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SIMULTANEOUS SMALL COVERINGS BY SMOOTH FUNCTIONS UNDER THE COVERING PROPERTY AXIOM

Abstract

The covering property axiom CPA is consistent with ZFC: it is satisfied in the iterated perfect set model. We show that CPA implies that for every $\nu \in \omega \cup \{\infty\}$ there exists a family $\mathcal{F}_\nu \subset C^\nu(\mathbb{R})$ of cardinality $\omega_1 < \mathfrak{c}$ such that for every $g \in D^\nu(\mathbb{R})$ the set $g \setminus \bigcup \mathcal{F}_\nu$ has cardinality $\leq \omega_1$. Moreover, we show that this result remains true for partial functions g (i.e., $g \in D^\nu(X)$ for some $X \subset \mathbb{R}$) if, and only if, $\nu \in \{0, 1\}$. The proof of this result is based on the following theorem of independent interest (which, for $\nu \neq 0$, seems to have been previously unnoticed): for every $X \subset \mathbb{R}$ with no isolated points, every ν-times differentiable function $g \colon X \to \mathbb{R}$ admits a ν-times differentiable extension $\bar{g} \colon B \to \mathbb{R}$, where $B \supset X$ is a Borel subset of $\mathbb{R}$. The presented arguments rely heavily on a Whitney's Extension Theorem for the functions defined on perfect subsets of $\mathbb{R}$, which short but fully detailed proof is included. Some open questions are also posed.

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1 Preliminaries, the main result, and discussion

For an $X \subset \mathbb{R}$, let $X' \subset \mathbb{R}$ denote, as usual, the set of all accumulation points of X. A function $f\colon X \to \mathbb{R}$ is differentiable, provided it admits a derivative, that is, a map $f'\colon X' \cap X \to \mathbb{R}$ defined, for every $p \in X' \cap X$, as

$$f'(p) = \lim_{x \to p,\ x \in X} \frac{f(x) - f(p)}{x - p}.$$

Note that $X' \cap X$ consists of all non-isolated points of X. We will be mainly interested in the perfect sets X when, of course, $X' \cap X = X$.

For $n \in \omega = \{0, 1, 2, \ldots\}$ and $X \subset \mathbb{R}$, we will use symbol $D^n(X)$ to denote the class of all functions $f\colon X \to \mathbb{R}$ which are n-times differentiable (with all derivatives being finite) and symbol $C^n(X)$ for the class of all $f \in D^n(X)$ whose n-th derivative $f^{(n)}$ is continuous. In particular, $C^0(X) = D^0(X)$ is the the class $C(X)$ of all continuous maps from X to $\mathbb{R}$. Also, $C^\infty(X) = D^\infty(X)$ denote all infinitely many times differentiable maps $f\colon X \to \mathbb{R}$. Recall that

$$C^\infty(X) \subset \cdots \subset C^n(X) \subset D^n(X) \subset \cdots \subset C^1(X) \subset D^1(X) \subset C^0(X)$$

and that for $X = \mathbb{R}$ all these inclusions are proper.

In what follows we identify functions with their graphs. We will write $A \subset^\star B$ to denote $|A \setminus B| \le \omega_1$ which, clearly, defines a partial order relation. Note that

$$\text{if } A_\xi \subset^\star B_\xi \text{ for all } \xi < \omega_1, \text{ then also } \textstyle\bigcup_{\xi<\omega_1} A_\xi \subset^\star \bigcup_{\xi<\omega_1} B_\xi. \tag{1}$$

Symbol $\mathrm{CPA}_{\mathrm{prism}}$ denotes a consequence, and a simpler part, of the full version of the covering property axiom CPA.

The main aim of this paper is to present, prove, and discuss the following Main Theorem. The proof of its part (i) for $\nu = 0$ can be also found in [6]. The results in the direction of partitioning the Euclidean space into subsets with certain smoothness properties can also be found in [17].

Main Theorem. $\mathrm{CPA}_{\mathrm{prism}}$ *implies that for every* $\nu \in \omega \cup \{\infty\}$ *there exists a family* $\mathcal{F}_\nu \subset C^\nu(\mathbb{R})$ *of cardinality* $\omega_1 < \mathfrak{c}$ *such that*

(i) $g \subset^\star \bigcup \mathcal{F}_\nu$ *for every* $g \in D^\nu(\mathbb{R})$.

Moreover, for $n \in \{0, 1\}$ *we also have*

(ii) $g \subset^\star \bigcup \mathcal{F}_n$ *for every* $g \in D^n(X)$*, where* $X \subset \mathbb{R}$ *is arbitrary.*

The Main Theorem generalizes part (a) of the following result of Ciesielski and Pawlikowski from [8]. (See [7, theorems 4.1.1(b) and 4.1.6].) It is also closely related to its part (b), where f^{-1} stands for $\{\langle f(x), x\rangle \colon x \in \mathbb{R}\}$.

Proposition 1.1. $\mathrm{CPA}_{\mathrm{prism}}$ *implies that* $\mathfrak{c} = \omega_2$ *and*

(a) For every $0 < n < \omega$ *and* $g \in D^n(\mathbb{R})$ *there is a family* $\mathcal{F}_g \subset C^n(\mathbb{R})$ *of cardinality* ω_1 *such that* $g \subset \bigcup \mathcal{F}_g$.

(b) There is a $\mathcal{G} \subset C^1(\mathbb{R})$ *of cardinality* ω_1 *such that* $\mathbb{R}^2 = \bigcup_{f\in\mathcal{G}}(f \cup f^{-1})$.

We restate Proposition 1.1(a) as Theorem 2.2, since its proof presented in [8] and [7] is not correct for the case of $n > 1$, as it is based on the following proposition (L), which is false for $n > 1$. Thus, our proof of Theorem 2.2 constitutes the first correct argument for this result. In (L) and below we use the notation $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\Delta = \{\langle x, x\rangle : x \in \mathbb{R}\}$.

(L) *For* $n \in \mathbb{N}$ *let* $f \in C^{n-1}(\mathbb{R})$ *and let* $P \subset \mathbb{R}$ *be a perfect set for which the map* $F\colon P^2 \setminus \Delta \to \mathbb{R}$ *defined by* $F(x, y) = \frac{f^{(n-1)}(x)-f^{(n-1)}(y)}{x-y}$ *is uniformly continuous and bounded. Then* $f \restriction P$ *can be extended to an* $\bar{f} \in C^n(\mathbb{R})$.

The falsehood of (L) is justified by the following example.

Example 1.1. *Let* $\mathfrak{C}$ *be the Cantor ternary set. There exists an* $f \in C^1(\mathbb{R})$ *such that* $f' \restriction \mathfrak{C} \equiv 0$ *and for no perfect set* $P \subset \mathfrak{C}$ *there is an extension* $\bar{f} \in C^2(\mathbb{R})$ *of* $f \restriction P$. *In particular,* $f \restriction \mathfrak{C}$ *contradicts (L) for* $n = 2$.

CONSTRUCTION. For $n \in \mathbb{N}$ let $\mathcal{J}_n$ be the family of all connected components of $\mathbb{R} \setminus \mathfrak{C}$ of length 3^{-n}. Define $f_0 \colon \mathbb{R} \to \mathbb{R}$ as

$$f_0(x) = \begin{cases} \frac{2^{-n}}{3^{-n}} \operatorname{dist}(x, \mathfrak{C}) & \text{if } x \in J, \text{ where } J \in \mathcal{J}_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that f_0 is continuous, as $f_0[J] \subset [0, 2^{-n}]$ for every $J \in \mathcal{J}_n$. Define $f \colon \mathbb{R} \to \mathbb{R}$ as $f(x) = \int_0^x f_0(t)\,dt$. Clearly we have $f \in C^1(\mathbb{R})$ and $f' \restriction \mathfrak{C} = f_0 \restriction \mathfrak{C} \equiv 0$. We just need to verify the statement about the extension.

To see this, notice that for every $n \in \mathbb{N}$ and distinct $a, b \in \mathfrak{C}$

$$\text{if } |b - a| < 3^{-n}, \text{ then } \frac{|f(b) - f(a)|}{(b - a)^2} > \frac{1}{36}\left(\frac{3}{2}\right)^n. \tag{2}$$

Indeed, if $m \in \mathbb{N}$ is the smallest such that there is $J = (p, q) \in \mathcal{J}_m$ between a and b, then $m > n$, $|b - a| \le 3 \cdot 3^{-m}$, and $|f(b) - f(a)| \ge \int_p^q f_0(t)dt = \frac{1}{2}3^{-m}\frac{1}{2}2^{-m}$. So, $\frac{|f(b)-f(a)|}{(b-a)^2} \ge \frac{\frac{1}{4}3^{-m}2^{-m}}{(3\cdot3^{-m})^2} = \frac{1}{36}\left(\frac{3}{2}\right)^m > \frac{1}{36}\left(\frac{3}{2}\right)^n$. But this means that for every perfect $P \subset \mathfrak{C}$ the map $f \restriction P$ does not satisfy condition (W_2) from Theorem 3.3, our version of Whitney's Extension Theorem, which is necessary for admitting extension $\bar{f} \in C^2(\mathbb{R})$. More specifically, either $(f \restriction P)''(a)$ does not exist or else

$$\begin{aligned}\left|q^2_{f\restriction P}(a,b)\right| &= \frac{\left|f(b)-f(a)-\frac{1}{2}(f\restriction P)''(a)(b-a)^2\right|}{(b-a)^2}\\ &\geq \frac{|f(b)-f(a)|}{(b-a)^2} - \frac{1}{2}(f\restriction P)''(a),\end{aligned}$$

that is, $q^2_{f\restriction P}$ is not continuous at the point $\langle a,a\rangle$, since, by (2), we have $\lim_{b\to a, b\in P} \frac{|f(b)-f(a)|}{(b-a)^2} = \infty$. □

Notice also, that if f is from Example 1.1, then $g = f \restriction \mathfrak{C}$ shows that the property (ii) from the Main Theorem is false for $n = 2$ even in case when $X = \mathfrak{C}$ is perfect, $g \in C^\infty(X)$, and g has an extension $f \in C^1(\mathbb{R})$.

Remark 1.2. *In the Main Theorem is also possible to ensure that have part (b) from Proposition 1.1:* $\mathbb{R}^2 = \bigcup_{f\in\mathcal{F}_1}(f \cup f^{-1})$. *This can be achieved by replacing its family* $\mathcal{F}_1$ *with* $\mathcal{G} \cup \mathcal{F}_1$*, where* $\mathcal{G}$ *is from Proposition 1.1.*

It is clear that the Main Theorem implies immediately Proposition 1.1(a). Indeed, it implies that for every $g \in D^n(\mathbb{R})$ we have $g \subset^\star \bigcup \mathcal{F}_n$. In particular, for every $g \in D^n(\mathbb{R})$ there exists a family $\mathcal{C}_g$ of cardinality $\leq \omega_1$ consisting of constant functions such that $\mathcal{F}_g = \mathcal{F}_n \cup \mathcal{C}_g$ satisfies Proposition 1.1(a).

