumulate at x through both sides of S.

There is an important fact that we will be leveraging. From Lemmas 4.1,4.2,4.3 of [2] it follows that given a rectangle R there cannot be infinitely many points of a level set on each line which meets R and is parallel to a side of R. From this we obtain the following lemma.

Lemma 4. *For any $\epsilon>0$ and any x in the level set $\{y:f(y)=c\}$ there are at most a finite number of disjoint arcs in $S(x,\epsilon,\frac{\pi}{4},\nabla f(x)^{\perp})\cap\{y:f(y)=c\}$ which contain a point on both boundary curves of the annulus $A(x,\dfrac{\epsilon}{2},\epsilon)$.*

PROOF. Suppose that there exists an $\epsilon > 0$ such that there are infinitely many pairwise disjoint arcs $\{\Gamma_i\}$ in $S = S(x, \epsilon, \frac{\pi}{4}, \nabla f(x)^{\perp})$ which are contained in the level set $\{y : f(y) = c\}$ and intersect both boundary curves of the annulus $A(x, \frac{\epsilon}{2}, \epsilon)$. Let R_1, R_2 be rectangles described by the following:

- one side is parallel to the direction of $\nabla f(x)$ and tangent to the boundary of $B(x, \dfrac{\epsilon}{2})$
- this point of tangency is the midpoint of the side and the length of this side is $\sqrt{2}\epsilon$
- another side, perpendicular to the first side, has length $\dfrac{\epsilon}{\sqrt{2}} - \dfrac{\epsilon}{2}$ and has an endpoint on $B(x, \epsilon)$.

Then, at least one of these rectangles has infinitely many points of $\cup_i \Gamma_i$ on each line which is parallel to $\nabla f(x)$ and meets the rectangle, a contradiction. $\square$

Lemma 5. *Let α_0 be an arc in $\{f = c\}$ connecting points u and v. Then for any $R > 0$, α_0 may be extended in an arcwise manner to an arc α which intersects the boundary of $B(x, R)$.*

PROOF. Let A_0 be an arc in $\{f = c\}$ which connects points x and y_0. Furthermore, assume that all arcs which extend A_0 and are contained in $\{f = c\}$, must be contained in some ball $B(x, R)$. We first observe that any arc between two points y and z in $Z = \{f = c\}$ may be extended in an arcwise manner by an arc contained in Z. For suppose $Z \cap B(z, \delta) \subset S = S(z, \delta, \frac{\pi}{4}, \nabla f(z)^{\perp})$. By continuity the arc can only approach z through one of the halves of S. Applying local arcwise connectedness one may extend the arc by concatenating with an arc from z to a point of the level set in the other half of S. For the remainder of the proof an extended arc will be assumed to be contained in Z. Let

$$S_1 = \{y \,:\, \exists \,\text{an extended arc of}\, A_0 \,\text{from}\, x \,\text{to}\, y\}$$

and let $y_1 \in S_1$ be a point such that

$$|y_0 - y_1| \geq \frac{1}{2} \sup_{y \in S_1} \{|y_0 - y|\} \,.$$

Call A_1 the arc from y_0 to y_1. Inductively, we define for $n \geq 2$

$$S_n = \{y \,:\, \exists \,\text{an extended arc of}\, A_0 + A_1 + \cdots + A_{n-1} \,\text{from}\, x \,\text{to}\, y\}$$

and $y_n \in S_n$ a point such that

$$|y_n - y_{n-1}| \geq \frac{1}{2} \sup_{y \in S_n} \{|y_{n-1} - y|\}$$

and let A_n denote the arc from y_{n-1} to y_n. Note that $A_i, A_j, i < j$ are disjoint except when $j = i + 1$ in which case their intersection is the point y_i.
Let $d_n = |y_{n-1} - y_n|$. It is clear that all points of A_n are within $2d_n$ of y_{n-1}, since they are in S_n, and thus the diameter of A_n is at most $4d_n$. Suppose $\{y_{n_k}\}$ is a subsequence converging to a point z, that there is some other point of accumulation z' of the sequence $\{y_n\}$ and set $w = |z - z'|$. This leads to a contradiction of the fact that Z contains no loops as we may (using the local arcwise connectedness of Z at z) connect two points $y_{n_{k_1}}$ and $y_{n_{k_2}}$, when n_{k_1} and n_{k_2} are large, to z by arcs in Z contained in $B(z, w/4)$, while there is another arc in Z between them which must intersect $B\left(z', \frac{w}{4}\right)$. Thus we must have $y_n \to z$, and $d_n \to 0$. Define $A = \sum_i A_i$, and let $\epsilon > 0$, and $\alpha : [0, 1) \to \mathbb{R}^2$ be a parametrization of A such that $\alpha\left(\left[1 - \frac{1}{2^n}, 1 - \frac{1}{2^{n+1}}\right]\right)$ is equal to A_n. It is clear that α is injective. Let $\psi : [0, 1] \to \mathbb{R}^2$ extend α by defining $\psi(1) = z$, and let Ψ be the image of ψ. Let n be large enough so that $|y_m - z|$, $d_m < \frac{\epsilon}{100}$ for all $m > n$. If $t \in \left(1 - \frac{1}{2^{n+10}}, 1\right]$, then $\psi(t) \in A_m$ for some $m \geq n + 10$. We have

