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RESTRICTED FAMILIES OF PROJECTIONS AND RANDOM SUBSPACES

Abstract

We study the restricted families of orthogonal projections in $\mathbb{R}^3$. We show that there are families of random subspaces which admit Marstrand-Mattila type projection theorem.

1 Introduction

A fundamental problem in fractal geometry is to determine how the projections affect dimension. Recall the classical Marstrand-Mattila projection theorem: Let $E \subset \mathbb{R}^n, n \geq 2$, be a Borel set with Hausdorff dimension s.

- (dimension part) If $s \leq m$, then the orthogonal projection of E onto almost all m-dimensional subspaces has Hausdorff dimension s.
- (measure part) If $s > m$, then the orthogonal projection of E onto almost all m-dimensional subspaces has positive m-dimensional Lebesgue measure.

In 1954 J. Marstand [11] proved this projection theorem in the plane. In 1975 P. Mattila [12] proved this for general dimension via 1968 R. Kaufman's [9] potential theoretic methods. We refer to the recent survey of P. Mattila [15], K. Falconer, J. Fraser, and X. Jin [5] for more backgrounds. For monographs which are related to orthogonal projections of fractal sets, we refer to K. Falconer [4], P. Mattila [13], [14].

In this paper, we study the restricted families of projections in Euclidean spaces. Let $G(n, m)$ denote the collection of all the m-dimensional linear

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subspaces of $\mathbb{R}^n$. For $V \in G(n,m)$, let $\pi_V : \mathbb{R}^n \to V$ stand for the orthogonal projection onto V. For $G \subset G(n,m)$, we call $(\pi_V)_{V\in G}$ a restricted family of projections. One of the problems is to look for some "small" subset $G \subset G(n,m)$ such that the Marstrand-Mattila type theorem holds for this restricted family of projections $(\pi_V)_{V\in G}$.

The best possible lower bounds for general restricted families of projections $(\pi_V)_{V\in G}$ (here G is a smooth subset of $G(n,m)$) were obtained by E. Järvenpää, M. Järvenpää, T. Keleti, F. Ledrappier and M. Leikas, see [7] and [8].

What kind of subset $G \subset G(n,m)$ admits a better lower bound or even more such that the Marstrand-Mattila type theorem holds? K. Fässler and T. Orponen [6, Conjecture 1.6] conjectured that if G has "curvature condition" then G admits a Marstrand-Mattila type theorem. The following description of T. Orponen [19] is helpful. "Informally speaking, one could conjecture that any (smooth) subset $G \subset G(n,m)$ such that no "large part" of G contained in a single non-trivial subspace, should satisfy the Marstrand-Mattila projection theorem". A prototypical example of a curve with curvature condition is given by

$$\Gamma = \{\frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, 1), \theta \in [0, 2\pi)\}.$$

Recently, A. Käenmäki, T. Orponen, and L. Venieri [10] proved that the dimensional part of Marstrand-Mattila type theorem holds for the restricted families of projections for the curve Γ, which partially answered a conjecture of [6, Conjecture 1.6] for the curve Γ. We refer to [10] for more details and references therein. For the restricted families of projections $\{\pi_{V_e}\}_{e\in\Gamma}$ where $V_e := e^{\perp}$ the orthogonal complement space of e, we refer to [22] for more details and new improvement. We note that D. Oberlin and R. Oberlin [18] applied the Fourier restricted estimates to these restricted families of projections with the "curvature condition".

We note that the subsets G which were mentioned in the former results are always some smooth subsets of $G(n,m)$. T. Orponen [21] talked about the restricted families of projections over general subsets of $G(n,m)$ and the random subsets of $G(2,1)$. In this paper, inspired by T. Orponen's talk, we study the restricted families of projections over a random subset of $G(n,m)$. We show that there exists non-smooth (fractal) subset of $G(3,1)$ such that the Marstrand-Mattila type theorem holds on this restricted family of projections. Here, the random sets play the same role as the sets with the curvature condition.

