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ON THE GROWTH OF REAL FUNCTIONS AND THEIR DERIVATIVES

Abstract

We show that for any k-times differentiable function $f : [a,\infty) \longrightarrow \mathbb{R}$, any integer $q \geq 0$ and any $\alpha > 1$ the inequality

$$\liminf_{x\to\infty} \frac{x^k \cdot \log x \cdot \log_2 x \cdot \ldots \cdot \log_q x \cdot |f^{(k)}(x)|}{1 + |f(x)|^\alpha} = 0$$

holds and that this result is best possible in the sense that $\log_q x$ cannot be replaced by $(\log_q x)^\beta$ with any $\beta > 1$.

1 Introduction and Statement of Results

Many classical and more recent inequalities deal with relations between a real-valued function and its derivatives, for example the Landau-Hadamard-Kolmogorov inequalities

$$\| f^{(k)} \|_\infty \leq C_{k,n} \| f \|_\infty^{1-k/n} \cdot \| f^{(n)} \|_\infty^{k/n}$$

for n-times differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ (where $k \in \{1, \ldots, n-1\}$) and their numerous variations, see [8, pp. 138-140]. In this paper we prove a different fundamental growth estimate for real-valued functions on unbounded intervals which, to our best knowledge, hasn't been studied so far and which turns out to be best possible. Here, $\log_q x$ denotes the q-times iterated natural logarithm, defined recursively by $\log_0 x := x$ and $\log_q x := \log(\log_{q-1} x)$ for $q \geq 1$.

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Theorem 1. *Let $k \geq 1$ and $q \geq 0$ be integers, $\alpha > 1$, $a \in \mathbb{R}$ and $f : [a, \infty) \longrightarrow \mathbb{R}$ a k-times differentiable function. Then*

$$\liminf_{x \to \infty} \frac{x^k \cdot \log x \cdot \log_2 x \cdot \ldots \cdot \log_q x \cdot f^{(k)}(x)}{1 + |f(x)|^\alpha} \leq 0 \tag{1}$$

and

$$\liminf_{x \to \infty} \frac{x^k \cdot \log x \cdot \log_2 x \cdot \ldots \cdot \log_q x \cdot |f^{(k)}(x)|}{1 + |f(x)|^\alpha} = 0. \tag{2}$$

(Here, of course, for $q = 0$, the product $\log x \cdot \log_2 x \cdot \ldots \cdot \log_q x$ is understood to be the empty product, i.e. 1.)

2 Remarks

(1) This result is best possible in the sense that it is no longer valid if $\log_q x$ is replaced by $(\log_q x)^\beta$ with any $\beta > 1$. This can be seen by considering the function $f : [a, \infty) \longrightarrow \mathbb{R}$ defined by

$$f(x) := (-1)^{k-1} \cdot \int_a^x \int_{x_k}^\infty \cdots \int_{x_2}^\infty \frac{1}{x_1^k \cdot \log x_1 \cdot \ldots \cdot \log_q x_1} \, dx_1 \ldots dx_k,$$

where $a > 0$ is chosen sufficiently large. (For $k = 1$ the iterated integral reduces to the one-dimensional integral from a to x.) Indeed, for $x \geq a$ we have

$$\begin{aligned} & |f(x)| \\ \leq \; & \int_a^x \frac{1}{\log x_k \cdot \ldots \cdot \log_q x_k} \left(\int_{x_k}^\infty \cdots \int_{x_2}^\infty \frac{1}{x_1^k} \, dx_1 \ldots dx_{k-1} \right) dx_k \\ = \; & \frac{1}{(k-1)!} \int_a^x \frac{1}{\log x_k \cdot \ldots \cdot \log_q x_k} \cdot \frac{1}{x_k} \, dx_k \\ = \; & \frac{1}{(k-1)!} \cdot \left(\log_{q+1} x - \log_{q+1} a \right) \end{aligned}$$

and of course

$$f^{(k)}(x) = \frac{1}{x^k \cdot \log x \cdot \ldots \cdot \log_q x},$$

hence for any $\alpha, \beta > 1$

$$\begin{aligned} & \frac{x^k \cdot \log x \cdot \log_2 x \cdot \ldots \cdot (\log_q x)^\beta \cdot f^{(k)}(x)}{1 + |f(x)|^\alpha} \\ \geq \quad & \frac{(\log_q x)^{\beta-1}}{1 + \left(\frac{1}{(k-1)!} \cdot \log_{q+1} x + C\right)^\alpha} \overset{x\to\infty}{\longrightarrow} \infty, \end{aligned}$$

where C is a constant. So (1) does not hold, and neither does (2).