The Main Theorem generalizes Proposition 1.1(a) twofolds: (1) for $n = 1$ it concerns all partial functions, rather than just functions defined on $\mathbb{R}$; (2) each family $\mathcal{F}_n$ covers, in the $\subset^\star$ sense, every $g \in D^n(\mathbb{R})$, while each family $\mathcal{F}_g$ in Proposition 1.1 concerns only a single function g.

Notice that the generalization (2) is essential, as shown in the following Example 1.2. (The details and constructions needed in order to justify Examples 1.2 and 1.3 shall be presented in Section 6.)

Example 1.2. *Under* $\mathrm{CPA}_{\mathrm{prism}}$*, there exists a collection* $\{\bar{\mathcal{F}}_g \colon g \in D^1(\mathbb{R})\}$ *such that each* $\bar{\mathcal{F}}_g$ *satisfies (a) of Proposition 1.1, while no single* $\bar{\mathcal{F}}_g$ *can serve as the family* $\mathcal{F}_1$ *in the Main Theorem.*

It may also look, at a first glance, that the family $\mathcal{G}$ from Proposition 1.1 could always be used as the family $\mathcal{F}_1$ (or $\mathcal{F}_0$) in the Main Theorem—after all, it works for any constant function g. However, this is not the case, as shown in the next example.

Example 1.3. *Under* $\mathrm{CPA}_{\mathrm{prism}}$*, there exists a family* $\mathcal{G}$ *satisfying (b) of Proposition 1.1 such that if* $\mathcal{G}_1 = \mathcal{G} \cup \{g^{-1} \in \mathbb{R}^\mathbb{R} \colon g \in \mathcal{G}\}$ *and* $h \colon \mathbb{R} \to \mathbb{R}$ *is given as* $h(x) = x + 2$*, then* $h \setminus \bigcup \mathcal{G}_1$ *has cardinality* $\mathfrak{c}$.

2 Obtaining the Main Theorem via three key theorems

Our proof of Main Theorem is based on the following three theorems, which will be proved in Sections 4 and 5, respectively. The first of these theorems generalizes [6, theorem 3.6] from the case of $n = 0$ to an arbitrary $\nu \in \omega \cup \{\infty\}$. The second is presented in Ciesielski and Pawlikowski [8] and [7, theorems 4.1.1(b) and 4.1.6]. However the proof presented there is incorrect, as indicated above. The third theorem for the case of $n = 0$ follows quite easily from [7, theorem 4.1.1(a)]. However, the case of $n = 1$ is a bit more involved.

Theorem 2.1. $\mathrm{CPA}_{\mathrm{prism}}$ *implies that for every* $\nu \in \omega \cup \{\infty\}$ *and every compact interval* $I = [a, b] \subset \mathbb{R}$ *there exists a family* $\mathcal{F}_\nu^I \subset C^\nu(\mathbb{R})$ *of cardinality* $\omega_1 < \mathfrak{c}$ *such that* $g \subset^\star \bigcup \mathcal{F}_\nu^I$ *for every* $g \in C^\nu(I)$.

Theorem 2.2. $\mathrm{CPA}_{\mathrm{prism}}$ *implies that for every* $n \in \mathbb{N}$ *and* $g \in D^n(\mathbb{R})$ *there exists a family* $\mathcal{F}_g \subset C^n(\mathbb{R})$ *of cardinality* $\omega_1 < \mathfrak{c}$ *such that* $g \subset \bigcup \mathcal{F}_g$.

Theorem 2.3. $\mathrm{CPA}_{\mathrm{prism}}$ *implies that for every* $n \in \{0, 1\}$ *and* $g \in D^n(X)$ *with* $X \subset \mathbb{R}$ *there exists a family* $\mathcal{F}_g \subset C^n(\mathbb{R})$ *of cardinality* $\omega_1 < \mathfrak{c}$ *such that* $g \subset \bigcup \mathcal{F}_g$.

PROOF OF MAIN THEOREM. For every $\nu \in \omega \cup \{\infty\}$ let $\mathcal{F}_\nu = \bigcup_{n=1}^\infty \mathcal{F}_\nu^{[-n,n]}$, where each family $\mathcal{F}_\nu^{[-n,n]}$ is from Theorem 2.1. Clearly $\mathcal{F}_\nu$ has cardinality ω_1. To see that it is as desired, choose a $g \in D^\nu(X)$ such that $X \subset \mathbb{R}$ and $X = \mathbb{R}$ unless $\nu < 2$. We need to show that $g \subset^\star \bigcup \mathcal{F}_\nu$.

For this, notice that there exists a family $\mathcal{F}_g \subset C^\nu(\mathbb{R})$ of cardinality $\leq \omega_1 < \mathfrak{c}$ such that $g \subset \bigcup \mathcal{F}_g$. For $\nu < 2$ this follows from Theorem 2.3, for $\nu = \infty$ this is justified by $\mathcal{F}_g = \{g\} \subset D^\infty(\mathbb{R}) = C^\infty(\mathbb{R})$, while for the remaining cases this follows from Theorem 2.2. For each $n \in \mathbb{N}$ and $f \in \mathcal{F}_g$ we have $f \restriction [-n, n] \subset^\star \bigcup \mathcal{F}_\nu^{[-n,n]} \subset \bigcup \mathcal{F}_\nu$. Therefore

$$g \subset \bigcup \mathcal{F}_g = \bigcup_{f \in \mathcal{F}_g} \bigcup_{n=1}^{\infty} f \restriction [-n, n] \subset^\star \bigcup \mathcal{F}_\nu,$$

as needed. $\square$

3 Whitney's Extension Theorem and other preliminaries

Definition 3.1. For $n<\omega$, perfect set $P\subset\mathbb{R}$, $f\in D^n(P)$, and $a\in P$ let $T^n_a f(x)$ denote the n-th degree Taylor polynomial of f at a, that is,

$$T^n_a f(x)=\sum_{i=0}^{n}\frac{f^{(i)}(a)}{i!}(x-a)^i.$$

We define the map $q^n_f\colon P^2\to\mathbb{R}$ as

$$q^n_f(a,b)=\begin{cases}\dfrac{T^n_b f(b)-T^n_a f(b)}{(b-a)^n} & \text{if } a\neq b,\\ 0 & \text{if } a=b.\end{cases}$$

Notice that, for $a,x\in P$, the quantity $T^n_x f(x)-T^n_a f(x)=f(x)-T^n_a f(x)$ is the remainder $R^n_a f(x)$ of $T^n_a f(x)$. In particular, $q^n_f(a,b)=\frac{R^n_a f(b)}{(b-a)^n}$ for all $a\neq b$.

We will use the following simple facts about $q^n_f\colon\mathbb{R}^2\to\mathbb{R}$, which can be found in many advance calculus texts.

Proposition 3.2. *Let $n<\omega$.*

(i) If $f\in D^n(\mathbb{R})$, then $q^n_f\colon\mathbb{R}^2\to\mathbb{R}$ is continuous with respect to the second variable, that is, the map $\mathbb{R}\ni x\mapsto q^n_f(a,x)\in\mathbb{R}$ is continuous for every $a\in\mathbb{R}$.

(ii) If $f\in C^n(\mathbb{R})$, then $q^n_f\colon\mathbb{R}^2\to\mathbb{R}$ is continuous.

Proof. For $n=0$ both statements are obvious. So assume that $n>0$.

(i). Let $a\in\mathbb{R}$. It is enough to show that $q^n_f(a,\cdot)$ is continuous at a, that is, that the limit

$$L=\lim_{x\to a}q^n_f(a,x)=\lim_{x\to a}\frac{T^n_x f(x)-T^n_a f(x)}{(x-a)^n}$$

exists and is equal to 0.

In order to see this, notice that, for every $i<n$, the limit

$$L_i=\lim_{x\to a}\frac{\frac{d^i}{dx^i}\left(T^n_x f(x)-T^n_a f(x)\right)}{\frac{d^i}{dx^i}\left((x-a)^n\right)}$$

is of indeterminate form $\frac{0}{0}$ and that $L = L_0$. Thus, using L'Hôpital's Rule $(n-1)$-times we obtain

$$\begin{aligned} L &= \lim_{x\to a} \frac{\frac{d^{n-1}}{dx^{n-1}}\left(T^n_x f(x) - T^n_a f(x)\right)}{\frac{d^{n-1}}{dx^{n-1}}\left((x-a)^n\right)} \\ &= \lim_{x\to a} \frac{f^{(n-1)}(x) - \left(f^{(n-1)}(a) + f^{(n)}(a)(x-a)\right)}{n!(x-a)} \\ &= \lim_{x\to a} \frac{1}{n!}\left(\frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x-a} - f^{(n)}(a)\right) = 0, \end{aligned}$$

as needed.

(ii). Clearly q^n_f is continuous on $\mathbb{R}^2 \setminus \Delta$. We need to show that q^n_f is continuous at every $\langle a, a\rangle$. To see this, choose a sequence $\langle a_k, b_k\rangle_{k\in\mathbb{N}}$ converging to $\langle a, a\rangle$. We need to show that $\lim_{k\to\infty} q^n_f(a_k, b_k) = 0$.

By the standard formula for the Taylor polynomial reminder, for every $k \in \mathbb{N}$ there is ξ_k between a_k and b_k with $f(b_k) - T^{n-1}_{a_k} f(b_k) = \frac{f^{(n)}(\xi_k)}{n!}(b_k - a_k)^n$. Thus, since $T^n_{b_k} f(b_k) - T^n_{a_k} f(b_k) = f(b_k) - \left(T^{n-1}_{a_k} f(b_k) + \frac{f^{(n)}(b_k)}{n!}(b_k - a_k)^n\right)$,

$$q^n_f(a_k, b_k) = \frac{\frac{f^{(n)}(\xi_k)}{n!}(b_k - a_k)^n - \frac{f^{(n)}(b_k)}{n!}(b_k - a_k)^n}{(a_k - b_k)^n} = \frac{f^{(n)}(\xi_k) - f^{(n)}(b_k)}{n!}$$

converges to 0, as $k \to \infty$, since $f^{(n)}$ is continuous and $\langle a_k, b_k\rangle \xrightarrow{k} \langle a, a\rangle$. Therefore,

$$\lim_{k\to\infty} q^n_f(a_k, b_k) = 0 = q^n_f(a, a),$$

as needed. □

Our proof of Main Theorem will make use the following version of Whitney's Extension Theorem.

Theorem 3.3. *Let $P \subset \mathbb{R}$ be perfect, $n \in \mathbb{N}$, and $f\colon P \to \mathbb{R}$. There exists an extension $\bar{f} \in C^n(\mathbb{R})$ of f if, and only if,*

(W_n) $f \in C^n(P)$ and the map $q^{n-i}_{f^{(i)}}\colon P^2 \to \mathbb{R}$ is continuous for every $i \le n$.

Also, f admits an extension $\bar{f} \in C^\infty(\mathbb{R})$ if, and only if, (W_n) holds for every $n \in \mathbb{N}$.