We note that the random sets play the same role as the curvature condition in some other situations also, e.g., the restricted Fourier transform, see T.

Mitsis [16] and G. Mockenhaupt [17].

Definition 1 (MMP spaces)**.** *Let $G \subset G(n,m)$ and γ be a nonzero finite Borel measure on G. We call the pair $((\pi_V)_{V\in G}, \gamma)$ a MMP space if the Marstrand-Mattila projection theorem holds for the restricted families of projections $(\pi_V)_{V\in G}$ with respect to the measure γ.*

By "mapping" a class of random Cantor sets of P. Shmerkin and V. Suomala [23] onto the sphere $\mathbb{S}^2$ and combining some classical potential theoretical arguments for orthogonal projections, we obtain the following Theorem 2.

Let $x \neq 0$. Denoted by L_x the line through zero and the point x, and $L_x^\perp$ the orthogonal complement of L_x. For convenience, we may identify a subset of $\mathbb{S}^2$ with subset of $G(3,1)$ which makes no confusion.

Theorem 2. *For any $1 < \alpha \leq 2$ there exists an α- Ahlfors regular set $G \subset \mathbb{S}^2$ such that $((\pi_{L_x})_{x\in G}, \mathcal{H}^\alpha)$ and $((\pi_{L_x^\perp})_{x\in G}, \mathcal{H}^\alpha)$ are MMP spaces.*

Recall that $E \subset \mathbb{R}^n$ is called α-Ahlfors regular for $0 < \alpha \leq n$, if there exists a positive constant C such that

$$r^\alpha/C \leq \mathcal{H}^\alpha(E \cap B(x,r)) \leq Cr^\alpha$$

for all $x \in E$ and $0 < r < \operatorname{diam}(E)$, where $\operatorname{diam}(E)$ denotes the diameter of E. Note that for the case $\alpha = 2$, Theorem 2 follows from the Marstrand-Mattila projection theorem. Thus, we only consider the case $\alpha \in (1,2)$.

I thank Tuomas Orponen for pointing out that the technique in the proof of Theorem 2 and the random sets in papers [1], [20] imply the following result.

Theorem 3. *For any $0 < \alpha \leq 1$ there exists a set $G \subset \mathbb{S}^2$ with $0 < \mathcal{H}^\alpha(G) < \infty$ such that $((\pi_{L_x})_{x\in G}, \mathcal{H}^\alpha)$ admits a (dimension part) Marstrand-Mattila type theorem i.e., for any subset $E \subset \mathbb{R}^3$ with $\dim_H E \leq \alpha$,*

$$\dim_H \pi_{L_x}(E) = \dim_H E \text{ for } \mathcal{H}^\alpha \text{ a.e. } x \in G.$$

We note that there has been a growing interest in studying finite fields version of some classical problems arising from Euclidean spaces. In [2] the author studied the projections in vector spaces over finite fields, and obtained the Marstrand-Mattila type projection theorem in this setting. For finite fields version of restricted families of projection, the author [3] obtained that a random collection of subspaces admit a Marstrand-Mattila type theorem with high probability. For more details see [2] and [3].

2 Preliminaries

In this section we show some known lemmas for later use. The proofs of the following lemmas are based on the potential theoretical arguments. For clearness of our conditions in the following two lemmas, we show the proofs here. Specially, for Lemma 5, we provide an different approach to [14, Chapter 5]. For more details we refer to [4, Chapter 6], [13, Chapter 9], and [14, Chapter 5].

We write $f \lesssim g$ if there is a positive constant C such that $f \leq Cg$, $f \gtrsim g$ if $g \lesssim f$, and $f \approx g$ if $f \lesssim g$ and $f \gtrsim g$.