Another, related counterexample is $f(x) := \log_{q+1} x$. However, it is more difficult to verify that it has the desired properties than for the example given above.

(2) The denominator $1+|f(x)|^\alpha$ cannot be replaced by $|f(x)|^\alpha$ (which might appear as a more natural choice at first sight), not even if one skips the term x^k and the logarithmic terms and assumes that $f^{(k)}$ and f don't have common zeros. This is demonstrated by the functions $f(x) := \frac{1}{x^m}$, where $m > \frac{k}{\alpha-1}$; here, the quotient $\frac{f^{(k)}(x)}{|f(x)|^\alpha}$ tends to ∞ if $x \to \infty$.

(3) Of course, the appearance of the terms $\log x \cdot \log_2 x \cdot \ldots \cdot \log_q x$ in Theorem 1 and the fact that $\log_q x$ cannot be replaced by $(\log_q x)^\beta$ with $\beta > 1$ are reminescent of the well-known fact from basic calculus that for any natural number q the infinite series

$$\sum_{k=k_0}^{\infty} \frac{1}{k \log k \cdot \ldots \cdot \log_{q-1} k \cdot (\log_q k)^\beta}$$

(where k_0 is chosen sufficiently large) is convergent for $\beta > 1$ and divergent for $0 < \beta \leq 1$ and that a corresponding result holds for the improper integral

$$\int_{x_0}^{\infty} \frac{1}{x \cdot \log x \cdot \ldots \cdot \log_{q-1} x \cdot (\log_q x)^\beta}\, dx.$$

This resemblance seems to be more than coincidence as Case 3 of the proof of (1) reveals: It makes crucial use of the divergence of $\int_{x_0}^{\infty} (x \cdot \log x \cdot \ldots \cdot \log_q x)^{-1}\, dx$.

(4) For $k = 1$, the geometric idea behind our main result is the following simple one: If (1) should be violated, then f is growing so rapidly that it cannot exist on the whole interval $[a, \infty)$; it tends to ∞ within a finite time.

Accordingly, the case $k=1$ of assertion (1) can be easily deduced from a standard comparison principle for differential inequalities. Indeed, if $k=1$ and f is as in Theorem 1 and if (1) does not hold, then there is an $\varepsilon>0$ and an $a_0\ge 0$ such that

$$x\cdot\log x\cdot\log_2 x\cdot\ldots\cdot\log_q x\cdot f'(x)\ge\varepsilon\cdot(1+|f(x)|^\alpha)\qquad\text{for all } x\ge a_0.$$

In particular, $f'(x)\ge\varepsilon/(x\cdot\log x\cdot\log_2 x\cdot\ldots\cdot\log_q x)$ for all $x\ge a_0$. In view of the divergence of $\int_{a_0}^\infty 1/(x\cdot\log x\cdot\ldots\cdot\log_q x)\,dx$ this implies $\lim_{x\to\infty} f(x)=+\infty$. Therefore we can conclude that there exists an $x_0\ge a_0$ such that for all $x\ge x_0$ we have $f(x)>0$ and

$$f'(x)\ge\frac{\varepsilon}{x\cdot\log x\cdot\log_2 x\cdot\ldots\cdot\log_q x}\cdot f^\alpha(x).$$

However, the solution of the initial value problem

$$y'(x)=\frac{\varepsilon}{x\cdot\log x\cdot\log_2 x\cdot\ldots\cdot\log_q x}\cdot y^\alpha(x),\qquad y(x_0)=f(x_0)$$

does not exist on the whole interval $[x_0,\infty)$; there is some $b<+\infty$ such that $\lim_{x\to b-} y(x)=+\infty$. So by the afore-mentioned comparison principle (see for example [9, Chapter II.8]) we obtain $f(x)\ge y(x)$ for all admissible $x\ge x_0$, a contradiction. – Without using the comparison principle the same can be obtained even more immediately by integrating

$$\frac{f'(x)}{f^\alpha(x)}\ge\frac{\varepsilon}{x\cdot\log x\cdot\log_2 x\cdot\ldots\cdot\log_q x}.$$

However, we don't see a feasible way to extend this reasoning to the case $k\ge 2$.