Proof. To see the necessity of the property (W_n) assume that $f\colon P \to \mathbb{R}$ has an extension $\bar{f} \in C^n(\mathbb{R})$. Then, clearly $f = \bar{f} \restriction P \in C^n(P)$. Moreover,

$\bar{f}^{(i)} \in C^{n-i}(\mathbb{R})$ for all $i \leq n$. So by Proposition 3.2, each function $q^{n-i}_{\bar{f}^{(i)}}$ is continuous and so is $q^{n-i}_{f^{(i)}} = q^{n-i}_{\bar{f}^{(i)}} \restriction P^2$.

The sufficiency of (W_n) can be deduced from [9, theorem 3.1.14].[1] However, this and other proofs of Whitney's Extension Theorem (see, e.g., [1, 2, 10]) are quite long, mainly due to their generality and framework considered (of dealing with partial functions on $\mathbb{R}^n$). Thus, for a sake of simplicity and completeness of this article, we include a simple proof of this theorem, in our particular form, in Section 7.

In the case of C^∞ extensions, the necessity of all conditions (W_n) follows from the finite cases. They are sufficient, since then f satisfies the assumptions of C^∞ version of Whitney's extension theorem from [14, theorem 3][2] and so, f admits an extension $\bar{f} \in C^\infty(\mathbb{R})$. $\square$

Definition 3.4. Let $\tau \colon [0,\infty] \to [0,1]$ be an increasing homeomorphism. For example, we can take $\tau(x) = \frac{2}{\pi} \arctan x$. For a perfect set $P \subset \mathbb{R}$ and $f \in D^n(P)$ let $\varphi^n_f \colon P^2 \to \mathbb{R}$ be defined as

$$\varphi^n_f(a,b) = \sum_{k=0}^{n} |q^{n-k}_{f^{(k)}}(a,b)| + \sum_{k=0}^{n} |q^{n-k}_{f^{(k)}}(b,a)|.$$

Also, for $\nu \in \omega \cup \{\infty\}$ let $C^\nu_*(P)$ be the class of all $f \in C(P)$ for which

(I) there exists a $k \leq \nu$ such that $f \in C^k(P)$, the map $\tau \circ \varphi^k_f \restriction P^2 \setminus \Delta$ is uniformly continuous, and either $k = \nu$ or else $k < \nu$ and, for all $x \in P$,

$$f^{(k+1)}(x) \stackrel{\text{df}}{=} \lim_{y \to x,\ y \in P} \frac{f^{(k)}(x) - f^{(k)}(y)}{x - y} = \pm\infty.$$

Notice that $C^0_*(P) = C(P)$.

Lemma 3.5. *Let $P \subset \mathbb{R}$ be perfect, $n \in \mathbb{N}$, and $f \in C^n(P)$. Then φ^n_f is continuous, if, and only if, $q^{n-i}_{f^{(i)}}$ is continuous for every $i \leq n$.*

[1]For every $i \leq n$ and compact $C \subset P$, the restriction $q^{n-i}_{f^{(i)}} \restriction C^2$ is uniformly continuous. So, if $\rho_i(C,\delta) = \sup\{|q^{n-i}_{f^{(i)}}(a,b) - q^{n-i}_{f^{(i)}}(a',b')| \colon a,a',b,b' \in C \ \&\ \max\{|a-a'|,|b-b'|\} \leq \delta\}$ for $\delta > 0$, then $\lim_{\delta \to 0} \rho_i(C,\delta) = 0$. Therefore, for every $a,b \in C$ with $0 < |a-b| \leq \delta$, we have $\left|\frac{(T^n_a f)^{(i)}(b) - (T^n_b f)^{(i)}(b)}{(a-b)^n}\right| = \left|q^{n-i}_{f^{(i)}}(a,b)\right| = \left|q^{n-i}_{f^{(i)}}(a,b) - q^{n-i}_{f^{(i)}}(a,a)\right| \leq \rho(C,\delta)$, the assumption of Whitney's Extension Theorem from [9, theorem 3.1.14].

[2]This result can be also deduced from Whitney's papers [20] and [19, §12]. See also the 1998 paper [15], where it is shown that the analogous result for functions on $\mathbb{R}^k$, $k \geq 2$, does not hold.

PROOF. Clearly continuity of all maps $q^{n-i}_{f^{(i)}}$ implies continuity of φ^n_f.

To prove the other implication, assume that φ^n_f is continuous and fix an $i \leq n$. We need to show that $q^{n-i}_{f^{(i)}}$ is continuous. Since $q^{n-i}_{f^{(i)}}$ is clearly continuous on $P^2 \setminus \Delta$ and $q^{n-i}_{f^{(i)}}(a,a) = 0$ for all $a \in P$, it remains to show that $\lim_{p\to\langle a,a\rangle} \left|q^{n-i}_{f^{(i)}}(p)\right| = 0$. But this follows from the continuity of φ^n_f, as

$$0 \leq \lim_{p\to\langle a,a\rangle} \left|q^{n-i}_{f^{(i)}}(p)\right| \leq \lim_{p\to\langle a,a\rangle} \varphi^n_f(p) = \varphi^n_f(a,a) = 0,$$

finishing the proof. □

As in [7], given a subset Y of a Polish space X, we let $\mathrm{Perf}(Y)$ denote the collection of all subsets of Y homeomorphic to the Cantor set $\mathfrak{C}$ (rather than all perfect subsets of Y). The key notion behind the axiom $\mathrm{CPA}_{\mathrm{prism}}$ is that of $\mathcal{F}_{\mathrm{prism}}$-density of the families of $\mathcal{E} \subset \mathrm{Perf}(X)$. The definition of this notion is based on the family $\mathbb{P} = \bigcup_{0<\alpha<\omega_1} \mathbb{P}_\alpha$ of compact perfect sets (each $\mathbb{P}_\alpha$ consisting of subsets of $\mathfrak{C}^\alpha$), whose precise description is not described here, since it is not essential to what follows. (It can be found in [7].)

We will refer to a set $P \in \mathrm{Perf}(X)$ as a *prism* if it comes with a continuous injection h (possibly given only implicitly) from an $E \in \mathbb{P}$ onto P. We say that $Q \in \mathrm{Perf}(X)$ is a *subprism* of a prism P provided $Q = h[E']$ for some $E' \subset E$ from $\mathbb{P}$. A family $\mathcal{E} \subset \mathrm{Perf}(X)$ is $\mathcal{F}_{\mathrm{prism}}$-dense provided for every prism P in $\mathrm{Perf}(X)$ there exists a subprism Q of P with $Q \in \mathcal{E}$. Now, we can state $\mathrm{CPA}_{\mathrm{prism}}$. (See [8] or [7].)

$\mathrm{CPA}_{\mathrm{prism}}$**:** $\mathfrak{c} = \omega_2$ and for every Polish space X and every $\mathcal{F}_{\mathrm{prism}}$-dense family $\mathcal{E} \subset \mathrm{Perf}(X)$ there is an $\mathcal{E}_0 \subset \mathcal{E}$ such that $|\mathcal{E}_0| \leq \omega_1$ and $|X \setminus \bigcup \mathcal{E}_0| \leq \omega_1$.

To use of $\mathrm{CPA}_{\mathrm{prism}}$ we will need the following two lemmas on $\mathcal{F}_{\mathrm{prism}}$-density.

Lemma 3.6. *Let X and Y be Polish spaces.*

(a) *Every prism $P \in \mathrm{Perf}(X \times Y)$ admits a subprism Q such that either $\pi_1 \restriction Q$ or $\pi_2 \restriction Q$ is one-to-one, where π_1 and π_2 are the projections of $X\times Y$ onto the first and the second coordinate, respectively. In particular,*

$$\mathcal{E}_0 = \{P \in \mathrm{Perf}(X \times Y)\colon \text{ either } \pi_1 \restriction P \text{ or } \pi_2 \restriction P \text{ is one-to-one}\}$$

is $\mathcal{F}_{\mathrm{prism}}$-dense.

(b) *Let $P \in \mathrm{Perf}(\mathbb{R}\times Y)$ be a prism and $\psi\colon P \to \mathbb{R}$ be a continuous function. Assume that $\pi_1 \restriction P$ is one-to-one, that is, that $\psi_P \stackrel{\mathrm{df}}{=} \psi \circ (\pi_1 \restriction P)^{-1} \in C(\pi_1[P])$. Then for every $n < \omega$ there exists a subprism Q of P such that $\psi_Q = \psi \circ (\pi_1 \restriction Q)^{-1} \in C^n_*(\pi_1[Q])$.*

(c) *If $X \subset \mathbb{R}$ is an interval and $\psi\colon X \times Y \to \mathbb{R}$ is continuous, then for every $n < \omega$ the families*

$$\mathcal{E}_n = \{P \in \operatorname{Perf}(X \times Y)\colon\ \pi_2 \restriction P \textit{ is one-to-one or } \pi_1 \restriction P \textit{ is one-to-one and } \psi_P \in C^n_*(\pi_1[P])\}$$

and $\mathcal{E}_\infty = \bigcap_{n<\omega} \mathcal{E}_n$ are $\mathcal{F}_{\text{prism}}$-dense.

Proof. (a). For $X = Y = \mathbb{R}$ this is proved in [7, proposition 4.1.3(b)]. (The condition [7, (4.5) p. 100] implies that either π_1 or π_2 is one-to-one on $h[E]$.) The same argument works for arbitrary Polish spaces X and Y.

(b). We will prove this by induction on $n < \omega$. Let P and ψ be as in the assumptions. For $n = 0$ this holds for $Q = P$, as $\psi \circ P \in C(\pi_1[P]) = C^0_*(\pi_1[P])$. So, assume that for some $n < \omega$ there is a subprism Q of P such that $\psi_Q \in C^n_*(\pi_1[Q])$. We need to find a subprism R of Q with $\psi_R \in C^{n+1}_*(\pi_1[R])$. Let $k \le n$ be the number from (I) justifying that $f = \psi_Q$ belongs to $C^n_*(\pi_1[Q])$. If $k < n$, then $R = Q$ is already our desired subprism of P belonging to $C^{n+1}_*(\pi_1[Q])$, as this is justified by the same k. So, we can assume that $k = n$, that is, that $f = \psi_Q \in C^n(Q)$ and $f^{(n)} \in C(Q)$.

Put $g = f^{(n)} = \psi_Q^{(n)}$ and let $\hat{h}\colon E \to Q$, with $E \in \mathbb{P}$, be the bijection making Q a prism. By [7, lemma 4.2.2] applied to g and $h = \pi_1 \circ \hat{h}$, there exists a subset $E' \in \mathbb{P}$ of E such that either $g \restriction \pi_1[h[E']]$ has the derivative with a constant value in $\{\infty, -\infty\}$, or else $g \restriction \pi_1[h[E']]$ admits an extension $\hat{f} \in C^1(\mathbb{R})$. Consider the subprism $R' = \hat{h}[E']$ of Q.

If $g' = f^{(n+1)} \equiv \pm\infty$ on $\pi_1[h[E']]$, then $\psi_{R'} \in C^{n+1}_*(\pi_1[R'])$, as this is justified by $k = n$. In particular, $R = R'$ is as needed.