Lemma 4. *Let $G \subset G(n,m)$ and γ be a positive finite Borel measure on G. If for any unit vector $\xi \in \mathbb{R}^n$,*

$$\gamma(\{V \in G : |\pi_V(\xi)| \leq \rho\}) \lesssim \rho^m,$$

then $((\pi_V)_{V\in G}, \gamma)$ is a MMP space.

Lemma 5. *Let $G \subset G(n,m)$ and γ be a positive finite Borel measure on G. If for any unit vector $\xi \in \mathbb{R}^n$,*

$$\gamma(\{V \in G : d(\xi, V) \leq \rho\}) \lesssim \rho^{n-m},$$

then $((\pi_V)_{V\in G}, \gamma)$ is a MMP space.

The proofs depend on the following energy characterization of Hausdorff dimension. For a Borel set $E \subset \mathbb{R}^n$,

$$\dim_H E = \sup\{s : \mathcal{I}_s(\mu) < \infty, \quad \mu \text{ is a nonzero Radon measure with compact support on } E\ \}$$

where $\mathcal{I}_s(\mu) = \int\int |x-y|^{-s} d\mu x d\mu y$. We also need the following identity which connects the fractal geometry and Fourier analysis,

$$\mathcal{I}_s(\mu) \approx \int_{\mathbb{R}^n} |x|^{s-n} |\widehat{\mu}(x)|^2 dx.$$

Here $\widehat{\mu}(x) = \int e^{-2\pi i\langle x,y\rangle} d\mu(y)$ the Fourier transform of the measure μ at x. For more connections between fractal geometry and Fourier analysis, we refer to [14].

Proof of Lemma 4. Let $E \subset \mathbb{R}^n$ with $\dim_H E \leq m$. Then, for any t with $0 < t < \dim_H E$ there exists a Radon measure μ on E with compact support and $\mathcal{I}_t(\mu) < \infty$. For $V \in G$, let μ_V be the image measure of μ under the map π_μ, i.e.,

$$\mu_V(A) = \mu(\pi_V^{-1}(A)) \text{ for } A \subset V\ .$$

Changing variables in integral and applying Fubini's theorem, we obtain

$$\begin{aligned}\int_G \mathcal{I}_t(\mu_V) d\gamma V &= \int_G \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\pi_V(x) - \pi_V(y)|^{-t} d\mu x d\mu y d\gamma V \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_G |\pi_V(\frac{x-y}{|x-y|})|^{-t} |x-y|^{-t} d\gamma V d\mu x d\mu y \\ &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |x-y|^{-t} d\mu x d\mu y < \infty.\end{aligned}$$

The last second estimate holds, since for any unit vector ξ,

$$\begin{aligned}\int_G |\pi_V(\xi)|^{-t} d\gamma V &= \int_0^\infty \gamma(\{V \in G : |\pi_V(\xi)|^{-t} \geq u\}) du \\ &= t \int_0^\infty \gamma\{V \in G : |\pi_V(\xi)| \leq \rho\} \rho^{-t-1} d\rho \\ &\lesssim \int_0^1 \rho^{m-t-1} d\rho + \gamma(G) \int_1^\infty \rho^{-1-t} d\rho < \infty.\end{aligned}$$

It follows that $\mathcal{I}_t(\mu_V) < \infty$, and hence $\dim_H \pi_V(E) \geq t$ for γ almost all $V \in G$. This is true for any $t < \dim_H E$, therefore $\dim_H \pi_V(E) \geq \dim_H E$ for γ almost all $V \in G$.