(5) This paper is related to (and was partially motivated by) our previous work in [3], [2], [1], [4], [6], [7] and [5] where we had studied differential inequalities in the context of complex analysis, more precisely with respect to the question whether they constitute normality (or at least quasi-normality) in the sense of Montel. In [2] it was shown that a family $\mathcal{F}$ of meromorphic functions in some domain D in the complex plane such that

$$\frac{|f^{(k)}|}{1+|f|^\alpha}(z)\ge C\qquad\text{for all } z\in D\text{ and all } f\in\mathcal{F}\tag{3}$$

(where $\alpha>1$, $C>0$ and $k\ge 1$) has to be normal. This doesn't hold any longer if $\alpha>1$ is replaced by $\alpha=1$ as easy examples demonstrate.

However, for $\alpha = 1$ condition (3) implies at least quasi-normality [7, 5]. Furthermore, in [1] we had shown that the condition

$$\frac{|f^{(k)}|}{1+|f^{(j)}|^\alpha}(z) \ge C \qquad \text{for all } z \in D \tag{4}$$

(where $k > j \ge 0$ are integers, $\alpha > 1$ and $C > 0$) implies quasi-normality.

As to *entire* functions, it is almost obvious that they cannot satisfy a differential inequality like (3). Indeed, if f is entire and $|f^{(k)}|(z) \ge C \cdot (1 + |f(z)|^\alpha)$ for all $z \in \mathbb{C}$, then in particular $|f^{(k)}(z)| \ge C$ for all $z \in \mathbb{C}$, so $f^{(k)}$ is constant by Picard's (or Liouville's) theorem. But then f is a non-constant polynomial, and one obtains a contradiction for $z \to \infty$ provided that $\alpha > 0$.

In view of Theorem 1 and the fact that the exponential function grows larger than every polynomial, the following fact certainly doesn't come as a big surprise:

For every continuously differentiable function $g : [a, \infty) \longrightarrow \mathbb{R}$ we have

$$\liminf_{x\to\infty} \frac{g'(x)}{e^{g(x)}} \le 0. \tag{5}$$

Indeed, otherwise there would be an $\varepsilon > 0$ and an $x_0 \ge a$ such that $g'(x) \ge \varepsilon \cdot e^{g(x)}$ for all $x \ge x_0$. In particular, g' is positive on $[x_0, \infty)$, so g is increasing there, hence $g'(x) \ge \varepsilon \cdot e^{g(x_0)}$ for all $x \ge x_0$, which implies $\lim_{x\to\infty} g(x) = \infty$. This enables us to conclude that $\frac{e^{g(x)}}{g^2(x)} \to \infty$ for $x \to \infty$. Combining this with the fact that $\liminf_{x\to\infty} \frac{g'(x)}{1+|g(x)|^2} \le 0$ by Theorem 1 gives the assertion.

However, it might be a bit surprising that this no longer holds if g' is replaced by higher derivatives of g, i.e. for $k \ge 2$ in general the estimate $\liminf_{x\to\infty} \frac{g^{(k)}(x)}{e^{g(x)}} \le 0$ does not hold. This is demonstrated by the function $g(x) := -x^{k-3/2}$ which satisfies

$$\frac{g^{(k)}(x)}{e^{g(x)}} = C \cdot \frac{x^{-3/2}}{\exp(-x^{k-3/2})} \longrightarrow \infty \qquad \text{for } x \to \infty$$

with some $C > 0$.

On the other hand, for every k times continuously differentiable function $g : [a, \infty) \longrightarrow \mathbb{R}$ ($k \ge 1$) we have

$$\liminf_{x\to\infty} \frac{g^{(k)}(x)}{1+e^{g(x)}} \le 0 \qquad \text{and} \qquad \liminf_{x\to\infty} \frac{g^{(k)}(x)}{e^{|g(x)|}} \le 0.$$

Both inequalities are proved by a similar reasoning as in the proof of (5), applying Theorem 1 with (for example) $\alpha = 2$ and keeping in mind that $g^{(k)}(x) \geq \varepsilon$ for all $x \geq x_0$ would imply $g(x) \longrightarrow \infty$ for $x \to \infty$ resp. that $x \mapsto \frac{e^{|g(x)|}}{1+|g(x)|^2}$ is bounded away from zero.