Thus, assume that $g \restriction \pi_1[R']$ admits an extension $\hat{f} \in C^1(\mathbb{R})$. Then, the restriction of $g = f^{(n)} = \psi_Q^{(n)}$ to $\pi_1[R']$ has continuous derivative, that is, $\tilde{f} = \psi_{R'}$ is in $C^{n+1}(\hat{P})$, where $\hat{P} = \pi_1[R']$. In particular, the map

$$G = \tau \circ \varphi^{n+1}_{\tilde{f}} \circ \langle \pi_1, \pi_1 \rangle\colon (h[E'])^2 \setminus \Delta \to [0,1]$$

is well defined, symmetric, and continuous. Therefore, by [7, proposition 4.2.1], there exists a subset $E'' \in \mathbb{P}$ of E' such that G is uniformly continuous on $(h[E''])^2 \setminus \Delta$. Hence, for a subprism $R = h[E'']$ of Q, we have $\psi_R \in C^{n+1}_*(\pi_1[R])$ with $k = n+1$, since $\psi_R \in C^{n+1}(\pi_1[R])$ and $\tau \circ \varphi^k_{\psi_R} \restriction \pi_1[R]^2 \setminus \Delta$ is uniformly continuous.

(c). For $n < \omega$ this immediately follows from (a) and (b).

To see the $\mathcal{F}_{\text{prism}}$-density of $\mathcal{E}_\infty$ fix a prism $P \in \operatorname{Perf}(X \times Y)$. We need to find its subprism $Q \in \mathcal{E}_\infty$. If there is $n < \omega$ and a subprism $Q \in \mathcal{E}_n$ of P for which either $\pi_2 \restriction P$ is 1-to-1 or both $\pi_1 \restriction P$ is 1-to-1 and $\psi_P \in$

$C^n_*(\pi_1[P])\setminus C^n(\pi_1[P])$, then $Q\in\mathcal{E}_\infty$ is as desired. So, assume that this is not the case.

Let $h\colon E\to P$, with $E\in\mathbb{P}_\alpha$ and $0<\alpha<\omega_1$, be the bijection making P a prism. For every $n<\omega$ let

$$\mathcal{D}_n=\{E'\in\mathbb{P}_\alpha\colon\ E'\subset E \text{ and } \psi\circ(\pi_1\restriction h[E'])^{-1}\in C^n(\pi_1[h[E']])\}$$

and notice that for every $E'\in\mathbb{P}_\alpha$ contained in E there exists an $E''\in\mathcal{D}_n$ contained in E'. Indeed, $P'=h[E']$ is a subprism of P. Since $\mathcal{E}_n$ is $\mathcal{F}_{\rm prism}$-dense, there exists a subprism P'' of P' with $P''\in\mathcal{E}_n$. Then $P''=h[E'']$ for some $E''\in\mathbb{P}_\alpha$ contained in E'. Also, by our assumption, $P''\in\mathcal{E}_n$ must be justified by the fact that $\pi_1\restriction P''$ is 1-to-1 and $\psi_{P''}\in C^n(\pi_1[P''])$, ensuring that E'' is indeed in $\mathcal{D}_n$.

The above argument shows that the families $\mathcal{D}_n$ satisfy the assumptions of [7, corollary 3.1.3]. Therefore, by [7, corollary 3.1.3], there exists an $E_0\in\bigcap_{n<\omega}\mathcal{D}_n$. In particular, $Q=h[E_0]$ is a subprism of P with $\psi\circ(\pi_1\restriction Q)^{-1}\in\bigcap_{n<\omega}C^n(\pi_1[Q])\subset\bigcap_{n<\omega}C^n_*(\pi_1[Q])$, that is, $Q\in\mathcal{E}_\infty$, as needed. $\square$

Fix a $\nu\in\omega\cup\{\infty\}$ and an interval $I=[a,b]$. We will considered $C^\nu(I)$ with the metric:

$$\rho(f,g)=\sum_{i\le n}\|f^{(i)}-g^{(i)}\|_\infty$$

when $\nu<\omega$, and with metric

$$\rho(f,g)=\sum_{i<\omega}2^{-i}\min\{1,\|f^{(i)}-g^{(i)}\|_\infty\}$$

when $\nu=\infty$. Notice that $C^\nu(I)$, with such metric, is a Polish space (for finite ν see, e.g., [11, example 5.4]. For the general case we refer the interested reader to the calculations from either [18, pages 550 and 562] or [16, example 1.46], or [22, §1]). In particular, $I\times C^\nu(I)$ is a Polish space. Define function $\psi^\nu_I\colon I\times C^\nu(I)\to\mathbb{R}$ by $\psi^\nu_I(x,g)=g(x)$ and notice that it is continuous. Also, for $S\subset I\times C^\nu(I)$ and $g\in C^\nu(I)$ let

$$S^g=\{x\in I\colon\langle x,g\rangle\in S\}.$$

Lemma 3.7. *Let $I=[a,b]$, $\nu\in\omega\cup\{\infty\}$, and $\mathcal{E}_\infty$ be as in Lemma 3.6(c) used with $Y=C^\nu(I)$ and $\psi=\psi^\nu_I$. Then, for every $Q\in\mathcal{E}_\infty$ there exists an $f_Q\in C^\nu(\mathbb{R})$ such that*

($\star$) *$g\restriction Q^g\subset^\star f_Q$ for every $g\in C^\nu(I)$.*

PROOF. Fix a $Q \in \mathcal{E}_\infty$. If Q^g is countable for every $g \in C^\nu(I)$, then every function $f_Q \in C^\nu(\mathbb{R})$ satisfies $(\star)$. So, assume that there exists a $g_0 \in C^\nu(I)$ such that Q^{g_0} is uncountable. In particular, $\pi_2 \restriction Q$ is not one-to-one, so $\pi_1 \restriction Q$ is one-to-one and $\psi_Q \in C^\nu_*(\pi_1[Q])$. In fact,

$$\psi_Q \in C^\nu(\pi_1[Q]).$$

To see this, first note that

$$g \restriction Q^g = \psi_Q \restriction Q^g \text{ for every } g \in C^\nu(I),$$

since for every $x \in Q^g$ we have $\psi_Q(x) = \psi(x, g) = g(x)$. In particular, if P is a perfect subset of Q^{g_0}, then, for every number $k < \nu$, we have $(\psi_Q \restriction P)^{(k+1)} = (g_0 \restriction P)^{(k+1)} \in C(P)$. Thus, for every $x \in P$, $(\psi_Q)^{(k+1)}(x) = g_0^{(k+1)}(x) \in \mathbb{R}$ so that $(\psi_Q)^{(k+1)}(x)$ cannot be equal to $\pm\infty$. But, by the definition of $C^\nu_*(\pi_1[Q])$, this can happen only when $\psi_Q \in C^\nu(\pi_1[Q])$.

Next, let

$$\hat{Q} = \{q \in Q \colon (\forall \text{ open } U \subset I \times C^\nu(I))\ q \in U \Rightarrow (\exists g \in C^\nu(I))\ |(U \cap Q)^g| = \mathfrak{c}\}.$$

Notice, that $\hat{Q}$ is closed in Q, $\pi_1[\hat{Q}] \subset I$ is perfect, and $\psi_{\hat{Q}} \in C^\nu(\pi_1[\hat{Q}])$. Moreover,

$$g \restriction Q^g = \psi_Q \restriction Q^g \subset^\star \psi_{\hat{Q}} \restriction \hat{Q}^g \text{ for every } g \in C^\nu(I),$$

since $Q^g \setminus \hat{Q}^g$ is at most countable, as $(I \times C^\nu(I)) \setminus \hat{Q}$ is a countable union of sets W for which W^g is countable. We will show that $\psi_{\hat{Q}}$ satisfies the assumptions of Theorem 3.3, our version of Whitney's Extension Theorem; that is, that for every $n \in \mathbb{N}$ with $n \leq \nu$ the property (W_n) holds. Notice, that this will finish the proof of the lemma, since then there exists an $f_Q \in C^\nu(\mathbb{R})$ extending $\psi_{\hat{Q}}$ and so,

$$g \restriction Q^g = \psi_Q \restriction Q^g \subset^\star \psi_{\hat{Q}} \restriction \hat{Q}^g \subset \psi_{\hat{Q}} \subset f_Q \text{ for every } g \in C^\nu(I),$$

the desired property $(*)$.

So, fix an $n \in \mathbb{N}$ with $n \leq \nu$. To see that (W_n) holds, by Lemma 3.5, it is enough to show that $\varphi^n_{\psi_{\hat{Q}}}$ is continuous.

To see this, first notice that $\psi_Q \in C^n(\pi_1[Q]) \cap C^n_*(\pi_1[Q])$. Thus, the map $\tau \circ \varphi^n_{\psi_Q} \restriction (\pi_1[Q])^2 \setminus \Delta$ is uniformly continuous. In particular, it has a unique continuous extension $F \colon (\pi_1[Q])^2 \to [0,1]$ and $\tau^{-1} \circ F \colon (\pi_1[Q])^2 \to [0,\infty]$ is a continuous extension of $\varphi^n_{\psi_Q} \restriction (\pi_1[Q])^2 \setminus \Delta$. Therefore, $\varphi = \tau^{-1} \circ F \restriction (\pi_1[\hat{Q}])^2$

is a continuous extension of $\varphi^n_{\psi_{\hat{Q}}} \restriction (\pi_1[\hat{Q}])^2 \setminus \Delta$. It remains to show that $\varphi = \varphi^n_{\psi_{\hat{Q}}}$, that is, that $\varphi(a,a) = 0 = \varphi^n_{\psi_{\hat{Q}}}(a,a)$ for every $a \in \pi_1[\hat{Q}]$.

To see this, let $g \in C^\nu(I)$ be such that $\langle a, g\rangle \in \hat{Q}$. Since $\psi_{\hat{Q}} \in C^\nu(\pi_1[\hat{Q}])$, it is enough to show that for every open $U \subset I \times C^\nu(I)$ containing $\langle a, g\rangle$ there exists a $\langle \hat{a}, \hat{g}\rangle \in U \cap \hat{Q}$ such that $\varphi(\hat{a}, \hat{a}) = 0$.

So, fix an open $U \subset I \times C^\nu(I)$ containing $\langle a, g\rangle$. By the definition of $\hat{Q}$, there exists a $\hat{g} \in C^\nu(I)$ such that $(U \cap Q)^{\hat{g}}$ is uncountable. Let $P \subset (U \cap Q)^{\hat{g}}$ be a perfect set and notice that $\psi_{\hat{Q}} \restriction P = \hat{g} \restriction P$ has an extension $\hat{g} \in C^\nu(I)$, which clearly can be further extended to a $\bar{g} \in C^\nu(\mathbb{R})$. In particular, by Theorem 3.3, the map $\varphi^n_{\hat{g}\restriction P}\colon P^2 \to \mathbb{R}$ is continuous. So, for every $\hat{a} \in P$ we have $\langle \hat{a}, \hat{g}\rangle \in U \cap \hat{Q}$ and $\varphi(\hat{a}, \hat{a}) = \lim_{b\to\hat{a}, b\in P} \varphi(\hat{a}, b) = \lim_{b\to\hat{a}, b\in P} \varphi^n_{\psi_{\hat{Q}}}(\hat{a}, b) = \lim_{b\to\hat{a}, b\in P} \varphi^n_{\hat{g}\restriction P}(\hat{a}, b) = \varphi^n_{\hat{g}\restriction P}(\hat{a}, \hat{a}) = 0$, as needed. $\square$

The following lemma will be used in the proof of Theorem 2.2.