Now we turn to the "measure part" of Marstrand-Mattila theorem. Let $\dim_H E > m$, then there exists a measure μ on E such that $\mathcal{I}_m(\mu) < \infty$. Applying Fatou's lemma and Fubini's theorem,

$$\begin{aligned}&\int_G \int_V \liminf_{\rho \to 0} \frac{\mu_V(B(x,\rho))}{\rho^m} d\mu_V x d\gamma V \\ &\quad \leq \liminf_{\rho \to 0} \frac{1}{\rho^m} \int_G \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbf{1}_{\{(u,u') : |\pi_V(u) - \pi_V(u')| \leq \rho\}}(x,y) d\mu x d\mu y d\gamma V \\ &\quad \lesssim \mathcal{I}_m(\mu) < \infty.\end{aligned} \tag{1}$$

We use the fact that for any $x \neq y$,

$$\begin{aligned}&\int_G \mathbf{1}_{\{(u,u') : |\pi_V(u) - \pi_V(u')| \leq \rho\}}(x,y) d\gamma V \\ &= \gamma(\{V \in G : |\pi_V(x) - \pi_V(y)| \leq \rho\}) \lesssim \left(\frac{\rho}{|x-y|}\right)^m.\end{aligned}$$

The estimate (1) implies that for γ almost all $V \in G$,

$$\int_V \liminf_{r\to 0} \frac{\mu_V(B(x,r))}{r^m} d\mu_V x < \infty,$$

and hence for μ_V almost all $x \in V$,

$$\liminf_{r\to 0} \frac{\mu_V(B(x,r))}{r^m} < \infty.$$

Together with [13, Theorem 2.12 (3)], we obtain that μ_V is absolutely continuous to $\mathcal{H}^m$ (and hence $\mathcal{H}^m(\pi_V(E)) > 0$) for γ almost all $V \in G$. $\square$

Proof of Lemma 5. Let $\dim_H E = s \leq m$. Then for any $0 < t < s$ there exists a Radon measure μ on E with compact support and

$$\mathcal{I}_t(\mu) \approx \int_{\mathbb{R}^n} |\widehat{\mu}(x)|^2 |x|^{t-n} dx d\gamma V < \infty. \tag{2}$$

It is sufficient to prove

$$\int_G \int_V |\widehat{\mu_V}(x)|^2 |x|^{t-m} d\mathcal{H}^m x d\gamma V < \infty.$$

Note that for any $V \in G(n,m)$ and $x \in V$,

$$\widehat{\mu_V}(x) = \widehat{\mu}(x).$$

Since the measure μ is finite, there is a positive constant C such that for any $V \in G(n,m)$,

$$\int_{B(0,1)} |\widehat{\mu_V}(x)|^2 |x|^{t-m} d\mathcal{H}^m x \leq C < \infty.$$

Thus, it is sufficient to prove

$$\int_G \int_{V\cap B(0,1)^c} |\widehat{\mu}(x)|^2 |x|^{t-m} d\mathcal{H}^m x d\gamma V < \infty,$$

where $B(0,1)^c$ is the complement set of $B(0,1)$. Let $0 < \rho < 1/10\sqrt{n}$. Define

$$\mathcal{Q}_\rho := \{[k_1\rho, (k_1+1)\rho] \times \cdots \times [k_n\rho, (k_n+1)\rho] : k_j \in \mathbb{Z}, 1 \leq j \leq n\} = \{Q_j\}_{j=1}^\infty.$$

We consider the cubes which intersects $B(0,1)^c$. Thus, we define

$$J = \{j : Q_j \cap B(0,1)^c \neq \emptyset, Q_j \in \mathcal{Q}_\rho\}.$$

Since μ is a Radon measure with compact support, $\widehat{\mu}$ is a bounded Lipschitz continuous function, i.e.,

$$|\widehat{\mu}(x) - \widehat{\mu}(y)| \lesssim |x - y|.$$

It follows that for each $Q_j, j \in J$ and any $x, x' \in Q_j$,

$$|\widehat{\mu}(x)|^2 |x|^{t-m} \lesssim |\widehat{\mu}(x')|^2 |x'|^{t-m} + \rho |x'|^{t-m}. \tag{3}$$