3 Proof of Theorem 1

Our main efforts are required to prove (1). Then (2) will be an easy consequence from (1).

We want to prove (1) by induction w.r.t. q. However, the start of our induction is to consider $\frac{f^{(k)}(x)}{1+|f(x)|^\alpha}$ rather than $\frac{x^k \cdot f^{(k)}(x)}{1+|f(x)|^\alpha}$ (which would be the case $q = 0$). So we have to introduce a unifying notation first. For given $k \geq 1$, we set

$$P_{-1}(x) := 1, \quad P_0(x) := x^k \qquad \text{and} \qquad P_q(x) := x^k \cdot \prod_{j=1}^{q} \log_j x \quad \text{for } q \geq 1.$$

Then our assertion (1) has the form

$$\liminf_{x\to\infty} \frac{P_q(x) \cdot f^{(k)}(x)}{1 + |f(x)|^\alpha} \leq 0. \tag{6}$$

First we consider the case $q = -1$ in (6). Let's assume the assertion is wrong. Then there is an $\varepsilon > 0$ and an $a_0 \geq 0$ such that

$$f^{(k)}(x) \geq \varepsilon \cdot (1 + |f(x)|^\alpha) \qquad \text{for all } x \geq a_0. \tag{7}$$

From $f^{(k)}(x) \geq \varepsilon$ for all $x \geq a_0$ one easily sees that there is some $a_1 \geq a_0$ such that

$$f^{(k)}(x) > 0,\ f^{(k-1)}(x) > 0, \dots, f'(x) > 0,\ f(x) > 0 \quad \text{for all } x \geq a_1.$$

In particular, f is strictly increasing (i.e. one-to-one) on $[a_1, \infty)$ and $\lim_{x\to\infty} f(x) = \infty$. We choose a natural number n such that $(\alpha-1)\cdot n > k-1$. Then there is a natural number j_0 such that $f([a_1, \infty))$ contains the interval $[j_0^n, \infty)$. For $j \geq j_0$ we set

$$r_j := f^{-1}(j^n).$$

Then $(r_j)_j$ is strictly increasing and unbounded, and by the mean value theorem, applied to $\varphi(t) := t^n$, we have

$$f(r_{j+1}) - f(r_j) = (j+1)^n - j^n \leq n \cdot (j+1)^{n-1} \qquad \text{for all } j \geq j_0. \tag{8}$$

On the other hand, for $j \geq j_0$ we deduce from the fundamental theorem of calculus

$$\begin{aligned}
f(r_{j+1}) - f(r_j) &= \int_{r_j}^{r_{j+1}} f'(x_1)\, dx_1 \\
&= \int_{r_j}^{r_{j+1}} \left(f'(r_j) + \int_{r_j}^{x_1} f''(x_2)\, dx_2 \right) dx_1 \\
&\geq \int_{r_j}^{r_{j+1}} \int_{r_j}^{x_1} f''(x_2)\, dx_2\, dx_1 \\
&\geq \dots \\
&\geq \int_{r_j}^{r_{j+1}} \int_{r_j}^{x_1} \dots \int_{r_j}^{x_{k-2}} f^{(k-1)}(x_{k-1})\, dx_{k-1} \dots dx_2 dx_1;
\end{aligned}$$

here again in the case $k = 1$ the iterated integrals are understood to reduce to a one-dimensional integral. From (7) we obtain

$$f^{(k-1)}(x) \geq f^{(k-1)}(r_j) + \varepsilon \cdot \int_{r_j}^{x} (1 + f^{\alpha}(t))\, dt$$

for all $x \geq r_j$ and $j \geq j_0$. (Observe that this cannot be deduced from the fundamental theorem of calculus since $f^{(k)}$ might be not integrable. However it follows by an easy monotonicity argument.) Therefore we arrive at

$$\begin{aligned}
f(r_{j+1}) - f(r_j) &\geq \varepsilon \cdot \int_{r_j}^{r_{j+1}} \int_{r_j}^{x_1} \dots \int_{r_j}^{x_{k-1}} (1 + f^{\alpha}(x_k))\, dx_k \dots dx_2 dx_1 \\
&\geq \varepsilon \cdot \int_{r_j}^{r_{j+1}} \int_{r_j}^{x_1} \dots \int_{r_j}^{x_{k-1}} f^{\alpha}(r_j)\, dx_k \dots dx_2 dx_1 \\
&= \varepsilon \cdot j^{\alpha n} \cdot \frac{1}{k!} \cdot (r_{j+1} - r_j)^k.
\end{aligned}$$