Lemma 3.8. *Let $\psi\colon Q^2 \setminus \Delta \to [-\infty, \infty]$ be continuous, where $Q \subset \mathbb{R}$ is perfect. If*

(a) $\delta_1, \delta_2\colon Q^2 \cap \Delta \to [-\infty, \infty]$ are continuous,

(b) $\psi_1 = \psi \cup \delta_1$ is continuous with respect to the first variable, and

(c) $\psi_2 = \psi \cup \delta_2$ is continuous with respect to the second variable,

then $\delta_1 = \delta_2$.

Proof. Replacing $[-\infty, \infty]$ with its homeomorphic bounded copy, say $[-1, 1]$, if necessary, we can always assume that the ranges of ψ, δ_1, and δ_2 have only finite values.

By way of contradiction, assume that $\delta_1 \neq \delta_2$. Then, there exists an $\varepsilon > 0$ and an open non-empty set $U \subset Q$ such that[3]

$$|\delta_1(p,p) - \delta_2(q,q)| > \varepsilon \text{ for every } p, q \in U. \tag{3}$$

Since ψ_1 is continuous with respect to the first variable (by item (b) above), we have that for every $q \in U$ there is an $n_q \in \mathbb{N}$ such that $|\psi_1(q,q) - \psi_1(p,q)| < \varepsilon/2$ for every $p \in Q$ with $|p - q| < 1/n_q$. On the other hand, since U is of second category in Q, there is an $n \in \mathbb{N}$ such that $Z = \{q \in U\colon n_q = n\}$ is dense in some non-empty open subset V of U. Now, choose $p \in V$. Since ψ_2 is continuous with respect to the second variable (by item (c) above),

[3]Indeed, if $r \in Q$ is such that $\delta_1(r,r) \neq \delta_2(r,r)$ and $\varepsilon = |\delta_1(r,r) - \delta_2(r,r)|/3$, then the set $U = \{p \in Q\colon |\delta_1(p,p) - \delta_1(r,r)| < \varepsilon \ \&\ |\delta_2(p,p) - \delta_2(r,r)| < \varepsilon\}$ is as needed.

there exists a non-empty open subset W of V containing p and such that $|\psi_2(p,p)-\psi_2(p,q)|<\varepsilon/2$ for every $q\in W$. Next, choose $q\in W\cap Z$ such that $0<|p-q|<1/n$. Then $|\psi_1(q,q)-\psi_1(p,q)|<\varepsilon/2$, as $|p-q|<1/n=1/n_q$. In particular,

$$|\delta_1(p,p)-\delta_2(q,q)|\le|\psi_1(p,p)-\psi_1(p,q)|+|\psi_2(p,q)-\psi_2(q,q)|<\varepsilon,$$

contradicting (3). □

4 Proofs of Theorems 2.1 and 2.2

Proof of Theorem 2.1. Fix a $\nu\in\omega\cup\{\infty\}$ and an interval $I=[a,b]$. We will use $\mathrm{CPA}_{\mathrm{prism}}$ for the Polish space $I\times C^\nu(I)$. The idea of this proof comes from the proof of the existence of a family $\mathcal{G}$ from Proposition 1.1(b), see [7, theorem 4.1.1].

Let $\psi=\psi_I^\nu$ be as in Lemma 3.7. By Lemma 3.6(c) used with this ψ, the family $\mathcal{E}_\infty$ is $\mathcal{F}_{\mathrm{prism}}$-dense. Therefore, by $\mathrm{CPA}_{\mathrm{prism}}$, there is an $\mathcal{E}\subset\mathcal{E}_\infty$ such that $|\mathcal{E}|\le\omega_1$ and $|(I\times C^\nu(I))\setminus\bigcup\mathcal{E}|\le\omega_1$, that is, $I\times C^\nu(I)\subset^\star\bigcup\mathcal{E}$.

By Lemma 3.7, for every $Q\in\mathcal{E}$ there exists an $f_Q\in C^\nu(\mathbb{R})$ satisfying $(\star)$. We claim that the family

$$\mathcal{F}_\nu^I=\{f_Q\colon Q\in\mathcal{E}\}$$

is as needed.

Indeed, for every $g\in C^\nu(I)$ we have $I\times\{g\}\subset I\times C^\nu(I)\subset^\star\bigcup\mathcal{E}=\bigcup_{Q\in\mathcal{E}}Q$. So, using $(\star)$, the fact that $|\mathcal{E}|\le\omega_1$, and (1), we obtain

$$g\subset^\star g\restriction\left(\bigcup_{Q\in\mathcal{E}}Q^g\right)=\bigcup_{Q\in\mathcal{E}}(g\restriction Q^g)\subset^\star\bigcup_{Q\in\mathcal{E}}f_Q=\bigcup\mathcal{F}_\nu^I,$$

as desired. □

Proof of Theorem 2.2. Fix an $n\in\mathbb{N}$ and a $g\in D^n(\mathbb{R})$. Let

$$\mathcal{E}_n=\left\{Q\in\mathrm{Perf}(\mathbb{R})\colon g\restriction Q\in C^n(Q)\ \&\ \varphi^n_{g\restriction Q}\in C(Q^2)\right\}$$

and notice that it is $\mathcal{F}_{\mathrm{prism}}$-dense in the Polish space $X=\mathbb{R}$. To see this, choose an arbitrary prism $P\in\mathrm{Perf}(\mathbb{R})$. Since $g^{(n)}\colon\mathbb{R}\to\mathbb{R}$ is Borel (in fact, Baire class one), using [7, lemma 4.2.2] we can find a subprism P' of P such that $g^{(n)}\restriction P'\in C(P')$. Then, $g\restriction P'\in C^n(P')$. Furthermore, similarly as in the proof of the part (b) of Lemma 3.6, we can find a subprism Q of P'

such that $\varphi^n_{g\restriction Q} \in C(Q^2)$. More specifically, since $g \restriction P' \in C^n(P')$, the map $G = \tau \circ \varphi^n_{g\restriction P'} \colon (P')^2 \setminus \Delta \to [0,1]$ is well defined, symmetric, and continuous. Hence, by [7, proposition 4.2.1], there exists a subprism Q of P' such that G is uniformly continuous on $Q^2 \setminus \Delta$. In particular, it has a unique continuous extension $\bar{G} \colon Q^2 \to [0,1]$ and $\tau^{-1} \circ \bar{G} \colon Q^2 \to [0,\infty]$ is a continuous extension of $\varphi^n_{g\restriction Q} \restriction Q^2 \setminus \Delta$. Thus, to prove that $\mathcal{E}_n$ is $\mathcal{F}_{\text{prism}}$-dense, it remains to show that $\varphi^n_{g\restriction Q}$ equals to the continuous map $\tau^{-1} \circ \bar{G}$ that is, that

$$\tau^{-1} \circ \bar{G} \restriction \Delta \equiv 0. \tag{4}$$

To see this, define function $\Psi \colon Q^2 \to \mathbb{R}$ as

$$\Psi(a,b) = \sum_{k=0}^{n} |q^{n-k}_{(g\restriction Q)^{(k)}}(a,b)|.$$

Then, we have that

$$\tau^{-1} \circ \bar{G}(a,b) = \varphi^n_{g\restriction Q}(a,b) = \Psi(a,b) + \Psi(b,a),$$

for every $\langle a,b\rangle \in Q^2 \setminus \Delta$. We will prove (4) by applying Lemma 3.8 to the maps $\psi = \Psi \restriction Q^2 \setminus \Delta$ and $\delta_1, \delta_2 \colon Q^2 \cap \Delta \to \mathbb{R}$, where $\delta_2 \equiv 0$ and $\delta_1 = \tau^{-1} \circ \bar{G} \restriction Q^2 \cap \Delta$. The map $\psi_2 = \psi \cup \delta_2 = \Psi$ is continuous with respect to the second variable (by (i) from Proposition 3.2). The map $\psi_1 = \psi \cup \delta_1$ is continuous with respect to the first variable, since $\tau^{-1} \circ \bar{G} - \psi_2$ is continuous with respect to the second variable and $\psi_1(a,b) = (\tau^{-1} \circ \bar{G} - \psi_2)(b,a)$ for every $\langle a,b\rangle \in Q^2$. Hence, by Lemma 3.8, $\delta_1 = \delta_2$. Thus, for every $a \in Q$ we have

$$\tau^{-1} \circ \bar{G}(a,a) = \lim_{b \to a,\, b \in Q} (\psi(a,b) + \psi(b,a)) = \delta_1(a,a) + \delta_2(a,a) = 0,$$

obtaining, as desired, (4) and $\mathcal{F}_{\text{prism}}$-density of $\mathcal{E}_n$.

Since $\mathcal{E}_n$ is $\mathcal{F}_{\text{prism}}$-dense, by $\text{CPA}_{\text{prism}}$ there is an $\mathcal{E} \subset \mathcal{E}_n$ such that $|\mathcal{E}| \le \omega_1$ and $|\mathbb{R} \setminus \bigcup \mathcal{E}| \le \omega_1$, that is, $\mathbb{R} \subset^\star \bigcup \mathcal{E}$. Therefore, $g \subset^\star \bigcup_{Q \in \mathcal{E}_n} g \restriction Q$.

Now, for every $Q \in \mathcal{E}_n$, $g \restriction Q \in C^n(Q)$ and $\varphi^n_{g\restriction Q}$ is continuous. Thus, property (W_n) holds and, by Theorem 3.3, there is an $f_Q \in C^n(\mathbb{R})$ extending $g \restriction Q$. Hence, if $\mathcal{F} = \{f_Q \colon Q \in \mathcal{E}\}$, then

$$g \subset^\star \bigcup_{Q \in \mathcal{E}_n} g \restriction Q \subset \bigcup_{Q \in \mathcal{E}_n} f_Q = \bigcup \mathcal{F}.$$

For every $p \in g \setminus \bigcup \mathcal{F}$ choose an $f_p \in C^n(\mathbb{R})$ containing p. Then the family $\mathcal{F}_g = \mathcal{F} \cup \{f_p \colon p \in g \setminus \bigcup \mathcal{F}\}$ is as needed. This finishes the proof. $\square$

5 Proof of Theorem 2.3

We start with the following lemma. It is well known for $X=\mathbb{R}$ (see [21, 4] and [3, §14]). However, we need it for arbitrary subsets X of $\mathbb{R}$. Note that the $F_{\sigma\delta}$ Borel rank of $\mathrm{Dif}(g)$ is the best possible even for $X=\mathbb{R}$, as shown by Zahorski in 1941 (see [21] for the 1946 French edition of his work).