For each $Q_j, j \in J$ let $x_j \in Q_j$ such that

$$|\widehat{\mu}(x_j)| \leq |\widehat{\mu}(x)| \text{ for any } x \in Q_j.$$

For any $R > 1$ let $\rho = \rho_R = R^{-m}$. Then the estimate (3) implies that for any $V \in G(n, m)$,

$$\begin{aligned}
&\int_{V \cap B(0,1)^c \cap B(0,R)} |x|^{t-m} |\widehat{\mu}(x)|^2 d\mathcal{H}^m x \\
&\lesssim \sum_{j \in J} |\widehat{\mu}(x_j)|^2 |x_j|^{t-m} \mathcal{H}^m(V \cap Q_j \cap B(0,R)) + \rho \mathcal{H}^m(V \cap B(0,R)) \\
&\lesssim \sum_{j \in J} |\widehat{\mu}(x_j)|^2 |x_j|^{t-m} \mathcal{H}^m(V \cap Q_j) + 1.
\end{aligned} \tag{4}$$

For any $Q_j, j \in J$, we have

$$\begin{aligned}
&\int_G \mathcal{H}^m(V \cap Q_j) d\gamma V \\
&\lesssim \operatorname{diam}(Q_j)^m \gamma(\{V \in G : d(x_j, V) \leq \operatorname{diam}(Q_j)\}) \\
&\lesssim \operatorname{diam}(Q_j)^m \left(\frac{\operatorname{diam}(Q_j)}{|x_j|}\right)^{n-m} \lesssim \operatorname{diam}(Q_j)^n |x_j|^{m-n}.
\end{aligned} \tag{5}$$

Combining with the estimates (4), (5), and the condition (2), we obtain

$$\begin{aligned}
&\int_G \int_{V \cap B(0,1)^c \cap B(0,R)} |x|^{t-m} |\widehat{\mu}(x)|^2 d\mathcal{H}^m x d\gamma V \\
&\lesssim \int_G \sum_{j \in J} |x_j|^{t-m} |\widehat{\mu}(x_j)|^2 \mathcal{H}^m(Q_j \cap V) d\gamma V + 1 \\
&\lesssim \sum_{j \in J} |x_j|^{t-m} |\widehat{\mu}(x_j)|^2 \int_G \mathcal{H}^m(V \cap Q_j) d\gamma V + 1 \\
&\lesssim \sum_{j \in J} |x_j|^{t-n} |\widehat{\mu}(x_j)|^2 \operatorname{diam}(Q_j)^n + 1 \\
&\lesssim \mathcal{I}_t(\mu) + 1 < \infty.
\end{aligned}$$

Together with Fatou's lemma, we obtain

$$\begin{aligned}
&\int_G \int_{V\cap B(0,1)^c} |x|^{t-m}|\widehat{\mu}(x)|^2 d\mathcal{H}^m x d\gamma V \\
&= \int_G \int_{V\cap B(0,1)^c} \lim_{R\to\infty} \mathbf{1}_{B(0,R)}(x)|x|^{t-m}|\widehat{\mu}(x)|^2 d\mathcal{H}^m x d\gamma V \\
&\le \liminf_{R\to\infty} \int_G \int_{V\cap B(0,1)^c} \mathbf{1}_{B(0,R)}(x)|x|^{t-m}|\widehat{\mu}(x)|^2 d\mathcal{H}^m x d\gamma V \\
&\lesssim \mathcal{I}_t(\mu) + 1 < \infty.
\end{aligned}$$

Thus, we complete the proof of the dimension part of Marstrand-Mattila type theorem.