Combining this estimate with (8) yields

$$n \cdot (j+1)^{n-1} \geq \frac{\varepsilon}{k!} \cdot j^{\alpha n} \cdot (r_{j+1} - r_j)^k,$$

hence

$$r_{j+1} - r_j \leq \left(\frac{n \cdot k!}{\varepsilon} \cdot \frac{(j+1)^{n-1}}{j^{\alpha n}} \right)^{1/k} \leq \left(\frac{n \cdot k! \cdot 2^{n-1}}{\varepsilon} \right)^{1/k} \cdot \frac{1}{j^{((\alpha-1)\cdot n+1)/k}}.$$

Here, by our choice of n, $((\alpha-1)\cdot n+1)/k > 1$ which ensures that the series $\sum_{j=j_0}^{\infty} 1/j^{((\alpha-1)\cdot n+1)/k}$ converges. Hence also the telescope series $\sum_{j=j_0}^{\infty}(r_{j+1}-r_j) = \lim_{j\to\infty} r_j - r_{j_0}$ converges, contradicting $\lim_{j\to\infty} r_j = \infty$. This proves (1) for $q=-1$.

Now let some $q \geq 0$ be given and assume that (1) is true for $q-1$ instead of q and for all k-times differentiable functions $f : [a,\infty) \longrightarrow \mathbb{R}$. We assume there is a k-times differentiable function $f : [0,\infty) \longrightarrow \mathbb{R}$ and an $\varepsilon > 0$ such that

$$P_q(x) \cdot f^{(k)}(x) \geq \varepsilon \cdot (1 + |f(x)|^\alpha) \tag{9}$$

holds for all x large enough. Then in particular $f^{(k)}(x) > 0$ for all large enough x, so $f^{(k-1)}$ is increasing, and we easily see by induction that $f^{(k-1)}, f^{(k-2)}, \ldots, f', f$ are strictly monotonic on an appropriate interval $[a_2,\infty)$ where a_2 is large enough. So the limits

$$L_j := \lim_{x\to\infty} f^{(j)}(x) \qquad (j = 0,\ldots,k-1)$$

exist. (They might be $+\infty$ or $-\infty$.)

In the following we will apply the induction hypothesis to the function

$$g(t) := f(e^t)$$

and will use that

$$g^{(k)}(t) = f^{(k)}(e^t)\cdot e^{kt} + \sum_{j=1}^{k-1} c_j f^{(j)}(e^t)\cdot e^{jt} \tag{10}$$

for certain constants $c_j \geq 0$. (This is easily seen by induction.)

By the mean value theorem, for all $n \in \mathbb{N}$ there is a $\zeta_n \in [n, 2n]$ such that

$$n \cdot |f^{(k)}(\zeta_n)| = |f^{(k-1)}(2n) - f^{(k-1)}(n)|. \tag{11}$$

Here of course we have $\lim_{n\to\infty} \zeta_n = \infty$.

Now we consider several cases.

Case 1: $L_{k-1} \neq 0$.

Since $f^{(k-1)}$ is increasing, we either have $L_{k-1} \in \mathbb{R}$ or $L_{k-1} = +\infty$.

Case 1.1: $L_{k-1} \in \mathbb{R}$, w.l.o.g. $L_{k-1} > 0$.

Then we have

$$\frac{1}{2}\cdot L_{k-1} \leq f^{(k-1)}(x) \leq 2L_{k-1} \qquad \text{for large enough } x,$$

hence

$$\frac{1}{3(k-1)!}\cdot L_{k-1}\cdot x^{k-1} \leq f(x) \leq \frac{3}{(k-1)!} L_{k-1}\cdot x^{k-1} \qquad \text{for large enough } x.$$

Using the lower estimate, we conclude that for large enough x

$$0 \leq P_q(x) \cdot \frac{1}{x} \cdot \frac{1}{1+|f(x)|^\alpha} \leq \frac{x^{(k-1)(1+\alpha)/2}}{1+|f(x)|^\alpha} \longrightarrow 0 \quad (x \to \infty). \tag{12}$$