Lemma 5.1. *For every subset X of $\mathbb{R}$ with no isolated points and every $g\in C(X)$ the set $\mathrm{Dif}(g)$ of points of differentiability of g is an $F_{\sigma\delta}$ subset of X.*

Proof. It is not difficult to see that an $x\in X$ belongs to $\mathrm{Dif}(g)$ (that is, the finite derivative $g'(x)=\lim_{y\to x,\ y\in X}\frac{g(y)-g(x)}{y-x}$ exists) if, and only if, x belongs to $S\stackrel{\text{df}}{=}\bigcap_{k\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}S^n_k$, where S^n_k is defined as the set of all $x\in X$ such that

$$\left|\tfrac{g(x)-g(y)}{x-y}-\tfrac{g(x)-g(y')}{x-y'}\right|\le 2^{-k} \text{ for all } y,y'\in X\cap\left(x-\tfrac{1}{n},x+\tfrac{1}{n}\right)\setminus\{x\}.$$

Indeed, the inclusion $\mathrm{Dif}(g)\subset S$ is obvious. To see the other inclusion, take an $x\in S$. For every $k\in\mathbb{N}$ the sequence $\langle S^n_k\rangle_n$ is ascending. So, there is an $n_k\ge k$ for which $x\in S^{n_k}_k$. Pick an $x_k\in X\cap\left(x-\frac{1}{n_k},x+\frac{1}{n_k}\right)\setminus\{x\}$. Then $x\in S$ implies that the sequence $\langle\frac{g(x)-g(x_k)}{x-x_k}\rangle_k$ is Cauchy, so we can define $g'(x)=\lim_{k\to\infty}\frac{g(x)-g(x_k)}{x-x_k}$. This value does not depend on the choice of the sequence since $x\in S$. So, indeed $x\in\mathrm{Dif}(g)$.

To finish the proof, it is enough to show that each set S^n_k is closed in X. Indeed, $y,y'\in X$ belong to $\left(x-\frac{1}{n},x+\frac{1}{n}\right)\setminus\{x\}$ if, and only if, $x\in X$ belongs to the X-open set $U^n_{y,y'}\stackrel{\text{df}}{=}X\cap\left(y-\frac{1}{n},y+\frac{1}{n}\right)\cap\left(y'-\frac{1}{n},y'+\frac{1}{n}\right)\setminus\{y,y'\}$. Therefore S^n_k is equal to $\bigcap_{y,y'\in X}S^n_k(y,y')$, where

$$\begin{aligned}S^n_k(y,y') &= \left\{x\in X\colon x\in U^n_{y,y'}\Rightarrow\left|\tfrac{g(x)-g(y)}{x-y}-\tfrac{g(x)-g(y')}{x-y'}\right|\le 2^{-k}\right\}\\ &= \left\{x\in X\colon \left|\tfrac{g(x)-g(y)}{x-y}-\tfrac{g(x)-g(y')}{x-y'}\right|\le 2^{-k}\right\}\cup\left(X\setminus U^n_{y,y'}\right)\end{aligned}$$

is closed in X, as a union of two closed sets. So, indeed $S^n_k=\bigcap_{y,y'\in X}S^n_k(y,y')$ is closed in X. □

We will also need the following two lemmas.

Lemma 5.2. *For every with $X\subset\mathbb{R}$ with no isolated points, if $f\in D^1(X)$, then the derivative $f'\colon X\to\mathbb{R}$ is Borel. In fact it is of Baire class 2.*

PROOF. Let $\{x_k\colon k<\omega\}$ be an enumeration, with no repetition, of a dense subset of X. For every $n\in\mathbb{N}$ let $j_n\colon X\to\omega$ be defined, for every $x\in X$, as

$$j_n(x)=\min\{j<\omega\colon 0<|x-x_j|<1/n\}.$$

Notice that, for every $k<\omega$,

$$j_n^{-1}(\{0,\ldots,k\})=X\cap\bigcup_{i\le k}\big((x_i-1/n,x_i+1/n)\setminus\{x_i\}\big)$$

is open in X. Thus, $j_n^{-1}(k)=j_n^{-1}(\{0,\ldots,k\})\setminus j_n^{-1}(\{0,\ldots,k-1\})$ is both F_σ- and G_δ-set in X.

Let $q_n\colon X\to\mathbb{R}$ be defined, for every $x\in X$, as $q_n(x)=\dfrac{f(x)-f(x_{j_n(x)})}{x-x_{j_n(x)}}$. Notice that, for every open set $U\subset\mathbb{R}$,

$$q_n^{-1}(U)=\bigcup_{k\in\mathbb{N}}\left(j_n^{-1}(k)\cap\left\{x\in X\setminus\{x_k\}\colon \frac{f(x)-f(x_k)}{x-x_k}\in U\right\}\right)$$

is an F_σ-set in X. Finally it is clear that $f'(x)=\lim\limits_{n\to\infty}q_n(x)$ for all $x\in X$. In particular, since every open set $U\subset\mathbb{R}$ can be represented as $U=\bigcup_{j<\omega}B_j=\bigcup_{j<\omega}\mathrm{cl}(B_j)$ for some open intervals B_j, the set

$$(f')^{-1}(U)=\bigcup_{j<\omega}\bigcup_{m\in\mathbb{N}}\bigcap_{n>m}q_n^{-1}(\mathrm{cl}(B_j))$$

is a $G_{\delta\sigma}$-set in X. Therefore, indeed, f' is a Borel and a Baire class 2 map. $\square$

Lemma 5.3. *Assume that $X\subset\mathbb{R}$ has no isolated points and that $X\subset B\subset\mathrm{cl}(X)$. Let $\bar\gamma\in C(B)$ be an extension of $\gamma\in C(X)$. Then $\mathrm{Dif}(\gamma)\subset\mathrm{Dif}(\bar\gamma)$. In particular, if $\gamma\in D^1(X)$, then $X\subset\mathrm{Dif}(\bar\gamma)$.*

PROOF. Assume that $x\in\mathrm{Dif}(\gamma)$. Then $\gamma'(x)=\lim_{y\to x,y\in X}\frac{\gamma(x)-\gamma(y)}{x-y}$ exists and we need to show that $\lim_{b\to x,b\in B}\frac{\bar\gamma(x)-\bar\gamma(b)}{x-b}=\gamma'(x)$. To see this, fix an $\varepsilon>0$ and let $\delta>0$ be such that $\left|\frac{\gamma(x)-\gamma(y)}{x-y}-\gamma'(x)\right|\le\varepsilon$ for all $y\in X$ with $0<|y-x|<\delta$. Then, for every $b\in B$ with $0<|b-x|<\delta$ we have

$$\left|\frac{\bar\gamma(x)-\bar\gamma(b)}{x-b}-\gamma'(x)\right|=\lim_{y\to b,y\in X}\left|\frac{\gamma(x)-\gamma(y)}{x-y}-\gamma'(x)\right|\le\varepsilon.$$

Thus, indeed,

$$\lim_{b\to x,b\in B}\frac{\bar\gamma(x)-\bar\gamma(b)}{x-b}=\gamma'(x)$$

and $x\in\mathrm{Dif}(\bar\gamma)$. $\square$

In the proof of Theorem 2.3 we will use the following result only for $\nu = 1$. However the result for all values of ν seem to be unknown and of independent interest, so we include it in its full generality.

Theorem 5.4. *Let $X \subset \mathbb{R}$ be with no isolated points, $g \in C(X)$, and let $\bar{g} \in C(G)$ be an extension of g, where G is a G_δ-set in $\mathbb{R}$ with $X \subset G \subset \mathrm{cl}(X)$. For every $\nu \in \omega \cup \{\infty\}$, if $g \in D^\nu(X)$, then there is a Borel set $B_\nu \subset G$ containing X such that $\bar{g} \restriction B_\nu \in D^\nu(B_\nu)$.*

Proof. First, we will prove the theorem, by induction, for $\nu = n < \omega$.

For $n = 0$ the theorem is true, since by a well known result (see [12, §3.B]) we have: *for any $g \in D^0(X) = C(X)$, where $X \subset \mathbb{R}$, there exists a G_δ-set $G \subset \mathbb{R}$ with $X \subset G \subset \mathrm{cl}(X)$ and an extension $\bar{g} \in C(G) = D^0(G)$ of g.* Then, $B_0 = G$ is as desired.

Next, choose an $n < \omega$ for which the theorem is true. We need to show that the theorem holds for $n + 1$. For this, fix an $X \subset \mathbb{R}$ with no isolated points and $g \in D^{n+1}(X)$. By the inductive assumption, there exists a Borel set $B_n \subset G$ containing X for which $\bar{g} \restriction B_n \in D^n(B_n)$. We shall find a Borel set $B_{n+1} \subset B_n$ containing X for which $\bar{g} \restriction B_{n+1} \in D^{n+1}(B_{n+1})$. Note that, by $\bar{g} \restriction B_n \in D^n(B_n)$, we have $\bar{g}^{(i)} \restriction X = g^{(i)}$ for all $i \leq n$.

To find B_{n+1} first notice that

$$\text{there is a Borel } \hat{B}_n \subset B_n \text{ containing } X \text{ such that } \bar{g} \restriction \hat{B}_n \in C^n(\hat{B}_n). \tag{5}$$

For $n = 0$ the property (5) holds with $\hat{B}_n = B_n$, since $D^0(B_0) = C^0(B_0)$. Thus, let us assume that $n > 0$.

Let $\gamma = g^{(n-1)}$. Then $\gamma \in C^1(X)$ and $\gamma' = g^{(n)} \in C(X)$. Next, let $h \in C(\hat{G})$ be an extension of γ' such that $\hat{G} \subset G$ is a G_δ-set with

$$X \subset \hat{G} \subset \mathrm{cl}(X).$$

If we now make $\tilde{B}_n = \hat{G} \cap B_n$, then $\tilde{B}_n$ is Borel and

$$X \subset \tilde{B}_n \subset \mathrm{cl}(X).$$

Put $\bar{\gamma} = \bar{g}^{(n-1)} \restriction \tilde{B}_n$. Then $\bar{\gamma} \in D^1(\tilde{B}_n)$ extends $\gamma \in C^1(X)$. Thus, both $h \restriction \tilde{B}_n$ and, by Lemma 5.2, $\bar{\gamma}'$ are Borel maps. Therefore, the set

$$\hat{B}_n = \{x \in \tilde{B}_n : \bar{\gamma}'(x) = h(x)\} \supset X$$

is Borel. In particular, $\bar{g} \restriction \hat{B}_n \in C^n(\hat{B}_n)$ and (5) holds, since

$$(\bar{g} \restriction \hat{B}_n)^{(n)} = (\bar{g}^{(n-1)})' \restriction \hat{B}_n = \bar{\gamma}' \restriction \hat{B}_n = h \restriction \hat{B}_n \in C(\hat{B}_n).$$

To finish induction notice that, by Lemma 5.1, $B_{n+1} = \mathrm{Dif}((\bar{g} \restriction \hat{B}_n)^{(n)})$ is Borel in $\hat{B}_n$ and so is in $\mathbb{R}$. This $\hat{B}_n$ contains X by Lemma 5.3, since $(\bar{g} \restriction \hat{B}_n)^{(n)} \in C(\hat{B}_n)$ extends $g^{(n)} \in D^1(X)$. In particular,

$$(\bar{g} \restriction B_{n+1})^{(n)} = (\bar{g} \restriction \hat{B}_n)^{(n)} \restriction B_{n+1} \in D^1(B_{n+1}),$$

that is, $\bar{g} \restriction B_{n+1} \in D^{n+1}(B_{n+1})$. This finishes the induction.