Now we turn to the measure part of Marstrand-Mattila type theorem. Let $\dim_H E = s > m$, then there exists a Radon measure μ on E with compact support $\mathcal{I}_m(\mu) < \infty$. A variant of the former argument implies that (using the same notation as above)

$$\begin{aligned}
&\int_G \int_{V\cap B(0,1)^c \cap B(0,R)} |\widehat{\mu}(x)|^2 d\mathcal{H}^m x d\gamma V \\
&\lesssim \int_G \sum_{j\in J} |\widehat{\mu}(x_j)|^2 \mathcal{H}^m(Q_j \cap V) d\gamma V + 1 \\
&\lesssim \sum_{j\in J} |\widehat{\mu}(x_j)|^2 \int_G \mathcal{H}^m(V\cap Q_j) d\gamma V + 1 \\
&\lesssim \sum_{j\in J} |\widehat{\mu}(x_j)|^2 |x_j|^{m-n} \mathrm{diam}(Q_j)^n + 1 \\
&\lesssim \mathcal{I}_m(\mu) + 1 < \infty.
\end{aligned}$$

It follows that

$$\int_G \int_{V\cap B(0,1)^c} |\widehat{\mu_V}(x)|^2 d\mathcal{H}^m x d\gamma V < \infty.$$

Recall that if $\int_{\mathbb{R}^n} |\widehat{\mu}(x)|^2 dx < \infty$, then μ is absolutely continuous to $\mathcal{H}^n$, see [14, Theorem 3.3]. Thus, we obtain that μ_V is absolutely continuous to $\mathcal{H}^m$ for γ almost all $V \in G$, and hence $\mathcal{H}^m(\pi_V(E)) > 0$ for γ almost all $V \in G$. $\square$

3 Proofs of Theorems 2-3

P. Shmerkin and V. Suomala [23] constructed the following sets. A tube $T \subset \mathbb{R}^2$ with width δ is a δ neighborhood of some line in $\mathbb{R}^2$.

Theorem 6. *For any $\alpha \in (1,2)$ there exists an α-Ahlfors regular compact set $E \subset \mathbb{R}^2$, such that for any tube T with width $w(T)$,*

$$\mathcal{H}^\alpha(E \cap T) \lesssim w(T). \tag{6}$$

By "mapping" the sets in Theorem 6 to sphere $\mathbb{S}^2$, we obtain the following Lemma 7.

Lemma 7. *For any $\alpha \in (1,2)$ there exists an α-Ahlfors regular compact set $G \subset \mathbb{S}^2$, such that for any unit vector $\xi \in \mathbb{R}^3$ and $\rho > 0$,*

$$\mathcal{H}^\alpha(\{L \in G : |\pi_L(\xi)| \leq \rho\}) \lesssim \rho. \tag{7}$$

It follows that for any unit vector $\xi \in \mathbb{R}^3$ and $\rho > 0$,

$$\mathcal{H}^\alpha(\{L \in G : d(\xi, L^\perp) \leq \rho\}) \lesssim \rho. \tag{8}$$

Proof. By Theorem 6 there exists an α-Ahlfors regular compact set $E \subset [-1/10, 1/10]^2$ such that for any tube T,

$$\mathcal{H}^\alpha(E \cap T) \lesssim w(T).$$

Let $\tilde{E} = E + (0,0,1/2)$ and $G := \{\frac{x}{|x|} : x \in \tilde{E}\}$. We intend to prove that G satisfy our need. Note that G is the image of $\tilde{E}$ under the map $F : x \to \frac{x}{|x|}$ for $x \neq 0$. In the following, we restrict the map F to the set $[-1/10, 1/10]^2 + (0,0,1/2) := S$. Then F is a bi-Lipschitz map, i.e.,

$$|F(x) - F(y)| \approx |x - y|, \ x, y \in S.$$

Furthermore, F^{-1} maps the "big circle" to some "segment" on S, i.e., for any plane $W \in G(3,2)$ there exists a line ℓ_W such that

$$F^{-1}(W \cap \mathbb{S}^2) = \ell_W \cap S.$$

Combining with the bi-Lipschitz of the map F, we conclude that

$$F^{-1}(\{L \in G : |\pi_L(\xi)| \leq \rho\}) \subset \{x \in \tilde{E} : d(x, \ell_{\xi^\perp}) \lesssim \rho\}$$

where $\ell_{\xi^\perp} = F^{-1}(\xi^\perp)$. Therefore,

$$\mathcal{H}^\alpha(\{L \in G : |\pi_L(\xi)| \leq \rho\}) \approx \mathcal{H}^\alpha(F^{-1}(\{L \in G : |\pi_L(\xi)| \leq \rho\})) \lesssim \rho.$$