(Here it is crucial that $1 < \frac{1}{2} \cdot (1+\alpha) < \alpha$.) Furthermore,

$$0 \leq \zeta_n \cdot |f^{(k)}(\zeta_n)| \leq 2n \cdot |f^{(k)}(\zeta_n)| = 2 \cdot |f^{(k-1)}(2n) - f^{(k-1)}(n)| \overset{n\to\infty}{\longrightarrow} 0 \tag{13}$$

since L_{k-1} is finite. Multiplying (12) and (13) gives

$$0 \leq P_q(\zeta_n) \cdot \frac{|f^{(k)}(\zeta_n)|}{1+|f(\zeta_n)|^\alpha} \longrightarrow 0 \quad (n \to \infty).$$

This is a contradiction to (9).

Case 1.2: $L_{k-1} = +\infty$.

Then for large enough x we have $f^{(k-1)}(x) \geq 1, f^{(k-2)}(x) \geq 1, \ldots, f'(x) \geq 1, f(x) \geq 1$ (and $L_{k-2} = \cdots = L_1 = L_0 = +\infty$). By applying the induction hypothesis to g, using (10) and substituting $t = \log x$ we obtain for $q \geq 1$

$$\begin{aligned}
0 &\geq \liminf_{t\to+\infty} P_{q-1}(t) \cdot \frac{|g^{(k)}(t)|}{1+|g(t)|^\alpha} \\
&= \liminf_{t\to+\infty} \prod_{j=1}^{q-1} \log_j t \cdot t^k \cdot \frac{f^{(k)}(e^t)\cdot e^{kt} + \sum_{j=1}^{k-1} c_j f^{(j)}(e^t)\cdot e^{jt}}{1+|f(e^t)|^\alpha} \\
&= \liminf_{x\to+\infty} \prod_{j=1}^{q-1} \log_{j+1} x \cdot (\log x)^k \cdot \frac{f^{(k)}(x)\cdot x^k + \sum_{j=1}^{k-1} c_j f^{(j)}(x)\cdot x^j}{1+|f(x)|^\alpha} \\
&\geq \liminf_{x\to+\infty} \prod_{j=2}^{q} \log_j x \cdot \log x \cdot \frac{f^{(k)}(x)\cdot x^k}{1+|f(x)|^\alpha} \\
&= \liminf_{x\to+\infty} \frac{P_q(x)\cdot f^{(k)}(x)}{1+|f(x)|^\alpha},
\end{aligned}$$

as desired. This remains valid for $q = 0$ if we replace $\prod_{j=1}^{q-1} \log_j t \cdot t^k$ by 1 in the second line of this calculation and make similar modifications in the following lines.

Case 2: $L_{k-1} = \cdots = L_{m+1} = 0$, but $L_m \neq 0$ for some integer $m \geq 0$, $m \leq k-2$. (In particular, this case can occur only for $k \geq 2$.)

Then for $j = k-1, k-2, \ldots, m+1$ and all large enough x by the mean value theorem we find a $\zeta_x \in [x, 2x]$ such that

$$x \cdot |f^{(j)}(2x)| \leq x \cdot |f^{(j)}(\zeta_x)| = |f^{(j-1)}(2x) - f^{(j-1)}(x)| \leq |f^{(j-1)}(x)|; \tag{14}$$

here we have used that $|f^{(j)}|$ is decreasing (since $f^{(j)}$ is monotonic and $L_j = 0$) and that $f^{(j-1)}(2x)$ and $f^{(j-1)}(x)$ have the same sign forlarge enough x.

By induction we obtain for all x large enough

$$\begin{aligned} x^{k-1} \cdot |f^{(k-1)}(2^{k-1-m}x)| &\leq \frac{1}{2^{(k-1-m)(k-2-m)/2}} \cdot x^m \cdot |f^{(m)}(x)| \\ &\leq x^m \cdot |f^{(m)}(x)| \end{aligned} \tag{15}$$

Case 2.1: $L_m \neq \pm\infty$, i.e. $L_m \in \mathbb{R}$.