$$|\psi(t) - z| \leq |\psi(t) - y_{m-1}| + |y_{m-1} - z| < 4d_m + \frac{\epsilon}{100} < \epsilon\,.$$

Thus ψ is continuous and Ψ is an arc from x to z as z cannot also be in A. This arc may then be extended non-trivially which contradicts $d_n \to 0$.

□

2 Proof of Theorem 2

Proof. Fix $x \in Z$. By local arcwise connectedness there is a bounded open set U containing x such that $Z_0 = Z \cap U$ is arcwise connected. Let Γ_0 be an arc in Z_0 that passes from x to the boundary of U. Such an arc exists since x is a limit point of Z and Lemma 5 ensures that any arc in Z may be extended to an unbounded arc. Extend $\{\Gamma_0\}$ to a maximal family $\{\Gamma_k\}_{k=0}^{n}$ of arcs which are disjoint except for x and end at the boundary of U intersecting bd$U$ at a single point. By Lemma 4 this family must be finite. Suppose $\{z_n\}$ is a sequence of distinct points $z_n \to x$, $z_n \in Z_0 \setminus \cup_{k=0}^n \Gamma_k$. For each z_i let A_i be an arc in Z_0 beginning at x, passing through z_i and ending on the boundary of bd$U$ such that $A_i \cap \mathrm{bd}U$ is a single point. Let $\alpha_i : [0,1] \to \mathbb{R}^2$ denote a parametrization of A_i. We may write $A_i = A_{i_0} + A_{i_1}$ where $A_{i_0} = \alpha_i|_{[0,t_i]}$, $A_{i_1} = \alpha_i|_{(t_i,1]}$, and $t_i = \sup\{s \in [0,1] \,:\, \alpha_i(s) \in \cup_{k=0}^n \Gamma_k\}$. By the maximality of $\{\Gamma_k\}_{k=0}^n$ and the fact that Z contains no loops $0 < t_i < 1$, $A_{i_0} \subset \cup_{k=0}^n \Gamma_k$, $A_{i_1} \cap \cup_{k=0}^n \Gamma_k = \emptyset$, and $A_{i_1} \cap B(x, \eta_i) = \emptyset$ for some $\eta_i > 0$. By the arcwise-connectedness of Z_0 there exists a $m_i > i$ such that if $j > m_i$ there is an arc from x to z_j which is contained in $B(x, \frac{\eta_i}{2}) \cap A_j$, and as such we must have, $z_j \notin A_i$, and $A_{i_1} \cap A_{j_1} = \emptyset$ (this intersection is empty since any point in it would not be in $B(x, \eta_i)$ and thus there would be two distinct paths from x to such a point, one with z_j and one without z_j). This clearly contradicts Lemma 4. Hence, there is a ball $V \subset U$ containing x such that $Z \cap V \subset \cup_{i=0}^n \Gamma_i$. If γ_i is a parametrization of Γ_i and $s_i = \inf\{s \,:\, \gamma_i(s) \in \mathrm{bd}V\}$, we let W be the connected component of $V \setminus \cup \gamma_i[s_i, 1]$ containing x. We note that the number of arcs is even, since f must have opposite sign on two adjacent connected components of $W \setminus \cup_{i=0}^n \Gamma_i$ or we would have a critical point. Thus we may pair the paths of the collection $\{\gamma_i[0, s_i)\}_{i=0}^n$ to get the required pieces, which are homeomorphic to open intervals. If $q \in W$ with $q \neq x$, and $q \in \Gamma_i \cap W$ for some i, we may clearly find a neighborhood W_q of q contained in W such that $W_q \cap Z$ is homeomorphic to an open interval. Thus, branch points are isolated and the claim follows. $\square$

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