The estimate (8) follows by $d(\xi, L^\perp) = \pi_L(\xi)$, as required. $\square$

Then Theorem 2 follows by combining Lemma 7 with Lemmas 4-5. More precisely, the estimate (7) and the Lemma 4 imply that $((\pi_{L_x})_{x\in G}, \mathcal{H}^\alpha)$ is a MMP space. The estimate (8) and the lemma 5 imply that $((\pi_{L_x^\perp})_{x\in G}, \mathcal{H}^\alpha)$ is a MMP space.

Now we turn to the proof of Theorem 3. The method is similar to the proof of Theorem 2. We "map" some random sets of the plane to the sphere $\mathbb{S}^2$, and then we apply the classical potential theoretical arguments for these restricted families of projections. First note that the classical potential theoretical arguments imply the following Lemma 8, see the proof of Lemma 4. For more details we refer to [5, Section 3], [14, Theorem 5.1].

Lemma 8. *Let $G \subset G(n,m)$ and γ be a positive finite Borel measure on G. If for any unit vector $\xi \in \mathbb{R}^n$,*

$$\gamma(\{V \in G : |\pi_V(\xi)| \le \rho\}) \lesssim \rho^\alpha,$$

where α is a positive constant, then $((\pi_V)_{V\in G}, \gamma)$ admits a (dimension part) Marstrand-Mattila type theorem i.e., for any subset $E \subset \mathbb{R}^n$ with $\dim_H E \le \min\{\alpha, m\}$, we have

$$\dim_H \pi_V(E) = \dim_H E \text{ for } \gamma \text{ a.e. } V \in G.$$

T. Orponen [20] constructed the following sets.

Theorem 9. *For any $0 < \alpha < 1$ there exists a compact set $E \subset [0,1]^2$ with $0 < \mathcal{H}^\alpha(E) < \infty$, such that such that for any tube T with width $w(T)$,*

$$\mathcal{H}^\alpha(E \cap T) \lesssim w(T)^\alpha.$$

Note that for any subset $E \subset \mathbb{R}^2$ with $0 < \mathcal{H}^1(E) < \infty$,

$$\sup_T \frac{\mathcal{H}^1(E\cap T)}{w(T)} = \infty$$

where the supremum is over all tubes T with width $w(T) > 0$. For more details, see [20]. For the case $\alpha = 1$, the author [1] constructed the following set which settles a question of T. Orponen. There exists a compact set $E \subset [0,1]^2$ with $0 < \mathcal{H}^1(E) < \infty$ such that for any $s < 1$, and for any tube T with width $w(T)$,

$$\mathcal{H}^1(E \cap T) \lesssim_s w(T)^s. \tag{9}$$

Here $\lesssim_s$ means that the constant depends on s.

We map the above sets to the sphere $\mathbb{S}^2$ in the same way as Lemma 7, and the similar arguments imply the following result.

Corollary 10. *For any* $0 < \alpha \leq 1$ *there exists a compact set* $G \subset \mathbb{S}^2$ *with* $0 < \mathcal{H}^\alpha(G) < \infty$*, such that for any* $s < \alpha$*, and for any unit vector* $\xi \in \mathbb{R}^3$*,*

$$\mathcal{H}^\alpha(\{L \in G : |\pi_L(\xi)| \leq \rho\}) \lesssim_s \rho^s.$$

Note that for the case $0 < \alpha < 1$*, we have the following stronger estimate*

$$\mathcal{H}^\alpha(\{L \in G : |\pi_L(\xi)| \leq \rho\}) \lesssim \rho^\alpha.$$

Theorem 3 follows by combining Corollary 10 and Lemma 8.

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