Then for all x large enough we have

$$|f(x)| \geq \frac{x^m}{2 \cdot m!} \cdot L_m,$$

hence

$$0 \leq \prod_{j=1}^{q} \log_j x \cdot \frac{x^m}{1 + |f(x)|^\alpha} \leq \prod_{j=1}^{q} \log_j x \cdot \frac{x^m}{1 + \left(\frac{x^m}{2m!} \cdot L_m\right)^\alpha} \overset{x\to\infty}{\longrightarrow} 0. \tag{16}$$

From (11) and (15) we conclude that for all n large enough

$$\begin{aligned} n^k \cdot |f^{(k)}(\zeta_n)| &= n^{k-1}|f^{(k-1)}(2n) - f^{(k-1)}(n)| \\ &\leq n^{k-1}|f^{(k-1)}(n)| \\ &= 2^{(k-1-m)(k-1)} \cdot \left(\frac{n}{2^{k-1-m}}\right)^{k-1} |f^{(k-1)}(n)| \\ &\leq 2^{(k-1-m)(k-1)} \cdot \left(\frac{n}{2^{k-1-m}}\right)^{m} |f^{(m)}\left(\frac{n}{2^{k-1-m}}\right)|. \end{aligned}$$

If we combine this estimate with (16) and observe that $f^{(m)}$ is bounded (since $L_m \in \mathbb{R}$), we obtain (with $C_m := 2^{(k-1-m)^2+k}$)

$$\begin{aligned} 0 &\leq \prod_{j=1}^{q} \log_j \zeta_n \cdot \frac{\zeta_n^k \cdot |f^{(k)}(\zeta_n)|}{1 + |f(\zeta_n)|^\alpha} \\ &\leq \prod_{j=1}^{q} \log_j \zeta_n \cdot 2^k \cdot \frac{n^k \cdot |f^{(k)}(\zeta_n)|}{1 + |f(\zeta_n)|^\alpha} \\ &\leq C_m \cdot \prod_{j=1}^{q} \log_j \zeta_n \cdot \frac{n^m}{1 + |f(\zeta_n)|^\alpha} \cdot |f^{(m)}\left(\frac{n}{2^{k-1-m}}\right)| \\ &\leq C_m \cdot \prod_{j=1}^{q} \log_j \zeta_n \cdot \frac{\zeta_n^m}{1 + |f(\zeta_n)|^\alpha} \cdot |f^{(m)}\left(\frac{n}{2^{k-1-m}}\right)| \longrightarrow 0 \quad (n \to \infty) \end{aligned}$$

for all n large enough. This settles Case 2.1.

Case 2.2: $L_m = \pm\infty$, w.l.o.g. $L_m = +\infty$.

Then for all x large enough we have

$$f^{(m)}(x) \geq m! + 1,\ f^{(m-1)}(x) \geq m! \cdot x + 1, \ldots, f'(x) \geq m \cdot x^{m-1} + 1$$

and finally

$$f(x) \geq x^m, \tag{17}$$

hence

$$\prod_{j=1}^{q} \log_j x \cdot \frac{x^m}{1 + |f(x)|^\alpha} \longrightarrow 0 \quad (x \to \infty).$$

For $j = 1, \ldots, m$ and all x large enough there are numbers $\zeta_x \in [x, 2x]$ such that

$$f^{(j-1)}(2x) = f^{(j-1)}(x) + x \cdot f^{(j)}(\zeta_x) \geq 0 + x \cdot f^{(j)}(x),$$

and by induction we conclude that

$$f(2^m x) \geq 2^{m(m-1)/2} x^m \cdot f^{(m)}(x) \geq x^m \cdot f^{(m)}(x), \tag{18}$$

provided that x is large enough. On the other hand, $f^{(m+1)}$ is positive and decreases to 0, so for a suitably chosen $a_3 \geq 0$ and all $x \geq 2a_3$ we obtain

$$\begin{aligned} f^{(m)}(2^m x) &\leq f^{(m)}(a_3 + 2^m x) = f^{(m)}(a_3) + \int_{a_3}^{a_3 + 2^{m+1} \cdot \frac{x}{2}} f^{(m+1)}(t)\, dt \\ &\leq f^{(m)}(a_3) + 2^{m+1} \cdot \int_{a_3}^{a_3 + \frac{x}{2}} f^{(m+1)}(t)\, dt \\ &= 2^{m+1} \cdot f^{(m)}\left(a_3 + \frac{x}{2}\right) - (2^{m+1} - 1) \cdot f^{(m)}(a_3) \\ &\leq 2^{m+1} \cdot f^{(m)}(x) + 0. \end{aligned}$$