Finally notice that if $g \in D^\infty(X)$, then the sets B_n are well defined and $B_\infty = \bigcap_{n<\omega} B_n$ is as needed. □

PROOF OF THEOREM 2.3. Let $n \in \{0,1\}$, $X \subset \mathbb{R}$, and $g \in D^n(X)$. If X is countable, then any family $\mathcal{F}_g \subset C^n(X)$ of right cardinality satisfies the requirements. So, we can assume that X is uncountable. Also, removing from X a countable set, if necessary, we can assume that that X has no isolated points. Then, by Theorem 5.4, there exist a Borel set B with $X \subset B \subset \mathbb{R}$ and an extension $\hat{g} \in D^n(B)$ of g.

By $\mathrm{CPA}_{\mathrm{prism}}$, see [7, fact 1.1.7], there exists a family $\mathcal{P}$ of cardinality $\leq \omega_1$ of compact subsets of B such that $B = \bigcup \mathcal{P}$. Then, for every $P \in \mathcal{P}$, we have $\hat{g} \restriction P \in D^n(P)$ and there exists an extension $g_P \in D^n(\mathbb{R})$ of $\hat{g} \restriction P$: for $n = 0$ this is ensured by Tietze Extension theorem, while for $n = 1$ by a theorem of Jarník from [5]. Therefore, by Theorem 2.2, there exists a family $\mathcal{F}_P \subset C^n(\mathbb{R})$ of cardinality $\omega_1 < \mathfrak{c}$ such that $g_P \subset \bigcup \mathcal{F}_P$. Then, for the family $\mathcal{F}_g = \bigcup_{P \in \mathcal{P}} \mathcal{F}_P$ we have

$$g \subset \hat{g} = \bigcup_{P \in \mathcal{P}} \hat{g} \restriction P \subset \bigcup_{P \in \mathcal{P}} g_P \subset \bigcup_{P \in \mathcal{P}} \bigcup \mathcal{F}_P = \bigcup \mathcal{F}_g$$

as needed. □

6 Constructions justifying Examples 1.2 and 1.3

CONSTRUCTION JUSTIFYING EXAMPLE 1.2. For $g \in D^1(\mathbb{R})$ let $\mathcal{F}_g$ be as in (a) of Proposition 1.1. Let $M \in \mathbb{R}$ be such that $g(x) < M$ for all $x \in [-2,2]$. For every $f \in \mathcal{F}_g$ let $U_f = \{x \in (-2,2) \colon f(x) > M\}$, put $P_f = \mathbb{R} \setminus U_f$, and notice that

- there exists an extension $\bar{f} \in C^1(\mathbb{R})$ of $f \restriction P_f$ such that $\bar{f}(x) < M+1$ for all $x \in [-1,1]$.

This follows from a version of Whitney's extension theorem (see [19, 20]), proved in [5]. First let $\hat{f} \colon [-1,1] \to (-\infty, M+1)$ be a C^1 extension of $f \restriction P_f \cap [-1,1] \to (-\infty, M]$. Then there exists an extension $\bar{f} \in C^1(\mathbb{R})$ of $(f \restriction P_f) \cup \hat{f}$ which satisfies property •.

Now, the collection of all $\bar{\mathcal{F}}_g = \{\bar{f}\colon f \in \mathcal{F}_g\}$ is as needed. Indeed, $g \subset \bigcup \bar{\mathcal{F}}_g$ since $g \subset \bigcup \mathcal{F}_g$ and for every $f \in \mathcal{F}_g$ we have $g \cap f \subset g \cap \bar{f}$.[4] On the other hand, if $h \equiv M+1$, then $h \setminus \bigcup \bar{\mathcal{F}}_g \supset h \restriction [-1,1]$. In particular, no single $\bar{\mathcal{F}}_g$ can serve as the family $\mathcal{F}_1$ in Main Theorem. $\square$

SKETCH OF THE CONSTRUCTION JUSTIFYING EXAMPLE 1.3. This argument is a refinement of the construction of the family $\mathcal{G}$ from part (b) of Proposition 1.1, as presented in [8] and [7, theorem 4.1.1(b)]. Let $\mathfrak{C} \subset \mathbb{R}$ be the Cantor ternary set and choose a non-injective $H \in C^1(\mathbb{R})$ containing $(h \restriction \mathfrak{C})^{-1}$ and disjoint with $h \restriction \mathfrak{C}$. Note that $Y = H \cup H^{-1}$ is a closed subset of $\mathbb{R}^2$ containing the set $K = (h \restriction \mathfrak{C})^{-1} \cup (h \restriction \mathfrak{C})$. Using $\mathrm{CPA}_{\mathrm{prism}}$ to the Polish space $X = \mathbb{R}^2 \setminus Y$ (and the family of perfect subsets of X that belong to the family $\mathcal{E}$ considered in the proof of [7, theorem 4.1.1(b)]) we can find a cover $\mathcal{E}_0$ of X of size ω_1 consisting of pairwise disjoint compact sets such that for each $P \in \mathcal{E}_0$ there is a $\hat{P} \in \{P, P^{-1}\}$ which can be extended to a map $f_P \in C^1(\mathbb{R})$. By a C^1 version of Whitney's extension theorem from [5], we can assume that each extension f_P of $\hat{P}$ is not one-to-one and is disjoint with the zero-dimensional set K. (Since $\hat{P}$ is disjoint with K, we can first find a finite non-injective extension g_P of $\hat{P}$ whose linear interpolation $\bar{g}_P$ is disjoint with K. Then there exists a small modification $\bar{f}_P$ of $\bar{g}_P$ – of size smaller than the distance between $\bar{f}_P$ and K – which is an extension of $g_P \supset \hat{P}$ and disjoint with K.) The family $\mathcal{G} = \{f_P\colon P \in \mathcal{E}_0\} \cup \{H\}$ is as needed.

Indeed, clearly $\mathcal{G} \subset C(\mathbb{R})$ has cardinality ω_1. Also $\mathbb{R}^2 = \bigcup_{f\in\mathcal{G}}(f \cup f^{-1})$, since $\mathbb{R}^2 = H^{-1} \cup \bigcup_{P\in\mathcal{E}_0} P \subset H^{-1} \cup \bigcup_{P\in\mathcal{E}_0}(f_P \cup f_P^{-1}) \subset \bigcup_{f\in\mathcal{G}}(f \cup f^{-1})$. The fact that neither H nor any f_P is injective implies that $\mathcal{G}_1 = \mathcal{G}$. Finally, $h \setminus \bigcup \mathcal{G}_1 = h \setminus \bigcup \mathcal{G}$ has cardinality $\mathfrak{c}$, as it contains $h \restriction \mathfrak{C}$. This is the case, since $H \cup \bigcup_{f\in\mathcal{G}}(f \cup f^{-1}) \supset \bigcup \mathcal{G}$ is disjoint with $h \restriction \mathfrak{C}$. $\square$

7 Proof of sufficiency part of Theorem 3.3

Let perfect $P \subset \mathbb{R}$, $n \in \mathbb{N}$, and $f\colon P \to \mathbb{R}$ be such that

(W_n) $f \in C^n(P)$ and the map $q^{n-i}_{f^{(i)}}\colon P^2 \to \mathbb{R}$ is continuous for every $i \leq n$.

Let H be the convex hull of P. We will construct an extension $\bar{f} \in C^n(H)$ of f. This will finish the proof since, in an event when the interval H is not the entire $\mathbb{R}$, a further extension of $\bar{f}$ to a function in $C^n(\mathbb{R})$ is an easy exercise.

[4]If $\langle x, y\rangle \in g \cap f$, then $x \in P_f$, as otherwise $g(x) = y = f(x) > M$ and $x \in (-2,2)$, contradicting the choice of M. So, $\bar{f}(x) = f(x)$ and $\langle x, y\rangle \in g \cap \bar{f}$.

Let $\{(a_j, b_j) \colon j \in J\}$ be the family of all connected components of $H \setminus P$. Let $\psi \in C^\infty(\mathbb{R})$ be non-decreasing such that $\psi = 1$ on $[2/3, \infty)$ and $\psi = 0$ on $(-\infty, 1/3]$. For every $j \in J$ define the following functions from $\mathbb{R}$ to $\mathbb{R}$:

- the linear map $L_j(x) = \frac{x-a_j}{b_j-a_j}$ (with $L_j(a_j) = 0$ and $L_j(b_j) = 1$);
- $\beta_j = \psi \circ L_j$ and $\alpha_j = 1 - \beta_j$;
- $\bar{f}_j = \alpha_j T^n_{a_j} f + \beta_j T^n_{b_j} f$.

Notice that $\alpha_j + \beta_j \equiv 1$ and that the values of α_j and β_j are in $(0, 1)$ only on the middle third portion of (a_j, b_j).

We define $\bar{f}$ on P as f and on every interval (a_j, b_j), $j \in J$, by a simple formula

$$\bar{f} = \bar{f}_j \restriction (a_j, b_j).$$

Clearly $\bar{f} \colon H \to \mathbb{R}$ extends f and is C^∞ on $H \setminus P$. Also, $\bar{f}$ is C^n on the interior of P. Thus, it remains to show that for every $i \leq n$, $i \in \mathbb{N}$, $\bar{f}^{(i)}$ is well defined and continuous at every $p \in \mathrm{bd}(P)$, the boundary of P in H. Of course, $\bar{f}^{(i)}$ can be well defined on P only when

(P_i) $\bar{f}^{(i)}(p) = f^{(i)}(p)$ for every $p \in \mathrm{bd}(P)$.

We will prove (P_i) by induction on $i \leq n$.