From this estimate and (18) we conclude that for all x large enough

$$2^{m+1} \cdot f(2^m x) \geq x^m \cdot f^{(m)}(2^m x),$$

hence (by replacing $2^m x$ with x)

$$2^{m^2+m+1} \cdot f(x) \geq x^m \cdot f^{(m)}(x). \tag{19}$$

If we combine this estimate with (11), (15) and (17), as in Case 2.1 we obtain

$$\begin{aligned} 0 &\le P_q(\zeta_n)\cdot\frac{|f^{(k)}(\zeta_n)|}{1+|f(\zeta_n)|^\alpha} \\ &\le C_m\cdot\prod_{j=1}^q \log_j\zeta_n\cdot\frac{n^m\cdot\left|f^{(m)}\left(\frac{n}{2^{k-1-m}}\right)\right|}{1+|f(\zeta_n)|^\alpha} \\ &\overset{(19)}{\le} C'_m\cdot\prod_{j=1}^q \log_j\zeta_n\cdot\frac{\left|f\left(\frac{n}{2^{k-1-m}}\right)\right|}{1+|f(\zeta_n)|^\alpha} \\ &\le C'_m\cdot\prod_{j=1}^q \log_j\zeta_n\cdot|f(\zeta_n)|^{1-\alpha} \\ &\overset{(17)}{\le} C'_m\cdot\prod_{j=1}^q \log_j\zeta_n\cdot\zeta_n^{m(1-\alpha)}\longrightarrow 0 \qquad (n\to\infty), \end{aligned}$$

where C'_m is an appropriate constant. This settles this case as well.
Case 3: $L_{k-1}=\cdots=L_0=0$

In this case, (15) holds as well (with $m=1$), i.e.

$$|f'(x)|\ge x^{k-2}\cdot|f^{(k-1)}\left(2^{k-2}x\right)|$$

for all x large enough. Now we use

$$|f^{(k)}(x)|\ge\frac{\varepsilon}{x^k\prod_{j=1}^q\log_j x}$$

(which is valid for all large enough x) and once more the mean value theorem to deduce for all large enough x

$$\begin{aligned} |f'(x)| &\ge x^{k-2}\cdot|f^{(k-1)}(2^{k-2}x)-f^{(k-1)}(2^{k-1}x)| \\ &= 2^{k-2}\cdot x^{k-1}\cdot|f^{(k)}(\zeta_x)| \qquad (\text{where } 2^{k-2}x\le\zeta_x\le 2^{k-1}x) \\ &\ge \frac{2^{k-2}\cdot x^{k-1}\cdot\varepsilon}{\zeta_x^k\cdot\prod_{j=1}^q\log_j\zeta_x} \\ &\ge \frac{2^{k-2}\cdot x^{k-1}\cdot\varepsilon}{(2^{k-1}x)^k\cdot\prod_{j=1}^q\log_j(2^{k-1}x)} \\ &\ge c\cdot\frac{1}{x\cdot\prod_{j=1}^q\log_j x} \end{aligned}$$

with a suitable constant $c>0$, hence by integration

$$|f(x)|\ge c\cdot\log_{q+1}x+d\to\infty \qquad (x\to\infty)$$

for some $d > 0$, since $f'(x)$ doesn't change its sign for x large enough. This contradicts $L_0 = 0$, i.e. this case cannot occur.

This completes the proof of (1).

In fact, Case 3 is the only part of the proof where it is crucial that in the assertion only the factor $\log_q x$ and not $(\log_q x)^\beta$ with $\beta > 1$ occurs. It would fail for $\beta > 1$ since the improper integral

$$\int_{x_0}^{\infty} 1/(x \log x \cdot \ldots \cdot \log_{q-1} x \cdot (\log_q x)^\beta)\, dx \quad \text{(with } x_0 \text{ large enough)}$$

converges.

Now (2) is an easy consequence from (1) and from Darboux' intermediate value theorem for derivatives. Indeed, if there exists an x_0 such that $f^{(k)}(x) \geq 0$ for all $x \geq x_0$ or $f^{(k)}(x) \leq 0$ for all $x \geq x_0$, (2) follows immediately from (1), applied to either f or $-f$. Otherwise, by Darboux's theorem there is a sequence $\{x_n\}_n$ tending to ∞ such that $f^{(k)}(x_n) = 0$ for all n, and (2) holds as well.

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