This clearly holds for $i = 0$. So, assume that (P_i) holds for some $i < n$. We need to show (P_{i+1}), that is, that $\lim_{x\to p} \left| \frac{\bar{f}^{(i)}(x)-\bar{f}^{(i)}(p)}{x-p} - f^{(i+1)}(p) \right| = 0$. However, by (P_i), for every $x, p \in P$ we have $\bar{f}^{(i)}(x) = f^{(i)}(x)$ and $\bar{f}^{(i)}(p) = f^{(i)}(p)$. Therefore,

$$\lim_{\substack{x\to p \\ x\in P}} \left| \frac{\bar{f}^{(i)}(x) - \bar{f}^{(i)}(p)}{x - p} - f^{(i+1)}(p) \right| = \lim_{\substack{x\to p \\ x\in P}} \left| \frac{f^{(i)}(x) - f^{(i)}(p)}{x - p} - f^{(i+1)}(p) \right| = 0.$$

So, to prove (P_{i+1}), we need $\lim_{x\to p,\, x\in H\setminus P} \left| \frac{\bar{f}^{(i)}(x)-\bar{f}^{(i)}(p)}{x-p} - f^{(i+1)}(p) \right| = 0$, that is,

$$\lim_{\substack{x\to p \\ x\in H\setminus P}} \left| \frac{\bar{f}^{(i)}(x) - f^{(i)}(p)}{x - p} - f^{(i+1)}(p) \right| = 0 \quad \text{for every } p \in \mathrm{bd}(P). \tag{6}$$

Next notice that

$$
\begin{aligned}
&\left|\frac{\bar{f}^{(i)}(x)-f^{(i)}(p)}{x-p}-f^{(i+1)}(p)\right| \\
&\quad\leq\left|\frac{\bar{f}^{(i)}(x)-(T_p^n f)^{(i)}(x)}{x-p}\right|+\left|\frac{(T_p^n f)^{(i)}(x)-f^{(i)}(p)}{x-p}-f^{(i+1)}(p)\right| \\
&\quad=\left|\frac{\bar{f}^{(i)}(x)-(T_p^n f)^{(i)}(x)}{x-p}\right|+\left|\frac{\sum_{k=0}^{n-i}\frac{f^{(i+k)}(p)}{k!}(x-p)^k-f^{(i)}(p)}{x-p}-f^{(i+1)}(p)\right| \\
&\quad=\left|\frac{\bar{f}^{(i)}(x)-(T_p^n f)^{(i)}(x)}{x-p}\right|+\left|\sum_{k=2}^{n-i}\frac{f^{(i+k)}(p)}{k!}(x-p)^{k-1}\right|
\end{aligned}
$$

and $\lim_{x\to p}\left|\sum_{k=2}^{n-i}\frac{f^{(i+k)}(p)}{k!}(x-p)^{k-1}\right|=0$. Hence, to prove (6) it is enough to show that

$$
\lim_{\substack{x\to p\\ x\in H\setminus P}}\left|\frac{\bar{f}^{(i)}(x)-(T_p^n f)^{(i)}(x)}{(x-p)^{n-i}}\right|=0 \qquad \text{for every } i\leq n \text{ and } p\in \mathrm{bd}(P). \tag{7}
$$

To argue for (7) we need to know that, for every $p,q\in P$ and $x\in\mathbb{R}$,

$$
(T_p^n f)^{(i)}(x)-(T_q^n f)^{(i)}(x)=\sum_{k=0}^{n-i}\frac{q_f^{n-i-k}(q,p)}{k!}(x-q)^k(q-p)^{n-i-k}. \tag{8}
$$

Indeed, if $m=n-i$, then $g(x)=(T_p^n f)^{(i)}(x)-(T_q^n f)^{(i)}(x)$ is a polynomial of degree $\leq m$. Therefore, it is equal to its m-th degree Taylor polynomial:

$$
\begin{aligned}
g(x) &= \sum_{k=0}^{m}\frac{g^{(k)}(q)}{k!}(x-q)^k \\
&= \sum_{k=0}^{m}\frac{(x-q)^k}{k!}\left((T_p^n f)^{(i+k)}(q)-(T_q^n f)^{(i+k)}(q)\right) \\
&= \sum_{k=0}^{m}\frac{(x-q)^k}{k!}\left(T_p^{m-k}\left(f^{(i+k)}\right)(q)-T_q^{m-k}\left(f^{(i+k)}\right)(q)\right) \\
&= \sum_{k=0}^{n-i}\frac{(x-q)^k}{k!}(q-p)^{n-i-k}q_f^{n-i-k}(q,p)
\end{aligned}
$$

giving desired (8).

Coming back to the proof of (7), for $x \in (a_j, b_j)$ let $q_x \in \{a_j, b_j\}$ be the closest among these points to x, and notice that

$$|x - q_x| \leq |x - p| \quad \text{and} \quad |q_x - p| \leq 2|x - p|.$$

Indeed, if q_x is between p and x, then $|x - q_x| \leq |x - p|$ and $|q_x - p| \leq |x - p|$. Otherwise, $|x - q_x| \leq \frac{|b_j - a_j|}{2} \leq |x - p|$ and $|q_x - p| = |x - q_x| + |x - p| \leq 2|x - p|$, as needed. Therefore, using (8),

$$\begin{aligned}
&\left| \frac{\bar{f}^{(i)}(x) - (T_p^n f)^{(i)}(x)}{(x-p)^{n-i}} \right| \\
&\quad = \left| \frac{\bar{f}^{(i)}(x) - (T_{q_x}^n f)^{(i)}(x)}{(x-p)^{n-i}} + \frac{(T_{q_x}^n f)^{(i)}(x) - (T_p^n f)^{(i)}(x)}{(x-p)^{n-i}} \right| \\
&\quad = \left| \frac{\bar{f}^{(i)}(x) - (T_{q_x}^n f)^{(i)}(x)}{(x-p)^{n-i}} + \frac{\sum_{k=0}^{n-i} \frac{q_f^{n-i-k}(q_x,p)}{k!}(x-q_x)^k (q_x - p)^{n-i-k}}{(x-p)^k (x-p)^{n-i-k}} \right| \\
&\quad \leq \left| \frac{\bar{f}^{(i)}(x) - (T_{q_x}^n f)^{(i)}(x)}{(x-p)^{n-i}} \right| + \sum_{k=0}^{n-i} \left| q_f^{n-i-k}(q_x,p) \right| \frac{2^{n-i-k}}{k!}.
\end{aligned}$$

Since, by (W_n), $\lim_{x\to p,\, x\in H\setminus P} \sum_{k=0}^{n-i} \left| q_f^{n-i-k}(q_x,p) \right| \frac{2^{n-i-k}}{k!} = 0$, to prove (7) it is enough to show that $\lim_{x\to p,\, x\in H\setminus P} \left| \frac{\bar{f}^{(i)}(x) - (T_{q_x}^n f)^{(i)}(x)}{(x-p)^{n-i}} \right| = 0$. Moreover, if U is the union of all middle thirds of the intervals (a_j, b_j), then we have $\bar{f}^{(i)}(x) = (T_{q_x}^n f)^{(i)}(x)$ for all $x \in H \setminus P$ not in U. In particular, (6) holds for any p which is not a limit point of U. So, we may assume that p is a limit point of U. To finish the proof of (7) and (6) for such p, it is enough to show that

$$\lim_{\substack{x\to p \\ x\in U}} \left| \frac{\bar{f}^{(i)}(x) - (T_{q_x}^n f)^{(i)}(x)}{(x-p)^{n-i}} \right| = 0. \tag{9}$$

Now, if $x \in (a_j, b_j) \cap U$, then $\bar{f}(x) = \alpha_j(x) T_{a_j}^n f(x) + \beta_j(x) T_{b_j}^n f(x)$ and

$$|x - q| \leq \frac{|b_j - a_j|}{2} = \frac{3}{2}\frac{|b_j - a_j|}{3} \leq \frac{3}{2}|x - p|.$$

Next, assume that $q_x = a_j$, the case when $q_x = b_j$ being similar. Then, by (8),

$$\begin{aligned}
\left|\bar{f}^{(i)}(x) - (T^n_{q_x} f)^{(i)}(x)\right| &= \left|(\bar{f} - T^n_{a_j} f)^{(i)}(x)\right| = \left|\left(\beta_j \cdot (T^n_{b_j} f - T^n_{a_j} f)\right)^{(i)}(x)\right| \\
&= \left|\sum_{k=0}^{i} \binom{i}{k} \beta_j^{(i-k)}(x) \left[(T^n_{b_j} f)^{(k)}(x) - (T^n_{a_j} f)^{(k)}(x)\right]\right| \\
&= \left|\sum_{k=0}^{i} \binom{i}{k} \frac{\psi^{(i-k)}(L_j(x))}{(b_j - a_j)^{i-k}} \left[\sum_{s=0}^{n-k} \frac{q_f^{n-k-s}(a_j, b_j)}{s!} (x - a_j)^s (a_j - b_j)^{n-k-s}\right]\right| \\
&\le \sum_{k=0}^{i} \binom{i}{k} \frac{\left|\psi^{(i-k)}(L_j(x))\right|}{s!} \left[\sum_{s=0}^{n-k} \left|q_f^{n-k-s}(a_j, b_j)\right| |x - a_j|^s |a_j - b_j|^{n-i-s}\right].
\end{aligned}$$

From this, and the fact that

$$3|x - a_j| \le 3|a_j - b_j| \le |x - p|$$

we obtain

$$\left|\frac{\bar{f}^{(i)}(x) - (T^n_{q_x} f)^{(i)}(x)}{(x-p)^{n-i}}\right| \le \sum_{k=0}^{i} \binom{i}{k} \frac{\left|\psi^{(i-k)}(L_j(x))\right|}{s!} \left[\sum_{s=0}^{n-k} \left|q_f^{n-k-s}(a_j, b_j)\right| 3^{n-i}\right].$$

Hence, by the assumption (W_n), the right hand side converges to 0, as $x \in (a_j, b_j) \cap U$ converges to p. This completes the proof of (9), (6), and differentiability of $\bar{f}$.

To finish the proof, it remains to show that $\bar{f}^{(n)}$ is continuous at the points $p \in \mathrm{bd}(P)$. But clearly

$$\lim_{\substack{x \to p \\ x \in P}} \left(\bar{f}^{(n)}(x) - \bar{f}^{(n)}(p)\right) = \lim_{\substack{x \to p \\ x \in P}} \left(f^{(n)}(x) - f^{(n)}(p)\right) = 0$$

while, using $\bar{f}^{(n)}(p) = (T^n_p f)^{(n)}(p)$ and (7),

$$\lim_{\substack{x \to p \\ x \in H \setminus P}} \left|\bar{f}^{(n)}(x) - \bar{f}^{(n)}(p)\right| = \lim_{\substack{x \to p \\ x \in H \setminus P}} \left|\bar{f}^{(n)}(x) - (T^n_p f)^{(n)}(p)\right| = 0$$

giving the desired continuity.

8 Final remarks and open questions

It is worth to notice that, for no $\nu \in \omega \cup \{\infty\}$, a family $\mathcal{F}_\nu$ from the Main Theorem can be countable. Indeed, if $|\mathcal{F}_\nu| < \mathfrak{c}$, then there is a constant (so, C^∞) function g such that $[f = g] \stackrel{\mathrm{df}}{=} \{x \in \mathbb{R}\colon f(x) = g(x)\}$ is nowhere dense in $\mathbb{R}$. In particular, if $|\mathcal{F}_\nu| \le \omega$, then $g \setminus \bigcup \mathcal{F}_\nu$ is co-meager in g, so it has cardinality $\mathfrak{c}$, a contradiction. The same argument shows that under Martin's Axiom the property (i) from Main Theorem implies that $|\mathcal{F}_\nu| = \mathfrak{c}$. In particular, for no $\nu \in \omega \cup \{\infty\}$ the existence $\mathcal{F}_\nu$ as in Main Theorem can be proved in ZFC.

Problem 8.1. *What is the lowest Baire class of the extension in Lemma 5.2?*

Problem 8.2. *What is the lowest Borel rank of the sets B_n in Theorem 5.4?*

Problem 8.3. *Is there a model of ZFC in which there is a family $\mathcal{G}$ as in Proposition 1.1 with $\mathcal{G} \subset \mathcal{C}(\mathbb{R})$ but there is no such family in $\mathcal{C}^1(\mathbb{R})$?*

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